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Compatible spreads of symmetry in near polygons

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Abstract In De Bruyn [7] it was shown that spreads of symmetry of near polygons give rise to many other near polygons, the so-called glued near polygons. In the present paper we will study spreads of symmetry in product and glued near polygons. Spreads of symmetry in product near polygons do not lead to new glued near polygons. The study of spreads of symmetry in glued near polygons gives rise to the notion of 'compatible spreads of symmetry'. We will classify all pairs of compatible spreads of symmetry for the known classes of dense near polygons. All these pairs of spreads can be used to construct new glued near polygons.

Keywords Near polygon · Generalized quadrangle · Spread

1. Elementary definitions

A *near polygon* [19] is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$ there exists a unique point on L nearest to p. Here distances $\mathrm{d}(\cdot, \cdot)$ are measured in the point graph or collinearity graph Γ. If $d = \mathrm{diam}(\mathcal{S})$ denotes the diameter of Γ (or of $\mathcal{S}$), then the near polygon is called a near $2d$-gon. A near 0-gon is a point and a near 2-gon is a line.

If X_1 and X_2 are two sets of points, then $\mathrm{d}(X_1, X_2)$ denotes the minimal distance between a point of X_1 and a point of X_2. If $X_1 = \{x\}$, then we also write $\mathrm{d}(x, X_2)$ instead of $\mathrm{d}(\{x\}, X_2)$. For every $i \in \mathbb{N}$, $\Gamma_i(X_1)$ denotes the set of all points y for which $\mathrm{d}(y, X_1) = i$. If $X_1 = \{x\}$, we also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A near $2d$-gon, $d \geq 2$, is called a *generalized* $2d$*-gon* [20] if $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = 1$ for every $i \in \{1, \ldots, d-1\}$ and every two points x and y at distance i from each other. A generalized $2d$-gon is called *degenerate* if it does not contain ordinary $2d$-gons as

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subgeometries, or equivalently, if it contains a point which has distance at most $d-1$ from any other point. The near quadrangles are precisely the generalized quadrangles (GQ's, [18]). A degenerate generalized quadrangle consists of a number of lines through a point.

Let X be a nonempty set of points of a near polygon $\mathcal{S}$. The set X is called a *subspace* if every line meeting X in at least two points is completely contained in X. The set X is called *geodetically closed* if it is a subspace and if every point on a shortest path between two points of X is as well contained in X. If X is a subspace, then we can define a subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ by considering only those points and lines of $\mathcal{S}$ which are completely contained in X. If X is geodetically closed, then $\mathcal{S}_X$ clearly is a sub near polygon of $\mathcal{S}$. If $\mathcal{S}_X$ is a nondegenerate generalized quadrangle, then X and often also $\mathcal{S}_X$ will be called a *quad*. If $X_1, \dots, X_k$ are nonempty sets of points, then $\mathcal{C}(X_1, \dots, X_k)$ denotes the minimal geodetically closed sub near polygon through $X_1 \cup \dots \cup X_k$, i.e. the intersection of all geodetically closed sub near polygons through $X_1 \cup \dots \cup X_k$. If x and y are two different points of $\mathcal{S}$, then we denote $\mathcal{C}(\{x, y\})$ also by $\mathcal{C}(x, y)$.

A near polygon is said to have *order* (s, t) if every line is incident with exactly $s+1$ points and if every point is incident with exactly $t+1$ lines. A near polygon is called *thin* if every line is incident with precisely two points. The thin near polygons are precisely the bipartite graphs (if one regards the edges as lines). A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. Dense near polygons satisfy several nice properties. By Lemma 19 of [2], every point of a dense near polygon $\mathcal{S}$ is incident with the same number of lines. We denote this number by $t_{\mathcal{S}}+1$. If x and y are two points of a dense near polygon at distance δ from each other, then by Theorem 4 of [2], there exists a unique geodetically closed sub near 2δ-gon through x and y which necessarily coincides with $\mathcal{C}(x, y)$. So, if x and y are two points at distance 2 in a dense near polygon, then these points are contained in a unique quad.

A geodetically closed sub near polygon F of a near polygon $\mathcal{S}$ is called *classical* if for every point x there exists a (necessarily unique) point $\pi_F(x)$ in F such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. Obviously, every line of a near polygon is classical. If F_1 and F_2 denote two classical sub near polygons, then we denote by π_{F_1,F_2} the restriction of π_{F_2} to the point set of F_1. Two classical sub near polygons F_1 and F_2 are called *parallel* if π_{F_1,F_2} is an isomorphism. If this is the case, then also π_{F_2,F_1} is an isomorphism and $\pi_{F_2,F_1}^{-1} = \pi_{F_1,F_2}$. The parallel relation is not necessarily transitive. We denote the set of all partitions of $\mathcal{S}$ in mutually parallel classical geodetically closed sub near polygons by $\Upsilon(\mathcal{S})$.

A *spread* of a near polygon $\mathcal{S}$ is a set of lines partitioning the point set. A spread is called *admissible* if it belongs to $\Upsilon(\mathcal{S})$, or equivalently, if every two lines of it are parallel. Obviously, every spread of a generalized quadrangle is admissible. A spread S is called *regular* if it is admissible and if the following holds for any two lines $K, L \in S$ with $\mathrm{d}(K, L) = 1$: (i) $\{K, L\}^{\perp\perp}$ and $\{K, L\}^{\perp}$ cover the same set of points of $\mathcal{S}$, (ii) every line of $\{K, L\}^{\perp\perp}$ belongs to S. ($\{K, L\}^{\perp}$ is the set of lines of $\mathcal{S}$ meeting K and L, $\{K, L\}^{\perp\perp}$ is the set of lines of $\mathcal{S}$ meeting each line of $\{K, L\}^{\perp}$.) A spread S is called *a spread of symmetry* if for every line $K \in S$ and for every two points k_1 and k_2 on K there exists an automorphism of $\mathcal{S}$ fixing each line of S and mapping k_1 to k_2. Obviously, every spread of symmetry is regular. If S is an admissible spread (a spread of symmetry) of a near polygon $\mathcal{S}$ and if F is a geodetically closed sub near polygon of $\mathcal{S}$, then the set S_F of all lines of S which are contained in F is either empty or an admissible spread (a spread of symmetry) of F, see e.g. Theorem 5 of [7].

