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Homotopy theory of graphs

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1. Introduction

Recently a new homotopy theory for graphs and simplicial complexes was defined (cf. [3, 4]). The motivation for the definition came initially from a desire to find invariants for dynamic processes that could be encoded via (combinatorial) simplicial complexes. The invariants were supposed to be topological in nature, but should at the same time be sensitive to the combinatorics encoded in the complex, in particular to the level of connectivity of simplices (see [7]). Namely, let Δ be a simplicial complex of dimension d, let $0 \leq q \leq d$ be an integer, and let $\sigma_0 \in \Delta$ be a simplex of dimension greater than or equal to q. One obtains a family of groups

$$A_n^q(\Delta, \sigma_0), \quad n \geq 1,$$

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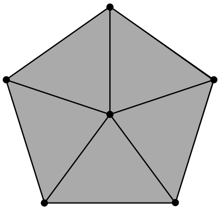


Fig. 1 A 2-dimensional complex Δ with nontrivial A_1^1.

the A-groups of Δ, based at σ_0. These groups differ from the classical homotopy groups of Δ in a significant way. For instance, the group $A_1^1(\Delta, \sigma_0)$, for the 2-dimensional complex Δ in Figure 1 is isomorphic to $\mathbb{Z}$, measuring the presence of a "connectivity" hole in its center. (See the example on p. 101 of [4].) The theory is based on an approach proposed by R. Atkin [1, 2]; hence the letter "A." This is not to be mistaken with the $\mathbb{A}^1$-homotopy theory of schemes by Voevodsky [9].

The computation of these groups proceeds via the construction of a graph, $\Gamma^q(\Delta)$, whose vertices represent simplices in Δ. There is an edge between two simplices if they share a face of dimension greater than or equal to q. This construction suggested a natural definition of the A–theory of graphs, which was also developed in [4]. Proposition 5.12 in that paper shows that A_1 of the complex can be obtained as the fundamental group of the space obtained by attaching 2-cells into all 3- and 4-cycles of $\Gamma^q(\Delta)$.

The goal of the present paper is to generalize this result. Let Γ be a simple, undirected graph, with distinguished base vertex v_0. We will construct an infinite cell complex X_Γ together with a homomorphism

$$A_n(\Gamma, v_0) \longrightarrow \pi_n(X_\Gamma, v_0).$$

Moreover, we can show this homomorphism to be an isomorphism if a (plausible) cubical analog of the simplicial approximation theorem holds.

There are several reasons for this generalization. One reason is the desire for a homology theory associated to the A-theory of a graph. A natural candidate is the singular homology of the space X_Γ. This will be explored in a future paper.

Another reason is a connection to the homotopy of the complements of certain subspace arrangements. While computing A_1^{n-3} of the order complex of the Boolean lattice B_n, it became clear that this computation was equivalent to computing the fundamental group of the complement of the 3-equal arrangement [6]. (This result for the k-equal arrangement was proved independently by A. Björner [5].) To generalize this connection to a wider class of subspace arrangements, a topological characterization of A-theory is needed.

The content of the paper is as follows. After a brief review of the definition of A-theory, we construct the model space X_Γ, followed by a proof of the main result (Theorem 5.2). The main result refers to a yet unknown analog of a simplicial approximation theorem in the cubical world (Property 5.1). The last section introduces the loop graph of a graph, and we prove that the $(n+1)$-st A-group of the graph is isomorphic to the n-th A-group of the loop graph, in analogy to a standard result about classical homotopy.

2. *A*-theory of Graphs

We first recall the definition given in Sect. 5 of [4].

Definition 2.1. Let $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ be simple graphs, that is, graphs without loops and multiple edges.

(1) The *Cartesian product (cf. [10])* $\Gamma_1 \Box \Gamma_2$ is the graph with vertex set $V_1 \times V_2$. There is an edge between (u_1, u_2) and (v_1, v_2) if either $u_1 = v_1$ and $u_2 v_2 \in E_2$ or $u_2 = v_2$ and $u_1 v_1 \in E_1$. The product can also be viewed as the 1-skeleton of the product cell complex, where Γ_1 and Γ_2 are viewed as 1-dimensional cell complexes.
(2) Slightly different from standard notation we define a (*simplicial*) *graph homomorphism* $f : \Gamma_1 \longrightarrow \Gamma_2$ to be a set map $V_1 \longrightarrow V_2$ such that, if $uv \in E_1$, then either $f(u) = f(v)$ or $f(u)f(v) \in E_2$. A graph homomorphism viewed as a map of cell complexes is the same as a cellular map.
(3) Let $\Gamma_1' \subset \Gamma_1$ and $\Gamma_2' \subset \Gamma_2$ be subgraphs. A (*relative*) *graph homomorphism* $f : (\Gamma_1, \Gamma_1') \longrightarrow (\Gamma_2, \Gamma_2')$ is a graph homomorphism $f : \Gamma_1 \longrightarrow \Gamma_2$ which restricts to a graph homomorphism $f|_{\Gamma_1'} : \Gamma_1' \longrightarrow \Gamma_2'$. In the case that a graph Γ' is a single vertex graph with vertex v, we will denote Γ' by v. In particular, we will deal with homomorphisms such as $f : (\Gamma_1, v_1) \longrightarrow (\Gamma_2, v_2)$, where $v_i \in V_i$, $i = 1, 2$, is a vertex.
(4) Let $\mathbf{I}_n$ be the graph with $n + 1$ vertices labeled $0, 1, \ldots, n$, and n edges $(i - 1)i$ for $i = 1, \ldots, n$.

