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Properties of Some Character Tables Related to the Symmetric Groups

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Abstract. We determine invariants like the Smith normal form and the determinant for certain integral matrices which arise from the character tables of the symmetric groups S_n and their double covers. In particular, we give a simple computation, based on the theory of Hall-Littlewood symmetric functions, of the determinant of the regular character table $\mathcal{X}_{RC}$ of S_n with respect to an integer $r \geq 2$. This result had earlier been proved by Olsson in a longer and more indirect manner. As a consequence, we obtain a new proof of the Mathas' Conjecture on the determinant of the Cartan matrix of the Iwahori-Hecke algebra. When r is prime we determine the Smith normal form of $\mathcal{X}_{RC}$. Taking r large yields the Smith normal form of the full character table of S_n. Analogous results are then given for spin characters.

Keywords: symmetric group, character, spin character, Smith normal form

1. Introduction

In this paper we determine invariants like the Smith normal form or the determinant for certain integral matrices which come from the character tables of the finite symmetric groups S_n and their double covers $\hat{S}_n$. The matrices in question are the so-called regular and singular character tables of S_n and the reduced spin character table of $\hat{S}_n$.

In Section 2 we calculate the determinants of the r-regular and r-singular character tables of S_n for arbitrary integers $r \geq 2$, using symmetric functions and some bijections involving regular partitions. The knowledge of these determinants is equivalent to the knowledge of the determinants of certain "generalized Cartan matrices" of S_n as considered in [9]. In particular we obtain a new proof of a conjecture of Mathas about the Cartan matrix of an Iwahori-Hecke algebra of S_n at a primitive rth root of unity which is simpler than the

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original proof given by Brundan and Kleshchev in [3]. In Section 3 we determine the Smith normal form of the regular character table in the case where r is a prime. As a special case the Smith normal form of the character table of S_n may be calculated. We also determine the Smith normal form of the reduced spin character table for $\hat{S}_n$. The paper also presents some open questions.

2. The determinant of the regular part of the character table of S_n

We fix positive integers n, r, where $r \geq 2$.

If $\mu = (\mu_1, \mu_2, \ldots)$ is a partition of n we write $\mu \in \mathcal{P}$ and denote by $\ell(\mu)$ the number of (non-zero) parts of μ. We let z_μ denote the order of the centralizer of an element of (conjugacy) type μ in S_n. Suppose $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \ldots)$, is written in exponential notation. Then we may factor $z_\mu = a_\mu b_\mu$, where

$$a_\mu = \prod_{i \geq 1} i^{m_i(\mu)}, \quad b_\mu = \prod_{i \geq 1} m_i(\mu)!$$

Whenever $\mathcal{Q} \subseteq \mathcal{P}$ we define

$$a_{\mathcal{Q}} = \prod_{\mu \in \mathcal{Q}} a_\mu, \quad b_{\mathcal{Q}} = \prod_{\mu \in \mathcal{Q}} b_\mu.$$

Let $\mu \in \mathcal{P}$. We write $\mu \in R$ and call μ *regular* if $m_i(\mu) \leq r - 1$ for all $i \geq 1$. We write $\mu \in C$ and call μ *class regular* if $m_i(\mu) = 0$, whenever $r \mid i$.

We are particularly interested in the integers a_C and b_C. By [12, Theorem 4] there is a connection between a_C and b_C given by

$$b_C = r^{d_C} a_C, \tag{1}$$

where the class regular defect number d_C is defined by

$$d_C = \sum_{\mu \in C} d(\mu), \quad d(\mu) = \sum_{i,k \geq 1} \left\lfloor \frac{m_i(\mu)}{r^k} \right\rfloor.$$

Here $\lfloor \cdot \rfloor$ is the floor function, i.e., $\lfloor x \rfloor$ denotes the integral part of x. Note that for $r > n$ we have $R = C = \mathcal{P}$ and then $d_{\mathcal{P}} = 0$ and thus $a_{\mathcal{P}} = b_{\mathcal{P}}$.

Let $\mathcal{X}_{RC}$ denote the regular character table of S_n with respect to r. It is a submatrix of the character table $\mathcal{X}$ of S_n. The subscript RC indicates that the rows of $\mathcal{X}_{RC}$ are indexed by the set R of regular partitions of n, and the columns by the set C of class regular partitions of n. We want to present a proof of the following result:

Theorem 1 *We have*

$$|\det(\mathcal{X}_{RC})| = a_C.$$

This result was first proved in [12], but the proof relied on results of [9] for which the work of Donkin [4] and Brundan and Kleshchev [3] was used in a crucial way. Our proof of Theorem 1 does not use [4] or [3]; it is direct and thus much shorter.