2. Motivation and short overview

Spreads of symmetry give rise to new near polygons, the so-called *glued near polygons*, see [5] and [7]. For all known classes of indecomposable dense near polygons (these are dense near polygons which are not glued and not a product near polygon), all spreads of symmetry have been determined (see Section 5 for an overview). Something which has not yet been done is the study of spreads of symmetry in glued near polygons themselves. This study, see Theorems 4.4 and 4.5, led to the notion of compatible spreads of symmetry which we will discuss in Section 3. The known examples of compatible spreads of symmetry of indecomposable dense near polygons are listed in Section 5. Each such pair of spreads will give rise to new glued near polygons. In Theorem 4.7, we will describe how compatible spreads of symmetry in glued near polygons are obtained. This theorem can be used to construct further examples of glued near polygons. In order to be complete, we also study (compatible) spreads of symmetry in product near polygons (Theorems 4.4, 4.5 and 4.6), but this will essentially not lead to new glued near polygons.

3. Compatible spreads of symmetry

Theorem 3.1. *Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon, let S_1 and S_2 denote two different spreads of symmetry in $\mathcal{A}$ and let G_i, $i \in \{1, 2\}$, denote the group of automorphisms of $\mathcal{A}$ which fix each line of S_i. Then the following are equivalent:*

(i) $[G_1, G_2] = 0$;
(ii) *for every line $l \in S_1$ and every $g \in G_2$, $l^g \in S_1$;*
(iii) *for every line $l \in S_2$ and every $g \in G_1$, $l^g \in S_2$;*
(iv) *the partial linear space $\mathcal{B} = (\mathcal{P}, S_1 \cup S_2, \mathrm{I}_{|\mathcal{P}\times(S_1\cup S_2)})$ is a disjoint union of lines and grids.*

Proof: $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$: By symmetry, it suffices to prove the implication $(i) \Rightarrow (ii)$. Let l denote an arbitrary line of S_1, let x denote an arbitrary point of l and let g denote an arbitrary element of G_2. Then $l^g = (x^{G_1})^g = (x^g)^{G_1} \in S_1$.

$(ii) \Rightarrow (iv)$ and $(iii) \Rightarrow (iv)$: By symmetry, it suffices to prove the implication $(ii) \Rightarrow (iv)$. Suppose that the lines $K_1 \in S_1$ and $K_2 \in S_2$ intersect in a point x^*. For all $x_1 \in K_1$, $x_1^{G_2} \in S_2$ and for all $x_2 \in K_2$, $x_2^{G_1} \in S_1$. We will now prove that the lines $x_1^{G_2}$, $x_1 \in K_1$, and $x_2^{G_1}$, $x_2 \in K_2$, define a subgrid of $\mathcal{B}$. Obviously, $x_1^{G_2} \cap x_1'^{G_2} = \emptyset$ for all $x_1, x_1' \in K_1$ with $x_1 \neq x_1'$ and $x_2^{G_1} \cap x_2'^{G_1} = \emptyset$ for all $x_2, x_2' \in K_2$ with $x_2 \neq x_2'$. Now, consider arbitrary points $x_1 \in K_1$ and $x_2 \in K_2$ and let g_2 denote an arbitrary element of G_2 such that $x_2 = (x^*)^{g_2}$. The point x_2 lies on the line $K_1^{g_2}$ which, by our assumption, belongs to the spread S_1. So, $x_2^{G_1} = K_1^{g_2}$ and $x_2^{G_1} \cap x_1^{G_2} = K_1^{g_2} \cap x_1^{G_2} = \{x_1^{g_2}\}$. As a consequence, every two different intersecting lines of $S_1 \cup S_2$ are contained in a subgrid of $\mathcal{B}$. The implication now follows from the fact that every point of $\mathcal{B}$ is contained in at most two lines of $\mathcal{B}$.

$(iv) \Rightarrow (i)$: Let x be an arbitrary point of $\mathcal{A}$, let g_1 be an arbitrary element of G_1 and let g_2 be an arbitrary element of G_2. We will prove that $x^{g_1g_2} = x^{g_2g_1}$. We distinguish the following cases.

- Suppose that x is contained in a subgrid G of $\mathcal{B}$. Let l_i, $i \in \{1, 2\}$, denote the unique line of S_i through x. Since $x^{g_i} \in l_i$, the unique line m_{3-i} of S_{3-i} through x^{g_i} is contained in G. Let y be the common point of the lines m_1 and m_2. Since $x \sim x^{g_1}$, $x^{g_2} \sim x^{g_1 g_2}$. So, $x^{g_1 g_2}$ is the unique point of m_2 collinear with x^{g_2}. Hence, $y = x^{g_1 g_2}$. In a similar way, one proves that $y = x^{g_2 g_1}$. As a consequence, $x^{g_1 g_2} = x^{g_2 g_1}$.
- Suppose that x is not contained in a subgrid of $\mathcal{B}$, i.e. x is contained in a line L of $S_1 \cap S_2$. Since $S_1 \neq S_2$, $\mathcal{B}$ has a subgrid G. Every line of G is parallel with L. Let $y \in G$ such that x is the unique point of L nearest to x. Then $x^{g_1 g_2}$ (respectively $x^{g_2 g_1}$) is the unique point of $L = L^{g_1 g_2} = L^{g_2 g_1}$ nearest to $y^{g_1 g_2}$ (respectively $y^{g_2 g_1}$). Since $y^{g_1 g_2} = y^{g_2 g_1}$, it follows that $x^{g_1 g_2} = x^{g_2 g_1}$.

□

Definition. Let $\mathcal{A}$ be a near polygon, let S_1 and S_2 be two (possibly equal) spreads of symmetry of $\mathcal{A}$ and let G_i, $i \in \{1, 2\}$, denote the group of automorphisms of $\mathcal{A}$ which fix each line of S_i. Then the spreads S_1 and S_2 are called *compatible* if $[G_1, G_2] = 0$. In the case that S_1 and S_2 are different, Theorem 3.1 allows us to give some equivalent definitions.

