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Algebraic and combinatorial properties of zircons

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Abstract In this paper we introduce and study a new class of posets, that we call zircons, which includes all Coxeter groups partially ordered by Bruhat order. We prove that many of the properties of Coxeter groups extend to zircons often with simpler proofs: in particular, zircons are Eulerian posets and the Kazhdan-Lusztig construction of the Kazhdan-Lusztig representations can be carried out in the context of zircons. Moreover, for any zircon Z, we construct and count all balanced and exact labelings (used in the construction of the Bernstein-Gelfand-Gelfand resolutions in the case that Z is the Weyl group of a Kac-Moody algebra).

Keywords Bruhat order · Special matchings · Coxeter groups

1 Introduction

Coxeter group theory has a wide range of applications in different areas of mathematics such as algebra, combinatorics, and geometry (see e.g. [2, 5, 12, 14]). Bruhat order arises in Coxeter group theory in several contexts such as in connection with the Bruhat decomposition, with inclusions among Schubert varieties, with the Verma modules of a complex semisimple Lie algebra, and in Kazhdan-Lusztig theory. Coxeter groups partially ordered by Bruhat order have a rich combinatorial structure which has been the object of several studies. In this paper, we introduce a new class of partially ordered sets, that we call zircons, which properly includes the class of finite

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and infinite Coxeter groups partially ordered by Bruhat order. Many of the properties of the Coxeter groups extend to zircons: in particular, we prove that zircons are Eulerian posets, that open intervals in zircons are isomorphic to spheres, and that the Kazhdan-Lusztig construction of the Kazhdan-Lusztig representations can be carried out in the context of zircons. It is often the case that the proofs for zircons are simpler than the corresponding proofs for Coxeter groups: in particular, the proof of Theorem 3.4, as far as we know, is the shortest among the many different arguments which prove the Eulerianity of Coxeter groups (see [3, 9, 15, 20] and the recent paper by J. Stembridge [18]). The definition of zircon is based on the concept of special matchings. These are particular matchings of the Hasse diagram that play a fundamental role in the proof of Lusztig's Conjecture on the combinatorial invariance of Kazhdan-Lusztig polynomials for lower Bruhat intervals (see [7] or [17]).

For every Coxeter group W, D. Kazhdan and G. Lusztig [15] define certain polynomials indexed by pairs of elements in W which are now known as the Kazhdan-Lusztig polynomials. These polynomials are introduced in order to construct certain representations of the Hecke algebra associated to W. In [8], the authors show that Kazhdan and Lusztig's construction can be carried out in a more general (and entirely combinatorial) context. Here we produce a further generalization showing that all results in [8], which cannot be extended to arbitrary zircons, are indeed valid in the category of well refined zircons, which are zircons with the additional structure given by specifying certain special matchings. More precisely, we can define a family of polynomials indexed by pairs of elements in any well refined zircon which reduce to the Kazhdan-Lusztig polynomials in the case that the zircon is a Coxeter group. We then associate to every well refined zircon a Coxeter group and hence a Hecke algebra, and show that this Coxeter group and the corresponding Hecke algebra act on certain subsets of the zircon (the zircon cells). These representations are the usual Kazhdan-Lusztig representations when the zircon is a Coxeter group and the Hecke algebra is the wanted one.

I. N. Bernstein, I. M. Gelfand and S. I. Gelfand [1] construct certain resolutions, now called the BGG resolutions, of a finite-dimensional irreducible module of a complex semisimple Lie algebra $\mathfrak{g}$ by Verma modules (see also [16]). The differential maps of the BGG resolutions are explicitly given in terms of certain labelings of the Hasse diagram of the Weyl group W associated to $\mathfrak{g}$ (partially ordered by Bruhat order). I. N. Bernstein, I. M. Gelfand and S. I. Gelfand [1, Lemma 10.4] prove the existence of such labelings for any finite Coxeter group W. Here, for any finite or infinite zircon Z, we give an algorithm to construct all such labelings, and we count their number producing explicit bijections with the subsets of $Z \setminus \{\text{minimal elements}\}$. The proof of this result achieved using special matchings is simpler than the proof of the more particular result for Coxeter groups.

This work is organized as follows. In Sect. 2, we recall some basic definitions and results that are needed in the sequel. In Sect. 3, we introduce the class of zircons and we derive their first properties, including the fact that they are Eulerian posets. In Sect. 4, we show how to develop Kazhdan and Lusztig's theory in the context of zircons. Sections 5 and 6 are devoted to the labelings used to construct the BGG complexes and the BGG resolutions. We call such labelings balanced labelings and exact labelings. In Sect. 5, we give some general results on balanced and exact labelings. In Sect. 6, we first prove that the set of balanced labelings of a zircon Z is in bijection with the set of the subsets of $Z \setminus \{\text{minimal elements}\}$. Then we show that the concepts of balanced labeling and exact labeling coincide for zircons. The results in Sect. 6 are new also in the case of Coxeter groups and imply Lemma 10.4 of [1].

2 Notation and background

This section reviews the background material on posets, Coxeter systems and special matchings that is needed in the rest of this work. We refer the reader to [2, 14] and [17] for a more detailed treatment. We write ":=" if we are defining the left hand side by the right hand side. We let $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$, and for $a, b \in \mathbb{N}$ we let $[a, b] := \{a, a+1, a+2, \ldots, b\}$ and $[a] := \{1, 2, \ldots, a\}$. The cardinality of a set A will be denoted by $|A|$. The disjoint union of two sets A and $\tilde{A}$ will be denoted by $A \uplus \tilde{A}$.

