J Algebr Comb (2006) 24:285–297
DOI 10.1007/s10801-006-0019-2

Partial geometries pg(s, t, 2) with an abelian Singer group and a characterization of the van Lint-Schrijver partial geometry

S. De Winter

Received: 8 September 2005 / Accepted: 13 February 2006 / Published online: 11 July 2006

Abstract Let $\mathcal{S}$ be a proper partial geometry pg$(s, t, 2)$, and let G be an abelian group of automorphisms of $\mathcal{S}$ acting regularly on the points of $\mathcal{S}$. Then either $t \equiv 2$ (mod $s + 1$) or $\mathcal{S}$ is a pg(5, 5, 2) isomorphic to the partial geometry of van Lint and Schrijver (Combinatorica **1** (1981), 63–73). This result is a new step towards the classification of partial geometries with an abelian Singer group and further provides an interesting characterization of the geometry of van Lint and Schrijver.

Keywords Partial geometry · Abelian Singer group · Geometry of van Lint-Schrijver

1. Introduction and motivation

One of the most important and longstanding conjectures in the theory of finite projective planes is certainly the following:

> Any finite projective plane admitting an abelian group acting regularly on its points must be Desarguesian.

Although with the techniques of today it seems impossible to prove this conjecture, several weaker versions have been proved and an extensive literature on the subject exists (we refer to [9] for a recent survey containing extensive bibliographic information).

More generally it is now natural to wonder what happens in the case of other geometries admitting a(n) (abelian) Singer group. This motivated Ghinelli [8] to study generalized quadrangles (GQs) admitting a (not necessarily abelian)

The author is Postdoctoral Fellow of the Fund for Scientific Research Flanders (FWO-Vlaanderen).

S. De. Winter (✉)
Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281-S22, B-9000 Gent, Belgium
e-mail: sgdwinte@cage.UGent.be

Singer group. At this point we want to mention that looking at GQs is natural as both projective planes and GQs are members of the larger class of generalized polygons.

Recently a question of J. C. Fisher at the 2004 conference "Incidence Geometry" (May, 2004, La Roche, Belgium) inspired the author and K. Thas to study this subject in the case of GQs. In [7] they obtained that every finite GQ which admits an abelian Singer group necessarily arises as the generalized linear representation of a generalized hyperoval. Further it was noted in that paper that no finite generalized n-gon with $n > 4$ can admit an abelian Singer group.

In the present paper we will study partial geometries pg$(s, t, 2)$ admitting an abelian Singer group. This is a natural generalization since partial geometries generalize projective planes as well as GQs. Further this study is also motivated by the existence of an interesting example: the so-called van Lint-Schrijver partial geometry [12].

Before proceeding we will now provide some definitions and notation.

A *partial geometry* pg(s, t, α) is a finite partial linear space $\mathcal{S}$ of order (s, t), $s, t \geq 2$, such that

- for every antiflag (p, L) of $\mathcal{S}$ there are exactly $\alpha > 0$ lines through p intersecting L.

Partial geometries were introduced in 1963 by Bose [3]. It is easily seen that these geometries have a strongly regular point graph srg$(v = (s + 1)(st + \alpha)/\alpha, s(t + 1), s - 1 + t(\alpha - 1), \alpha(t + 1))$. Partial geometries for which $\alpha = 1$ are known as *generalized quadrangles* (GQs). A partial geometry will be called proper if $1 < \alpha < \min(s, t)$. If two distinct points x and y of $\mathcal{S}$ are collinear this will be denoted by $x \sim y$, while the line determined by these points will be denoted by $\langle x, y\rangle$. The maximum size of a clique in the point graph of $\mathcal{S}$ equals $s + 1$, and if C is a clique of this size, then every point p not belonging to C is adjacent to exactly α elements of C (this is Lemma 2 of [10]). Finally we mention two constructions of partial geometries.

Let $\mathcal{R} = \{\mathrm{PG}^{(0)}(m, q), \mathrm{PG}^{(1)}(m, q), \ldots, \mathrm{PG}^{(t)}(m, q)\}$, $t \geq 1$, be a set of mutually disjoint PG(m, q) in PG(n, q). We say that $\mathcal{R}$ is a *PG-regulus* if and only if the following condition is satisfied.

- If PG$(m + 1, q)$ contains $\mathrm{PG}^{(i)}(m, q)$, $i = 0, 1, \ldots, t$, then it has a point in common with $\alpha > 0$ elements of $\mathcal{R} \setminus \{\mathrm{PG}^{(i)}(m, q)\}$.

PG-reguli are a special case of the more general class of *SPG-reguli* which were introduced in 1983 by J. A. Thas [16] (these SPG-reguli give rise to so-called semipartial geometries). Now suppose that $\mathcal{R}$ is a PG-regulus in $\Pi := \mathrm{PG}(n, q)$ and embed Π as a hyperplane in PG$(n + 1, q)$. Define $\mathcal{S}$ to be the geometry with as point set the set of all points of $\mathrm{PG}(n + 1, q) \setminus \Pi$, with as line set the set of all $\mathrm{PG}(m + 1, q) \subset \mathrm{PG}(n + 1, q)$ that are not contained in Π and intersect Π in an element of $\mathcal{R}$, and for which the incidence relation is the natural one. Then $\mathcal{S}$ is a partial geometry pg$(q^{m+1} - 1, t, \alpha)$ (see Thas [16]).