4. Importance of compatible spreads of symmetry

In this section we will show that compatible spreads of symmetry give rise to spreads of symmetry in glued near polygons and hence also to new glued near polygons.

4.1. Product and glued near polygons

Let $\mathcal{A}$ be a dense near polygon. For every $i \in \{0, 1\}$, let $\Delta_i(\mathcal{A})$ denote the set of all pairs $\{T_1, T_2\}$ satisfying the following properties:

(1) T_j, $j \in \{1, 2\}$, is a partition of $\mathcal{A}$ in geodetically closed sub near polygons of diameter at least $i + 1$;
(2) every element of T_1 intersects every element of T_2 in a point (if $i = 0$) or a line (if $i = 1$);
(3) every line of $\mathcal{A}$ is contained in at least one element of $T_1 \cup T_2$;
(4) if $i = 1$, then the spread induced in every element $F \in T_j$, $j \in \{1, 2\}$, by intersecting it with the elements of T_{3-j} is an admissible spread.

If $\{T_1, T_2\} \in \Delta_i(\mathcal{A})$, then by Lemma 2 of [11] and Corollary 4.7 of [9], there exist near polygons $\mathcal{A}_1$ and $\mathcal{A}_2$ such that every element of T_j, $j \in \{1, 2\}$, is isomorphic to $\mathcal{A}_j$. If $j = 0$, then $\mathcal{A}$ is the product near polygon $\mathcal{A}_1 \times \mathcal{A}_2$. If $i = 1$, then $\mathcal{A}$ is a so-called glued near polygon and we will say that $\mathcal{A}$ is a glued near polygon of type $\mathcal{A}_1 \otimes \mathcal{A}_2$. We refer to [7] for the precise definition of glued near polygon. A spread S of $\mathcal{A}$ is said to be *trivial* if there exists a $T \in \Upsilon(\mathcal{A})$ such that $\{S, T\} \in \Delta_0(\mathcal{A})$. If $\{T_1, T_2\} \in \Delta_1(\mathcal{A}) \setminus \Delta_0(\mathcal{A})$, then the spread induced in every element $F \in T_j$, $j \in \{1, 2\}$, by intersecting it with the elements of T_{3-j} is a spread of symmetry. So, glued near polygons give rise to spreads of symmetry. Also the converse is true. If S_1 and S_2 are spreads of symmetry of near polygons $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfying certain properties (see Theorem 14 of [7]), then they give rise to glued near polygons of type $\mathcal{A}_1 \otimes \mathcal{A}_2$. In this paper we study spreads of symmetry in product and glued near polygons. Spreads of symmetry in glued near polygons will give rise to new glued near polygons. Spreads of symmetry in product near polygons will essentially not give rise to new near polygons, because by Lemma 8 of [13], every near polygon of type $(\mathcal{A}_1 \times \mathcal{A}_2) \otimes \mathcal{A}_3$ is also of type $(\mathcal{A}_j \otimes \mathcal{A}_3) \times \mathcal{A}_{3-j}$ for a certain $j \in \{1, 2\}$.

4.2. The sets $\Upsilon_0(\mathcal{A})$ and $\Upsilon_1(\mathcal{A})$ for a dense near polygon $\mathcal{A}$

Let $\mathcal{A}$ be a dense near polygon. We say that an element $T \in \Upsilon(\mathcal{A})$ belongs to $\Upsilon_i(\mathcal{A}), i \in \{0, 1\}$, if there exists a $T' \in \Upsilon(\mathcal{A})$ such that $\{T, T'\} \in \Delta_i(\mathcal{A})$.

Proposition 4.1. *If $T \in \Upsilon_0(\mathcal{A})$, then there exists a unique $T' \in \Upsilon(\mathcal{A})$ such that $\{T, T'\} \in \Delta_0(\mathcal{A})$.*

Proof: Let x denote an arbitrary point of $\mathcal{A}$, let F denote the unique element of T through x and let $L_1, \ldots, L_k$ denote all the lines through x not contained in T. If T' is an element of $\Upsilon(\mathcal{A})$ such that $\{T, T'\} \in \Delta_0(\mathcal{A})$, then the unique element of T' through x coincides with $\mathcal{C}(L_1, \ldots, L_k)$. This proves the proposition. □

Proposition 4.2. *If $T \in \Upsilon_1(\mathcal{A}) \setminus \Upsilon_0(\mathcal{A})$, then there exists a unique $T' \in \Upsilon(\mathcal{A})$ such that $\{T, T'\} \in \Delta_1(\mathcal{A})$.*

Proof: Let x denote an arbitrary point of $\mathcal{S}$, let F_x denote the unique element of T through x and let $L_1, \ldots, L_k$ denote the lines through x not contained in F_x. Let T' be an element of $\Upsilon(\mathcal{A})$ such that $\{T, T'\} \in \Delta_1(\mathcal{A})$ and let F'_x denote the unique element of T' through x. Clearly, $\mathrm{diam}(F_x) + \mathrm{diam}(F'_x) = \mathrm{diam}(\mathcal{A}) + 1$, $\mathrm{diam}(F_x \cap F'_x) = 1$ and $L_1, \ldots, L_k \subseteq F'_x$. There are two possibilities.

(a) $\mathcal{C}(L_1, \ldots, L_k) \cap F_x = \{x\}$.
In this case, we have $\mathrm{diam}\,(\mathcal{C}(L_1, \ldots, L_k)) = \mathrm{diam}(F'_x) - 1$ and hence $\mathrm{diam}(\mathcal{A}) = \mathrm{diam}(F_x) + \mathrm{diam}(\mathcal{C}(L_1, \ldots, L_k))$. By Lemma 3 of [11], $T \in \Upsilon_0(\mathcal{A})$, a contradiction.
(b) $\mathcal{C}(L_1, \ldots, L_k) \cap F_x$ is a line.
In this case, we have $\mathcal{C}(L_1, \ldots, L_k) = F'_x$.