Let P be a partially ordered set (or poset for short). An order ideal of P is a subset $S \subseteq P$ such that, if $x \in S$ and $y \leq x$, then $y \in S$. An element $x \in P$ is *maximal* (respectively *minimal*) if there is no element $y \in P \setminus \{x\}$ such that $x \leq y$ (respectively $y \leq x$). We say that P has a *bottom element* $\widehat{0}$ if there exists an element $\widehat{0} \in P$ satisfying $\widehat{0} \leq x$ for all $x \in P$. Similarly, P has a *top element* $\widehat{1}$ if there exists an element $\widehat{1} \in P$ satisfying $x \leq \widehat{1}$ for all $x \in P$. If $x \leq y$ we define the *(closed) interval* $[x, y] = \{z \in P : x \leq z \leq y\}$ and the *open interval* $(x, y) = \{z \in P : x < z < y\}$. If every interval of P is finite, then P is called a *locally finite poset*. We say that x is covered by y if $x < y$ and $[x, y] = \{x, y\}$, and we write $x \lhd y$ or $y \rhd x$. If P has a $\hat{0}$ then an element $x \in P$ is an *atom* of P if $\hat{0} \lhd x$. Similarly, if P has a $\hat{1}$ then an element $x \in P$ is a *coatom* of P if $x \lhd \hat{1}$. Given $p \in P$, the coatoms of p are the coatoms of $\{x \in P : x \leq p\}$. A *chain* of P is a totally ordered subset of P. A chain c with top element y and bottom element x is *saturated* if it is a maximal chain of the interval $[x, y]$.

A standard way of depicting a poset P is by its *Hasse diagram*. This is the digraph with P as node set and having an upward-directed edge from x to y if and only if $x \lhd y$. We say that P is *connected* if its Hasse diagram is connected as a graph. A morphism of posets is a map $\phi : P \to Q$ from the poset P to the poset Q which is *order-preserving*, namely such that $x \leq y$ in P implies $\phi(x) \leq \phi(y)$ in Q, for all $x, y \in P$. Two posets P and Q are *isomorphic* if there exists an order-preserving bijection $\phi : P \to Q$ whose inverse is also order-preserving. In this case ϕ is an isomorphism of posets.

A poset P is *ranked* if there exists a (rank) function $\rho : P \to \mathbb{N}$ such that $\rho(y) = \rho(x) + 1$ whenever $x \lhd y$. A poset P is *pure* of length $\ell(P) = n$ if all maximal chains are of the same length n. A poset P with bottom element $\widehat{0}$ is *graded* if every interval $[\widehat{0}, x]$ is pure. A poset P is a Boolean algebra if it is isomorphic to the poset of all subsets of a certain set S, partially ordered by inclusion. In this case, if $|S| = n$, then we say that P is the Boolean algebra of rank n. We say that a ranked poset P with rank function ρ is *thin* if for all ordered pairs $x \leq y \in P$ with $\rho(y) - \rho(x) = 2$, the interval $[x, y]$ consists of exactly 4 elements. In this case we say that $[x, y]$ is a square. The order complex $\Delta(P)$ of a poset P is the simplicial complex whose simplices are the chains in P. We denote by $||\Delta(P)||$ its geometric realization. A poset P is a *piecewise linear sphere*, or a *PL-sphere*, if $\Delta(P)$ admits a subdivision which is a subdivision of the boundary of a simplex.

The *Möbius function* of P assigns to each ordered pair $x \leq y$ an integer $\mu(x, y)$ according to the following recursion:

$$\mu(x, y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x, z), & \text{if } x < y. \end{cases}$$

A graded poset P, with rank function ρ, is *Eulerian* if $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$ for all $x, y \in P$, $x \leq y$. Equivalently, P is Eulerian if and only if

$$|\{z \in [x, y] \, : \, \rho(z) \text{ is even}\}| = |\{z \in [x, y] \, : \, \rho(z) \text{ is odd}\}|$$

for all $x, y \in P$, $x \leq y$.

Given a Coxeter system (W, S) and $w \in W$, we denote by $l(w)$ the length of w, we call any product of $l(w)$ elements of S which represents w a *reduced expression for* w, and we let

$$\begin{aligned} D_R(w) &:= \{s \in S : \, l(ws) < l(w)\} = D_L(w^{-1}), \\ D_L(w) &:= \{s \in S : \, l(sw) < l(w)\} = D_R(w^{-1}). \end{aligned}$$

We call $D_R(w)$ and $D_L(w)$ respectively the *right* and the *left descent set of* w. We denote by e the identity of W, and we let $T := \{wsw^{-1} : w \in W, \; s \in S\}$ be the set of reflections of W. We denote the symmetric group on n elements by $\mathfrak{S}(n)$ and the transpositions in $\mathfrak{S}(n)$ by (i, j), where $1 \leq i < j \leq n$. Let $S := \{s_1 = (1, 2), s_2 = (2, 3), \ldots, s_{n-1} = (n-1, n)\}$. It is well known that $(\mathfrak{S}(n), S)$ is a Coxeter system of rank $n - 1$. We call an interval $[u, v]$ in a poset P *dihedral* if it is isomorphic to a finite Coxeter system of rank ≤ 2 ordered by Bruhat order.