To end this section we provide a construction of the above mentioned van Lint-Schrijver partial geometry [12]. Let β be a primitive element of GF(81) and define $\gamma := \beta^{16}$. Define $\mathcal{S}$ to be the geometry with point set the set of elements of GF(81), with line set the set of 6-tuples $(b, 1+b, \gamma+b, \gamma^2+b, \gamma^3+b, \gamma^4+b), b \in \mathrm{GF}(81)$, and with as incidence relation symmetrized containment. Then van Lint and Schrijver have shown that $\mathcal{S}$ is a partial geometry pg(5, 5, 2). It is worthwhile to notice that the point graph of this partial geometry is a cyclotomic graph.

Finally we note that throughout this paper $\mathbb{N}$ will denote the set of natural numbers (including 0) and that $\mathbb{N}_0$ will denote the set of natural numbers without 0.

2. A Benson-type theorem for partial geometries

In this section we will provide an analogue for partial geometries of the theorem of Benson [2] on automorphisms of generalized quadrangles. The proof of this theorem is analogous to the proof of Benson's theorem as given in [14] (Section 1.9, page 23), so we will only give a sketch of the proof here.

Theorem 2.1. *Let $\mathcal{S}$ be a partial geometry* pg(s, t, α)*, and let θ be any automorphism of $\mathcal{S}$. Denote by f the number of fixed points of $\mathcal{S}$ under θ and by g the number of points x of $\mathcal{S}$ for which $x \neq x^\theta \sim x$. Then*

$$(1+t)f + g \equiv (1+s)(1+t) \pmod{s+t-\alpha+1}.$$

Proof: Let $\mathcal{P} = \{x_1, x_2, \ldots, x_v\}$ be the point set of $\mathcal{S}$. If we denote by A the adjacency matrix of $\mathcal{S}$, then $A^2 + (t-s+\alpha+1)A + (t+1)(\alpha-s)I = \alpha(t+1)J$, with I the identity matrix and J the matrix with all entries equal to 1. Consequently A has eigenvalues $s(t+1)$, $-1-t$ and $s-\alpha$, with respective multiplicities $m_0 = 1$, m_1 and m_2, where m_1 and m_2 can easily be computed. Next, if we denote by D the incidence matrix of $\mathcal{S}$, then $M := DD^T = A + (t+1)I$, and so M has eigenvalues $(1+s)(1+t), 0$ and $t+1+s-\alpha$, with respective multiplicities $m_0 = 1, m_1$ and m_2. Now let Q be the matrix with the ij-th entry 1 if $x_i^\theta = x_j$ and 0 otherwise. Then Q is a permutation matrix and one can show that $QM = MQ$ (see [14]). Consequently, if θ has order n, then $(QM)^n = Q^n M^n = M^n$. One deduces that the eigenvalues of QM are the eigenvalues of M multiplied with the appropriate roots of unity. From $QMJ = MJ = (s+1)(t+1)J$ it follows that $(s+1)(t+1)$ will be an eigenvalue of QM. By $m_0 = 1$, this eigenvalue will have multiplicity 1. Further 0 will be an eigenvalue of QM with multiplicity m_1. Let d be a divisor of n. Then the multiplicity of the eigenvalue $\xi_d(s+t-\alpha+1)$ of QM, will only depend on d and not on the primitive dth rooth of unity ξ_d. Denote this multiplicity by a_d. Further, as the sum of all dth primitive roots of unity is an integer U_d [11] we obtain that $\mathrm{tr}(QM) = \sum_{d|n} a_d(s+t+\alpha-1)U_d + (s+1)(t+1)$. On the other hand it is clear that $\mathrm{tr}(QM) = (t+1)f + g$. The theorem follows. □

We will apply this theorem to the case of partial geometries admitting a regular abelian group of automorphisms, but first we shall obtain an easy lemma which will be used frequently in the rest of this paper without further notice.

Lemma 2.2. *Let $\mathcal{S}$ be a partial geometry admitting a regular abelian group of automorphisms G. Suppose $g \in G$. If $x^g \sim x$ for some point x of $\mathcal{S}$, then $y^g \sim y$ for every y of $\mathcal{S}$.*

Proof: By the regularity of G there exists an $h \in G$ such that $x^h = y$. As h is an automorphism of $\mathcal{S}$ and as G is abelian, we obtain $x^h \sim x^{gh} = x^{hg} = y^g$, that is, $y \sim y^g$. □

Corollary 2.3. *If $\mathcal{S}$ is a partial geometry* $\mathrm{pg}(s, t, \alpha)$*, $\alpha \neq s+1$, and $\mathcal{S}$ admits a regular abelian group G of automorphisms, then*

$$(s+1)\frac{st+\alpha}{\alpha} \equiv (1+s)(1+t) \equiv 0 \pmod{s+t-\alpha+1}.$$

Proof: First notice that if an element of G has a fixed point, then it is necessarily the identity. To obtain the first equivalence, choose any $g \in G \setminus \{id\}$ for which there exists a point x such that $x \sim x^g$, and apply Theorem 2.1. In order to obtain the second equivalence, choose a $g \in G$ for which there does not exist an x such that $x \sim x^g$ (notice that g exists as $\alpha \neq s+1$). □

3. Subdivision into three classes

The following basic observation will yield a natural subdivision into three classes of the pairs $(\mathcal{S}, G)$, with $\mathcal{S}$ a partial geometry and G an abelian automorphism group of $\mathcal{S}$ acting regularly on the points of $\mathcal{S}$.