In [9], an r-analogue of the modular representation theory for S_n was developed systematically, and in particular, an r-analogue of the Cartan matrix for the symmetric groups (and the corresponding r-blocks) was introduced.

In [2] the explicit value of this latter determinant was conjectured to be r^{d_C} in the notation above; this was proved in [9, Proposition 6.11] using [4] and [3]. This result is now a consequence of our theorem:

Corollary 2 *Let $\mathcal{C}$ be the r-analogue of the Cartan matrix of S_n as defined in* [9]. *Then we have*

$$\det(\mathcal{C}) = r^{d_C}.$$

Proof: As is shown in [12] there is a simple equation connecting the determinants of $\mathcal{C}$ and $\mathcal{X}_{RC}$, namely

$$\det(\mathcal{X}_{RC})^2 \det(\mathcal{C}) = a_C b_C.$$

Thus in view of Eq. (1) Theorem 1 implies the Corollary. □

Mathas conjectured that the determinant of the Cartan matrix of an Iwahori-Hecke algebra of S_n at a primitive rth root of unity should be a power of r; via [4], the conjecture in [2] mentioned above predicted the explicit value of this determinant, thus providing a strengthening of Mathas' conjecture. Mathas' conjecture was proved by Brundan and Kleshchev [3]; in fact, they also gave an explicit formula for this determinant for blocks of the Hecke algebra. We can now provide an alternative proof of these conjectures.

Corollary 3 *The strengthened Mathas' conjecture is true.*

Proof: Donkin [4] has shown that the Cartan matrix for the Hecke algebra has the same determinant as the Cartan matrix $\mathcal{C}$ considered in Corollary 2. □

Based on this and the results on r-blocks in [9], the results in [2] then also give the determinants of Cartan matrices of r-blocks of S_n explicitly, without the use of [3].

Let us finally mention that in [9, Section 6] there is an explicit conjecture about the Smith normal form of $\mathcal{C}$. In the case where r is a prime, this is known to be true by the general theory of R. Brauer. One may also ask about the Smith normal form of $\mathcal{X}_{RC}$; we answer this question in this article in the prime case.

We now proceed to describe the proof of Theorem 1. It is obtained by combining Theorems 4 and 5 below. Theorem 4 evaluates $\det(\mathcal{X}_{RC})^2$ using symmetric functions as an expression involving a primitive rth root of unity. Theorem 5 shows that this expression equals ${a_C}^2$. It is based on general bijections involving regular partitions.

Define

$$z_\mu(t) = z_\mu \prod_j (1 - t^{\mu_j})^{-1} = z_\mu \prod_i (1 - t^i)^{-m_i(\mu)}$$

where the product ranges over all j for which $\mu_j > 0$, and

$$b_\lambda(t) = \prod_i (1 - t)(1 - t^2) \cdots \left(1 - t^{m_i(\lambda)}\right).$$

Let $\omega = e^{2\pi i/r}$, a primitive rth root of unity.

We use notation from the theory of symmetric functions from [10] or [14]. In particular, m_λ, s_λ, and p_λ denote the monomial, Schur, and power sum symmetric functions, respectively, indexed by the partition λ.

Theorem 4 *We have*

$$\det(\mathcal{X}_{RC})^2 = \prod_{\mu \in C} z_\mu(\omega) \cdot \prod_{\lambda \in R} b_\lambda(\omega).$$

Proof: Let $Q_\lambda(x; t)$ denote a Hall-Littlewood symmetric function as in [10, p. 210]. It is immediate from the definition of $Q_\lambda(x; t)$ that $Q_\lambda(x; \omega) = 0$ unless $\lambda \in R$. Moreover (see [10, Exam. III.7.7, p. 249]) when $Q_\lambda(x; \omega)$ is expanded in terms of power sums p_μ, only class regular μ appear. Thus [10, (7.5), p. 247] for $\lambda \in R$ we have

$$Q_\lambda(x; \omega) = \sum_{\mu \in C} z_\mu(\omega)^{-1} X_\mu^\lambda(\omega) p_\mu(x),$$

where $X_\mu^\lambda(t)$ is a Green's polynomial.