Next we define homotopy of graph homomorphisms and homotopy equivalence of graphs.

Definition 2.2. (1) Let $f, g : (\Gamma_1, v_1) \longrightarrow (\Gamma_2, v_2)$ be graph homomorphisms. We call f and g *A–homotopic*, denoted by $f \simeq_A g$, if there is an integer n and a graph homomorphism

$$\phi : \Gamma_1 \Box \mathbf{I}_n \longrightarrow \Gamma_2,$$

such that $\phi(-, 0) = f$, and $\phi(-, n) = g$, and such that $\phi(v_1, i) = v_2$ for all i.

Definition 2.3. (1) Let

$$\mathbf{I}_m^n = \mathbf{I}_m \Box \cdots \Box \mathbf{I}_m$$

be the n-fold Cartesian product of $\mathbf{I}_m$ for some m. We will call $\mathbf{I}_m^n$ an *n-cube* of *height m*. Its *distinguished base point* is $\mathbf{O} = (0, \ldots, 0)$.
(2) Define the *boundary* $\partial \mathbf{I}_m^n$ of a cube $\mathbf{I}_m^n$ of height m to be the subgraph of $\mathbf{I}_m^n$ containing all vertices with at least one coordinate equal to 0 or m.

It is easy to show that any graph homomorphism from $\mathbf{I}_m^n$ to Γ can be extended to a graph homomorphism from $\mathbf{I}_{m'}^n$ to Γ for any $m' \geq m$. In other words, there is an inclusion $\mathrm{Hom}(\mathbf{I}_m^n, \Gamma) \hookrightarrow \mathrm{Hom}(\mathbf{I}_{m'}^n, \Gamma)$, and hence each element $f \in \mathrm{Hom}(\mathbf{I}_m^n, \Gamma)$ defines an element $f \in \varinjlim \operatorname{Hom}(\mathbf{I}_m^n, \Gamma)$ in the colimit. This in mind, we will sometimes omit the subscript m.

Definition 2.4. Let $A_n(\Gamma, v_0)$, $n \geq 1$, be the set of homotopy classes of graph homomorphisms

$$f : (\mathbf{I}^n, \partial \mathbf{I}^n) \longrightarrow (\Gamma, v_0).$$

For $n = 0$, we define $A_0(\Gamma, v_0)$ to be the pointed set of connected components of Γ, with distinguished element the component containing v_0. We will denote the equivalence class of a homomorphism f in $A_n(\Gamma, v_0)$ by $[f]$.

We can define a multiplication on the set $A_n(\Gamma, v_0)$, $n \geq 1$, as follows. Given elements $[f]$, $[g] \in A_n(\Gamma, v_0)$, represented by

$$f, g : \left(\mathbf{I}_m^n, \partial \mathbf{I}_m^n\right) \longrightarrow (\Gamma, v_0),$$

defined on a cube of height m, we define $[f] * [g] \in A_n(\Gamma, v_0)$ as the homotopy class of the map

$$h : \left(\mathbf{I}_{2m}^n, \partial \mathbf{I}_{2m}^n\right) \longrightarrow (\Gamma, v_0),$$

defined on a cube of height $2m$ as follows.

$$h(i_1, \ldots, i_n) = \begin{cases} f(i_1, \ldots, i_n) & \text{if } i_j \leq m \text{ for all } j, \\ g(i_1 - m, \ldots, i_n) & \text{if } i_1 > m \text{ and } i_j \leq m \text{ for } j > 1, \\ v_0 & \text{otherwise.} \end{cases}$$

Alternatively, using Theorem 5.16 in [4], one can describe the A-theory of graphs using multidimensional "grids" of vertices as follows. Let Γ be a graph with distinguished vertex v_0. Let $\mathcal{A}_n(\Gamma, v_0)$ be the set of functions

$$\mathbb{Z}^n \longrightarrow V(\Gamma),$$

from the lattice $\mathbb{Z}^n$ into the set of vertices of Γ which take on the value v_0 almost everywhere, and for which any two adjacent lattice points get mapped into either the same or adjacent vertices of Γ. We define an equivalence relation on this set as follows. Two functions f and g are equivalent, if there exists

$$h : \mathbb{Z}^{n+1} \longrightarrow V(\Gamma),$$

in $\mathcal{A}_{n+1}(\Gamma, v_0)$ and integers k and l, such that

$$h(i_1, \ldots, i_n, k) = f(i_1, \ldots, i_n),$$
$$h(i_1, \ldots, i_n, l) = g(i_1, \ldots, i_n)$$

for all $i_1, \ldots, i_n \in \mathbb{Z}$. For a definition of a group operation on the set of equivalence classes see Prop. 3.5 of [4]. Then it is straightforward to see that $A_n(\Gamma, v_0)$ is isomorphic to the group of equivalence classes of elements in $\mathcal{A}_n(\Gamma, v_0)$. It will be useful to think of $A_n(\Gamma, v_0)$ in those terms.