Lemma 3.1. *Let $\mathcal{S}$ be a partial geometry* $\mathrm{pg}(s, t, \alpha)$ *and G an abelian Singer group of $\mathcal{S}$. Let L be any line of $\mathcal{S}$. Then either* $|\mathrm{Stab}_G(L)| = 1$*, or* $|\mathrm{Stab}_G(L)| = s+1$.

Proof: Let x be any point on L and suppose that $g \in G \setminus \{id\}$ stabilizes L. Let x^h, $h \in G$, be any point of $L \setminus \{x, x^g\}$. Clearly x^{hg} is a point of L, distinct from x^h. It follows that $L^{h^{-1}} = \langle x^h, x^{hg}\rangle^{h^{-1}} = \langle x, x^g\rangle = L$. Consequently h stabilizes L and $|\mathrm{Stab}_G(L)| = s+1$.

□

Based on the above observation we can introduce the following three classes of partial geometries $\mathcal{S}$ with an abelian Singer group G:

- the pair $(\mathcal{S}, G)$ is of *spread type* if $|\mathrm{Stab}_G(L)| = s+1$ for each line L of $\mathcal{S}$;
- the pair $(\mathcal{S}, G)$ is of *rigid type* if $|\mathrm{Stab}_G(L)| = 1$ for each line L of $\mathcal{S}$;
- the pair $(\mathcal{S}, G)$ is of *mixed type* otherwise.

Remark 3.2. Notice that such a division into classes of spread, rigid and mixed type can be introduced for any partial linear space with an abelian Singer group. (Lemma 3.1 only uses the fact that $\mathcal{S}$ is a partial linear space).

4. Pairs $(\mathcal{S}, G)$ of spread-type

In [7] it is shown that for every partial geometry $\mathcal{S}$ with $\alpha = 1$, that is every finite generalized quadrangle, which admits an abelian Singer group G, the pair $(\mathcal{S}, G)$ is of spread-type, yielding that $\mathcal{S}$ is the generalized linear representation of a generalized hyperoval.

Next suppose that $\alpha \geq 2$ and that $(\mathcal{S}, G)$ is of spread type. Choose any point x in $\mathcal{S}$ and denote the $t + 1$ lines through x by $L_0, L_1, \ldots, L_t$. It readily follows that L_i^G determines a spread (of symmetry) of $\mathcal{S}$, explaining the introduced terminology. Let S_i be the stabilizer in G of the line L_i, $i = 0, 1, \ldots, t$. Then it is also easily seen that the pair $(G, J = \{S_0, S_1, \ldots, S_t\})$ is an SPG-family in the sense of [6]. From Theorem 2.5 of that same paper [6] it then follows that $\mathcal{S}$ is isomorphic to a partial geometry constructed from a PG-regulus. We will now have a closer look at the parameters in the case $\alpha = 2$.

Theorem 4.1. *Let $\mathcal{S}$ be a partial geometry* $\mathrm{pg}(s, t, 2)$ *and let G be an abelian regular automorphism group of $\mathcal{S}$. If the pair $(\mathcal{S}, G)$ is of spread type, then $\mathcal{S}$ is constructed from a PG-regulus and hence $s = q^{m+1} - 1$ for some prime power q and some positive integer m. Moreover, either $t = 2$, in which case $\mathcal{S}$ is a translation net, or $t = 2(s + 2)$. In the latter case, q is necessarily a power of* 3.

Proof: By Theorem 2.5 of [6] $\mathcal{S}$ is constructed from an (S)PG-regulus $\mathcal{R}$, say an (S)PG-regulus consisting of m-dimensional spaces in $\mathrm{PG}(n, q)$, implying $s = q^{m+1} - 1$ for a certain prime power q. Consider two distinct intersecting lines. The stabilizers in G of these lines have size $s + 1$ and clearly generate a subgroup of G of size $(s + 1)^2$. Consequently $v = (s + 1)(st + 2)/2$ is divisible by $(s + 1)^2$. It follows that $(s + 1)$ divides $(st + 2)$, that is, $(s + 1)$ divides $(s + 1)t - t + 2$. Consequently $t = z(s + 1) + 2$, $z \in \mathbb{N}$. If $z = 0$, then $\mathcal{S}$ is a translation net. Next suppose that $z \geq 1$. Let $L_0, \ldots, L_t$ denote the $t + 1$ lines through a fixed point x. Denote the stabilizers in G of L_i by S_i, $i = 0, 1, \ldots, t$. The lines L_0 and L_1 determine in a natural way an $(s + 1) \times (s + 1)$-grid L_0L_1, which corresponds to the subgroup S_0S_1 of G. As $\mathcal{S}$ is a $\mathrm{pg}(s, t, 2)$, at most s of the lines $L_2, \ldots, L_t$ can intersect this grid in a point distinct from x. Consequently there exists a line, say without loss of generality L_2, intersecting L_0L_1 only in x. It follows that $S_0S_1S_2$ is a subgroup of order $(s + 1)^3$ of G. Hence $(s + 1)^2$ should divide $st + 2 = (s + 1)t - t + 2$. As we already know that $s + 1$ divides $t - 2$, we easily see that $s + 1$ must divide $(t - 2)/(s + 1) - t$. Since $t - 2$ is divisible by $s + 1$ we obtain that $s + 1$ divides $(t - 2)/(s + 1) - 2$. If $(t - 2)/(s + 1) - 2 = 0$, then $t = 2(s + 2)$. Now assume that $s + 1 \leq (t - 2)/(s + 1) - 2$. This implies $t \geq s^2 + 4s + 5$. As on the other hand for partial geometries $(s + 1 - 2\alpha)t \leq (s + 1 - \alpha)^2(s - 1)$ (see Chapter 13 of [14]), we see that $(s - 3)t \leq (s - 1)^3$. Consequently $s = 3$ (the case $s = 2$ yields a dual net, which we do not consider). Substitution in the identity $(s + 1)(t + 1) \equiv 0 \pmod{s + t - 1}$ from Corollary 2.3 yields $4(t + 1) \equiv 0 \pmod{t + 2}$, and so $t + 2$ must divide 4. Hence $t \in \{0, 2\}$, a contradiction. Finally to obtain that $q = 3^h$, it is sufficient to substitute $t = 2(s + 2)$ in $(s + 1)(t + 1) \equiv 0 \pmod{s + t - 1}$ from Corollary 2.3. This proves the theorem. $\square$

