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A basis for the non-crossing partition lattice top homology

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Abstract We find a basis for the top homology of the non-crossing partition lattice T_n. Though T_n is not a geometric lattice, we are able to adapt techniques of Björner (A. Björner, On the homology of geometric lattices. Algebra Universalis 14 (1982), no. 1, 107–128) to find a basis with C_{n-1} elements that are in bijection with binary trees. Then we analyze the action of the dihedral group on this basis.

Keywords Non-crossing partition· Binary trees· Homology group· Catalan numbers· Representation matrix· Dihedral group· Stack-sortable permutations

1. Preliminaries

Let Π_n be the set of all partitions of the set $[n] = \{1, 2, \ldots, n\}$. Elements Y of Π_n are denoted by $Y = B_1/\cdots/B_k$, where the subsets $B_1, \ldots, B_k$ partition $[n]$ and are called the blocks of Y. With the refinement ordering, $X \leq Y$ if each block of X is contained in a block of Y, and the rank function, $r_{\Pi_n}(B_1/\cdots/B_k) = n - k$, the set Π_n is a ranked lattice. Two disjoint subsets A and B of $[n]$ are said to be crossing if there are $a, b \in A$ and $x, y \in B$ such that $a < x < b < y$ or $x < a < y < b$. A partition $Y = B_1/\cdots/B_k$ of $[n]$ is called non-crossing if no two of its blocks cross. Let T_n denote the ranked lattice of all non-crossing partitions of $[n]$ ordered by refinement and with the same rank function as for Π_n.

Let $n \geq 3$. Kreweras shows in [8] that $\mu(T_n) = \mu_{T_n}(\mathbf{0}, \mathbf{1}) = (-1)^{n-1} C_{n-1}$ where $\mathbf{1} = /12\cdots n/$, $\mathbf{0} = 1/2/\cdots/n$ and C_k is the k-th Catalan number:

$$C_k = \frac{1}{k+1}\binom{2k}{k}.$$

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The lattice T_n is EL-shellable [3] and thus Cohen-Macaulay. Then $\tilde{H}_{n-3}(T_n)$ is a free abelian group of rank $|\mu(T_n)| = C_{n-1}$.

2. NC-bases

An element M of the partition lattice Π_n is an atom if M has $n-1$ blocks. We write $M = /i\ j/$ to denote the atom with the block $\{i, j\}$ and all other blocks singletons.

Definition 2.1. *A set B of atoms of Π_n is an NC-base if $|B| = n-1$, the blocks of its elements do not cross pairwise and the equality $\bigvee B = \mathbf{1}$ holds in Π_n.*

Theorem 2.2. *If $B = \{b_1, \ldots, b_{n-1}\}$ is an NC-base then the equality $r_{T_n}(b_1 \vee b_2 \vee \cdots \vee b_i) = i$ holds.*

Proof: Since Π_n is semimodular, $r_{\Pi_n}(b_1 \vee b_2 \vee \cdots \vee b_i \vee b_{i+1}) \leq r_{\Pi_n}(b_1 \vee b_2 \vee \cdots \vee b_i) + 1$. Since B is an NC-base, $r_{\Pi_n}(b_1 \vee \cdots \vee b_{n-1}) = r_{\Pi_n}(\mathbf{1}) = n-1$. Therefore the numbers $r_{\Pi_n}(b_1), r_{\Pi_n}(b_1 \vee b_2), \ldots, r_{\Pi_n}(b_1 \vee \cdots \vee b_{n-1})$ are increasing by at most 1, starting with 1, and ending with $n-1$. This can only happen if $r_{\Pi_n}(b_1 \vee \cdots \vee b_i) = i$. It remains to prove that in Π_n, $b_1 \vee \cdots \vee b_i$ is a non-crossing partition.

We will show by induction on i that $b_1 \vee \cdots \vee b_i$ is non-crossing. For $i = 1$ it is clear. Suppose $x < y < z < w$ with $x, z \in B_1$ and $y, w \in B_2$ where B_1 and B_2 are blocks of $b_1 \vee \cdots \vee b_i$. By hypothesis, $b_1 \vee \cdots \vee b_{i-1}$ is non-crossing and so B_1 and B_2 cannot both be blocks of $b_1 \vee \cdots \vee b_{i-1}$. Assume that B_1 is the union of two blocks C_1, C_2 of $b_1 \vee \cdots \vee b_{i-1}$ with $x \in C_1$ and $z \in C_2$. There exists an atom $b_i = /x'z'/$ with $x' \in C_1$, $z' \in C_2$. Because the blocks in $b_1 \vee \cdots \vee b_{i-1}$ do not cross and $y < z < w$, it follows that $y < z' < w$ and so we can assume that $z = z'$. Similarly, we may assume that $x = x'$.