Remark 2.5. Even though the definition of $A_n^q(\Delta, \sigma_0)$ as mentioned in the introduction will not be needed in the sequel, we include it here for completeness. Let Δ be a simplicial complex, $0 \leq q \leq \dim \Delta$, and $\sigma_0 \in \Delta$ a simplex of $\dim \sigma_0 \geq q$. Let $\Gamma^q(\Delta)$ be the graph on the vertex set $\{\sigma \in \Delta : \dim \sigma \geq q\}$ and edges given by pairs of simplices that share a common q-face. Then we define $A_n^q(\Delta, \sigma_0) := A_n(\Gamma^q(\Delta), \sigma_0)$ (cf. Theorem 5.16 of [4]).

3. A cubical set setting for the A-theory of graphs

We now define a cubical set $C_*(\Gamma)$ associated to the graph Γ (see [8]). This gives the right setup in order to obtain a close connection to the space X_Γ which we define in the next section. Let $\mathbf{I}_\infty^n$ be the "infinite" discrete n-cube, that is, the infinite lattice labeled by $\mathbb{Z}^n$.

Definition 3.1. A graph homomorphism $f : \mathbf{I}_\infty^n \to \Gamma$ *stabilizes in direction* (i, ε)*,* $i = 1, \ldots, n$*,* $\varepsilon \in \{\pm 1\}$ if there exists an $m_0 = m_0(f, i, \varepsilon)$, s.t. for all $m \geq m_0$

$$f(a_1, \ldots, a_{i-1}, \varepsilon m_0, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, \varepsilon m, a_{i+1}, \ldots, a_n).$$

Let

$$C_n(\Gamma) = \mathrm{Hom}_s\left(\mathbf{I}_\infty^n, \Gamma\right),$$

the set of graph homomorphisms from the infinite n-cube to Γ that stabilize in each direction (i, ε).

For each "face" of $\mathbf{I}_\infty^n$, i.e., for each choice of (i, ε), $i = 1, \ldots, n$, $\varepsilon \in \{\pm 1\}$, we define *face maps*

$$\alpha'_{i,\varepsilon} : C_n(\Gamma) \longrightarrow C_{n-1}(\Gamma),$$

by

$$\alpha'_{i,\varepsilon}(f)(a_1, \ldots, a_{n-1}) = f(a_1, \ldots, a_{i-1}, \varepsilon m_0, a_i, \ldots, a_{n-1}),$$

where $m_0 = m_0(f, i, \varepsilon)$. In other words $\alpha'_{i,\varepsilon}(f)$ is the map in $C_{n-1}(\Gamma)$ whose values are equal to the stable values of f in direction (i, ε).

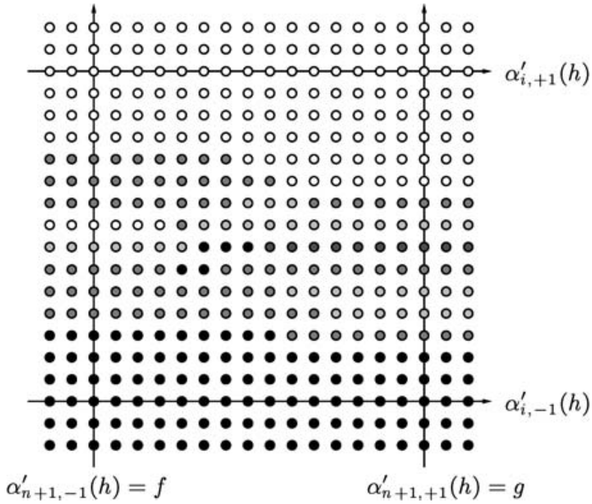
Degeneracy maps

$$\beta'_i : C_{n-1}(\Gamma) \longrightarrow C_n(\Gamma),$$

$i = 1, \ldots, n$, are defined as follows. Given a map $f \in C_{n-1}(\Gamma)$, extend it to a map on $\mathbf{I}_\infty^n$ by

$$\beta'_i(f)(a_1, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n),$$

Fig. 2 An illustration of a map h in the definition of $\sim$.



for each $(a_1, \ldots, a_n) \in \mathbf{I}^n_\infty$. It is straightforward to check that in this way $C_*(\Gamma)$ is a cubical set.

We now imitate the definition of combinatorial homotopy of Kan complexes; see, e.g. [8, Ch. 1.3].

Definition 3.2. We define a relation on $C_n(\Gamma)$, $n \geq 0$. Let $f, g \in C_n(\Gamma)$. Then $f \sim g$ if there exists $h \in C_{n+1}(\Gamma)$ such that for all $i = 1, \ldots, n$, $\varepsilon \in \{\pm 1\}$:

(1) $\alpha'_{i,\varepsilon}(f) = \alpha'_{i,\varepsilon}(g)$,
(2) $\alpha'_{i,\varepsilon}(h) = \beta'_n \alpha'_{i,\varepsilon}(f) = \beta'_n \alpha'_{i,\varepsilon}(g)$,
(3) $\alpha'_{n+1,-1}(h) = f$ and $\alpha'_{n+1,1}(h) = g$.