The Coxeter group W is partially ordered by (strong) *Bruhat order*, which will be denoted by $\leq$. Given $u, v \in W$, $u \leq v$ if and only if there exist $r \in \mathbb{N}$ and $t_1, \ldots, t_r \in T$ such that $t_r \ldots t_1 u = v$ and $l(t_i \ldots t_1 u) > l(t_{i-1} \ldots t_1 u)$ for $i = 1, \ldots, r$. It is well known that W, partially ordered by Bruhat order, is a graded poset having the length function l as its rank function and the identity e as bottom element. There is a well known characterization of Bruhat order on a Coxeter group (usually referred to as the *subword property*). By a *subword* of a word $s_1 s_2 \cdots s_q$ we mean a word of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq q$.

Theorem 2.1 *Let $u, v \in W$. Then the following are equivalent*:

(1) $u \leq v$,
(2) *every reduced expression for v has a subword that is a reduced expression for u*,
(3) *there exists a reduced expression for v which has a subword that is a reduced expression for u*.

Lemma 2.2 (Lifting Lemma) *Let $s \in S$ and $u, v \in W$, $u \leq v$. Then*

- *if $s \in D_R(v)$ and $s \in D_R(u)$ then $us \leq vs$*,
- *if $s \notin D_R(v)$ and $s \notin D_R(u)$ then $us \leq vs$*,
- *if $s \in D_R(v)$ and $s \notin D_R(u)$ then $us \leq v$ and $u \leq vs$*.

The *Hecke algebra* $\mathcal{H}(W)$ of W is the free $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module having the set $\{T_w : w \in W\}$ as a basis and multiplication uniquely determined by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w, \\ (q-1)T_w + qT_{sw} & \text{if } sw < w, \end{cases}$$

for all $w \in W$ and $s \in S$. This is an associative algebra having T_e as unity. Each basis element is invertible in $\mathcal{H}(W)$.

Proposition 2.3 *There exists a family of polynomials* $\{R_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ *satisfying*

$$(T_{w^{-1}})^{-1} = q^{-l(w)} \sum_{u \leq w} (-1)^{l(u,w)} R_{u,w}(q) T_u,$$

with $R_{w,w} = 1$ *for all* $w \in W$.

The polynomials $R_{u,v}$ are called the *R-polynomials* of W.

Define an involution $\iota : \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by $\iota(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$ and extend it to a $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-semilinear ring automorphism $\iota : \mathcal{H}(W) \to \mathcal{H}(W)$ satisfying $\iota(T_w) = (T_{w^{-1}})^{-1}$. The following result is due to D. Kazhdan and G. Lusztig [15].

Theorem 2.4 *There exists a unique basis* $\mathcal{C}' = \{C'_w : w \in W\}$ *of* $\mathcal{H}(W)$ *such that*:

1. $\iota(C'_w) = C'_w$;
2. $C'_w = q^{-\frac{l(w)}{2}} \sum_{u \leq w} P_{u,w}(q) T_u$;
3. $P_{u,w} \in \mathbb{Z}[q]$ *has degree at most* $\frac{1}{2}(l(w) - l(u) - 1)$ *if* $u < w$, *and* $P_{w,w} = 1$.

The polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ are the *Kazhdan-Lusztig polynomials* of W.

Recall that a matching of a graph $G = (V, E)$ is an involution $M : V \to V$ such that $\{M(v), v\} \in E$ for all $v \in V$. Let P be a partially ordered set. A matching M of the Hasse diagram of P is a *special matching of P* if

$$u \lhd v \Longrightarrow M(u) \leq M(v),$$

for all $u, v \in P$ such that $M(u) \neq v$.

Remark A special matching has certain rigidity properties. For example, if $u \lhd v$ and $M(v) \lhd v$, then $M(u) \lhd u$ and $M(u) \lhd M(v)$.

For the reader's convenience, we collect the following two results. The first one appears in [7] while the second one follows easily by Lemma 4.2 of [6].

Lemma 2.5 *Let P be a locally finite ranked poset, M be a special matching of P, and $u, v \in P$, $u \leq v$, be such that $M(u) \rhd u$ and $M(v) \lhd v$. Then M restricts to a special matching of $[u, v]$.*

Lemma 2.6 (Lifting Lemma for special matchings) *Let M be a special matching of a locally finite ranked poset P, and let $u, v \in P$, $u \leq v$. Then*

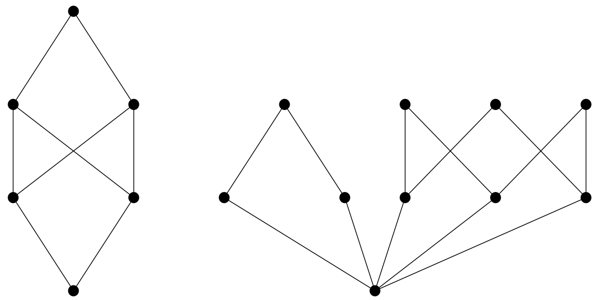


Fig. 1 A zircon

1. *if* $M(v) \lhd v$ *and* $M(u) \lhd u$ *then* $M(u) \leq M(v)$,
2. *if* $M(v) \rhd v$ *and* $M(u) \rhd u$ *then* $M(u) \leq M(v)$,
3. *if* $M(v) \lhd v$ *and* $M(u) \rhd u$ *then* $M(u) \leq v$ *and* $u \leq M(v)$.