Consequently, if $\mathcal{S}$ is a proper partial geometry with $\alpha = 2$ admitting an abelian Singer group G such that the pair $(\mathcal{S}, G)$ is of spread type, then $\mathcal{S}$ is a pg$(3^{h(m+1)} - 1, 2(3^{h(m+1)} + 1), 2)$ constructed from a PG-regulus consisting of m-dimensional subspaces of PG$(3m + 2, 3^h)$, for some positive integers m and h (the dimension $3m + 2$ of the projective space follows from the definition of a PG-regulus given the parameters s and t).

Remark 4.2. It is important to notice that there exists a very interesting example of a PG-regulus $\mathcal{R}$ in PG(5, 3) yielding a pg(8, 20, 2). This PG-regulus is due to Mathon [5] and is the only known PG-regulus yielding a pg$(s, 2(s + 2), 2)$. Further notice that such a partial geometry has the same parameters as the partial geometry $T_2^*(\mathcal{K})$, which would arise from a maximal 3-arc $\mathcal{K}$ in PG$(2, q)$; it is however well known that such a maximal arc does not exist, see Cossu [4], Thas [15], Ball, Blokhuis and Mazzocca [1].

5. Pairs $(\mathcal{S}, G)$ of mixed-type

Let $\mathcal{S}$ be a proper partial geometry pg$(s, t, 2)$ and G an abelian Singer group of $\mathcal{S}$. Suppose that the pair $(\mathcal{S}, G)$ is of mixed type. Then there are $x(s + 1)$, $x \in \mathbb{N}_0$, lines L through any given point such that the stabilizer of L in G is trivial (as the orbit of any such line contains exactly $s + 1$ lines through the given point), and p lines through any given point with a stabilizer in G of order $s + 1$. So $t + 1 = x(s + 1) + p$ for certain $x, p \in \mathbb{N}_0$.

Lemma 5.1. *Suppose that $\mathcal{S}$ is a proper partial geometry* pg$(s, t, 2)$ *and that G is an abelian Singer group of $\mathcal{S}$ such that the pair $(\mathcal{S}, G)$ is of mixed type. Then, with the above notation, $p \neq 1$.*

Proof: Suppose by way of contradiction that $p = 1$. From Corollary 2.3 we obtain that

$$x(s + 1)^2 + s + 1 \equiv 0 \pmod{(x + 1)(s + 1) - 2}.$$

Hence, after multiplying with $(x + 1)^2$ we obtain

$$x\,[(x + 1)(s + 1)]^2 + (x + 1)\,[(x + 1)(s + 1)] \equiv 0 \pmod{(x + 1)(s + 1) - 2},$$

that is,

$$6x + 2 \equiv 0 \pmod{(x + 1)(s + 1) - 2}.$$

Consequently $5x + 3 \geq xs + s$, and so $s < 5$. For $s = 3$ we find that $6x + 2$ must be divisible by $4x + 2$, yielding $x = 0$, a contradiction. If $s = 4$, then $6x + 2$ has to be divisible by $5x + 3$, from which follows that $x = 1$ and consequently $t = 5$. Substitution in the original identity $(s + 1)(t + 1) \equiv 0 \pmod{s + t - 1}$ yields $30 \equiv 0 \pmod{8}$, the final contradiction. □

Lemma 5.2. *If $\mathcal{S}$ and G are as in the previous lemma, then $p = y(s+1)+3$ for certain $y \in \{0, 1\}$. Further, if $y = 1$, then also $x = 1$.*

Proof: Consider two distinct intersecting lines with stabilizers in G of size $s+1$. As in Theorem 4.1 we obtain that $(s+1)^2$ divides $v = (s+1)(st+2)/2$ and consequently that $t = z(s+1)+2$, that is $p = y(s+1)+3$ for certain $y \in \mathbb{N}$. Suppose that $y \geq 1$. Again as in Theorem 4.1 we deduce that $(s+1)^3$ must divide v, which yields that $t = 2(s+2)$, that is $x = y = 1$. □

Remark 5.3. Although the author strongly tends to believe that no partial geometries pg$(s, t, 2)$ $\mathcal{S}$ with an abelian Singer group G exist such that $(\mathcal{S}, G)$ is of mixed type, he has not yet been able to prove this conjecture to be correct. In view of the existence of pg$(s, 2(s+2), 2)$, divisibility conditions will clearly not help excluding this case.

6. Pairs $(\mathcal{S}, G)$ of rigid type

In this section we will obtain that there is a unique partial geometry pg$(s, t, 2)$ $\mathcal{S}$ such that the pair $(\mathcal{S}, G)$ is of rigid type, namely the partial geometry pg$(5, 5, 2)$ of van Lint and Schrijver.