This is illustrated in Figure 2. A few vertices of the cube $\mathbf{I}^{n+1}_\infty$ are shown. They are differently shaded correspondingly to their image vertex in Γ. The vertical coordinate axis corresponds to the coordinates $1, \ldots, n$, the horizontal axis corresponds to the coordinate $n + 1$. The maps f and g are indicated by the two vertical lines. Relation (1) corresponds to the fact that they stabilize in all directions (i, ε), $i \leq n$, identically. In the picture this is indicated by the fact that in the north direction the image vertex is the constant white vertex, and in the south direction, it is the constant black vertex. Relation (2) says that h also stabilizes in the same way in these directions. Finally relation (3) yields that in the west direction h stabilizes identical to f and in the east direction h stabilizes identical to g.

Proposition 3.3. *The relation defined above is an equivalence relation.* □

Definition 3.4. Let $v_0 \in \Gamma$ be a distinguished vertex. Let $B_*(\Gamma, v_0) \subset C_*(\Gamma)$ be the subset of all maps that are equal to v_0 outside of a finite region of $\mathbf{I}^*_\infty$.

Observe that the equivalence relation $\sim$ restricts to an equivalence relation on $B_*(\Gamma, v_0)$, also denoted by $\sim$.

Proposition 3.5. *There is a group structure on the set* $B_n(\Gamma, v_0)/\sim$ *for all* $n \geq 1$, *and, furthermore,*

$$(B_n(\Gamma, v_0)/\sim) \cong A_n(\Gamma, v_0).$$

The proof is straightforward. For a definition of the group structure see Prop. 3.5 of [4].

4. Definition of X_Γ

Let Γ be a finite, simple (undirected) graph. In this section we define a cell complex X_Γ associated to Γ. This complex will be defined as the geometric realization of a certain cubical set $M_*(\Gamma)$. As before, let $\mathbf{I}_1^n$ be the discrete n-cube. Let

$$M_n(\Gamma) = \mathrm{Hom}\left(\mathbf{I}_1^n, \Gamma\right),$$

the set of all graph homomorphisms from $\mathbf{I}_1^n$ to Γ. We define face and degeneracy maps as follows.

First note that $\mathbf{I}_1^n$ has $2n$ faces $F_{i,\varepsilon}$, with $i = 1, \ldots, n$, and $\varepsilon \in \{\pm 1\}$, corresponding to the two faces for each coordinate. For $i = 1, \ldots, n$, $\varepsilon \in \{\pm 1\}$, let

$$\begin{aligned} a_{i,\varepsilon} : \mathbf{I}_1^{n-1} &\longrightarrow \mathbf{I}_1^n \\ (x_1, \ldots, x_{n-1}) &\longmapsto (x_1, \ldots, x_{i-1}, \tfrac{\varepsilon+1}{2}, x_i, \ldots x_{n-1}) \end{aligned}$$

be the graph homomorphism given by inclusion of $\mathbf{I}_1^{n-1}$ as the (i, ε)-face of $\mathbf{I}_1^n$. For $i = 1, \ldots, n$ define

$$\begin{aligned} b_i : \mathbf{I}_1^n &\longrightarrow \mathbf{I}_1^{n-1} \\ (x_1, \ldots, x_n) &\longmapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n) \end{aligned}$$

to be the projection in direction i.

Now let

$$\begin{aligned} \alpha_{i,\varepsilon} : M_n(\Gamma) &\longrightarrow M_{n-1}(\Gamma) \\ f &\longmapsto f \circ a_{i,\varepsilon} \end{aligned}$$

be the map induced by $a_{i,\varepsilon}$. Likewise,

$$\begin{aligned}\beta_i : M_{n-1}(\Gamma) &\longrightarrow M_n(\Gamma)\\ f &\longmapsto f \circ b_i\end{aligned}$$

to be the map induced by b_i. In this way we obtain a cubical set $M_*(\Gamma)$.

To each cubical set is associated a cell complex, namely its geometric realization. We recall the construction for $M_*(\Gamma)$. Let C^n be the geometric n-dimensional cube. We can define functions $a_{i,\varepsilon}$ and b_i on C^n in a fashion as above. Define the space

$$|M_*(\Gamma)| = \biguplus_{n\geq 0} M_n(\Gamma) \times C^n / \sim,$$

where $\sim$ is the equivalence relation generated by the following two types of equivalences:

$$(\alpha_{i,\varepsilon}(f), x_{n-1}) \sim (f, a_{i,\varepsilon}(x_{n-1})),\ f \in M_n(\Gamma),\ x_{n-1} \in C^{n-1} \quad (4.1)$$

$$(\beta_j(g), x_n) \sim (g, b_j(x_n)),\ g \in M_{n-1}(\Gamma),\ x_n \in C^n. \quad (4.2)$$

We will denote the cell complex $|M_*(\Gamma)|$ by X_Γ.

5. The main result

Before we state the main result we will introduce the following plausible property, which is a special case of a general cubical approximation theorem. We have not found it in the literature and have not been able to prove it yet.