The proposition now easily follows. □

Remark. The previous proposition is not necessarily valid if $T \in \Upsilon_0(\mathcal{A})$. Suppose that $\{T, T'\} \in \Delta_0(\mathcal{A})$, let $F \in T$ and let S be an admissible spread of F. For every point x of F, define $\tilde{F}_x := \mathcal{C}(L_x, F'_x)$, where L_x denotes the unique line of S through x and F'_x denotes the unique element of T' through x. Put $\tilde{T} := \{\tilde{F}_x | x \in F\}$. Then $\{T, \tilde{T}\} \in \Delta_1(\mathcal{A})$. So, if the near polygon F has two different admissible spreads, there exists at least two $\tilde{T} \in \Upsilon(\mathcal{A})$ such that $\{T, \tilde{T}\} \in \Delta_1(\mathcal{A})$.

Definitions. Let $T \in \Upsilon_0(\mathcal{A}) \cup \Upsilon_1(\mathcal{A})$. If $T \in \Upsilon_0(\mathcal{A})$, then we denote by T^C the unique element of $\Upsilon_0(\mathcal{A})$ such that $\{T, T^C\} \in \Delta_0(\mathcal{A})$. If $T \in \Upsilon_1(\mathcal{A}) \setminus \Upsilon_0(\mathcal{A})$, then we denote by T^C the unique element of $\Upsilon(\mathcal{A})$ such that $\{T, T^C\} \in \Delta_1(\mathcal{A})$. We call T^C the *complementary partition* of T. If $T \in \Upsilon_0(\mathcal{A})$, then $(T^C)^C = T$. If $T \in \Upsilon_1(\mathcal{A}) \setminus \Upsilon_0(\mathcal{A})$, then $(T^C)^C$ is not necessarily equal to T (see the previous remark). We denote by $\tilde{\Upsilon}_1(\mathcal{A})$ the set of all $T \in \Upsilon_1(\mathcal{A}) \setminus \Upsilon_0(\mathcal{A})$ for which $(T^C)^C = T$. We also define $\tilde{\Delta}_1(\mathcal{A}) := \Delta_1(\mathcal{A}) \cap \binom{\tilde{\Upsilon}_1(\mathcal{A})}{2}$.

4.3. Extensions of spreads and automorphisms

Let $\mathcal{A}$ be a dense near polygon, let T denote an element of $\Upsilon_0(\mathcal{A}) \cup \tilde{\Upsilon}_1(\mathcal{A})$, let F denote an arbitrary element of T and let F' denote an arbitrary element of T^C.

- For every spread S of F, we define $\bar{S} := \{\pi_{F,E}(L) \mid E \in T \text{ and } L \in S\}$. Obviously, $\bar{S}$ is a spread of $\mathcal{A}$. We call $\bar{S}$ the *extension* of S.
- For every automorphism θ of F and for every point x of $\mathcal{A}$, we define $\bar{\theta}(x) := \pi_{F,F_x} \circ \theta \circ \pi_{F_x,F}(x)$. Here F_x denotes the unique element of T through x. Obviously, $\bar{\theta}$ is a permutation of the point set of $\mathcal{A}$. We call $\bar{\theta}$ the extension of θ.

Proposition 4.3. (a) *Let* $T \in \Upsilon_0(\mathcal{A})$.

$(a1)$ *If* θ *is an automorphism of* F, *then* $\overline{\theta}$ *is an automorphism of* $\mathcal{A}$.
$(a2)$ *If* ϕ *is an automorphism of* $\mathcal{A}$ *fixing each element of* T, *then* $\phi = \bar{\theta}$ *for some automorphism* θ *of* F.
$(a3)$ *If* ϕ_1 *is an automorphism of* $\mathcal{A}$ *fixing each element of* T *and if* ϕ_2 *is an automorphism of* $\mathcal{A}$ *fixing each element of* T^C, *then* ϕ_1 *and* ϕ_2 *commute.*
(b) *Let* $T \in \tilde{\Upsilon}_1(\mathcal{A})$, *let* S^* *denote the spread of* F *obtained by intersecting* F *with every element of* T^C *and let* G^* *denote the group of automorphisms of* F *fixing each line of* S^*. *Let* θ *denote an automorphism of* F. *Then* $\bar{\theta}$ *is an automorphism of* $\mathcal{A}$ *if and only if* θ *commutes with every element of* G^*.

Proof: Properties (a1), (a2) and (a3) are straightforward. In order to prove property (b), it suffices to prove that $\bar{\theta}$ maps collinear points x and y to collinear points $\bar{\theta}(x)$ and $\bar{\theta}(y)$. There are two possibilities.

- $F_x = F_y$.
  The statement follows from the fact that the maps $\pi_{F_x,F}$, θ and π_{F,F_x} are isomorphisms.
- $F_x \neq F_y$.
  The points $\bar{\theta}(x)$ and $\bar{\theta}(y)$ are collinear if and only if $\pi_{F_x,F_y} \circ \bar{\theta}(x) = \bar{\theta}(y)$, i.e. if and only if $\pi_{F_x,F_y} \circ \pi_{F,F_x} \circ \theta \circ \pi_{F_x,F}(x) = \pi_{F,F_y} \circ \theta \circ \pi_{F_y,F} \circ \pi_{F_x,F_y}(x)$.

Hence, $\bar{\theta}$ is an automorphism if and only if $\pi_{F_x,F_y} \circ \pi_{F,F_x} \circ \theta = \pi_{F,F_y} \circ \theta \circ \pi_{F_y,F} \circ \pi_{F_x,F_y} \circ \pi_{F,F_x}$ for all $F_x, F_y \in T$, i.e. if and only if θ commutes with $\pi_{F_y,F} \circ \pi_{F_x,F_y} \circ \pi_{F,F_x}$ for all $F_x, F_y \in T$. Let S' denote the spread of F' obtained by intersecting F' with every element of T. Since $T \in \tilde{\Upsilon}_1(\mathcal{A})$, S' is not a trivial spread of F' and S^* is not a trivial spread of F. Put $H := \{\pi_{F_y,F} \circ \pi_{F_x,F_y} \circ \pi_{F,F_x} \mid F_x, F_y \in T\}$. Then $|H| = |\Pi_{S'}(F \cap F')|$ and $H \leq G^*$. By Theorem 11 of [7], $|H| = s + 1$ and $|G^*| = s + 1$. Hence $H = G^*$. As a consequence, $\bar{\theta}$ is an isomorphism if and only if θ commutes with every element of G^*. □

By the following two theorems, all spreads of symmetry in product and glued near polygons are characterized. The first theorem has been proved in [13] in the case $T \in \tilde{\Upsilon}_1(\mathcal{A})$. The result also holds if $T \in \Upsilon_0(\mathcal{A})$ (with a similar proof).