Hence by [10, (7.4)] the matrix $X(\omega)_{RC} = (X_\mu^\lambda(\omega))$, where $\lambda \in R$ and $\mu \in C$, satisfies

$$\det(X(\omega)_{RC})^2 = \prod_{\mu \in C} z_\mu(\omega) \prod_{\lambda \in R} b_\lambda(\omega). \tag{2}$$

Now consider the symmetric function $S_\lambda(x; t)$ as defined in [10, (4.5), p. 224]. It follows from the formula $S_\lambda(x; t) = s_\lambda(\xi)$ in [10, top of p. 225] that

$$S_\lambda(x; t) = s_\lambda(p_j \to (1 - t^j) p_j),$$

i.e., expand $s_\lambda(x)$ as a polynomial in the p_j's and substitute $(1 - t^j) p_j$ for p_j. Since

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi^\lambda(\mu) p_\mu,$$

we have

$$S_\lambda(x;\omega) = \sum_{\mu\in C} z_\mu(\omega)^{-1}\chi^\lambda(\mu)p_\mu.$$

The $S_\lambda(x;\omega)$'s thus lie in the space $A_{(r)}$ spanned over $\mathbb{Q}(\omega)$ by the p_μ's where $\mu \in C$. Since the $Q_\mu(x;\omega)$'s for regular μ span $A_{(r)}$ by [10, Exam. III.7.7, p. 249], the same is true of the $S_\lambda(s;\omega)$'s. Moreover, the transition matrix $M(S,Q)_{RR}$ between the $Q_\lambda(x;t)$'s and $S_\lambda(x;t)$'s is lower unitriangular by [10, top of p. 239] and [10, p. 241]. Hence

$$\det M(S,Q)_{RR} = 1. \tag{3}$$

Let $M(S,p)_{RC}$ denote the transition matrix from the p_μ's to S_λ's for $\mu \in C$ and $\lambda \in R$. Let $Z(t)_{CC}$ denote the diagonal matrix with entries $z_\lambda(t)$, $\lambda \in C$. By the discussion above we have

$$\begin{aligned}
\mathcal{X}_{RC} &= M(S,p)_{RC}\, Z(\omega)_{CC} \quad \text{(by the relevant definitions)}\\
&= M(S,Q)_{RR}\, M(Q,p)_{RC}\, Z(\omega)_{CC}\\
&= M(S,Q)_{RR}\, X(\omega)_{RC}\, Z(\omega)_{CC}^{-1}\, Z(\omega)_{CC}\\
&= M(S,Q)_{RR}\, X(\omega)_{RC}.
\end{aligned}$$

Taking determinants and using (2) and (3) completes the proof. □

Define

$$A_C(\omega) = \prod_{\mu\in C}\prod_i (1-\omega^i)^{-m_i(\mu)}$$

$$B_R(\omega) = \prod_{\lambda\in R} b_\lambda(\omega)^{-1} = \prod_{\lambda\in R}\left(\prod_i (1-\omega)(1-\omega^2)\cdots\left(1-\omega^{m_i(\lambda)}\right)\right)^{-1},$$

so that by Theorem 4

$$\det(\mathcal{X}_{RC})^2 = a_C b_C A_C(\omega) B_R(\omega)^{-1}.$$

In order to complete the proof of Theorem 1 we thus just need to show:

$$\frac{B_R(\omega)}{A_C(\omega)} = \frac{b_C}{a_C}\,.$$

As $\frac{b_C}{a_C} = r^{d_C}$ this is equivalent to showing

Theorem 5 *We have*

$$\frac{B_R(\omega)}{A_C(\omega)} = r^{d_C}.$$

Clearly the factors $1-\omega^j$ occurring on the left hand side in Theorem 5 depend only on the residue of j modulo r. Thus

$$A_C(\omega)^{-1} = \prod_{s=1}^{r-1}(1-\omega^s)^{\alpha_C^{(s)}}, \quad B_R(\omega)^{-1} = \prod_{s=1}^{r-1}(1-\omega^s)^{\beta_R^{(s)}},$$

where

$$\alpha_C^{(s)} = \sum_{\mu\in C}\sum_{\{i|i\equiv s(\mathrm{mod}\ r)\}} m_i(\mu)$$
$$\beta_R^{(s)} = \sum_{\rho\in R} |\{i|m_i(\rho)\geq s\}|.$$

We use the bijections $\kappa^{(s)}$ defined in Proposition 9 below to show the following:

Proposition 6 *For all* $s\in\{1,\dots,r-1\}$ *we have*

$$\alpha_C^{(s)} = \beta_R^{(s)} + d_C.$$

This shows then that

$$\frac{B_R(\omega)}{A_C(\omega)} = \left(\prod_{s=1}^{r-1}(1-\omega^s)\right)^{d_C}.$$

Then Theorem 5 follows from the fact that

$$\prod_{s=1}^{r-1}(1-\omega^s) = r.$$

(Simply substitute $x=1$ in the identity $1+x+\cdots+x^{r-1} = \prod_{s=1}^{r-1}(x-\omega^s)$.)