Property 5.1. *Let X be a cubical set, and let $f : C^n \longrightarrow |X|$ be a continuous map from the n-cube to the geometric realization of X, such that the restriction of f to the boundary of C^n is cubical. Then there exists a cubical subdivision D^n of C^n and a cubical map $f' : D^n \longrightarrow |X|$ which is homotopic to f and the restrictions of f and f' to the boundary of D^n are equal.*

Theorem 5.2. *There is a group homomorphism*

$$\phi : A_n(\Gamma, v_0) \longrightarrow \pi_n(X_\Gamma, v_0),$$

for all $n \geq 1$. If a cubical analog of the simplicial approximation theorem such as 5.1 holds, then ϕ is an isomorphism.

Proof: First we define ϕ. Let $[f] \in A_n(\Gamma, v_0) \cong B_n(\Gamma, v_0)/\sim$. Then a representative f is a graph homomorphism

$$f : \mathbf{I}_\infty^n \longrightarrow \Gamma,$$

whose value on vertices outside a finite region is equal to v_0, say for vertices outside of a cube with side length r. Our goal is to define a continuous map

$$\tilde{f} : C^n \longrightarrow X_\Gamma,$$

such that $\tilde{f}$ sends the boundary of C^n to v_0.

Let D^n be a cubical subdivision of C^n into cubes of side length $1/r$. The 1-skeleton of D^n can be identified with $\mathbf{I}_r^n$, which is contained in $\mathbf{I}_\infty^n$. And each subcube of $\mathbf{I}_r^n$ can be identified with $\mathbf{I}_1^n$. Hence, f restricts to a graph homomorphism on each cube in the 1-skeleton of D^n, that is, a graph homomorphism

$$\hat{f} : \mathbf{I}_1^n \longrightarrow \Gamma.$$

Thus, $\hat{f} \in \mathrm{Hom}(\mathbf{I}_1^n, \Gamma)$. Now define $\tilde{f}$ on each subcube of D^n by

$$\tilde{f}(x) = [(\hat{f}, x)] \in X_\Gamma = \Big(\biguplus_n \mathrm{Hom}(\mathbf{I}_1^n, \Gamma) \times C^n\Big) / \sim .$$

The equivalence relation $\sim$ guarantees that $\tilde{f}$ is well-defined on overlapping faces. Therefore, our definition extends to give a map

$$\tilde{f} : D^n \longrightarrow X_\Gamma.$$

So define

$$\phi([f]) = [\tilde{f}].$$

We need to show that ϕ is **well-defined.** Let $f \sim g$ be two maps in $B_n(\Gamma, v_0)$. Then there exists a homotopy $h \in B_{n+1}(\Gamma, v_0)$ such that $\alpha'_{n+1,-1}(h) = f$ and $\alpha'_{n+1,1}(h) = g$. We claim that $\phi(h)$ gives a homotopy between $\phi(f)$ and $\phi(g)$. From the definition of ϕ it is easy to see that

$$\phi\big(\big(\alpha'_{i,\varepsilon}(h)\big)(y) = [(\alpha_{i,\varepsilon}(\tilde{h}), y)],$$

for all i, ε. Therefore, the restriction of

$$\phi(h) : D^{n+1} \longrightarrow X_\Gamma$$

to the $(n+1, -1)$-face is equal to the map from D^n to X_Γ, sending x to $[(\alpha'_{n+1,-1}(h), x)]$, which is equal to $\phi(f)$; similarly for $\phi(g)$. It now follows that $\phi(h)$ is a homotopy between $\phi(f)$ and $\phi(g)$. This shows that ϕ is well-defined.

Now we show that ϕ is a **group homomorphism.** Recall [4, p. 111] that the multiplication in $A_n(\Gamma, v_0)$ is given by juxtaposing "grids." This carries over directly to $B_n(\Gamma, v_0)/\sim$. On the other hand, the multiplication in $\pi_n(X_\Gamma, v_0)$ is given by using the comultiplication on $(C^n, \partial C^n)$. It is then straightforward to check that ϕ preserves multiplication.

From here on we assume that Property 5.1 holds. Under this assumption we show that ϕ is **onto.** We first show that every element in $\pi_n(X_\Gamma, v_0)$ contains a cubical representative. Let $[f] \in \pi_n(X_\Gamma, v_0)$. Then $f : C^n \longrightarrow X_\Gamma$ sends the boundary of C^n to the base point v_0. Trivially then, the restriction of f to the boundary is a cubical map. By Property 5.1 f is homotopic to a cubical map on a cubical subdivision D^n of C^n, and agrees with f on the boundary. That is, $[f]$ contains a cubical representative. So we may assume that f is cubical on D^n.

Consider the restriction of f to the 1-skeleton of D^n. It induces in the obvious way a graph homomorphism $g : \mathbf{I}^n_\infty \longrightarrow \Gamma$, that is, an element $[g] \in B_n(\Gamma, v_0)/\sim$. We claim that $\phi(g) = [f]$, that is, $\tilde{g} \sim f$. We use induction on n. If $n = 1$, then we are done, since any two maps on the unit interval that agree on the end points are homotopic. Changing f up to homotopy we may assume that f and $\tilde{g}$ are equal on the 1-skeleton.