Write $u \sim v$ if $/u\,v/ = b_j$ for some $j = 1, \ldots, i$. There is a sequence of vertices $t_1, \ldots, t_k$ such that $y = t_0 \sim t_1 \sim \cdots \sim t_k \sim t_{k+1} = w$. Since the atoms $b_1, \ldots, b_i$ do not cross, $x < t_0 < z$ implies $x < t_1 < z$ and so on, so we conclude that $x < t_{k+1} < z$, a contradiction. $\square$

Definition 2.3. *Let $B = \{b_1, \ldots, b_{n-1}\}$ be an NC-base. If $\pi \in S_{n-1}$ let $\sigma_\pi(B)$ in T_n be the maximal chain given by*

$$\sigma_\pi(B) = [b_{\pi(1)}, b_{\pi(1)} \vee b_{\pi(2)}, \ldots, b_{\pi(1)} \vee b_{\pi(2)} \vee \cdots \vee b_{\pi(n-2)}].$$

By Theorem 2.2 this is in fact a maximal chain in $T_n \setminus \{\mathbf{0}, \mathbf{1}\}$ and therefore a simplex of dimension $n-3$ in $K(T_n)$.

Define

$$\rho_B = \sum_{\pi \in S_{n-1}} (-1)^\pi \sigma_\pi(B)$$

where $(-1)^\pi$ is the sign of the permutation π.

A simple calculation proves the following

Theorem 2.4. *If B is an NC-base then $\partial_{n-2}(\rho_B) = 0$ and thus $\rho_B \in \tilde{H}_{n-3}$.*

3. The construction of the basis

Recall from [12] that there is a bilinear form $\langle , \rangle$ defined on chains by

$$\langle c_1, c_2 \rangle = \begin{cases} 1 & \text{if } c_1 = c_2, \\ 0 & \text{otherwise} \end{cases}$$

and extended by linearity. Using this bilinear form, the homology and comology are dual spaces. The following lemma is a useful tool for finding bases of homology and cohomology.

Lemma 3.1. *Let P be a finite poset. If $H_r(P)$ has dimension m and there are elements $\rho_1, \rho_2, \ldots, \rho_m \in H_r(P)$ and $\gamma_1, \gamma_2, \ldots, \gamma_m \in H^r(P)$ such that $\langle \rho_i, \gamma_j \rangle = \delta_{i,j}$ for all $i, j = 1, \ldots, m$, then $\{\rho_1, \rho_2, \ldots, \rho_m\}$ is a basis for $H_r(P)$ with dual basis $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ for $H^r(P)$.*

We will use binary trees to construct a set of C_{n-1} elements of $H_{n-3}(T_n)$ and a set of C_{n-1} elements of $H^{n-3}(T_n)$ such that the previous lemma holds.

Definition 3.2. *A binary tree is an ordered rooted tree where each node has two subtrees, which can be empty. We distinguish between the left and the right subtree. The root of the left(right) subtree is the left(right) son of the root. The vertices of a binary tree are ordered recursively, with the left subtree ordered first, then the root, and finally the right subtree.*

Definition 3.3. *A binary tree is a right tree if its left subtree is empty. Let M_n be the set of all right trees.*

It is well-known that C_n is the number of binary trees with n vertices. Since the root of an ordered right tree has the label 1, there is an obvious bijection between the set M_n of right trees with n vertices and the set of binary trees with $n - 1$ vertices. Thus $|M_n| = C_{n-1}$.

Example 3.4. The five right trees with 4 vertices are shown in Figure 1.

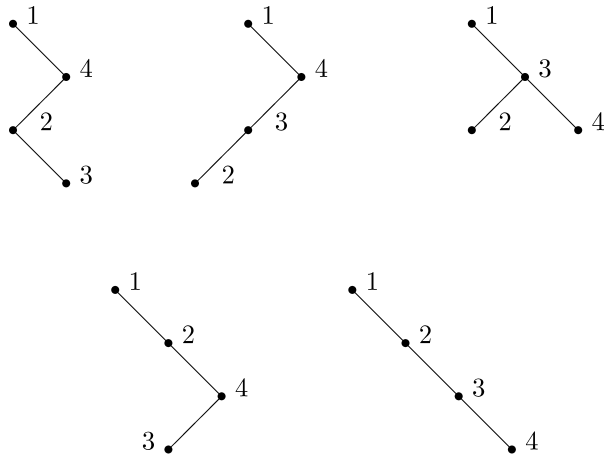

Fig. 1 All right trees with five vertices

Enumerate the vertices of a right tree as above. Each vertex will be identified with its label. For each $A \in M_n$ let B_A be the set of atoms such that $/i\ j/ \in B_A$ if and only if (i, j) is an edge of A. In the following we shall let $/i\ j/$ represent either the atom or the edge.