3 Zircons

In this section we introduce the main concept of this paper. This is a class of abstract partially ordered sets which includes Coxeter groups partially ordered by Bruhat order. Then we derive some of its basic properties including the fact that they are Eulerian.

Given a poset P, we denote the set of all special matchings of P by SM_P. Given an element $w \in P$, we say that M is a special matching of w if M is a special matching of the Hasse diagram of the subposet $\{x \in P : x \leq w\}$. We denote the set of all special matchings of w by SM_w.

Definition 3.1 *We say that a locally finite ranked poset* Z *is a* zircon *if* SM_w *is nonempty for all* $w \in Z$, w *not minimal.*

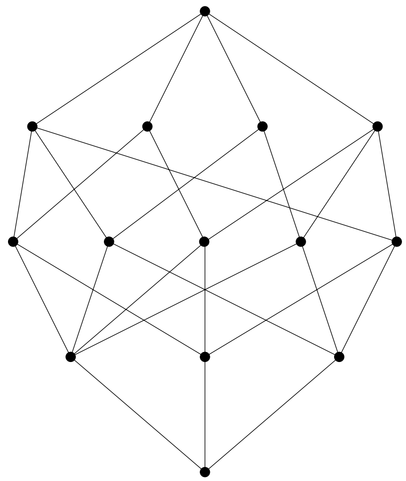
Note that the set SM_Z of all special matchings of the entire zircon Z may happen to be empty (see, for example, Fig. 1). For every element p in a locally finite ranked poset P, there exists at least one minimal element m which is $\leq p$. The following result says that, if P is a zircon, then such an element m is unique, and implies that connected zircons are graded posets.

Proposition 3.2 *Let* Z *be a zircon and let* $z \in Z$. *Then the subposet* $\{x \in Z : x \leq z\}$ *has a bottom element.*

Proof By contradiction, let m_1 and m_2 be two different minimal elements in $\{x \in Z : x \leq z\}$. Choose a minimal element w in the set $\{x \in Z : x \leq z, x \geq m_1, x \geq m_2\}$, which is not empty since it contains z. By the definition of a zircon, there exists a special matching M of w. Since m_1 and m_2 are minimal elements, $M(m_1) \rhd m_1$ and $M(m_2) \rhd m_2$, and so, by Lemma 2.6, $M(w) \geq m_1$ and $M(w) \geq m_2$. This contradicts the minimality of w. □

Corollary 3.3 *Any zircon is a disjoint union of graded posets* (*its connected components*).

Fig. 2 A zircon whose dual is not a zircon



Proof It is enough to prove that any connected zircon Z is a graded poset. Let us first show that Z has a bottom element. Suppose that m_1 and m_2 are two minimal elements in Z. Since Z is connected, there exists a sequence $(z_0 = m_1, z_1, \dots, z_{n-1}, z_n = m_2)$ of elements in Z such that, for all $i \in [n]$, either $z_{i-1} \lhd z_i$ or $z_{i-1} \rhd z_i$. The assertion follows by showing that $z_i \geq m_1$ for all $i = 0, \dots, n$ since this implies $m_1 = m_2$ by minimality. Let us proceed by induction on i, the case $i = 0$ being trivial. So assume $z_i \geq m_1$. If $z_i \lhd z_{i+1}$, then clearly $z_{i+1} \geq m_1$. If $z_i \rhd z_{i+1}$, then both z_{i+1} and m_1 are in the subposet $\{x \in Z : x \leq z_i\}$, which, by Proposition 3.2, has a bottom element $\hat{0}$. By the minimality of m_1, $m_1 = \hat{0}$ and hence $z_{i+1} \geq m_1$. The zircon Z is a graded poset since, given any $z \in Z$, the interval $[\hat{0}, z]$ is pure because it is a finite ranked poset with both bottom and top element. □

After Corollary 3.3, in the sequel we will often consider connected zircons, the generalization to arbitrary zircons being completely trivial. The class of zircons is closed under taking order ideals, disjoint unions (since $SM_{Z \uplus \tilde{Z}} \cong SM_Z \times SM_{\tilde{Z}}$ for all zircons Z and $\tilde{Z}$) and direct products (since $SM_{Z \times \tilde{Z}} = SM_Z \uplus SM_{\tilde{Z}}$, see [11], Example 2.8). Figure 2 shows that it is not closed under taking dual posets.

Remark Any Coxeter group partially ordered by Bruhat order is a connected zircon. In fact, let (W, S) be an arbitrary Coxeter system. The Coxeter group W is a locally finite ranked poset with the length function as rank function. Fix $w \in W \setminus \{e\}$ and $s \in D_R(w)$. Then, by Lemma 2.2, the involution $\rho_s : [e, w] \to [e, w]$ defined by $\rho_s(u) = us$ for all $u \in [e, w]$ is a special matching of w. Similarly, if $s \in D_L(w)$, the involution $\lambda_s : [e, w] \to [e, w]$ defined by $\lambda_s(u) = su$ for all $u \in [e, w]$ is a special matching of w.