Let $m\in\mathbb{N}$. We write m in its r-adic decomposition as $m=\sum_{j\geq 0} m_j r^j$, i.e., with $m_j\in\{0,\dots,r-1\}$ for all j. For $m\neq 0$, we can write $m=\sum_{j\geq k} m_j r^j$, with $m_k\neq 0$. In the power series convention, $k(m)=k$ is the degree of m and $\ell(m)=m_k$ its leading coefficient. We also set $h(m)=\sum_{j\geq k+1} m_j r^j = r^{k+1}q(m)$ for the higher terms of m. Thus

$$m = \ell(m)r^{k(m)} + q(m)r^{k(m)+1}.$$

For a given a, we define

$$h_a(m) = \sum_{j\geq a} m_j r^j = q_a(m)r^a, \quad q_a(m) = \left\lfloor \frac{m}{r^a}\right\rfloor.$$

We call $e \in \{1, \dots, m\}$ a *non-defect number for* m, if $h(e) = h_{k(e)+1}(m)$, otherwise e is a *defect number for* m (and then $h(e) < h_{k(e)+1}(m)$, and hence $q(e) < q_{k(e)+1}(m)$). Thus the non-defect numbers for m are of the form

$$e = e_a r^a + h_{a+1}(m), \quad e_a \in \{1, \dots, m_a\},$$

and thus there are $\sum_{j \geq 0} m_j$ such numbers. The defect numbers for m are of the form

$$e = e_a r^a + q r^{a+1}, \quad e_a \in \{1, \dots, r-1\}, \quad q \in \{0, \dots, q_{a+1}(m) - 1\}.$$

Their parameters (a, q) thus belong to the set

$$\mathcal{D}(m) = \{(a, q) \mid a \geq 0,\ 0 \leq q < q_{a+1}(m)\},$$

which is of cardinality

$$d(m) = \sum_{a \geq 1} \left\lfloor \frac{m}{r^a} \right\rfloor ,$$

called the *defect* of m. For each $s \in \{1, \dots, r-1\}$ there are exactly $d(m)$ defect numbers for m with leading coefficient s, namely $e = sr^a + qr^{a+1}$, where $(a, q) \in \mathcal{D}(m)$. Thus clearly we have $(r-1)d(m)$ defect numbers for m and

$$m = (r-1)d(m) + \sum_{j \geq 0} m_j .$$

For $\mu \in \mathcal{P}$, its defect (as defined at the beginning of this section) is then

$$d(\mu) = \sum_{i \geq 1} d(m_i(\mu)).$$

For $s \in \{1, \dots, r-1\}$ set

$$\mathcal{D}^{(s)}(\mu) = \{(i, a, q) \mid \ell(i) = s, (a, q) \in \mathcal{D}(m_i(\mu))\}$$

and

$$\mathcal{D}(\mu) = \bigcup_{s=1}^{r-1} \mathcal{D}^{(s)}(\mu) .$$

We have that

$$d^{(s)}(\mu) = |\mathcal{D}^{(s)}(\mu)| = \sum_{\{i \geq 1, \ell(i) = s\}} d(m_i(\mu))$$

and

$$d(\mu) = \sum_{s=1}^{r-1} d^{(s)}(\mu) = |\mathcal{D}(\mu)|.$$

Consider nonzero residues s, t modulo r, let $\mu = (i^{m_i(\mu)})$ and define

$$\mathcal{T}^{(st)}(\mu) = \{(i, j) \mid 1 \leq i,\ 1 \leq j \leq m_i(\mu), \ell(i) = s, \ell(j) = t\}.$$

Glaisher [6] defined a bijection between the sets C and R of class regular and regular partitions of n. Glaisher's map G is defined as follows. Suppose that $\mu = (i^{m_i(\mu)}) \in C$. Consider the r-adic expansion of each multiplicity $m_i(\mu)$:

$$m_i(\mu) = \sum_{j \geq 0} m_{ij}(\mu) r^j$$

where for all relevant i, j we have $m_{ij}(\mu) \in \{0, \ldots, r-1\}$. Then $G(\mu) = \rho$ where for all $i, j, r \nmid i$ we have $m_{ir^j}(\rho) = m_{ij}(\mu)$.

We show

Proposition 7 *If $\mu \in C$ then $|\mathcal{T}^{(st)}(\mu)| = |\mathcal{T}^{(st)}(G(\mu))| + d^{(s)}(\mu)$.*

Proof: We establish a bijection $\delta^{(st)}(\mu)$ between $\mathcal{T}^{(st)}(\mu)$ and the disjoint union $\mathcal{T}^{(st)}(G(\mu)) \cup \mathcal{D}^{(s)}(\mu)$. If $(i, j) \in \mathcal{T}^{(st)}(\mu)$ and $(k(j), q(j)) = (a, q)$, we have two possibilities

(i) j is a defect number for $m_i(\mu)$. Then we map (i, j) onto $(i, a, q) \in \mathcal{D}^{(s)}(\mu)$.
(ii) We have $j = tr^a + h_{a+1}(m_i(\mu))$ where $1 \leq t \leq m_{ia}(\mu)$. Then we map (i, j) onto $(r^a i, t) \in \mathcal{T}^{(st)}(G(\mu))$.