Theorem 3.5. *If $A \in M_n$ then B_A is an NC-base.*

Proof: Since A has $n-1$ edges, $|B_A| = n-1$. Since A is a tree every pair of vertices is joined by a sequence of edges. This implies that $\bigvee B_A = \mathbf{1}$ in Π_n.

We will show that the elements of B_A do not cross. Consider the atoms $/a, c/$ and $/b, d/$ and assume they cross. Without loss of generality suppose that $a < b < c < d$. Then A has edges (a, c) and (b, d). In such a binary tree the root is less than every vertex in the right subtree and greater than every vertex in the left subtree. There is an edge between a and c and therefore a is a son of c or vice versa. Suppose that c is a son of a. Then c must be the right son of a and, since b was enumerated after a and before c, b must be in the left subtree of c. But then every vertex connected to b must also be in this subtree and hence is less than or equal to c. Thus there cannot be an edge between b and d. The case when a is a son of c is similar. It follows that the atoms of B_A do not cross pairwise. □

For brevity, let ρ_A denote the simplex ρ_{B_A} associated to the NC-basis B_A. Recall that B_A is a set of atoms.

Example 3.6. Let A be the right tree in Figure 2.

The edges are: $b_1 = (1, 2)$, $b_2 = (2, 4)$, $b_3 = (3, 4)$.

$$\begin{aligned}\rho_A &= (b_1, b_1 \vee b_2) - (b_1, b_1 \vee b_3) - (b_2, b_2 \vee b_1) + (b_2, b_2 \vee b_3) + (b_3, b_3 \vee b_1) \\ &\quad - (b_3, b_3 \vee b_2) = (1\ 2/3/4, 1\ 2\ 4/3) - (1\ 2/3/4, 1\ 2/3\ 4) - (1/2\ 4/3, 1\ 2\ 4/3) \\ &\quad + (1/2\ 4/3, 1/2\ 3\ 4) + (1/2\ 4/3, 1\ 2/3\ 4) - (1/2\ 4/3, 1/2\ 3\ 4).\end{aligned}$$

Definition 3.7. *Enumerate the edges of a binary tree depth recursively with the edge joining the root with the left subtree first, then the edges of the left subtree, then the edge joining the root with the right subtree, and, finally, the edges of the right subtree.*

Let $b_1, \ldots, b_{n-1}$ be the edges of $A \in M_n$ in this order. The characteristic chain of A is $S_A = [b_1, b_1 \vee b_2, \ldots, b_1 \vee \cdots \vee b_{n-2}]$. By Theorem 2.2 it is a maximal chain in T_n.

The following lemma is clear.

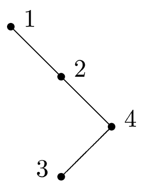

Fig. 2 A right tree

Lemma 3.8. *For every $j = 1, \ldots, n-1$ the edges $b_1, \ldots, b_j$ form a connected component of A, which contains the root.*

Lemma 3.9. *Every block of $b_1 \vee \cdots \vee b_k$ has only one element, except the one containing the vertex 1. If $a > 1$ belongs to this block, then the father of a also belongs to it.*

Proof: The first part is clear from Lemma 3.8. Suppose $a > 1$ belongs to the connected component but its father b does not. Then there is a path from a to 1 which does not contain the vertex b. But this implies that there is a cycle in A contradicting that A is a tree. □

Theorem 3.10. $\langle S_A, \rho_{A'} \rangle = \delta_{A,A'}$

Proof: It is clear from the definition of ρ_A that S_A is a chain of ρ_A and that it appears with the sign $+$. Assume that $\pm S_A$ is a chain of $\rho_{A'}$ and that the atoms of ρ_A and ρ'_A are $b_1, b_2, \ldots, b_{n-1}$ and $b'_1, b'_2, \ldots, b'_{n-1}$, respectively (ordered by depth). Then there is a permutation $k_1, k_2, \ldots, k_{n-1}$ of $1, 2, \ldots, n-1$ such that $b_1 \vee \cdots \vee b_m = b'_{k_1} \vee \cdots \vee b'_{k_m}$ for $m = 1, \ldots, n-1$.