The specialization of the following result to Coxeter groups was first conjectured [19] and later proved [20] by Verma. The Eulerianity of Coxeter groups can be shown with many different arguments (see [3, 9, 15, 20] and the recent paper by J. Stembridge [18]). The present one, as far as we know, is the shortest one.

Theorem 3.4 *Any connected zircon Z is an Eulerian poset.*

Proof We need to show that, for all $x, y \in Z$, $x < y$, we have

$$|\{z \in [x, y] : \rho(z) \text{ even}\}| = |\{z \in [x, y] : \rho(z) \text{ odd}\}|, \tag{1}$$

where $\rho : Z \to \mathbb{N}$ is the rank function. We proceed by induction on $\rho(y)$. The cases $\rho(y) = 0, 1$ are trivial.
So suppose $\rho(y) \geq 2$ and note that, if $[x, y]$ has a special matching, then (1) holds since an element of even rank is matched to an element of odd rank. Fix a special matching M of y. If $M(x) \rhd x$ then, by Lemma 2.5, M induces a special matching of $[x, y]$ and we are done. Otherwise, if $M(x) \lhd x$, we have

$$[x, y] = [x, M(y)] \uplus \{v \in [x, y] : v \not\leq M(y)\},$$

$$[M(x), y] = [M(x), M(y)] \uplus \{v \in [M(x), y] : v \not\leq M(y)\}.$$

We claim that $\{v \in [x, y] : v \not\leq M(y)\} = \{v \in [M(x), y] : v \not\leq M(y)\}$. This is equivalent to $\{v \in [M(x), y] : v \not\geq x \text{ and } v \not\leq M(y)\} = \emptyset$. Let $v \in [M(x), y]$. Then, by Lemma 2.6, we have that $v \leq M(y)$ if $M(v) \rhd v$, and $v \geq x$ if $M(v) \lhd v$. Hence the claim is proved.
Now,

$$|\{z \in [M(x), y] : \rho(z) \text{ even}\}| = |\{z \in [M(x), y] : \rho(z) \text{ odd}\}|,$$

$$|\{z \in [M(x), M(y)] : \rho(z) \text{ even}\}| = |\{z \in [M(x), M(y)] : \rho(z) \text{ odd}\}|,$$

respectively since M is a special matching of $[M(x), y]$ and by the induction hypothesis since $\rho(M(y)) < \rho(y)$. Hence

$$|\{v \in [M(x), y] : v \not\leq M(y) \text{ and } \rho(v) \text{ even}\}| = |\{v \in [M(x), y] : v \not\leq M(y) \quad \text{and} \\ \rho(v) \text{ odd}\}|,$$

and so, by the claim, we have

$$|\{v \in [x, y] : v \not\leq M(y) \text{ and } \rho(v) \text{ even}\}| = |\{v \in [x, y] : v \not\leq M(y) \text{ and } \rho(v) \text{ odd}\}|.$$

By the induction hypothesis

$$|\{z \in [x, M(y)] : \rho(z) \text{ even}\}| = |\{z \in [x, M(y)] : \rho(z) \text{ odd}\}|,$$

and (1) follows. □

From Theorem 3.4, we can derive some properties of the intervals in a zircon Z which are needed in the sequel. A regular CW complex Γ is a collection of balls in a Hausdorff space $\overline{\Gamma}$ such that the interiors of the balls partition $\overline{\Gamma}$ and the boundary of c is a union of some balls in Γ for all $c \in \Gamma$, $\dim c \geq 1$. If $\overline{\Gamma}$ is homeomorphic to the topological space X, then we say that Γ is a regular CW decomposition of X. The cell poset of Γ is the set of balls of Γ ordered by containment. Recall that we denote by $||\Delta((u, v))||$ the geometric realization of the order complex $\Delta((u, v))$ of the interval (u, v).

Corollary 3.5 *Let Z be a zircon with rank function ρ and let $u, v \in Z$, $u \leq v$, $\rho(v) - \rho(u) > 1$. Then the following assertions hold.*

1. $\Delta((u, v))$ *is a PL-sphere.*
2. *Consider* $||\Delta((u, v))||$ *and its subspaces* $c_z = ||\Delta((u, z])||$ *for all* $z \in (u, v)$, *and let* $\Gamma(u, v) := \{c_z : z \in (u, v)\}$. *Then* $\Gamma(u, v)$ *is a regular CW decomposition of* $||\Delta((u, v))||$ *which is homeomorphic to the sphere of dimension* $\rho(v) - \rho(u) - 2$.
3. *If* $\rho(v) - \rho(u) = 2$ *then* (u, v) *is a square* (*i.e.* Z *is thin*). *If* $\rho(v) - \rho(u) = 3$ *then* (u, v) *is a* k*-crown.*

Proof The first assertion follows by Theorem 3.4 and by Corollary 4.3 of [13] (which follows by results in [11] which, in turn, are special cases of unpublished results by Dyer [10]). After what we already proved, the proof of the second assertion is analogous to that for Bruhat intervals (see Theorem 2.7.12 of [2]). The third assertion is straightforward by Theorem 3.4 and by the first assertion. □

The following proposition deals with the structure of lower intervals in a zircon and implies that, as in the case of Coxeter groups, the only zircons which are lattices are the Boolean algebras.