Now let $n > 1$. Note that

$$f : D^n \longrightarrow X_\Gamma = \left(\biguplus_{n \geq 0} \mathrm{Hom}\left(\mathbf{I}^n_1, \Gamma\right) \right) \times C^n / \sim$$

is cubical, so each n-cube C^n in the cubical subdivision D^n is sent to an n-cube in X_Γ. The particular n-cube it is mapped to is determined by the image of the map on the 1-skeleton, since the map is cubical. This in turn determines an element in $\mathrm{Hom}(\mathbf{I}^n_1, \Gamma)$, serving as the label of the image cube. Hence, f and $\tilde{g}$ map each n-cube of the subdivision D^n to the same n-cube in X_Γ. By induction we may assume that f and $\tilde{g}$ are equal on the boundary of each n-cube. But observe that any two maps into C^n that agree on the boundary are homotopic, via a homotopy that leaves the boundary fixed. This shows f and $\tilde{g}$ are homotopic on each n-cube of the cubical subdivision D^n. Pasting these homotopies together along the boundaries, we obtain a homotopy between f and $\tilde{g}$, so that $[f] = [\tilde{g}]$.

To show that ϕ is **one-to-one** under the assumption of Property 5.1, suppose that $f, g \in B_n(\Gamma, v_0)/\sim$ such that $\phi(f) = \phi(g) \in \pi_n(\Gamma, v_0)$. Then there exists a homotopy $h : C^{n+1} \longrightarrow X_\Gamma$ such that the restrictions of h to the $(n+1)$-directional faces are $\phi(f)$ and $\phi(g)$, respectively. As above, we may assume that h is cubical on a subdivision D^{n+1} of C^{n+1}, providing a homotopy between cubical approximations of $\phi(f)$ and $\phi(g)$ on a subdivision D^n of C^n. Now observe that the restriction of h to the 1-skeleton of D^{n+1} induces a graph homomorphism $h' : \mathbf{I}^{n+1}_1 \longrightarrow \Gamma$ in $B_{n+1}(\Gamma, v_0)$, whose restrictions to the $(n+1)$-directional faces are refinements of f and g, respectively. But these refinements are equivalent to f and g, respectively. Thus, $[f] = [g] \in B_n(\Gamma, v_0)/\sim$. $\square$

6. Path- and loop graph of a graph

This section is devoted to further develop the connection between classical homotopy theory and A–theory. In classical homotopy theory the computation of the homotopy group $\pi_{n+1}(X)$ of a space X can be reduced to the computation of $\pi_n(\Omega X)$, the n-th homotopy group of the loop space ΩX of X. Here we want to introduce the path graph $P\Gamma$ and the loop graph $\Omega\Gamma$ of a graph Γ such that naturally $A_n(\Omega\Gamma) \cong A_{n+1}(\Gamma)$.

Definition 6.1. Let Γ be a graph with base vertex $*$. Define the *path graph* $P\Gamma = (V_{P\Gamma}, E_{P\Gamma})$ to be the graph on the vertex set

$$V_{P\Gamma} = \{\varphi : \mathbf{I}_m \to \Gamma : m \in \mathbb{N},\ \varphi \text{ a graph homomorphism with } \varphi(0) = *\}.$$

The edge set $E_{P\Gamma}$ is given as follows. Consider two vertices $\varphi_0 : \mathbf{I}_m \to \Gamma$ and $\varphi_1 : \mathbf{I}_{m'} \to \Gamma$. Assuming $m \leq m'$ extend φ_0 to a map $\varphi_0' : \mathbf{I}_{m'} \to \Gamma$ by repeating the last vertex $\varphi_0(m)$ at the end:

$$\varphi_0'(y) = \begin{cases} \varphi_0(y), & \text{if } y \leq m, \\ \varphi_0(m), & \text{otherwise.} \end{cases}$$

Define $\{\varphi_0, \varphi_1\}$ to be an edge if there exists a graph homomorphism $\Phi : \mathbf{I}_{m'} \Box \mathbf{I}_1 \to \Gamma$ such that $\Phi(\bullet, 0) = \varphi_0'$ and $\Phi(\bullet, 1) = \varphi_1$.

There is a graph homomorphism $p : P\Gamma \to \Gamma$ given by $p(\varphi) = \varphi(m)$ for a vertex $\varphi : \mathbf{I}_m \to \Gamma$ of $P\Gamma$.

Definition 6.2. For a graph Γ define the *loop graph* $\Omega\Gamma$ of Γ to be the induced subgraph of $P\Gamma$ on the vertex set $p^{-1}(*)$. We define the base vertex of $\Omega\Gamma$ to be the vertex $\varphi_0 : \mathbf{I}_0 \to \Gamma$, i.e., the map that sends the single vertex of $\mathbf{I}_0$ to $*$ in Γ. To avoid too much notation we will denote this map by $*$ as well.

Note that for a graph homomorphism $\psi : (\Gamma_1, *) \to (\Gamma_2, *)$ there is an induced map $\Omega\psi : (\Omega\Gamma_1, *) \to (\Omega\Gamma_2, *)$ defined by $\Omega\psi(\varphi)(y) = \psi(\varphi(y))$ where $\varphi : \mathbf{I}_m \to \Gamma_1$ and y is a vertex of $\mathbf{I}_m$.

Remark 6.3. Consider the constant loop $\varphi_m : \mathbf{I}_m \to \Gamma$ in $\Omega\Gamma$, i.e., $\varphi_m(x) = * \in \Gamma$ for all vertices x of $\mathbf{I}_m$. If a loop $\varphi : \mathbf{I}_m \to \Gamma$ is connected to φ_m via an edge, then it is also connected to $\varphi_0 = *$ via an edge.