6.1. The parameters

From now on let $\mathcal{S}$ be a proper partial geometry pg$(s, t, 2)$ admitting an abelian Singer group such that the pair $(\mathcal{S}, G)$ is of rigid type. One immediately observes that $t+1 = x(s+1)$ for certain $x \in \mathbb{N}_0$ (same argument as in the previous section). Further, Corollary 2.3 implies that $(s+1)(t+1) \equiv 0 \pmod{s+t-1}$, that is

$$x(s+1)^2 \equiv 0 \pmod{(x+1)(s+1)-3}.$$

Multiplying with $(x+1)^2$ yields

$$x\,[(x+1)(s+1)]^2 \equiv 0 \pmod{(x+1)(s+1)-3}$$

from which follows that

$$9x \equiv 0 \pmod{(x+1)(s+1)-3}.$$

Hence $9x \geq (x+1)(s+1)-3$ from which we deduce that $s < 8$.

Theorem 6.1. *If $\mathcal{S}$ is a proper partial geometry* pg$(s, t, 2)$ *and G is an abelian Singer group of $\mathcal{S}$ such that the pair $(\mathcal{S}, G)$ is of rigid type, then $(s, t) = (5, 5)$.*

Proof: We will one by one handle the cases $s = 3, 4, \ldots, 7$.

- If $s = 3$ we obtain that $9x$ must be divisible by $4x + 1$ which yields that $x = 2$ and hence $t = 7$. Substituting this in the original identity $2 + st \equiv 0 \pmod{s + t - 1}$ gives us a contradiction.
- In the case where $s = 4$ we obtain that $9x$ is divisible by $5x + 2$. This implies $x = 0$, a contradiction.
- For $s = 5$ we obtain that $9x$ is divisible by $6x + 3$, and so $x = 1$ and $t = 5$.
- Finally, the cases $s = 6$ and $s = 7$ can easily be excluded analogously as in the case $s = 4$.

$\square$

6.2. *G* is elementary abelian

Let $\mathcal{S}$ and G be as in the previous subsection, that is, $\mathcal{S}$ is a pg(5, 5, 2) and G is an abelian automorphism group of $\mathcal{S}$ acting regularly on the points of $\mathcal{S}$. We will show that G is elementary abelian. Although we know that $s = t = 5$, we will in this subsection always write s, as we believe that this makes general arguments easier to read.

Now choose any fixed point x in $\mathcal{S}$. We will identify the point y of $\mathcal{S}$ with the unique $g \in G$ for which $x^g = y$. Finally, D will denote the set of all points (elements of G), with exclusion of id, that are collinear with id. We notice that this notation comes from the theory of partial difference sets, as it is easily checked that D is a partial difference set in G (see e.g. [13]).

Lemma 6.2. *We have $D^2 \subset D$.*

Proof: Choose any $g \in D$, and consider the lines $L = \langle id, g\rangle$ and L^g. The line L^g consists of all points fg with $f \in L$. For any such f with $id \neq f \neq g$ the point fg is collinear with f and g on L. If g^2 were not collinear with id, then necessarily g^2 would be collinear with some $f \in L \setminus \{id, g\}$. However this f is already collinear with g and fg on L^g, implying that f would be collinear with more then 2 points on L^g, a contradiction. We conclude that $g^2 \in D$. (Notice that $g^2 \notin L$ since otherwise the stabilizer of L in G would not be trivial).

$\square$

Lemma 6.3. *Suppose L is a line through id. If we denote the points of L by $id, g_1, g_2, \ldots, g_s$, then $L^2 := \{id, g_1^2, g_2^2, \ldots, g_s^2\}$ is a set of two by two collinear points, no three on a line.*

Proof: From the previous lemma we know that $id \sim g_i^2$, for all $i \in \{1, 2, \ldots, s\}$. We now show that $g_i^2 \sim g_j^2$, for $i \neq j$. Consider the lines $M_i := \langle g_i, g_i^2\rangle$ and $M_j := \langle g_j, g_j^2\rangle$. Then $M_i \cap M_j = g_i g_j$. Note that $g_i g_j \notin D$ since the unique points on L collinear with $g_i g_j$ are g_i and g_j. From the fact that $g_i \sim g_j$ it follows that the point $g_l g_i$ on M_i is collinear with the point $g_l g_j$ on M_j, for all $l \notin \{i, j\}$. Now assume that g_i^2 would not be collinear with g_j^2. Then g_i^2 has to be collinear with either g_j or $g_l g_j$ for some $l \notin \{i, j\}$. If g_i^2 would be collinear with g_j, then it would be collinear with at least three points on L, a contradiction. Would g_i^2 be collinear with $g_l g_j$ for some

$l \notin \{i, j\}$, then $g_l g_j$ would be collinear with the distinct points $g_i g_j$, $g_l g_i$ and g_i^2 on M_i, again a contradiction. It follows that L^2 is indeed a set of two by two collinear points. Next, suppose there is a line containing at least three distinct points of L^2. From the fact that $\alpha = 2$ it is immediate that in this case L^2 is a line through id (distinct from L), say, without loss of generality, $L^{g_1{}^{-1}}$. Hence $g_2 g_1^{-1} = g_j^2$ for certain j. This implies that $j = 2$, as the point $g_2 g_1^{-1}$ is collinear with id, g_2 and g_j on L. But then $g_1^{-1} = g_2$, a contradiction. The lemma follows. □

Although the following lemma can be avoided, it provides interesting information on the structure of the geometry $\mathcal{S}$ with respect to the group G.

