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Conditions for Singular Incidence Matrices

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Abstract. Suppose one looks for a square integral matrix N, for which $NN^\top$ has a prescribed form. Then the Hasse-Minkowski invariants and the determinant of $NN^\top$ lead to necessary conditions for existence. The Bruck-Ryser-Chowla theorem gives a famous example of such conditions in case N is the incidence matrix of a square block design. This approach fails when N is singular. In this paper it is shown that in some cases conditions can still be obtained if the kernels of N and $N^\top$ are known, or known to be rationally equivalent. This leads for example to non-existence conditions for self-dual generalised polygons, semi-regular square divisible designs and distance-regular graphs.

Keywords: incidence matrix, Bruck-Ryser-Chowla theorem, generalised polygon, divisible design, distance-regular graph

1. Introduction

Consider a square 2-(v, k, λ) design with incidence matrix N. (We prefer the name 'square' to 'symmetric', since N is not necessarily symmetric.) Then $NN^\top = \lambda J_v + (k - \lambda)I_v$, where J_v is the $v \times v$ all-ones matrix and I_v is the identity matrix of size v. The Bruck-Ryser-Chowla theorem is based on two observations (see for example [7] p. 223). The first one is that $\det N = \det N^\top$ is an integer. Therefore $\det(\lambda J_v + (k-\lambda)I_v)$ is an integral square, hence $k - \lambda$ is a square if v is even. The other observation is that, since N is a non-singular rational matrix, $\lambda J_v + (k - \lambda)I_v$ is rationally congruent to I_v, and therefore these two matrices have the same Hasse-Minkowski invariants. These invariants can be expressed in terms of v, k and λ from which it follows that for odd v the Diophantine equation $(k-\lambda)X^2 + (-1)^{(v-1)/2}\lambda Y^2 = Z^2$ has an integral solution different from $X = Y = Z = 0$. Similar approaches work for other square incidence structures for which the determinant or the Hasse-Minkowski invariants of $NN^\top$ are known. See for example [7], Chapter 12. It is clear that this approach gives no conditions if N is singular. In the present paper we modify the mentioned approach such that we still find conditions for singular N. The key lemma is a simple trick that changes a singular N into a non-singular matrix M in such a way that for some types of designs it is still possible to compute the Hasse-Minkowski invariants or the (square free part of the) determinant of $MM^\top$.

Lemma 1 *Suppose N is a real $v \times v$ matrix of rank $v - m$. Let Z be a real $v \times v$ matrix of rank m, such that $N^\top Z = NZ^\top = O$. Define $M = N + Z$, then*
(i) $MM^\top = NN^\top + ZZ^\top$,

(ii) *the eigenvalues of $MM^\top$ are the positive eigenvalues of $NN^\top$ together with the positive eigenvalues of $ZZ^\top$*,

(iii) *$MM^\top$ is non-singular.*

Proof: Part (i) is staightforward. To prove (ii), first notice that $NN^\top$ and $ZZ^\top$ commute, so they have a common orthogonal basis of eigenvectors. Suppose $\mathbf{v}$ is such an eigenvector that corresponds to a positive eigenvalue of $NN^\top$. Then $\mathbf{v}$ is orthogonal to the kernel of $NN^\top$, which is the span of the columns of Z. Hence $Z^\top\mathbf{v} = \mathbf{0}$, so the corresponding eigenvalue of $ZZ^\top$ equals 0. Similarly, a positive eigenvalue of $ZZ^\top$ corresponds to an eigenvalue 0 of $NN^\top$. This proves (ii), since $NN^\top$ has $v - m$ positive eigenvalues, and $ZZ^\top$ has m positive eigenvalues. Statement iii follows because $MM^\top$ has only positive eigenvalues. □

For a given N, a matrix Z with the required properties always exists. One way to make such a Z is the following. Take rational $v \times m$ matrices L and R, whose columns form a basis for the left and the right kernel of N, respectively. Then $\operatorname{rank} L = \operatorname{rank} R = m$ and $N^\top L = NR = O$. Therefore $Z = LR^\top$ has the desired properties.

In the coming sections we will consider two kinds of square designs for which something new can be said: Self-dual designs and semi-regular square divisible designs.

2. Self-dual designs

Consider two m-dimensional subspaces V and W of the vectorspace $\mathbb{Q}^v$. Let L and R be rational $v \times m$ matrices whose columns span V and W, respectively. We call the subspaces V and W *rationally equivalent* if $L^\top L$ and $R^\top R$ are rationally congruent matrices, which means that $S^\top L^\top LS = R^\top R$ for some non-singular rational matrix S. Note that rational equivalence of vectorspaces does not depend on the choice of L and R.