Theorem 4.4 (Lemma 7 of [13]). *Every admissible spread (spread of symmetry)* S *of* $\mathcal{A}$ *is the extension of an admissible spread (spread of symmetry) in* F *or* F'.

Theorem 4.5. *Let* S *be a spread of symmetry of* F *and let* G_S *denote the group of automorphisms of* F *fixing each line of* S.

(a) *If* $T \in \Upsilon_0(\mathcal{A})$, *then* $\bar{S}$ *is a spread of symmetry of* $\mathcal{A}$.
(b) *If* $T \in \tilde{\Upsilon}_1(\mathcal{A})$, *let* S^* *denote the spread of* F *obtained by intersecting* F *with every element of* T^C *and let* G^* *denote the group of automorphisms of* F *fixing each line of* S^*. *Then* $\bar{S}$ *is a spread of symmetry of* $\mathcal{A}$ *if and only if* $[G^*, G_S] = 0$.

Proof:

(a) The group $\bar{G}_S := \{\bar{\theta} \mid \theta \in G_S\}$ fixes each line of $\bar{S}$ and acts transitively on each line of $\bar{S}$. So, $\bar{S}$ is a spread of symmetry.

(b) If $[G^*, G_S] = 0$, then by Proposition 4.3, $\bar{G}_S := \{\bar{\theta} \mid \theta \in G_S\}$ is a group of automorphisms of $\mathcal{A}$. Since $\bar{G}_S$ fixes each line of $\bar{S}$ and acts transitively on each line of $\bar{S}$, $\bar{S}$ is a spread of symmetry. Conversely, suppose that $\bar{S}$ is a spread of symmetry, let $G_{\bar{S}}$ denote the group of automorphisms of $\mathcal{A}$ fixing each line of $\bar{S}$. Then $G_{\bar{S}} = \bar{G}$ for some subgroup G of G_S. By Proposition 11, we have $[G^*, G] = 0$. If $G_S = G$, then we are done. If $G_S \neq G$, then $|G_S| > |G| \geq s + 1$. So, S is a trivial spread of F by the remark following Theorem 9 of [7]. Since $T \in \tilde{\Upsilon}_1(\mathcal{A})$, $S \neq S^*$. By Theorem 5 of [7], S and S^* have no line in common. By Proposition 4.3 (a3), it then follows that $[G^*, G_S] = 0$. □

4.4. Compatible spreads of symmetry in product and glued near polygons

Theorem 4.6. *Let $\mathcal{A}$ be a product near polygon, let $\{T_1, T_2\} \in \Delta_0(\mathcal{A})$, let $F_1 \in T_1$ and let $F_2 \in T_2$. Let S_0 and S_1 denote two spreads of symmetry of F_1 and let S_2 denote a spread of symmetry of F_2. Then*

(i) $\bar{S}_1$ and $\bar{S}_2$ are compatible spreads of symmetry of $\mathcal{A}$,
(ii) $\bar{S}_0$ and $\bar{S}_1$ are compatible spreads of symmetry of $\mathcal{A}$ if and only if S_0 and S_1 are compatible spreads of symmetry of F_1.

Proof: Property (i) follows from Proposition 4.3 (a3). If $\theta_0 \in G_0$ and $\theta_1 \in G_1$, then $\bar{\theta}_0\bar{\theta}_1 = \overline{\theta_0\theta_1}$ and $\bar{\theta}_1\bar{\theta}_0 = \overline{\theta_1\theta_0}$. So, θ_0 and θ_1 commute if and only if $\bar{\theta}_0$ and $\bar{\theta}_1$ commute. This proves (ii). □

Theorem 4.7. *Let $\mathcal{A}$ be a glued near polygon, let $\{T_1, T_2\} \in \tilde{\Delta}_1(\mathcal{A})$, let $F_1 \in T_1$ and let $F_2 \in T_2$. Let S_i^* denote the spread of symmetry of F_i obtained by intersecting F_i with the elements of T_{3-i}. Let S_0 and S_1 denote two spreads of symmetry of F_1 and let S_2 denote a spread of symmetry of F_2. Then*

(i) $\bar{S}_0$ and $\bar{S}_1$ are compatible spreads of symmetry if and only if the spreads S_0, S_1 and S_1^* are mutually compatible.
(ii) $\bar{S}_1$ and $\bar{S}_2$ are compatible spreads of symmetry of $\mathcal{A}$ if and only if for every $i \in \{1, 2\}$, S_i and S_i^* are compatible.

Proof:

(i) We may suppose that the pairs (S_0, S_1^*) and (S_1, S_1^*) are compatible (otherwise $\bar{S}_0$ and $\bar{S}_1$ would not be spreads of symmetry). For every $i \in \{0, 1\}$, let G_i denote the group of automorphisms of F_1 fixing each element of S_i. Since (S_i, S_1^*) is compatible, $\bar{G}_i = \{\bar{\theta} \mid \theta \in G_i\}$ is the full group of automorphisms of $\mathcal{A}$ fixing each element of $\bar{S}_i$. If $\theta_0 \in G_0$ and $\theta_1 \in G_1$, then as before θ_0 and θ_1 commute if and only if $\bar{\theta}_0$ and $\bar{\theta}_1$ commute. So, $\bar{S}_0$ and $\bar{S}_1$ are compatible spreads of symmetry of $\mathcal{A}$ if and only if S_0 and S_1 are compatible spreads of symmetry of F. This proves (i).