Proposition 3.6 *Let Z be a zircon, $z \in Z$, and $M \in SM_z$. Let $\hat{0}$ be the bottom element in $\{x \in Z : x \leq z\}$, and let J be the order ideal of $[\hat{0}, M(z)]$ defined by $J := \{x \in [\hat{0}, M(z)] : M(x) \in [\hat{0}, M(z)]\}$. Then $[\hat{0}, z] = [\hat{0}, M(z)] \uplus I$, where I is the set in bijection with $[\hat{0}, M(z)] \setminus J$ through the restriction of M. Furthermore, for all $x, y \in [\hat{0}, z]$, $y \neq M(x)$, we have*

$$x \lhd y \text{ in } [\hat{0}, z] \iff \begin{cases} x \lhd y \text{ in } [\hat{0}, M(z)], \\ \quad \text{if } x, y \in [\hat{0}, M(z)], \\ M(x) \lhd M(y) \text{ in } [\hat{0}, M(z)], \\ \quad \text{if } x \notin [\hat{0}, M(z)] \setminus J \text{ and } y \notin [\hat{0}, M(z)]. \end{cases}$$

Proof By Lemma 2.6, J is an order ideal. Again by Lemma 2.6, if $x \notin [\hat{0}, M(z)]$ then $M(x) \in [\hat{0}, M(z)]$ and so M restricts to a bijection from $[\hat{0}, M(z)] \setminus J$ to $[\hat{0}, z] \setminus [\hat{0}, M(z)]$. The last assertion follows by the definition of a special matching and its proof is left to the reader. □

Corollary 3.7 *Let Z be a zircon with rank function ρ and let $w \in Z$. Then $[\hat{0}, w]$ is a lattice if and only if it is a Boolean algebra.*

Proof We proceed by induction on $\rho(w)$, the assertion being clear if $\rho(w) = 1$. Let $\rho(w) > 1$ and M be a special matching of w. By the induction hypothesis, $[\hat{0}, M(w)]$ is a Boolean algebra of rank $\rho(w) - 1$. We need to show that $[\hat{0}, w]$ is the product of $[\hat{0}, M(w)]$ and a two element chain. By Proposition 3.6, this will follow if we prove that $J := \{z \in [\hat{0}, M(w)] : M(z) \in [\hat{0}, M(w)]\} = \emptyset$. By contradiction, let z be a maximal element in J. Clearly, $\rho(z) < \rho(w) - 1$ because $M(M(w)) = w \notin [\hat{0}, M(w)]$. So there exists $\tilde{z} \in [\hat{0}, M(w)]$ with $z \lhd \tilde{z}$. By maximality, $\tilde{z} \notin J$ and so $M(\tilde{z}) \notin [\hat{0}, M(w)]$ and $M(\tilde{z}) \rhd \tilde{z}$. Hence, by Lemma 2.5, M restricts to a special matching of the interval

$[M(z), M(\tilde{z})]$ of rank 3. Since it admits a special matching, the interval $[M(z), M(\tilde{z})]$ is a k-crown with $k = 2, 3$. On the other hand, $[M(z), M(\tilde{z})]$ cannot be a 3-crown since $M(M(z)) = z \lhd M(M(\tilde{z})) = \tilde{z}$. Hence $k = 2$, which is a contradiction since $[\hat{0}, w]$ is a lattice. □

Remarks 1. There exist intervals in zircons which are lattices but not Boolean algebras (for example, the k-crowns for all $k \geq 4$).

2. Proposition 3.6 implies that any zircon with a top element is an accessible poset in the sense of Du Cloux [11] (hence any interval in a zircon is an accessible poset by Proposition 3.3 of [11]).

3. Corollary 3.7 can also been obtained as a consequence of Corollary 1 of Sect. 5 of [11].

4 Kazhdan-Lusztig theory for zircons

Kazhdan and Lusztig [15] construct certain representations of the Hecke algebra of a Coxeter group W via a family of polynomials (in one variable, indexed by pairs of elements in W) which are strictly related to the Bruhat order on W. In this section we show how the Kazhdan-Lusztig construction can be carried out in the much more general context of zircons. The present construction generalizes also the construction in [8], where the authors consider the class of diamonds, which is a proper subclass of the class of zircons. The main difference from [8] consists in the fact that, in contrast with what is proved for diamonds, the analogues of the Kazhdan-Lusztig polynomials of an arbitrary zircon are not independent of the special matchings chosen to define them. Hence here we need to consider the new category of well refined zircons. Once found the right category to work with, most of the results appearing in [8] can be extended without substantial changes in proofs. We refer the reader to [2, 14] and [15] for all undefined notations concerning the classical Kazhdan-Lusztig theory.

Let Z be a connected zircon and $S \subseteq SM_Z$ be any set of special matchings of Z. We denote by (W_Z^S, S) the Coxeter system whose Coxeter generators are the special matchings in S and whose Coxeter matrix is given by $m(M, N) := o(MN)$, the period of MN as a permutation of Z (possibly ∞). We denote by $\mathcal{H}(Z, S)$ the Hecke algebra of W_Z^S and by $\mathcal{M}_Z$ the free $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module defined by

$$\mathcal{M}_Z := \bigoplus_{u \in Z} \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]u.$$

The natural action of W_Z^S on Z extends to an action of $\mathcal{H}(Z, S)$ on $\mathcal{M}_Z$.