Lemma 2 *Let N be a rational $v \times v$ matrix. If the left kernel and the right kernel of N are rationally equivalent then the product of the non-zero eigenvalues of $NN^\top$ is a rational square.*

Proof: Let L and R be rational $v \times m$ matrices whose columns form a basis for the left and the right kernel of N, respectively. Put $Z = LR^\top$. Then $ZZ^\top = LR^\top RL^\top = LS^\top L^\top LSL^\top$ (with S as above). The non-zero eigenvalues of $L(S^\top L^\top LSL^\top)$ coincide with the non-zero eigenvalues of $(S^\top L^\top LSL^\top)L$. But $\det(S^\top L^\top LSL^\top L) = (\det S)^2(\det L^\top L)^2$ which is a non-zero rational square. Thus we have that the product of the non-zero eigenvalues of $ZZ^\top$ is a square, and Lemma 1 finishes the proof. □

If N is the incidence matrix of a self-dual design (that is, N and $N^\top$ are isomorphic), then the left and right kernel of N are obviously rationally equivalent and Lemma 2 gives:

Theorem 1 *If N is the incidence matrix of a self-dual design, then the product of the positive eigenvalues of $NN^\top$ is an integral square.*

For example if N is the incidence matrix of a self-dual partial geometry with parameters s $(= t)$ and α (see [5]), the non-zero eigenvalues of $NN^\top$ are $(s + 1)^2$ of multiplicity 1, and $2s + 1 - \alpha$ of multiplicity $s^2(s + 1)^2/\alpha(2s + 1 - \alpha)$. So if the latter multiplicity is odd, $2s + 1 - \alpha$ is a square. In particular if $\alpha = 1$, the partial geometry is a generalised quadrangle of order s (denoted by $GQ(s)$) and we find:

Corollary 1 *There exists no self-dual $GQ(s)$ if $s \equiv 2 \pmod 4$ and $2s$ is not a square.*

For example no $GQ(6)$ is self-dual. Similarly, if N is the incidence matrix of a generalised hexagon of order s (denoted by $GH(s)$), the non-zero eigenvalues of $NN^\top$ are $(s + 1)^2$, s and $3s$ of multiplicity 1, $s(1 + s)^2(1 - s + s^2)/2$ and $s(1 + s)^2(1 + s + s^2)/6$, respectively (see for example [3] p. 203). Thus we find:

Corollary 2 *There exists no self-dual $GH(s)$ if $s \equiv 2 \pmod 4$.*

Stronger condition are known if the incidence matrix of a $GQ(s)$ or $GH(s)$ is symmetric (see [9] p. 309). A symmetric incidence matrix clearly implies that the structure is self-dual, but the converse is not true in general (see [2] for an easy counterexample). Weaker conditions for the existence of self-dual generalised quadrangles were already found by Payne and Thas [6].

3. Square divisible designs

Another case when Lemma 1 can be applied is when the left and right kernel of N are determined by the design requirements. Note that the left kernel of N is the kernel of $NN^\top$, and similarly, the right kernel of N is the kernel of $N^\top N$. So the lemma applies for square incidence matrices N for which $NN^\top$ and $N^\top N$ are prescribed. For example, consider a 2-(v, k, λ) design with a $v \times b$ incidence matrix where $b > v$. Extend the $v \times b$ incidence matrix with $b - v$ zero rows. For the $b \times b$ matrix N thus obtained $NN^\top$ is known, and so is its left kernel. The right kernel of N is in general not known, but there are some types of designs for which $N^\top N$ is prescribed. These include strongly resolvable designs and triangular designs. For these designs Bruck-Ryser-Chowla type conditions have been worked out; see [4, 7, 8], so we will not do it again.

In this section we consider semi-regular square divisible designs. A divisible design (also called group-divisible design) with parameters k, g, n, λ_1 and λ_2, is an incidence structure, denoted by $GD(k, g, n, \lambda_1, \lambda_2)$, for which the points can be ordered such that the incidence matrix N satisfies

$$NN^\top = \lambda_2 J_v + (\lambda_1 - \lambda_2)K_{n,g} + (r - \lambda_1)I_v \ , \quad \text{and} \quad N^\top J_v = kJ_v,$$

where $K_{n,g}$ is the block diagonal matrix $I_n \otimes J_g$, $v = ng$ is the number of points and $r = ((n - 1)g\lambda_2 + (g - 1)\lambda_1)/(k - 1)$ is the replication number. The eigenvalues of $NN^\top$ are easily seen to be kr, $r - \lambda_1$, and $g(\lambda_1 - \lambda_2) + r - \lambda_1$ with multiplicities 1, $n(g - 1)$ and $n - 1$, respectively. Assume that N is a square matrix. Then $r = k$, and the eigenvalues of

$NN^\top$ become k^2, $k-\lambda_1$ and $k^2-gn\lambda_2$. If N is non-singular, the divisible design is called regular, and necessary conditions for existence have been known for a long time, see [1], [7] p. 228, or [3] p. 23. If N is singular, either $k=\lambda_1$ and $N=N'\otimes J_n$, where N' is the incidence matrix of a square block design (then the divisible design is called singular), or $k^2=ng\lambda_2$ and the divisible design is called semi-regular.