Lemma 6.4. *Let the line L be as in the previous lemma. Suppose $h \in G$, with $h \notin D \cup \{id\}$. Then one, and only one, of the following cases occurs:*

- *$h \in \langle g_i^2, g_j^2 \rangle$ for unique i and j, $i \neq j$;*
- *$h \in \langle g_i, g_i^2 \rangle$ for exactly two values of i.*

Proof: Suppose $h = \langle g_i^2, g_j^2 \rangle \cap \langle g_l^2, g_k^2 \rangle$ or $h = \langle g_i^2, g_j^2 \rangle \cap \langle g_l, g_l^2 \rangle$, where i, j, k, l are pairwise distinct. In each case g_l^2 would be collinear with three points on $\langle g_i^2, g_j^2 \rangle$, a contradiction.

Any point $h \in \langle g_i, g_i^2 \rangle \setminus \{g_i, g_i^2\}$ can be written as $h = g_j g_i$ for certain $j \neq i$. Hence, such h also belongs to the line $\langle g_j, g_j^2 \rangle$. Now assume that there would be a third index k, with $i \neq k \neq j$, such that $h \in \langle g_k, g_k^2 \rangle$. Then g_k^2 would be collinear with the three distinct points $g_k g_i$, h and g_i^2 on $\langle g_i, g_i^2 \rangle$, a contradiction.

Denote by X_1 the number of elements of $G \setminus (D \cup \{id\})$ on lines of type $\langle g_i^2, g_j^2 \rangle$ and by X_2 the number of elements of $G \setminus (D \cup \{id\})$ on lines of type $\langle g_i, g_i^2 \rangle$. Easy counting shows

$$X_1 = s(s-1)^2/2$$

,

$$X_2 = s(s-1)/2$$

and hence

$$|D \cup \{id\}| + X_1 + X_2 = (s+1)(s^2+2)/2 = v.$$

The lemma now easily follows. □

Lemma 6.5. *Let the line L be as before. Then $g_i^3 = id$ for all i.*

Proof: First suppose that g_i^3 does not belong to $D \cup \{id\}$. Since L^2 is a clique of size $s+1$ (and so g_i^3 is collinear with exactly two points of L^2) the previous lemma implies that either $g_i^3 \in \langle g_i^2, g_k^2 \rangle$, or $g_i^3 = \langle g_i, g_i^2 \rangle \cap \langle g_k, g_k^2 \rangle$, $k \neq i$. In the first case we see that $g_k^2 \in \langle g_i^2, g_i^3 \rangle$ and hence $g_k^2 = g_f g_i^2$ for certain $f \in \{1, 2, \ldots, s\}$. Hence g_k^2 is collinear with id, g_k and g_f on L, which implies that $f = k$. But then $g_k = g_i^2$, a contradiction.

In the second case it follows that $g_i^3 = g_i g_k$, again yielding the contradiction $g_k = g_i^2$. We conclude that $g_i^3 \in D \cup \{id\}$.

Now suppose that $g_i^3 \neq id$. From $g_i^2 \sim g_i^3$ and $g_i^3 \in D$ it follows that $g_i^3 = g_k^2$ for certain $k \neq i$, or that $g_i^3 = g_i$. In the first case the point $g_i^3 = g_k^2$ would be collinear with three distinct points, id, g_i and g_k on L, a contradiction. The second case is absurd as well, as it would imply $g_i^2 = id$. The lemma is proved. □

Corollary 6.6. *For all $g \in G$ we have $g^3 = id$*

Proof: This follows immediately as each $g \in G$ can be written as $g = fh$, with $f, h \in D$. □

Corollary 6.7. *The group G is elementary abelian.*

Corollary 6.8. *The mapping $\beta : G \to G : g \mapsto g^2$ is an automorphism of G.*

Proof: It is clear that β is an endomorphism of G (G is abelian!). We need to show that β is bijective. Assume by way of contradiction that $g^2 = h^2$ for certain $g \neq h$ in G. By the above we obtain $id = g^3 = h^3 = g^2 h$. This implies that $g^3 = g^2 h$, that is $g = h$, a contradiction. □

Remark 6.9. Although β is an automorphism of G it does not induce an automorphism of our geometry $\mathcal{S}$. It has nevertheless an interesting geometric interpretation. Let $\mathcal{S}$ be a partial geometry pg(5, 5, 2) admitting an abelian Singer group. It is, using the above, easily seen that every two distinct collinear points (elements of G) g and h are contained in two cliques of size 6, namely the one defined by the line $L := \langle g, h \rangle$, and the one defined by M^β, where $M = \langle g^2, h^2 \rangle$, which is a set of 6 points of $\mathcal{S}$ no three of which are collinear (note that since $D^2 = D$, g^2 is indeed collinear with h^2). In fact every two distinct collinear points are contained in exactly two distinct cliques of order 6 as the number of points collinear with two given distinct collinear points equals 11, and two distinct cliques of size 6 intersect in at most two distinct points. Define C to be the set of all cliques that arise as the "squares" of lines of $\mathcal{S}$. Then the geometry with as point set the elements of G, with as line set the elements of C and with as incidence relation containment is a pg(5, 5, 2) $\mathcal{S}^*$ and β is an isomorphism from $\mathcal{S}$ to $\mathcal{S}^*$.