Theorem 4.1 *There exists a unique action of $\mathcal{H}(Z, S)$ on $\mathcal{M}_Z$ such that*

$$T_M(u) = \begin{cases} M(u), & \text{if } M(u) \rhd u, \\ qM(u) + (q-1)u, & \text{if } M(u) \lhd u, \end{cases} \tag{2}$$

for all $u \in Z$ and $M \in S$.

Proof The uniqueness part is clear. Let us prove that (2) defines an action of $\mathcal{H}(Z, S)$. The quadratic relations are satisfied since, for all $u \in Z$ and $M \in S$, we have

$$T_M^2(u) = \begin{cases} T_M(M(u)) = qu + (q-1)M(u), \\ \quad \text{if } M(u) \rhd u, \\ T_M(qM(u) + (q-1)u) = qu + (q-1)[qM(u) + (q-1)u], \\ \quad \text{if } M(u) \lhd u. \end{cases} \tag{3}$$

Fix $u \in Z$ and $M, N \in S$. We need to prove that

$$\underbrace{T_M(T_N(T_M(\cdots}_{m}(u)\cdots))) = \underbrace{T_N(T_M(T_N(\cdots}_{m}(u)\cdots))). \tag{4}$$

Let $\langle M, N\rangle$ be the group generated by M and N, $\langle M, N\rangle(u)$ the orbit of u under the action of $\langle M, N\rangle$, and $(W, \{s, t\})$ a dihedral Coxeter system of order $2d := |\langle M, N\rangle(u)|$. By Lemma 4.1 of [7], we can consider the isomorphism $\Phi : \langle M, N\rangle(u) \longrightarrow W$ sending $\underbrace{\cdots MNM}_{k}(u_0)$ to $\underbrace{\cdots sts}_{k}$, for all $k \in [2d]$, where u_0 is the smallest element in $\langle M, N\rangle(u)$. Let $\tilde{\mathcal{M}}$ be the submodule of $\mathcal{M}_Z$ generated by $\langle M, N\rangle(u)$. Extend Φ to a linear map $\Phi : \tilde{\mathcal{M}} \longrightarrow \mathcal{H}(W)$ by $\Phi(z) := T_{\Phi(z)}$ for all $z \in \langle M, N\rangle(u)$. Then $\Phi(T_M(z)) = T_s\Phi(z)$ and $\Phi(T_N(z)) = T_t\Phi(z)$ for all $z \in \langle M, N\rangle(u)$, and so

$$\begin{aligned} \Phi(\underbrace{T_M(T_N(T_M(\cdots}_{d}(z)\cdots))) &= \underbrace{T_sT_tT_s\cdots}_{d}\Phi(z) \\ &= \underbrace{T_tT_sT_t\cdots}_{d}\Phi(z) \\ &= \Phi(\underbrace{T_N(T_M(T_N(\cdots}_{d}(z)\cdots))). \end{aligned}$$

Hence $\underbrace{T_M(T_N(T_M(\cdots}_{d}(u)\cdots))) = \underbrace{T_N(T_M(T_N(\cdots}_{d}(u)\cdots)))$ and this proves (4) since d divides m because m is the least common multiple of the cardinalities of the orbits of $\langle M, N\rangle$. □

We want to construct some representations of $\mathcal{H}(Z, S)$ which are smaller than $\mathcal{M}_Z$. In order to do it, we must restrict our treatment to those sets $S \subseteq SM_Z$ satisfying a certain property. By definition, we can fix a family $\mathcal{M} = \{M_v\}_{v \in Z\setminus\hat{0}}$ of special matchings such that $M_v \in SM_v$ for all $v \in Z \setminus \hat{0}$. We call the pair $(Z, \mathcal{M})$ a *refined zircon*. We say that two refined zircons $(Z, \mathcal{M})$ and $(\tilde{Z}, \tilde{\mathcal{M}})$ are isomorphic if there exists a poset isomorphism $\phi : Z \to \tilde{Z}$ such that $\phi \circ M_z(z) = M_{\phi(z)} \circ \phi(z)$ for all $z \in Z$.

As a matter of fact, we are interested in a full subcategory of the category of refined zircons and in an equivalence relation which is weaker than isomorphism. Let $w \in Z$, and let $M, N \in SM_w$. We denote by $\langle M, N\rangle$ the subgroup of the symmetric group on $[\hat{0}, w]$ generated by M and N, and by $\langle M, N\rangle(z)$ the orbit of any element $z \in [\hat{0}, w]$

under the action of $\langle M, N\rangle$. Following [8] and [17], we say that M and N are strictly coherent if

$$|\langle M, N\rangle(x)| \text{ divides } |\langle M, N\rangle(w)| \tag{5}$$

for all $x \in [\hat{0}, w]$. We are interested in the transitive closure of this relation. We say that M and N are coherent if there exists a sequence $(M_0, M_1, \ldots, M_k)$ of special matchings in SM_w such that $M_0 = M$, $M_k = N$, and M_i and M_{i+1} are strictly coherent for all $i = 0, 1, \ldots, k-1$.