This establishes the desired bijection. □

Consider nonzero residues s, t modulo r, and define

$$\begin{aligned}
\mathcal{T}_C^{(st)} &= \{(\mu, i, j) \big| \mu \in C, (i, j) \in \mathcal{T}^{(st)}(\mu)\} \\
\mathcal{T}_R^{(st)} &= \{(\rho, i, j) \big| \rho \in R, (i, j) \in \mathcal{T}^{(st)}(\rho)\} \\
\mathcal{D}^{(s)} &= \{(\mu, i, a, q) | \mu \in C, (i, a, q) \in \mathcal{D}^{(s)}(\mu)\}.
\end{aligned}$$

Clearly the bijections $\delta^{(st)}(\mu),\ \mu \in C$, above induce a bijection

$$\delta^{(st)} : \mathcal{T}_C^{(st)} \leftrightarrow \mathcal{T}_R^{(st)} \cup \mathcal{D}^{(s)}.$$

Putting the bijections $\delta^{(ts)},\ t = 1, \ldots, r-1$ together we obtain a bijection

$$\delta^{(s)} : \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(ts)} \leftrightarrow \bigcup_{t=1}^{r-1} \mathcal{T}_R^{(ts)} \cup \mathcal{C},$$

where

$$\mathcal{C} = \bigcup_{t=1}^{r-1} \mathcal{D}^{(t)}.$$

In [12, proof of Theorem 4], an involution ι was defined on the set

$$\mathcal{T}_C = \{(\mu, i, j) \mid \mu \in C, i, j \geq 1, m_i(\mu) \geq j\}.$$

From the definition of ι it follows that it maps the subset $\mathcal{T}_C^{(st)}$ of $\mathcal{T}_C$ into $\mathcal{T}_C^{(ts)}$. Thus we conclude

Lemma 8 *For all $s \in \{1, \ldots, r-1\}$ there is a bijection*

$$\iota^{(s)} : \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(st)} \leftrightarrow \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(ts)} .$$

Composing the bijections $\iota^{(s)}$ and $\delta^{(s)}$ we see

Proposition 8 *For all $s \in \{1, \ldots, r-1\}$ there is a bijection*

$$\kappa^{(s)} : \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(st)} \leftrightarrow \bigcup_{t=1}^{r-1} \mathcal{T}_R^{(ts)} \cup \mathcal{C}.$$

Proof of Proposition 6: Just consider the cardinalities of the sets occurring in Proposition 9.

$$\left| \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(st)} \right| = \sum_{\mu \in C} \sum_{\{i \mid \ell(i) = s\}} m_i(\mu) = \alpha_C^{(s)}.$$

The latter equality holds because a class regular partition contains no parts divisible by r. Thus if $m_i(\mu) \neq 0$ then $\ell(i) = s$ if and only if $i \equiv s (\mathrm{mod}\ r)$.

$$\left| \bigcup_{t=1}^{r-1} \mathcal{T}_R^{(ts)} \right| = \sum_{\rho \in R} |\{i \mid m_i(\rho) \geq s\}| = \beta_R^{(s)}.$$

This is because parts in regular partitions have multiplicities $<r$. Finally

$$|\mathcal{C}| = \sum_{t=1}^{r-1} d^{(t)} = \sum_{\mu \in C} d(\mu) = d_C.$$

□

Remark There is of course also a *singular* character table for S_n, which we denote $\mathcal{X}_{R'C'}$. It is also a submatrix of the character table $\mathcal{X}$ of S_n. The subscript $R'C'$ indicates that the rows of $\mathcal{X}_{R'C'}$ are indexed by the set R' of singular (i.e. nonregular) partitions of n, and the columns by the set C' of class singular (i.e. non-class regular) partitions of n. For this we have

$$|\det(\mathcal{X}_{R'C'})| = b_{C'}. \tag{4}$$

There are different ways of proving this. In [12] there is a proof based on Theorem 1 and a result in [9].