We will show, by induction on m, that $b_m = b'_{k_m}$ and that the connected component formed by the edges $b_1, \ldots, b_m$ in A is equal to the connected component formed by the edges $b'_{k_1}, \ldots, b'_{k_m}$ in A'. For $m = 1$ we have $b_1 = b'_{k_1} = /1, i/$. Assume that $b_1 = b'_{k_1}, \ldots, b_{m-1} = b'_{k_{m-1}}$. Let U, V be blocks with more than one element in $b_1 \vee \cdots \vee b_{m-1} = b'_{k_1} \vee \cdots \vee b'_{k_{m-1}}$ and $b_1 \vee \cdots \vee b_m = b'_{k_1} \vee \cdots \vee b'_{k_m}$, respectively (they exist by Lemma 3.9). Then $V = U \cup \{x\}$, where $b_m = /x, y/, b'_{k_m} = /x, z/$, with $y, z \in U$. It must happen that, in the tree A', x is a son of z or vice versa. But $z \in U$, and if z is a son of x then $x \in U$, which is impossible. Therefore x is a son of z in A'. In the same way, x is a son of y in A. Let E be the tree formed by the edges $b_1, \ldots, b_{m-1}$. The vertices of this tree are the elements of U. Consider the trees F and F' whose edges are $b_1, \ldots, b_m$ and $b'_{k_1}, \ldots, b'_{k_m}$, respectively. These two trees are obtained by adding the edges b_m and b'_{k_m}, respectively, to the tree E and therefore they can only differ in the edge containing the vertex x. Assume that $y \neq z$. Let $1 = a_1, a_2, \ldots, a_t = x$, $1 = a'_1, a'_2, \ldots, a'_s = x$ be the paths from 1 to x in the trees F and F', respectively. We have $a_{t-1} = y$, $a'_{s-1} = z$. Let $j = \max\{l | a_l = a'_l\} < \min\{s-1, t-1\}$. Then $a_j = a'_j$, $a_{j+1} \neq a'_{j+1}$. This situation is shown in Figure 3.

Since $j + 1 < s, t$, $a_{j+1} \neq x$ and $a'_{j+1} \neq x$ and therefore a_{j+1} and a'_{j+1} are in the tree E, given that a_{j+1} and a'_{j+1} are sons of $a_j = a'_j$.

Without loss of generality we may assume that $x < a_j$. Then a_{j+1} must be the left son of a_j in E. In the same way, a'_{j+1} must be the left son of $a'_j = a_j$ in E, which contradicts that $a_{j+1} \neq a'_{j+1}$. We conclude that $y = z$ and therefore $b_m = b'_{k_m}$. Finally, when we adjoin the same edge to two equal connected components, we obtain the same connected component. This completes the induction. □

The next theorem follows from Lemma 3.1.

Theorem 3.11. $\{\rho_A | A \in M_n\}$ *is a basis of* $\tilde{H}_{n-3}(T_n)$.

Note that this base is different from the one that is obtained from an EL-labeling because there is no maximal chain that is shared by all the elements of the basis. In fact, every atom belongs to at least one chain.

In [12], several bases for $H(\Pi_n)$ are described, using node labeled trees. Although our basis is described in terms of node labeled trees and the construction itself is almost the same, the trees in Wachs' article are increasing (meaning that every vertex except the root is less than its father) while the binary trees used here are not. It is the fact that the trees we consider here have their vertices ordered that guarantees that the partitions are non-crossing.

4. The action of the dihedral group

Let D_n be the dihedral group. The elements of D_n have the form $\tau^a\gamma^b$, with $0 \leq a \leq n-1$, $0 \leq b \leq 1$, where τ is the rotation given by $\tau(i) = i-1$ for $2 \leq i \leq n$ and $\tau(1) = n$; and γ is the reflection defined by $\gamma(1) = 1$ and $\gamma(i) = n-i+2$. We want to analyze the action of D_n on the homology. To do this we analyze the action of τ and γ.

It is proved in [12] that, under the hypothesis of Lemma 3.1, the representation matrix $M(g)$ for g acting on $H_r(P)$ with respect to the basis $\{\rho_1, \dots, \rho_m\}$ has i, j component given by $M_{i,j}(g) = \langle g\rho_j, \gamma_i\rangle$. As a consequence, we have the following lemma.

Lemma 4.1. *For every element g in the dihedral group D_n, the entries $M_{i,j}(g)$ of its representation matrix with respect to the basis in Theorem 3.11 are -1, 0 or 1.*

In the following sections we will give a method for calculating the representation matrices $M(g)$ with respect to the basis in Theorem 3.11.