6.3. The uniqueness of $\mathcal{S}$

Throughout this section $\mathcal{S}$ will be a pg(5, 5, 2) and G the elementary abelian group of order 81 acting regularly on the points of $\mathcal{S}$. We will show that $\mathcal{S}$ is unique. We will again identify the points of $\mathcal{S}$ with elements of G. The set D will again be the set of all elements distinct from id collinear with id. Let L be any line through id. We denote the points of L by id, g_1, g_2, g_3, g_4 and g_5. Further denote by L_i, $i = 1, \ldots, 5$

the line through id containing g_i^2. The unique point on $L_i \setminus \{id\}$ collinear with g_j is the point $g_j g_i^2$.

Lemma 6.10. *We have $g_1 g_2 g_3 g_4 g_5 = id$.*

Proof: Consider the point $g_1 g_2$. On L_1 this point is collinear with g_1^2 and $g_2 g_1^2$ and we have $g_1 g_2 = (g_1^2)(g_2 g_1^2)$. Analogously the two points collinear with $g_1 g_2$ on L_2 are uniquely determined (g_2^2 and $g_1 g_2^2$). As $g_1 g_2$ is already collinear with g_1^2 and g_2^2 it cannot be collinear with g_i^2 for $i \notin \{1, 2\}$ (recall that a point exterior to a clique of order 6 is collinear with exactly 2 points of this clique). Suppose that $g_1 g_2$ would be collinear with $g_1 g_3^2$. Then, as $g_1 g_2 = (g_1 g_3^2)(g_2 g_3)$, we would obtain that $g_2 g_3$ belongs to D (recall Lemma 2.2), a contradiction. Hence, using an analogous reasoning, we obtain that the points collinear with $g_1 g_2$ on L_3, L_4 and L_5 are $g_4 g_3^2$, $g_5 g_3^2$, $g_3 g_4^2$, $g_5 g_4^2$, $g_3 g_5^2$ and $g_4 g_5^2$. There is a unique h such that $g_1 g_2 = (g_4 g_3^2)h$. Moreover h must be collinear with id (as it maps $g_4 g_3^2$ to a collinear point) and $g_1 g_2$ (as $g_4 g_3^2$ belongs to D). It follows that h must be one of the points collinear with $g_1 g_2$ on L_3, L_4 or L_5 (note that h cannot belong to L, L_1 or L_2 since $g_1 g_2 = (g_1)(g_2) = (g_1^2)(g_2 g_1^2) = (g_2^2)(g_1 g_2^2)$). We have that $h \neq g_4 g_3^2$ since otherwise this would imply that $g_1 g_2 = g_3 g_4^2$, a contradiction. Suppose that $h = g_5 g_3^2$. Then $g_1 g_2 = g_4 g_3^2 g_5 g_3^2 = g_3 g_4 g_5$, from which we deduce that $g_5 = (g_1 g_3^2)(g_2 g_4^2)$. Hence g_5 would be collinear with $g_2 g_4^2$, a contradiction. Further, as $g_1 g_2 \notin D \cup \{id\}$, $h \notin \{g_3 g_4^2, g_5 g_4^2, g_3 g_5^2\}$. It follows that $h = g_4 g_5^2$, that is, $g_1 g_2 = g_3^2 g_4^2 g_5^2$, that is, $g_1 g_2 g_3 g_4 g_5 = id$. □

Theorem 6.11. *The geometry $\mathcal{S}$ is unique.*

Proof: Supposes $\overline{\mathcal{S}}$ is a pg(5, 5, 2) with a regular abelian automorphism group $\overline{G}$. We need to show that $\mathcal{S} \cong \overline{\mathcal{S}}$. For the geometry $\mathcal{S}$ we will use the same notation as above. In the geometry $\overline{\mathcal{S}}$ we will identify the points with the elements of $\overline{G}$. Choose any line $\overline{L}$ of $\overline{\mathcal{S}}$ through $id_{\overline{G}}$ and denote its points distinct from $id_{\overline{G}}$ by $\overline{g_i}$, $i = 1, 2, \ldots, 5$. We define a mapping γ from G to $\overline{G}$ as follows: $\gamma(id_G) := id_{\overline{G}}$, $\gamma(g_i) := \overline{g_i}$, $i = 1, 2, \ldots, 5$, and $\gamma(\prod_i g_i) := \prod_i \overline{g_i}$. We will show that γ is well defined and is in fact an isomorphism between $\mathcal{S}$ and $\overline{\mathcal{S}}$. In order to show that γ is well defined we should show that $\prod_i g_i = \prod_j g_j$ implies that $\prod_i \overline{g_i} = \prod_j \overline{g_j}$. It is clear that the group G is generated by D, hence by $\{g_1, g_2, g_3, g_4, g_5\}$, and so, because of the previous lemma, by $\{g_1, g_2, g_3, g_4\}$. Hence every point of $\mathcal{S}$ (element of G) can be expressed in a unique way—as there are exactly 81 points—as $g_1^{i_1} g_2^{i_2} g_3^{i_3} g_4^{i_4}$, with $i_k \in \{0, 1, 2\}$, $k = 1, \ldots, 4$. From this one easily sees that γ is well defined. And from the definition of γ it is now also clear that γ is an isomorphism between G and $\overline{G}$. Further, as (with a little abuse of notation) $\gamma(L) = \overline{L}$, L^G is the set of all lines of $\mathcal{S}$, and $\overline{L}^{\overline{G}}$ is the set of all lines of $\overline{\mathcal{S}}$, is easily deduced that γ is an isomorphism between $\mathcal{S}$ and $\overline{\mathcal{S}}$. □