Another way of proving (4) is *via* an identity of Jacobi [5, p. 21]. Namely, suppose that A is an invertible $n \times n$ matrix, and write A and A^{-1} in the block form

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} B' & C' \\ D' & E' \end{bmatrix},$$

where B and B' are $k \times k$ matrices. Then

$$\det E' = \frac{\det B}{\det A}.$$

By the orthogonality of characters we have

$$\mathcal{X}^{-1} = \mathcal{X}^t \Delta\bigl(z_\mu^{-1}\bigr),$$

where $\Delta(z_\mu^{-1})$ is the diagonal matrix with the z_μ^{-1}, $\mu \in \mathcal{P}$, on the diagonal. Equation (4) follows immediately from this observation and Theorem 1.

Remark If we keep r fixed and let n vary, then the result of Proposition 6 may also be proved by calculating the generating functions for $\alpha_C^{(s)}$, $\beta_R^{(s)}$ and d_C. Indeed, if $P(q)$ is the generating function for the number of partitions of n, then $P_r(q) = \frac{P(q)}{P(q^r)}$ is the generating function for the number of regular partitions of n. We may then express the generating functions for $\alpha_C^{(s)}$, $\beta_R^{(s)}$ and d_C respectively by

$$\begin{aligned} A^{(s)}(q) &= P_r(q) \sum_{i \geq 0} \frac{q^{ir+s}}{1 - q^{ir+s}} \\ B^{(s)}(q) &= P_r(q) \sum_{j \geq 1} \frac{q^{js} - q^{jr}}{1 - q^{jr}} \\ D(q) &= P_r(q) \sum_{j \geq 1} \frac{q^{jr}}{1 - q^{jr}}. \end{aligned}$$

We omit the details. From this Proposition 6 may be deduced easily.

3. Smith normal forms of character tables related to S_n

For a partition λ of n, we denote by ξ^λ the permutation character of S_n obtained by inducing the trivial character of the Young subgroup S_λ up to S_n. First we explicitly describe the values of these permutation characters (this is included here as we have not been able to find a reference for it).

Proposition 10 *Let $\lambda, \mu \in \mathcal{P}$, $k = \ell(\lambda)$, $\ell = \ell(\mu)$. Then the value $\xi^\lambda(\mu)$ of the permutation character ξ^λ on the conjugacy class of cycle type μ equals the number of ordered set partitions $(B_1, \ldots, B_k)$ of $\{1, \ldots, \ell\}$ such that*

$$\lambda_j = \sum_{i \in B_j} \mu_i \quad \text{for } j \in \{1, \ldots, k\}.$$

Proof: Let σ_μ be a permutation of cycle type μ. Then (see [8]) $\xi^\lambda(\mu)$ is the number of λ-tabloids fixed by σ_μ. Now clearly, a λ-tabloid is fixed by σ_μ if and only if its rows are unions of complete cycles of σ_μ. Thus such a decomposition of rows corresponds to an ordered set partition $(B_1, \ldots, B_k)$ of the cycles of μ with the sum conditions in the statement of the Proposition. □

Remark One may also use a symmetric function argument for computing the values $R_{\lambda\mu} = \xi^\mu(\lambda)$. The complete homogeneous symmetric function h_λ is the (Frobenius) characteristic of the character ξ^λ (see [14, Cor. 7.18.3]), so $h_\lambda = \sum_\mu z_\mu^{-1} R_{\lambda\mu} p_\mu$. As the h_λ and m_μ are dual bases, as well as the p_λ and $z_\mu^{-1} p_\mu$, it then follows that $p_\lambda = \sum_\mu R_{\lambda\mu} m_\mu$. Using [14, Prop. 7.7.1] then also gives the formula in Proposition 10.

Corollary 11 *Let $\lambda, \mu \in \mathcal{P}$. Then we have*
(i) $\xi^\lambda(\mu) = 0$ *unless* $\lambda \geq \mu$ *(dominance order).*
(ii) $\xi^\lambda(\lambda) = b_\lambda = \prod_i m_i(\lambda)!$.
(iii) $\xi^\lambda(\lambda) \mid \xi^\lambda(\mu)$.

Proof: Using the remark above, parts (i) and (ii) follow immediately by [14, Cor. 7.7.2] (or one may also prove it directly using Proposition 10). For (iii), we use the combinatorial description given in Proposition 10. With notation as before, let $(B_1, \ldots, B_k)$ be an ordered partition of the set $\{1, \ldots, \ell\}$ contributing to $\xi^\lambda(\mu)$, i.e., satisfying the sum conditions. Now any permutation of $\{1, \ldots, k\}$ which interchanges only parts of λ of equal size leads to a permutation of the entries of $(B_1, \ldots, B_k)$ such that the corresponding ordered partition still satisfies the sum conditions. Hence $\xi^\lambda(\mu)$ is divisible by $\prod_i m_i(\lambda)! = b_\lambda$ and thus by $\xi^\lambda(\lambda)$. □

We can now determine the Smith normal form for the regular character table of S_n in the case where $r = p$ is prime.