Analogously to classical topology we have the following.

Proposition 6.4. *There is a natural isomorphism $A_n(\Omega\Gamma) \xrightarrow{\cong} A_{n+1}(\Gamma)$ for $n \geq 1$. Furthermore, there is a bijection $A_0(\Omega\Gamma) \xrightarrow{\cong} A_1(\Gamma)$.*

Proof: The case $n \geq 1$. Let $[f] \in A_n(\Omega\Gamma)$, i.e., f is a graph homomorphism $f : (\mathbf{I}_m^n, \partial\mathbf{I}_m^n) \to (\Omega\Gamma, *)$. For x a vertex of $\mathbf{I}_m^n$ there is an $m_f(x)$ such that $f(x)$ is a graph homomorphism $f(x) : (\mathbf{I}_{m_f(x)}, \partial\mathbf{I}_{m_f(x)}) \to (\Gamma, *)$. Let $m' = \max_x\{m_f(x), m\}$. We want to define a graph homomorphism $\alpha(f) : (\mathbf{I}_{m'}^{n+1}, \partial\mathbf{I}_{m'}^{n+1}) \to (\Gamma, *)$. For that

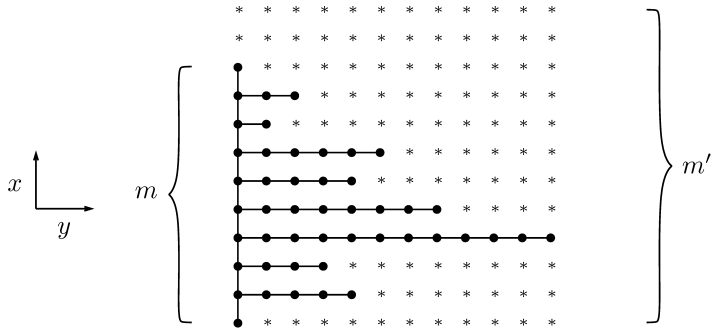


Fig. 3 The maps f and $\alpha(f)$.

reason write $\mathbf{I}^{n+1}_{m'} = \mathbf{I}^n_{m'} \square \mathbf{I}_{m'}$ and let (x, y) be a vertex of $\mathbf{I}^n_{m'} \square \mathbf{I}_{m'}$. Now let

$$\alpha(f)(x, y) = \begin{cases} f(x)(y), & \text{if } x \text{ is a vertex of } \mathbf{I}^n_m \subset \mathbf{I}^n_{m'} \text{ and } y \leq m_f(x), \\ *, & \text{otherwise.} \end{cases}$$

The construction is shown in Figure 3, where $n = 1, m = 10$, and $m' = 12$. The vertical line is $\mathbf{I}^n_m$, the horizontal lines indicate the paths $f(x)$, the whole square indicates $\alpha(f)$.

We claim that the map $[f] \to [\alpha(f)]$ is well defined and the desired natural isomorphism.

Well definedness: First of all it is easy to check that $\alpha(f)$ is a graph homomorphism $\alpha(f) : (\mathbf{I}^{n+1}_{m'}, \partial \mathbf{I}^{n+1}_{m'}) \to (\Gamma, *)$. Now let $[f] = [g] \in A_n(\Omega\Gamma)$, i.e., there exists an A–homotopy $H : \mathbf{I}^n_m \square \mathbf{I}_l \to \Omega\Gamma$ between f and g. Now let $m' = \max_{x,x'}\{m_f(x), m_g(x'), m\}$ and define $\bar{H} : \mathbf{I}^n_{m'} \square \mathbf{I}_{m'} \square \mathbf{I}_l \to \Gamma$ by

$$\bar{H}(x, y, t) = \begin{cases} H(x, t)(y), & \text{if } x \text{ is a vertex of } \mathbf{I}^n_m \text{ and } y \leq m_{H(x,t)}, \\ *, & \text{otherwise.} \end{cases}$$

Then $\bar{H}$ is a graph homomorphism and an A–homotopy between (possibly extended to a larger cube) $\alpha(f)$ and $\alpha(g)$.

Homomorphism: Is straightforward; similar techniques play a role that are needed to show that $A_n(\Gamma)$ is a group for $n \geq 1$.

Surjectivity: For $[h] \in A_{n+1}(\Gamma)$, say $h : (\mathbf{I}^{n+1}_m, \partial \mathbf{I}^{n+1}_m) \to (\Gamma, *)$, consider the map f defined by $f(x)(y) = h(x, y)$ for x a vertex of $\mathbf{I}^n_m$, y a vertex of $\mathbf{I}_m$. This map is not quite what we want since it is a map $f : (\mathbf{I}^n_m, \partial \mathbf{I}^n_m) \to (\Omega\Gamma, \varphi_m)$, where φ_m is the constant loop $\mathbf{I}_m \to \Gamma$ as in Remark 6.3. Now define $f' : (\mathbf{I}^n_m, \partial \mathbf{I}^n_m) \to (\Omega\Gamma, *)$ by $f'(x) = * \in \Omega\Gamma$ for x a vertex of $\partial \mathbf{I}^n_m$ and $f'(x) = f(x)$ for x a vertex of $\mathbf{I}^n_m \setminus \partial \mathbf{I}^n_m$. Thanks to Remark 6.3, f' is a well defined graph homomorphism and clearly $\alpha(f') = h$.