(ii) For $\bar{S}_i$, $i \in \{1, 2\}$, to be a spread of symmetry it is necessary that S_i and S_i^* are compatible. Conversely, suppose that for every $i \in \{1, 2\}$, S_i and S_i^* are compatible. If $S_1 = S_1^*$ and $S_2 = S_2^*$, then by (i) it follows that $\bar{S}_1 = \bar{S}_2$ is compatible with itself. We will therefore suppose that $S_1 \neq S_1^*$ or $S_2 \neq S_2^*$. Then $\bar{S}_1 \neq \bar{S}_2$. Let $L_1 \in \bar{S}_1$ and $L_2 \in \bar{S}_2$ be two lines intersecting in a point x. Let U_i, $i \in \{1, 2\}$, denote the unique element of T_i through x. Since $L_1 \subseteq U_1, L_2 \subseteq U_2, L_1 \neq L_2, \mathcal{C}(L_1, L_2)$ is a grid Q. By Theorem 5 of [7], the lines of Q disjoint from L_i, $i \in \{1, 2\}$, belong to $\bar{S}_i$. So, the spreads $\bar{S}_1$ and $\bar{S}_2$ satisfy property (iv) of Theorem 3.1 and hence are compatible. This proves (ii). $\square$

5. Known examples of compatible spreads in dense near polygons

In this section, we will list all known examples of compatible spreads of symmetry in dense near polygons. By Theorems 4.6 and 4.7, we may content ourselves to those dense near polygons $\mathcal{S}$ for which $\Delta_0(\mathcal{S}) = \emptyset = \Delta_1(\mathcal{S})$.

5.1. Near polygons with a linear representation

Let $\mathcal{K}$ be a set of points in PG(n, q) which satisfies the following properties:

(A) $\langle \mathcal{K} \rangle = \mathrm{PG}(n, q)$,

(B) for every point x of $\mathcal{K}$ and for every line L through x, there exists a unique point in $L \setminus \{x\}$ with smallest $\mathcal{K}$-index (the *$\mathcal{K}$-index* of a point y is the smallest number of points of $\mathcal{K}$ which are necessary to generate a subspace through y).

Now, embed PG(n, q) as a hyperplane Π_∞ in a projective space PG$(n + 1, q)$ and consider the point-line incidence structure $T_n^*(\mathcal{K})$ whose points are the affine points of PG$(n + 1, q)$ (i.e. the points of PG$(n + 1, q)$ not contained in Π_∞) and whose lines are the lines of PG$(n + 1, q)$ not contained in Π_∞ and intersecting Π_∞ in a point of $\mathcal{K}$ (natural incidence). By Theorem 4.4 of [10], the conditions (A) and (B) imply that $T_n^*(\mathcal{K})$ is a near polygon. For every point x of $\mathcal{K}$, let S_x denote the set of lines of PG$(n + 1, q)$ through x not contained in Π_∞. The group H of q elations of PG$(n + 1, q)$ with center x and axis Π_∞ determines a group G of automorphisms of $T_n^*(\mathcal{H})$. Since G fixes each line of S_x and acts regularly on each line of S_x, the spread S_x is a spread of symmetry. By the remark following Theorem 9 of [7], G is the full group of automorphisms of $T_n^*(\mathcal{K})$ fixing each line of S_x.

Proposition 5.1 (Theorem 7 of [7]). *If $q \geq 3$ (the case of dense near polygons), then every spread of symmetry of $T_n^*(\mathcal{K})$ is of the form S_x for some point x of $\mathcal{K}$.*

Since any two elations with axis Π_∞ commute, we have the following result.

Proposition 5.2. *For all $x_1, x_2 \in \mathcal{K}$, the spreads S_{x_1} and S_{x_2} are compatible.*

There are only two examples known of dense near polygons $T_n^*(\mathcal{K})$ with $\Delta_0(T_n^*(\mathcal{K})) = \Delta_1(T_n^*(\mathcal{K})) = \emptyset$. If $\mathcal{K}$ is a hyperoval in PG$(2, 2^h)$, $h \geq 2$, then $T_2^*(\mathcal{K})$ is a generalized quadrangle of order $(2^h - 1, 2^h + 1)$, see [18]. If $\mathcal{K}$ is the Coxeter-cap in PG$(5, 3)$ ([3]), then $T_5^*(\mathcal{K})$ is a near hexagon, see [10].

5.2. The near polygons $H^D(2n - 1, q^2)$, $n \geq 2$

Let the vector space $V(2n, q^2)$, $n \geq 2$, with base $\{\bar{e}_0, \ldots, \bar{e}_{2n-1}\}$, be equipped with a non-singular Hermitian form $(\cdot, \cdot)$, i.e., $(\sum a_i \bar{e}_i, \sum b_j \bar{e}_j) = \sum a_i b_j^q (\bar{e}_i, \bar{e}_j)$ for any two vectors $\sum a_i \bar{e}_i$ and $\sum b_j \bar{e}_j$ of $V(2n, q^2)$. Let ζ denote the corresponding Hermitian polarity of PG$(2n - 1, q^2)$ and let $H(2n - 1, q^2)$ denote the corresponding Hermitian variety in