For an integer matrix A we denote by $\mathcal{S}(A)$ its Smith normal form. If p is a prime, we write $A_{p'}$ for the matrix obtained by taking only the p'-parts of the entries. For a set of integers $M = \{r_1, \ldots, r_m\}$ we denote by $\mathcal{S}(M)$ or $\mathcal{S}(r_1, \ldots, r_m)$ the Smith normal form of the diagonal matrices with the entries $r_1, \ldots, r_m$ on the diagonal.

Theorem 12 *Let p be a prime, and let $\mathcal{X}_{RC}$ be the p-regular character table of S_n. Then we have*

$$\mathcal{S}(\mathcal{X}_{RC}) = \mathcal{S}(b_\mu \mid \mu \in C)_{p'} .$$

Proof: Let $\mathcal{Y} = \mathcal{Y}_{CC} = (\xi^\lambda(\mu))_{\lambda,\mu\in C}$ denote the part of the permutation character table of S_n with rows and columns indexed by the class p-regular partitions of n. Set $\mathcal{X} = \mathcal{X}_{RC}$.

As the characters χ^λ with λ in the set R of p-regular partitions of n form a basic set for the characters on the p-regular conjugacy classes by [9], we have a decomposition matrix $D = D_{CR}$ with integer entries such that

$$\mathcal{Y} = D \cdot \mathcal{X}.$$

Now by Corollary 11 the permutation character table $\mathcal{Y}$ is (with respect to a suitable ordering) a lower triangular matrix with the b_μ, $\mu \in C$, on the diagonal. Hence using [12, Theorem 4] and Theorem 1 we obtain

$$\det(\mathcal{Y})_{p'} = (b_C)_{p'} = a_C = |\det(\mathcal{X})| .$$

Thus $\det(D)$ is a p-power, and hence $\det(D)$ and $\det(\mathcal{X})$ are coprime. This implies by [11, Theorem II.15]

$$\mathcal{S}(\mathcal{Y}) = \mathcal{S}(D\,\mathcal{X}) = \mathcal{S}(D)\mathcal{S}(\mathcal{X}) .$$

Now using the divisibility property in Corollary 11 (iii) we can convert the triangular matrix $\mathcal{Y}$ by unimodular transformations to a diagonal matrix with the same entries b_μ, $\mu \in C$, on the diagonal, and hence $\mathcal{S}(\mathcal{Y}) = \mathcal{S}(b_\mu \mid \mu \in C)$. As $\mathcal{S}(D)$ is a diagonal matrix with only p-power entries on the diagonal, this yields the assertion in the Theorem. □

Remark Choosing $p > n$ in Theorem 12 shows in particular that the Smith normal form of the whole character table $\mathcal{X}$ is the same as that of the diagonal matrix with diagonal entries $b_\mu = R_{\mu\mu}$, $\mu \in \mathcal{P}$. One may also use the language of symmetric functions to prove this result. Here, one uses that the matrix $\mathcal{X}$ is the transition matrix from the Schur functions to the power sums [14, Cor. 7.17.4]. Since the transition matrix from the monomial symmetric functions to the Schur functions is an integer matrix of determinant 1 (in fact, lower unitriangular with respect to a suitable ordering on partitions [14, Cor. 7.10.6]), the transition matrix $R_n = (R_{\lambda\mu})_{\lambda,\mu\in\mathcal{P}}$ between the m_λ's and p_μ's has the same Smith normal form as $\mathcal{X}$. Then we use the same arguments as before to deduce the Smith normal form of R_n.

Remark We do not know at present how Theorem 12 should extend from the prime case to the case of general r. Some obvious guesses for r-versions do not hold. The following weaker version *might* be true. Let π be the set of primes of r, and for a number m let $m_{\pi'}$ denote its π'-part (the largest divisor of m coprime to r). Then

$$\mathcal{S}(\mathcal{X}_{RC})_{\pi'} = \mathcal{S}(b_\mu \mid \mu \in C)_{\pi'}.$$

Using Theorem 12 above for $p = 2$ also allows the determination of the Smith normal form of the reduced spin character table of the double covers of the symmetric groups. For the background on spin characters of S_n we refer to [7] and [13].