Corollary 6.12. *Let $\mathcal{S}$ be a proper partial geometry* pg$(s, t, 2)$ *and let G be an abelian group acting regularly on the points of $\mathcal{S}$. If the pair $(\mathcal{S}, G)$ is rigid then $\mathcal{S}$ is isomorphic to the partial geometry of van Lint and Schrijver.*

Corollary 6.13. *If $\mathcal{S}$ is a proper partial geometry* pg$(s, t, 2)$ *with* $t \not\equiv 2 \pmod{s+1}$ *admitting an abelian Singer group, then $\mathcal{S}$ is isomorphic to the partial geometry of van Lint and Schrijver.*

Proof: If $t \not\equiv 2 \pmod{s+1}$ the the pair $(\mathcal{S}, G)$ must be of rigid type. □

Remark 6.14. In the above proof we use the fact that there is a pg(5, 5, 2) with an abelian Singer group known. This is however not necessary, as our proof is implicitly constructive. Namely the following can be checked. Let G be the elementary abelian group of order 81. Let g_i, $i = 1, \ldots, 5$ be 5 distinct non-identity elements in G. Suppose that $g_1^{i_1} g_2^{i_2} g_3^{i_3} g_4^{i_4} g_5^{i_5} = id$, $i_k \in \{0, 1, 2\}$, if and only if all i_k are equal. Finally define $L := \{id, g_1, \ldots, g_5\}$. Then the geometry with as point set the elements of G, as line set the set $\{L^g \mid g \in G\}$ and with as incidence relation containment is a pg(5, 5, 2). Notice that the existence of such g_i is well known. For example consider the group $G \cong (\mathbb{Z}_3^4, +)$ as the set of all 4-tuples over $\mathbb{Z}_3$ and define L as the set $L := \{(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (-1, -1, -1, -1)\}$. This also gives us an easy geometric interpretation of the van Lint-Schrijver partial geometry: it is the geometry in AG(4, 3) of all translates of the set L.

7. Overview

We can now bring the results of the previous sections together in the following theorem.

Theorem 7.1. *Let $\mathcal{S}$ be a proper partial geometry* pg$(s, t, 2)$*, and suppose that there exists an abelian automorphism group G of $\mathcal{S}$ acting regularly on the points of $\mathcal{S}$. Then either $\mathcal{S}$ is a* pg$(3^{h(m+1)} - 1, 2(3^{h(m+1)} + 1), 2)$ *constructed from a PG-regulus in* PG$(3m + 2, 3^h)$*, or $\mathcal{S}$ is a* pg$(s, x(s + 1) + 2, 2)$ *with* $(\mathcal{S}, G)$ *of mixed type, or $\mathcal{S}$ is a* pg(5, 5, 2) *isomorphic to the partial geometry of van Lint and Schrijver.*

References

1. S. Ball, A. Blokhuis, and F. Mazzocca, "Maximal arcs in Desarguesian planes of odd order do not exist," *Combinatorica* **17** (1997), 31–41.
2. C.T. Benson, "On the structure of generalized quadrangles," *J. Algebra* **15** (1970), 443–454.
3. R.C. Bose, "Strongly regular graphs, partial geometries and partially balanced designs," *Pacific J. Math.* **13** (1963), 389–419.
4. A. Cossu, "Su alcune proprietá dei $\{k, n\}$-archidi un piano proiettivo sopra un corpo finito," *Rend. Mat. Appl.* **20** (1961), 271–277.
5. F. De Clerck, M. Delanote, N. Hamilton, and R. Mathon, "Perp-systems and partial geometries," *Adv. Geom.* **2** (2002), 1–12.
6. S. De Winter, "Elation and translation semipartial geometries," *J. Combin. Theory Ser. A* **108**(2) (2004), 313–330.
7. S. De Winter and K. Thas, "Generalized quadrangles with an abelian Singer group," *Des Codes Cryptogr.* **39**(1) (2006) 81–87.
8. D. Ghinelli, "Regular groups on generalized quadrangles and nonabelian difference sets with multiplier −1." *Geom. Dedicata* **41** (1992), 165–174.
9. D. Ghinelli and D. Jungnickel, "Finite projective planes with a large abelian group," in *Surveys in Combinatorics 2003*, London Mathematical Society, Lecture Notes Series **307**, 2003.

10. W. Haemers and J.H. van Lint, "A partial geometry pg(9, 8, 4)," *Ann. Discrete Math.* **15** (1982), 205–212.
11. N. Jacobson, *Basic Algebra II*, W. H. Freeman and Company, 1980.
12. J.H. van Lint and A. Schrijver, "Construction of strongly regular graphs, two-weight codes and partial geometries by finite fields," *Combinatorica* **1** (1981), 63–73.
13. S.L. Ma, "Regular automorphism groups on partial geometries," *Shanghai Conference Issue on Designs, Codes, and Finite Geometries, Part 1 (Shanghai, 1993). J. Statist. Plann. Inference* **51**(2) (1996), 215–222.
14. S.E. Payne and J.A. Thas, *Finite Generalized Quadrangles*, Research Notes in Mathematics, **110**, Pitman (Advanced Publishing Program), Boston MA, 1984, vi+312 pp.
15. J.A. Thas, "Some results concerning $\{(q+1)(n-1); n\}$-arcs and $\{qn-q+n; n\}$-arcs in finite projective planes of order q," *J. Combin. Theory Ser. A* **19** (1975), 228–232.
16. J.A. Thas, "Semipartial geometries and spreads of classical polar spaces," *J. Combin. Theory Ser. A* **35** (1983), 58–66.