PG$(2n-1,q^2)$. The dual polar space $H^D(2n-1,q^2)$ is the point-line incidence structure whose points, respectively lines, are the maximal, respectively next-to-maximal, subspaces of $H(2n-1,q^2)$. Let $p=\langle\bar{e}\rangle$ denote an arbitrary point of PG$(2n-1,q^2)$ not contained in $H(2n-1,q^2)$. Then p^ζ is a nontangent hyperplane which intersects $H(2n-1,q^2)$ in a $H(2n-2,q^2)$. Obviously, the set of all $(n-2)$-dimensional subspaces contained in p^ζ is a spread S_p of $H^D(2n-1,q^2)$. For every $\lambda\in$ GF(q^2) with $\lambda^{q+1}=1$, let $\theta_{\bar{e},\lambda}$ denote the following linear map of $V(2n,q^2)$: $\bar{x}\mapsto\bar{x}+(\lambda-1)\frac{(\bar{x},\bar{e})}{(\bar{e},\bar{e})}\bar{e}$. The map $\theta_{\bar{e},\lambda}$ induces an automorphism of PG$(2n-1,q^2)$ which fixes the Hermitian variety $H(2n-1,q^2)$ and every point of the hyperplane p^ζ. Since $\theta_{\bar{e},\lambda_1}\circ\theta_{\bar{e},\lambda_2}=\theta_{\bar{e},\lambda_1\lambda_2}$ for all $\lambda_1,\lambda_2\in$ GF(q^2) with $\lambda_1^{q+1}=\lambda_2^{q+1}=1$, $H:=\{\theta_{\bar{e},\lambda}\mid\lambda^{q+1}=1\}$ is a subgroup of GL$(2n,4)$. Let G denote the group of automorphisms of $H^D(2n-1,q^2)$ induced by the elements of H. Then G fixes each line of S_p. Since G acts regularly on each line of S_p, S_p is a spread of symmetry of $H^D(2n-1,q^2)$. By the remark following Theorem 9 of [7], G is the full group of automorphisms fixing each line of S_p. In [12], it was shown that every spread of symmetry is of the form S_p, where p is a point of PG$(2n-1,q^2)$ not contained in $H(2n-1,q^2)$.

Proposition 5.3. *Let p_1 and p_2 denote two points of* PG$(2n-1,q^2)$ *not contained in $H(2n-1,q^2)$. Then S_{p_1} is compatible with S_{p_2} if and only if either $p_1=p_2$ or $p_1\in p_2^\zeta$.*

Proof: Let $\bar{e}_1$ and $\bar{e}_2$ be vectors of $V(2n,q^2)$ such that $p_1=\langle\bar{e}_1\rangle$ and $p_2=\langle\bar{e}_2\rangle$ and let λ_1 and λ_2 denote arbitrary elements of GF$(q^2)\setminus\{1\}$ satisfying $\lambda_1^{q+1}=\lambda_2^{q+1}=1$. Then one calculates that $\theta_{\bar{e}_2,\lambda_2}\circ\theta_{\bar{e}_1,\lambda_1}$ is the following map:

$$\bar{x}\mapsto\bar{x}+(\lambda_1-1)\frac{(\bar{x},\bar{e}_1)}{(\bar{e}_1,\bar{e}_1)}\bar{e}_1+(\lambda_2-1)\frac{(\bar{x},\bar{e}_2)}{(\bar{e}_2,\bar{e}_2)}\bar{e}_2+(\lambda_1-1)(\lambda_2-1)\frac{(\bar{x},\bar{e}_1)(\bar{e}_1,\bar{e}_2)}{(\bar{e}_1,\bar{e}_1)(\bar{e}_2,\bar{e}_2)}\bar{e}_2.$$

Similarly, one calculates that $\theta_{\bar{e}_1,\lambda_1}\circ\theta_{\bar{e}_2,\lambda_2}$ is given by:

$$\bar{x}\mapsto\bar{x}+(\lambda_1-1)\frac{(\bar{x},\bar{e}_1)}{(\bar{e}_1,\bar{e}_1)}\bar{e}_1+(\lambda_2-1)\frac{(\bar{x},\bar{e}_2)}{(\bar{e}_2,\bar{e}_2)}\bar{e}_2+(\lambda_1-1)(\lambda_2-1)\frac{(\bar{x},\bar{e}_2)(\bar{e}_2,\bar{e}_1)}{(\bar{e}_1,\bar{e}_1)(\bar{e}_2,\bar{e}_2)}\bar{e}_1.$$

Hence $\theta_{\bar{e}_1,\lambda_1}$ and $\theta_{\bar{e}_2,\lambda_2}$ commute if and only if either $\bar{e}_1\parallel\bar{e}_2$ or $(\bar{e}_1,\bar{e}_2)=0$. The proposition now easily follows. □

As a consequence, there are many pairs of compatible spreads of symmetry in $H^D(2n-1,q^2)$, but not every two spreads of symmetry of $H^D(2n-1,q^2)$ are compatible. In particular, there are many pairs of compatible spreads of symmetry in the GQ $H^D(3,q^2)\cong Q(5,q)$.

5.3. The generalized quadrangle $AS(q)$, q odd

For every odd prime power q, we can construct the following generalized quadrangle $AS(q)$, see [1]. The points of $AS(q)$ are the points of the affine space $AG(3,q)$ and the lines are the following curves in AG$(3,q)$ (natural incidence):

(1) $x=\sigma$, $y=a$, $z=b$;
(2) $x=a$, $y=\sigma$, $z=b$;
(3) $x=c\sigma^2-b\sigma+a$, $y=-2c\sigma$, $z=\sigma$.

Here, the parameter σ ranges over the elements of GF(q) and a, b, c are arbitrary elements of GF(q). The lines of type (1) form a spread S.

Proposition 5.4. *The spread S is a spread of symmetry which is compatible with itself.*

Proof: For every $\lambda \in \mathrm{GF}(q)$, the translation $(x, y, z) \mapsto (x + \lambda, y, z)$ induces an automorphism θ_λ of $AS(q)$ which fixes each line of S. Obviously, the group $G := \{\theta_\lambda \mid \lambda \in \mathrm{GF}(q)\}$ acts regularly on every line of S. By the remark following Theorem 9 of [7], G is the full group of automorphisms fixing each line of S. The proposition now follows from the fact that any two elements of G commute. □

If $q \geq 5$, then S is the only spread of symmetry of $AS(q)$ by Theorem 6.3. of [4]. The generalized quadrangle $AS(3) \cong Q(5, 2) \cong H^D(3, 4)$ has more spreads of symmetry as we have seen in Section 5.2.

5.4. The generalized quadrangle $S^-_{x,y}$

The generalized quadrangle $S^-_{x,y}$ first occurred in [14], but we give the description taken from [15]. Let $\mathcal{H}$ be a hyperoval in PG($2, 2^h$), $h \geq 1$, which is embedded as a hyperplane π in PG($3, 2^h$) and let x and y be two different points of $\mathcal{H}$. The following generalized quadrangle of order $(2^h + 1, 2^h - 1)$ can then be constructed. The points of $S^-_{x,y}$ are of three types:

(1) points of PG($3, 2^h$) not contained in π;
(2) planes through x not containing y;
(3) planes through y not containing x.