4.1. Action of the reflection

Lemma 4.2. *For every tree $A \in M_n$ there exists a tree $D \in M_n$ such that $\gamma(\rho_A) = \pm\rho_D$.*

Proof: Let $A \in M_n$, with edges $a_1, \dots, a_{n-1}$, ordered by depth. Then $\gamma(a_1), \dots, \gamma(a_{n-1})$ are the edges of a tree $D = \gamma(A) \in M_n$. When these edges are enumerated by depth, they may appear in a different order $\gamma(a_{\beta(1)}), \dots, \gamma(a_{\beta(n-1)})$. Then, for every $\pi \in S_{n-1}$,

$$\begin{aligned}\gamma(\sigma_\pi(A)) &= [\gamma(a_{\pi(1)}), \dots, \gamma(a_{\pi(1)}) \vee \gamma(a_{\pi(2)}) \vee \cdots \vee \gamma(a_{\pi(n-2)})] \\ &= [\gamma(a_{\pi\circ\beta^{-1}\circ\beta(1)}), \cdots, \gamma(a_{\pi\circ\beta^{-1}\circ\beta(1)}) \vee \ldots \vee \gamma(a_{\pi\circ\beta^{-1}\circ\beta(n-2)})] = \sigma_{\pi\circ\beta^{-1}}(D),\end{aligned}$$

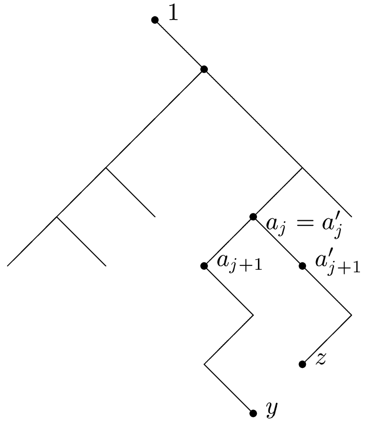

Fig. 3 An impossible situation

and therefore

$$\begin{aligned}\gamma(\rho_A) &= \sum_{\pi\in S_{n-1}} (-1)^{\pi}\gamma(\sigma_\pi(A))\\ &= \sum_{\pi\in S_{n-1}} (-1)^{\pi}\sigma_{\pi\circ\beta^{-1}}(D) = (-1)^{\beta}\sum_{\pi\in S_{n-1}} (-1)^{\pi\circ\beta^{-1}}\sigma_{\pi\circ\beta^{-1}}(D)\\ &= (-1)^{\beta}\sum_{\pi\in S_{n-1}} (-1)^{\pi}\sigma_\pi(D) = \pm\rho_D.\end{aligned}$$

□

As a consequence of this lemma, the representation matrix of γ is the direct sum of matrices of the forms

$$\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \begin{pmatrix}0&-1\\-1&0\end{pmatrix},\ (1),\ (-1).$$

Example 4.3. Let A be the tree on the left in Figure 4.

The edges of A are $a_1=(1,2),\ a_2=(2,4),\ a_3=(3,4),\ a_4=(4,5),\ a_5=(5,6)$. Then $\gamma(a_1)=(1,6),\ \gamma(a_2)=(4,6),\ \gamma(a_3)=(4,5),\ \gamma(a_4)=(3,4),\ \gamma(a_5)=(2,3)$. In order, the edges of D are $(1,6),\ (4,6),\ (3,4),\ (2,3),\ (4,5)$ and the corresponding permutation is $\beta = 1\,2\,4\,5\,3$. Since this permutation is even, we conclude that $\gamma(\rho_A)=\rho_D$.

Now we want to find the number of times that each of the matrices given above appears in the representation matrix $M(\gamma)$. The following lemma is clear from the proof of the Lemma 4.2.

Lemma 4.4. *For $A\in M_n$, $\gamma\,(\rho_A)=\pm\rho_A$ if and only if the right subtree of A (the tree with vertices $2,3,\ldots,n$) is symmetric, i.e., if it is invariant under the action of γ.*

Now we give a method to determine the sign in the expression $\gamma(\rho_A)=\pm\rho_D$.

In A, let k be the son of 1, and let A_l and A_r be the left and right subtrees of k, respectively. Then A_l has $k-2$ vertices and A_r has $n-k$ vertices. The edges in A_l, including the one that joins its root with k, are $a_2,\ldots,a_{k-1}$. As in the proof of the Lemma 4.2, let β_l be the permutation of the edges of A_l such that $\gamma(a_{\beta_l(2)}),\ldots,\gamma(a_{\beta_l(k-1)})$ are the edges of $\gamma(A_l)$, ordered by depth. Define β_r in an analogous way.