Injectivity: Consider $f : (\mathbf{I}^n_m, \partial \mathbf{I}^n_m) \to (\Omega\Gamma, *)$ and $g : (\mathbf{I}^n_{m'}, \partial \mathbf{I}^n_{m'}) \to (\Omega\Gamma, *)$ such that $[\alpha(f)] = [\alpha(g)]$, i.e., there is an A–homotopy $H : \mathbf{I}^n_{m''} \square \mathbf{I}_l \to \Gamma$ between (possibly extended to a larger cube) $\alpha(f)$ and $\alpha(g)$, where $m'' = \max_{x,x'}\{m_f(x), m_g(x'), m, m'\}$. Define $\bar{H} : \mathbf{I}^n_{m''}\Box\mathbf{I}_l \to \Omega\Gamma$ by $\bar{H}(x,t)(y) = H(x,y,t)$. Then $\bar{H}(x,t) : \mathbf{I}_{m''} \to \Omega\Gamma$ for all x and t. Furthermore $\bar{H}(x,t) = \varphi_m$ for x a vertex of $\partial\mathbf{I}^n_{m''}$. As before we replace $\bar{H}$ by $\bar{H}'$ by changing it only on the boundary and by replacing $\alpha(f)$ by f and $\alpha(g)$ by g.

$$\bar{H}'(x,t) = \begin{cases} \bar{H}(x,t), & \text{if } x \text{ a vertex of } \mathbf{I}^n_{m''} \setminus \partial\mathbf{I}^n_{m''} \text{ and } t \neq 0, m, \\ f(x), & \text{if } t = 0 \text{ and } x \text{ a vertex of } \mathbf{I}^n_m \subset \mathbf{I}^n_{m''}, \\ g(x), & \text{if } t = m \text{ and } x \text{ a vertex of } \mathbf{I}^n_{m'} \subset \mathbf{I}^n_{m''}, \\ \varphi_0, & \text{otherwise.} \end{cases}$$

Then by Remark 6.3 $\bar{H}'$ is a graph homomorphism and it yields an A–homotopy between (possibly extended to a larger cube) f and g.

Naturality: Let $\psi : (\Gamma_1, *_{\Gamma_1}) \to (\Gamma_2, *_{\Gamma_2})$ be a graph homomorphism and $f : (\mathbf{I}^n_m, \partial\mathbf{I}^n_m) \to (\Omega\Gamma_1, *)$. Then for a vertex x of $\mathbf{I}^n_m$ we obtain

$$\begin{aligned} \psi_\#(\alpha_{\Gamma_1}(f))(x,y) &= \begin{cases} \psi(f(x)(y)), & \text{if } y \leq m_f(x), \\ \psi(*_{\Gamma_1}), & \text{otherwise.} \end{cases} \\ &= \begin{cases} \Omega\psi(f)(x)(y), & \text{if } y \leq m_{\Omega\psi(f)}(x), \\ *_{\Gamma_2}, & \text{otherwise.} \end{cases} \\ &= \alpha_{\Gamma_2}((\Omega\psi)_\#(f)). \end{aligned}$$

The remaining case $n = 0$: Consider an element $[\varphi]$ of $A_0(\Omega\Gamma)$, i.e., a connected component of $\Omega\Gamma$ represented by a loop $\varphi : \mathbf{I}_m \to \Gamma$. This loop defines an element $[\varphi]$ (this time a homotopy class) of $A_1(\Gamma)$. Well definedness and bijectivity of this assignment is immediate. $\square$

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References

1. R. Atkin, "An algebra for patterns on a complex, I," *Internat. J. Man-Machine Stud.* **6** (1974), 285–307.
2. R. Atkin, "An algebra for patterns on a complex, II," *Internat. J. Man-Machine Stud.* **8** (1976), 483–448.
3. Héléne Barcelo and Reinhard Laubenbacher, "Perspectives on A–homotopy theory and its applications," to appear, special Issue of *Discr. Math.* **298** (2002) 39–61.
4. Héléne Barcelo, Xenia Kramer, Reinhard Laubenbacher, and Christopher Weaver, "Foundations of a connectivity theory for simplicial complexes," *Adv. Appl. Math.* **26** (2001), 97–128.
5. A. Björner, private communication.
6. A. Björner and V. Welker, "The homology of "k-Equal" manifolds and related partition lattices," *Adv. in Math.* **110** (1995), 277–313.

7. X. Kramer and R. Laubenbacher, "Combinatorial homotopy of simplicial complexes and complex information networks," in "Applications of computational algebraic geometry," D. Cox and B. Sturmfels (eds.), *Proc. Sympos. in Appl. Math.*, vol. 53, *Amer. Math. Soc.*, Providence, 1998.
8. P. May, *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, Chicago, 1967.
9. V. Voevodsky, $\mathbb{A}^1$-*homotopy theory*, Doc. Math., J. DMV, Extra Vol. ICM Berlin 1998, vol. I, 579–604 (1998).
10. D. West, *Introduction to Graph Theory,* second edition, Prentice-Hall, Upper Saddle River, 2001.