The lines of $S^-_{x,y}$ are those lines of PG($3, 2^h$) which are not contained in π and which intersect $\mathcal{H} \setminus \{x, y\}$. A point of $S^-_{x,y}$ and a line of $S^-_{x,y}$ are incident if and only if they are incident as objects of PG($3, 2^h$). One easily sees that the points of type (2) form an ovoid O_{xy} of $S^-_{x,y}$. Similarly, the points of type (3) form an ovoid O_{yx} of $S^-_{x,y}$. [An ovoid (of symmetry) is the dual notion of a spread (of symmetry), i.e., a set of points in a generalized quadrangle Q is called an ovoid (of symmetry) if it is a spread (of symmetry) in the point-line dual of Q.]

Proposition 5.5. *Each of the pairs (O_{xy}, O_{xy}), (O_{yx}, O_{yx}), (O_{xy}, O_{yx}), is compatible.*

Proof: The group H_x (respectively H_y) of the 2^h elations of PG($3, 2^h$) with center x (respectively y) and axis π determine a group G_x (respectively G_y) of automorphisms of $S^-_{x,y}$. The group G_x fixes the ovoid O_{xy} and acts regularly on any set of lines through a given point of O_{xy}. This proves that O_{xy} is an ovoid of symmetry. By the remark following Theorem 9 of [7], G_x is the full group of automorphisms of $S^-_{x,y}$ fixing each point of O_{xy}. Similarly, O_{yx} is an ovoid of symmetry and G_y is the full group of automorphisms of $S^-_{x,y}$ fixing each point of O_{yx}. The proposition now follows from the fact that any two elements of $G_x \cup G_y$ commute. □

If the point-line dual of $S^-_{x,y}$ is not isomorphic to $T_2^*(\mathcal{H})$ (see [16] when this precisely occurs), then by [17], O_{xy} and O_{yx} are the only regular ovoids and hence also the only ovoids of symmetry of $S^-_{x,y}$. If the point-line dual of $S^-_{x,y}$ is isomorphic to $T_2^*(\mathcal{H})$ then there are more than two ovoids of symmetry in $S^-_{x,y}$ as we have seen in Section 5.1. So, if $h \geq 2$ then the generalized quadrangle $[S^-_{x,y}]^D$ has either 2 or $q+2$ spreads of symmetry and all these spreads of symmetry are compatible.

5.5. The near polygons $\mathbb{G}_n$, $n \geq 2$

Let the vector space $V(2n, 4)$, $n \geq 2$, with base $\{\bar{e}_0, \ldots, \bar{e}_{2n-1}\}$ be equipped with the non-singular Hermitian form $(\bar{x}, \bar{y}) = x_0 y_0^2 + x_1 y_1^2 + \cdots + x_{2n-1} y_{2n-1}^2$, let ζ denote the corresponding Hermitian polarity of $PG(2n-1, 4)$ and let $H(2n-1, 4)$ denote the corresponding Hermitian variety in $PG(2n-1, 4)$. The *weight* of a point $\langle x_0\bar{e}_0 + x_1\bar{e}_1 + \cdots + x_{2n-1}\bar{e}_{2n-1}\rangle$ is defined as the number of $i \in \{0, \ldots, 2n-1\}$ for which $x_i \neq 0$. Let $\mathbb{G}_n$ be the incidence structure whose points are the $(n-1)$-dimensional subspaces of $H(2n-1, 4)$ containing precisely n points of weight 2 and whose lines are the $(n-2)$-dimensional subspaces of $H(2n-1, 4)$ containing at least $n-2$ points of weight 2 (incidence is reverse containment). It was shown in [8] that $\mathbb{G}_n$ is a dense near polygon of order $(2, \frac{3n^2-n-2}{2})$. For every $i \in \{0, \ldots, 2n-1\}$, let S_i denote the set of $(n-2)$-dimensional subspaces of $H(2n-1, 4)$ which are contained in $\langle \bar{e}_i \rangle^\zeta$ and which contain at least $n-2$ points of weight 2. By Lemma 15 of [8], S_i is a spread of symmetry of $\mathbb{G}_n$.

Proposition 5.6. *For all $i, j \in \{0, 1, \ldots, 2n-1\}$, S_i is compatible with S_j.*

Proof: For every $\lambda \in GF(4)^*$ and every $i \in \{0, \ldots, 2n-1\}$, let $\theta_{i,\lambda}$ denote the following linear map of $V(2n, 4)$: $\bar{e}_i \mapsto \lambda \bar{e}_i$ and $\bar{e}_j \mapsto \bar{e}_j$ for every $j \in \{0, \ldots, 2n-1\} \setminus \{i\}$. Obviously, the linear map $\theta_{i,\lambda}$ induces an automorphism $\tilde{\theta}_{i,\lambda}$ of $\mathbb{G}_n$. Now, let G_i, $i \in \{0, \ldots, 2n-1\}$, denote the group of automorphisms of $\mathbb{G}_n$ fixing each line of S_i. By the remark following Theorem 9 of [7], we know that $G_i := \{\tilde{\theta}_{i,\lambda} \mid \lambda \in GF(4)^*\}$. It is now easy to see that $[G_i, G_j] = 0$ for all $i, j \in \{0, \ldots, 2n-1\}$. □

If $n \geq 3$, then the spreads S_i, $i \in \{0, \ldots, 2n-1\}$, are the only spreads of symmetry of $\mathbb{G}_n$ (Corollary 3 of [8]). As a consequence, $\mathbb{G}_n$, $n \geq 3$, has $2n$ spreads of symmetry and all these spreads are mutually compatible. The generalized quadrangle $\mathbb{G}_2 \cong Q(5, 2) \cong H^D(3, 4)$ has more than $2n = 4$ spreads of symmetry and not every pair of such spreads is compatible, as we have seen in Section 5.2.

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