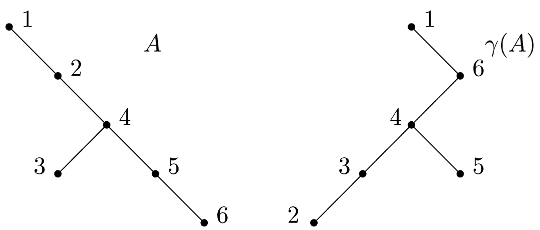

Fig. 4 A tree and its reflection

Then the edges of $\gamma(A)$, ordered by depth, are

$$\gamma(a_{\beta(1)}), \ldots, \gamma(a_{\beta(n-1)}) = a_1, \gamma(a_{\beta_r(k)}), \ldots, \gamma(a_{\beta_r(n)}), \gamma(a_{\beta_l(2)}), \ldots, \gamma(a_{\beta_l(k-1)}).$$

From this we conclude that the sign of β, which is also the sign in the expression $\gamma(\rho_A) = \pm\rho_D$, is $(-1)^{(k-2)(n-k)}(-1)^{\beta_l}(-1)^{\beta_r}$. We will use this fact in the proof of the following theorem.

Theorem 4.5. *The number of trees $A \in M_n$ such that $\gamma(\rho_A) = \rho_A$ is $C_{(n-2)/2}$ if n is congruent to 2 mod 4, and 0 otherwise.*

The number of trees $A \in M_n$ such that $\gamma(\rho_A) = -\rho_A$ is $C_{(n-2)/2}$ if n is multiple of 4, and 0 otherwise.

In other words, the multiplicity of the matrix $(-1)^{(n-2)/2}$ in the representation matrix for γ is $C_{(n-2)/2}$.

Proof: If n is odd, the trees in M_n cannot be symmetric, and the result follows from Lemma 4.4. If n is even, say $n = 2m$, there are C_{m-1} symmetric trees (the left subtree is determined by the right one). Let β, β_l and β_r be as in the paragraph before the statement of the theorem above. Then β_l and β_r are inverse permutations and so they have the same sign. Thus $(-1)^\beta = (-1)^{(m-1)^2} = (-1)^{m-1}$. We have concluded $\gamma(\rho_A) = (-1)^{(m-1)}\rho_A$ for every $A \in M_n$. □

The following result will enable us to determine the multiplicities of the 2×2 matrices in $M(\gamma)$.

Theorem 4.6. *Let x_n (resp. y_n) be the number of $A \in M_n$ so that $\gamma(\rho_A) = +\rho_D$ (resp. $\gamma(\rho_A) = -\rho_D$) for some $D \in M_n$. Then*

$$x_n = \begin{cases} \displaystyle\sum_{k=2}^{n}(x_{k-1}x_{n-k+1} + y_{k-1}y_{n-k+1}), & \text{if } n = 2m+1, \\ \displaystyle\sum_{p=1}^{m}(x_{2p-1}x_{n-2p+1} + y_{2p-1}y_{n-2p+1}) + 2\sum_{p=1}^{m-1}(x_{2p}y_{n-2p}), & \text{if } n = 2m, \end{cases}$$

and

$$y_n = \begin{cases} \displaystyle 2\sum_{k=2}^{n}x_{k-1}y_{n-k+1}, & \text{if } n = 2m+1, \\ \displaystyle 2\sum_{p=1}^{m}(2x_{2p-1}y_{n-2p+1}) + \sum_{p=1}^{m-1}(x_{2p}x_{n-2p} + y_{2p}y_{n-2p}), & \text{if } n = 2m. \end{cases}$$

Note that the multiplicity of the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in the representation matrix of γ is $\left(x_n - C_{(n-2)/2}\right)/2$ if n congruent to 2 mod 4, and $x_n/2$ otherwise. A similar expresion for the multiplicity of $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ can be found.

Proof: Using the formula $(-1)^\beta = (-1)^{(k-2)(n-k)}(-1)^{\beta_l}(-1)^{\beta_r}$ and taking into account the parity of n, we consider several cases. Assume $n = 2m$ is even. Then $\gamma(\rho_A) = +\rho_D$ if $k = 2p$

is even and β_l, β_r have the same sign, or if $k = 2p + 1$ is odd and β_l, β_r have different sign. In these cases

$$x_n = \sum_{p=1}^{m}(x_{2p-1}x_{n-2p+1} + y_{2p-1}y_{n-2p+1}) + \sum_{p=1}^{m-1}(x_{2p}y_{n-2p} + y_{2p}x_{n-2p})$$

and

$$y_n = 2\sum_{p=1}^{m}(x_{2p-1}y_{n-2p+1}) + \sum_{p=1}^{m-1}(x_{2p}x_{n-2p} + y_{2p}y_{n-2p}).$$

The other expressions are similar. □

4.2. Characteristic sequences

In order to find the representation matrix of the rotation τ we have to introduce the tool of the characteristic sequences.