Theorem 2 *Let D be a design with the property that both D and its dual are a semi-regular $GD(k, g, n, \lambda_1, \lambda_2)$. Then*
(i) *if g is even and n is odd, $k-\lambda_1$ is an integral square*,
(ii) *if g is even and $n\equiv 2 \pmod 4$ then $k-\lambda_1$ is the sum of two integral squares*,
(iii) *if g and n are odd, the equation $(k-\lambda_1)X^2+(-1)^{(g-1)/2}gY^2=Z^2$ has an integral solution different from $X=Y=Z=0$.*

Proof: Suppose N is the incidence matrix of D. We may assume that $NN^\top=N^\top N$, which implies that $N^\top$ and N have the same kernel, so by Lemma 2 the product of the non-zero eigenvalues of $NN^\top$ is a square, which proves i. Define $Z=(J_n-nI_n)\otimes J_g$. Then $\operatorname{rank} Z=n-1$, and $NN^\top Z=N^\top NZ=O$, so Z satisfies the requirement for Lemma 1. Hence

$$MM^\top=NN^\top+ZZ^\top=(\lambda_2-gn)J_v+(\lambda_1-\lambda_2+gn^2)K_{n,g}+(k-\lambda_1)I_v\ .$$

has eigenvalues k^2, $\rho=k-\lambda_1$ and $\sigma=g^2n^2$ of multiplicity 1, $n(g-1)$ and $n-1$ respectively. The Hasse-Minkowski invariant $C_p(MM^\top)$ with respect to the odd prime p of a matrix $MM^\top$ of the above form is known, see for example [1].

$$\begin{aligned}C_p(MM^\top)&=(\rho,-1)_p^{n(g-1)(n+g-1)/2}(\sigma,-1)_p^{n(n-1)/2}(\sigma,g)_p^n(\rho,g)_p^n(\sigma,\lambda_2-gn)_p\\&=(\rho,-1)_p^{n(g-1)(n+g-1)/2}(\rho,g)_p^n,\end{aligned}$$

where $(a,b)_p$ is the Hilbert norm residue symbol, defined by $(a,b)_p=1$ if for all t the congruence $aX^2+bY^2\equiv 1 \pmod{p^t}$ has a rational solution, and $(a,b)_p=-1$ otherwise. Since M is a non-singular rational matrix, $C_p(MM^\top)=C_p(I_v)=1$ for every odd prime p, and the conditions (ii) and (iii) follow. □

For example there exists no $GD(18, 4, 9, 6, 9)$ for which the dual is also such a design. Note that in case $n=1$, D is a square block design and the conditions are those of Bruck, Ryser and Chowla. The above theorem also has consequences for distance-regular graphs. Some putative distance-regular graphs imply the existence of square divisible designs (see [3] p. 22), and in case these divisible designs are semi-regular we obtain new conditions.

Corollary 3 *Suppose there exists a distance-regular graph of diameter 4 with $2g^2\mu$ vertices and intersection array $\{g\mu, g\mu-1, (g-1)\mu, 1\,;\, 1, \mu, g\mu-1, g\mu\}$. Then*
(i) *If μ is odd and $g\equiv 2 \pmod 4$ then $g\mu$ is the sum of two integral squares.*
(ii) *If μ and g are odd, then the equation $\mu X^2+(-1)^{(g-1)/2}Y^2=gZ^2$ has an integral solution different from $X=Y=Z=0$.*

Proof: Such a distance-regular graph is the incidence graph of a $GD(g\mu, g, g\mu, 0, \mu)$ for which the dual is also such a design. $\square$

For example a distance-regular graph with intersection array $\{15, 14, 12, 1\,;\ 1, 3, 14, 15\}$ does not exist. Note that a distance-regular graph with intersection array $\{g\mu - 1, (g - 1)\mu, 1\,;\ 1, \mu, g\mu - 1\}$ also gives rise to a semi-regular square divisible design; see [3], p. 24. But here we find no new restrictions.

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