We denote by $\mathcal{D}$ the set of partitions of n into distinct parts and by $\mathcal{O}$ the set of partitions of n into odd parts. Note that thus $\mathcal{D}$ is the set of 2-regular partitions of n and $\mathcal{O}$ is the set of class 2-regular partitions of n. For each $\lambda \in \mathcal{D}$ we have a spin character $\langle\lambda\rangle$ of S_n. If $n - \ell(\lambda)$ is odd, then there is an associate spin character $\langle\lambda\rangle' = \mathrm{sgn} \cdot \langle\lambda\rangle$ of S_n and λ is said to be of negative type; the corresponding subset of $\mathcal{D}$ is denoted by $\mathcal{D}^-$. The spin characters can have non-zero values only on the so-called doubling conjugacy classes of the double cover $\tilde{S}_n$ of S_n; these are labelled by the partitions in $\mathcal{O} \cup \mathcal{D}^-$. More precisely, for any such partition we have two conjugacy classes in $\tilde{S}_n$; one of these is chosen in accordance with [13], and we denote a corresponding representative by σ_μ. While the spin character values on the $\mathcal{D}^-$ classes are known explicitly (but they are in general not integers, and mostly not even real), for the values on the $\mathcal{O}$-classes we only have a recursion formula (due to A. Morris) which is analogous to the Murnaghan-Nakayama formula, and which shows that these are integers. We then define the reduced spin character table as the integral square matrix

$$Z_s = (\langle\lambda\rangle(\sigma_\mu))_{\substack{\lambda\in\mathcal{D}\\ \mu\in\mathcal{O}}}$$

For any integer $m \geq 0$, let $s(m)$ be the number of summands in the 2-adic decomposition of m. For $\alpha = (1^{m_1}, 3^{m_3}, \ldots) \in \mathcal{O}$ we define

$$k_\alpha = \sum_{i \text{ odd}} (m_i - s(m_i)) .$$

Then we have

Theorem 13 *The Smith normal form of the reduced spin character table Z_s of $\tilde{S}_n$ is given by*

$$\mathcal{S}(Z_s) = \mathcal{S}\big(2^{[k_\mu/2]}, \mu \in \mathcal{O}\big) \cdot \mathcal{S}(b_\mu, \mu \in \mathcal{O})_{2'} .$$

Proof: Let Φ denote the Brauer character table of $\tilde{S}_n$ at characteristic 2; this is equal to the Brauer character table of S_n. Then $Z_s = D_s \cdot \Phi$, where D_s is a "reduced" decomposition matrix at $p = 2$; the reduction corresponds to leaving out the associate spin characters $\langle\lambda\rangle'$ for $\lambda \in \mathcal{D}^-$. The matrix D_s is then an integral square matrix. In [1], the Smith normal form of D_s was determined:

$$\mathcal{S}(D_s) = \mathcal{S}\big(2^{[k_\mu/2]}, \mu \in \mathcal{O}\big) .$$

As this is a matrix of 2-power determinant and the determinant of the Brauer character table is coprime to 2, we have

$$\mathcal{S}(Z_s) = S(D_s) \cdot \mathcal{S}(\Phi) = \mathcal{S}\big(2^{[k_\mu/2]}, \mu \in \mathcal{O}\big) \cdot \mathcal{S}(\Phi) .$$

Now the Brauer characters and the characters χ^λ, $\lambda \in R = \mathcal{D}$, are both basic sets for the characters of S_n on 2-regular classes, hence $\mathcal{S}(\Phi) = S(\mathcal{X}_{RC})$. By Theorem 12 (for $p = 2$) we thus obtain

$$\mathcal{S}(\Phi) = \mathcal{S}(\mathcal{X}_{RC}) = \mathcal{S}(b_\mu \mid \mu \in \mathcal{O})_{2'} .$$

This proves the claim. □

Remark Let us finally mention some open questions. We have determined the Smith normal form for the whole reduced spin character table. It is natural to ask whether also a p-version (or even an r-version) of this holds, or at least, whether the determinant can be computed similarly as in the ordinary S_n case.

More precisely, for a prime p define

$$Z_{s,p} = (\langle\lambda\rangle(\sigma_\mu))_{\substack{\lambda\in\mathcal{D}_p \\ \mu\in\mathcal{O}_p}}$$

where $\mathcal{D}_p$ and $\mathcal{O}_p$ denote the sets of class p-regular partitions in $\mathcal{D}$ and $\mathcal{O}$, respectively. Some examples lead to the following conjecture:

$$\mathcal{S}(Z_{s,p}) = \mathcal{S}\big(2^{[k_\mu/2]}, \mu \in \mathcal{O}_p\big) \cdot \mathcal{S}(b_\mu, \mu \in \mathcal{O}_p)_{2'} .$$

Concerning the determinant, one may ask whether there is an analogue of Theorem 4 in the spin case.

For S_n as well as its double cover one may also try to look for sectional versions or block versions for the results on regular character tables.

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