Every tree $A \in M_n$ with edges $\{b_1, \dots b_{n-1}\}$ has a characteristic chain $S_A = [b_1, b_1 \vee b_2, \dots, b_1 \vee \dots \vee b_{n-2}]$. By Lemma 3.9, for $i = 1, \dots n-2$, the block U_i of $b_1 \vee \dots \vee b_i$ which contains the vertex 1 is the unique block with more than one element. Let $U_0 = \{1\}$. It is clear that $U_{i-1} \subset U_i$. Therefore we can order the vertices of A, such that $U_i = \{1, a_1, \dots a_{i-1}, a_i\}$. This ordering of the vertices is called a pre-order. It can be obtained recursively by first taking the root, then the vertices of the left subtree, and finally the vertices of the right subtree.

Definition 4.7. *The sequence $1, a_1, \dots a_{n-1}, a_n$ is called the characteristic sequence of the tree A.*

Example 4.8. The characteristic sequence of the tree A shown in Figure 4 is 1, 2, 4, 3, 5, 6.

Definition 4.9. *The stack-sorting S of a sequence of diferent numbers is defined recursively as follows: $S(\emptyset) = \emptyset$ and $S(A_1 x A_2) = S(A_1)S(A_2)x$ if x is the largest element of the sequence $A_1 x A_2$. A permutation $\sigma \in S_n$ (considered as a sequence) is called stack-sortable if $S(\sigma) = 1, 2, \dots, n$.*

There is a well-known correspondence between binary trees and stack-sortable permutations that can be found in [7]. As a consequence, we have the following result.

Theorem 4.10. *A sequence $1, a_1, \dots a_{n-1}$ is the characteristic sequence of some $A \in M_n$ if and only if it is stack-sortable and $\{a_1, \dots a_{n-1}\} = \{2, \dots, n\}$.*

Note that the number 1 at the beginning of the sequence forces the left subtree to be empty and guarantees that the tree is a right tree.

The following lemma, which can be found in [7], will be useful later.

Lemma 4.11. *If $a_1, \dots a_m$ is a stack-sortable permutation, then there is no triple of indices $i < j < k$ so that $a_k < a_i < a_j$.*

4.3. Action of the rotation

We are going to use the result from [12] mentioned at the beginning of this section. Let $A \in M_n$ with edges $b_1, \dots, b_n$, ordered by depth. Let $b_i' = \tau(b_i)$. Note that $b_1', \dots, b_n'$ are the edges of a binary tree A' with ordered vertices. The root is $\tau(1) = n$, the right subtree is empty, and the left subtree is obtained by subtracting 1 from the vertices of the right subtree of A. This is because τ switches the root with every other vertex and leaves the ordering of the other vertices unchanged. An example is shown in Figure 5.

If $t_G \neq 0$ and the edges of G (ordered by depth) are $d_1, \dots, d_{n-1}$ then S_G appears in

$$\tau(\rho_A) = \sum_{G \in M_n} t_G \rho_G = \tau\left(\sum_{\pi \in S_n} (-1)^\pi \sigma_\pi(A)\right) = \sum_{\pi \in S_n} (-1)^\pi \tau\left(\sigma_\pi(A)\right).$$

Thus there exists a permutation $\pi \in S_n$ with $t_G = (-1)^\pi$ and

$$\begin{aligned} S_G &= [d_1, d_1 \vee d_2, \dots, d_1 \vee \cdots \vee d_{n-1}] \\ &= \tau(\sigma_\pi(A)) = \tau([b_{\pi(1)}, b_{\pi(1)} \vee b_{\pi(2)}, \dots, b_{\pi(1)} \vee b_{\pi(2)} \vee \cdots \vee b_{\pi(n-1)}]) \\ &= [b'_{\pi(1)}, b'_{\pi(1)} \vee b'_{\pi(2)}, \dots, b'_{\pi(1)} \vee \cdots \vee b'_{\pi(n-1)}]. \end{aligned}$$

By Lemma 3.9, $d_1 \vee \cdots \vee d_j = b'_{\pi(1)} \vee \cdots \vee b'_{\pi(j)}$ has only one block with more than one element, and so $b'_{\pi(1)}, \dots, b'_{\pi(j)}$ form a connected component of the tree A'. Let W_j be the block with more than one element in $d_1 \vee \cdots \vee d_j = b'_{\pi(1)} \vee \cdots \vee b'_{\pi(j)}$ and let $1, g_1, \dots, g_{n-1}$ the characteristic sequence of G. Then $W_j = \{1, g_1, \dots, g_j\}$. From this we conclude the following.

Lemma 4.12. *Let $G \in M_n$ with characteristic sequence $1, g_1, \dots, g_{n-1}$. Then ρ_G appears in $\tau(\rho_A)$ with non-zero coefficient if and only if for every j the vertices $1, g_1, \dots, g_j$ form a connected component of $A' = \tau(A)$.*

Example 4.13. Let A be the left tree in Figure 5. The edges of A are $b_1 = (1,5), b_2 = (5,3), b_3 = (3,2), b_4 = (3,4), b_5 = (5,6)$, and the edges of A' are $b_1' = (6,4), b_2' = (4,2), b_3' = (2,1), b_4' = (2,3), b_5' = (4,5)$.

The characteristic sequences for possible G with $t_G \neq 0$ are: 1, 2, 3, 4, 5, 6; 1, 2, 4, 3, 5, 6; 1, 2, 3, 4, 6, 5; 1, 2, 4, 3, 6, 5; and the trees (that we call G_1, G_2, G_3 and G_4, respectively) for these sequences are as shown in Figure 6.

Other sequences, like 1, 2, 4, 6, 5, 3, are also obtained from A' but they are not characteristic sequences since they are not stack-sortable. Therefore $\tau(\rho_A) = \pm\rho_{G_1} \pm \rho_{G_2} \pm \rho_{G_3} \pm \rho_{G_4}$.

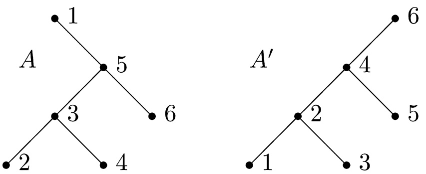

Fig. 5 A tree and its rotation

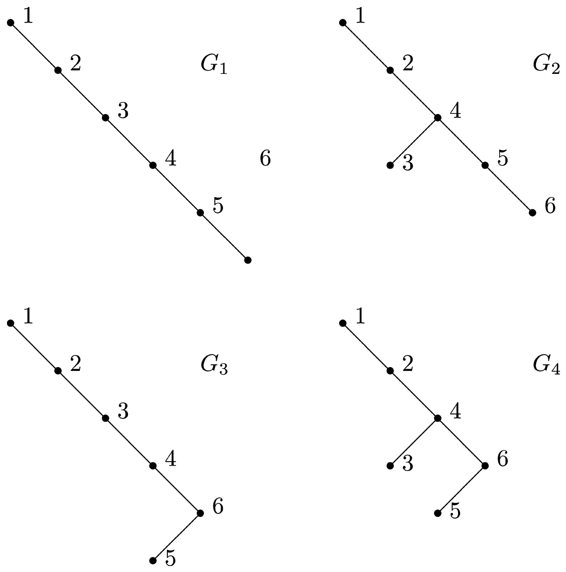

Fig. 6 The possible trees G

The signs can be calculated from the signs of the corresponding permutations of the edges as in the Example 4.3. Thus, in this example, $\tau(\rho_A) = \rho_{G_1} - \rho_{G_2} - \rho_{G_3} + \rho_{G_4}$.

The Lemma 4.15 give us a shortcut for calculations as the ones in the previous example.

Definition 4.14. *Let $y_1 = 1$ and, for $i \geq 0$, let y_{i+1} be the father of y_i in A'. The sequence $y_1, \ldots, y_s = n$ is called the main branch of A'. Let Y_i be the set containing the vertex y_i and every vertex in the right subtree of y_{i-1}.*

Since 1 has no left son in A', $Y_2 \cup \cdots \cup Y_s = \{2, \ldots, n\}$.

Lemma 4.15. *Let $(g_i) = 1, g_1, \ldots, g_{n-1}$ be as in Lemma 4.12. Then the first terms of the sequence $g_1, \ldots, g_{n-1}$ are the elements of Y_2 in some order, followed by the elements of Y_3 in some order, and so on.*

Proof: The vertex with the number $1 = y_1$ has as father the vertex y_2, its left subtree is empty, and the vertices in its right subtree are the other elements of Y_2. Therefore the vertices adjacent to 1 belong to Y_2, so $g_1 \in Y_2$. But after choosing a vertex, we must take every vertex less than it that has not been chosen before (according to Lemma 4.11), and those vertices are in $Y_1 \cup Y_2$. Similarly we show that the vertices in every Y_i appear consecutively because the vertices with label less than y_i are in $Y_1 \cup \cdots \cup Y_i$. □

Example 4.16. For the tree A in Example 4.13, $y_1 = 1$, $y_2 = 2$, $y_3 = 4$, $y_4 = 6$ and $Y_1 = \{1\}$, $Y_2 = \{2\}$, $Y_3 = \{4, 3\}$, $Y_4 = \{6, 5\}$, and the possible characteristic sequences are 1, 2, 3, 4, 5, 6; 1, 2, 3, 4, 6, 5;1, 2, 4, 3, 5, 6; and 1, 2, 4, 3, 6, 5.

In general, it is not true that every ordering of Y_i will work: by Theorem 4.10, the sequence must be stack-sortable.

Although this gives us an algorithm to find the representation matrix for τ, it would be interesting to find a more general way to do it.

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