Definition 4.2 *We say that a refined zircon* $(Z, \mathcal{M} = \{M_v\}_{v\in Z\setminus\hat{0}})$ *is a* well refined zircon *if the restriction of* M_v *to* $[\hat{0}, u]$ *is coherent to* M_u *for all* $u \le v \in Z \setminus \hat{0}$ *such that* $u \rhd M_v(u)$. *Two well refined zircons* $(Z, \mathcal{M} = \{M_v\}_{v\in Z\setminus\hat{0}})$ *and* $(\tilde{Z}, \tilde{\mathcal{M}} = \{\tilde{M}_v\}_{v\in Z\setminus\hat{0}})$ *are* coherent *if there exists a poset isomorphism* $\psi : Z \to \tilde{Z}$ *such that the special matchings* $\psi \circ M_z$ *and* $M_{\psi(z)}$ *are coherent for all* $z \in Z \setminus \hat{0}$.

Let us define the R-polynomials for any refined zircon.

Definition 4.3 *Let* $(Z, \mathcal{M})$ *be a refined zircon,* $\mathcal{M} = \{M_v\}_{v\in Z\setminus\hat{0}}$. *For all* $u, v \in Z$, *we inductively define the* R*-polynomial* $R_{u,v}(q)$ *by the following recursive property*:

$$R_{u,v}(q) = \begin{cases} R_{M_v(u),M_v(v)}(q), & \text{if } u \le v \text{ and } M_v(u) \lhd u, \\ qR_{M_v(u),M_v(v)}(q) + (q-1)R_{u,M_v(v)}(q), & \text{if } u \le v \text{ and } M_v(u) \rhd u, \\ 0, & \text{if } u \not\le v, \\ 1, & \text{if } u = v = \hat{0}. \end{cases} \tag{6}$$

The proof of Theorem 3.3 of [8] shows the two following useful facts.

1. If $(Z, \mathcal{M})$ is a well refined zircon, then (6) holds also replacing M_v with M_z for all $z \ge v$ such that $M_z(v) \lhd v$.
2. Two well refined zircons which are coherent have the same family of R-polynomials. More precisely, if $(Z, \mathcal{M})$ and $(\tilde{Z}, \tilde{\mathcal{M}})$ are two well refined zircons which are coherent through the poset isomorphism $\psi : Z \to \tilde{Z}$, then $R_{x,y}(q) = R_{\psi(x),\psi(y)}(q)$ for all $x, y \in Z$.

Now consider a well refined zircon $(Z, \mathcal{M} = \{M_v\}_{v\in Z\setminus\hat{0}})$ (and so the associated family of R-polynomials) and a set $S \subseteq SM_Z$ with the following property: if $M \in S$, $v \in Z$ and $v \rhd M(v)$, then the restriction of M to $[\hat{0}, v]$ is coherent to M_v. We want S to satisfy this property because we need (6) to hold also if we replace M_v with any $M \in S$ such that $M(v) \lhd v$. With these assumptions, it is not difficult to see that all results in Sects. 5 and 6 of [8] hold for the zircon Z too. In particular, we can introduce the family $\{P_{u,v}(q)\}_{u,v\in Z} \subseteq \mathbb{Z}[q]$ of analogues of the Kazhdan-Lusztig polynomials and we can construct what we shall call the *zircon graph*, the *zircon cells* and the *zircon cell representations* of $\mathcal{H}(Z, S)$.

The definitions of R-polynomials and Kazhdan-Lusztig polynomials of a zircon Z are consistent with the ones for Coxeter groups given in [15] and with the ones for diamonds given in [8]. In fact, suppose that Z is a zircon which is either isomorphic

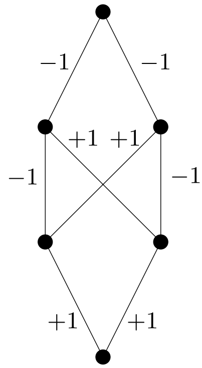


Fig. 3 A balanced labeling

to a lower Bruhat interval $[e, w]$ in a Coxeter group W (which is itself a diamond, see Theorem 3.8 of [8] or Theorem 7.2.5 of [8]) or isomorphic to a generic diamond D. Then we have that all refinements of Z give a structure of well refined zircon, and that any two such well refined zircons are coherent. Hence Z admits only one family of R-polynomials and one family of Kazhdan-Lusztig polynomials as a well refined zircon. These families coincide with the families of R-polynomials and Kazhdan-Lusztig polynomials of $[e, w]$ as a Coxeter group interval or of D as a diamond. More precisely, if ϕ is a poset isomorphism from Z to $[e, w]$ or to D, then $R_{x,y}(q) = R_{\phi(x),\phi(y)}(q)$ and $P_{x,y}(q) = P_{\phi(x),\phi(y)}(q)$ for all $x, y \in Z$. Note that this implies that every left, right or two-sided Kazhdan-Lusztig cell representation, as well as every diamond cell representation, is isomorphic to a zircon cell representation.

5 Balanced and exact labelings

In this section we give some definitions and results concerning balanced and exact labelings on a poset P. If $P = W$ is the Weyl group of a Kac-Moody algebra, these are the labelings needed to construct BGG complexes and BGG resolutions of a finite-dimensional irreducible module of a complex semisimple Lie algebra by Verma modules (see [1] or [16]). Throughout this section, let P be any thin graded poset with rank function ρ. Recall that this means that, for all ordered pairs $x \leq y \in P$ with $\rho(y) - \rho(x) = 2$, the interval $[x, y]$ is a square (i.e. consists of exactly 4 elements). Let $CovP := \{(u, v) \in P \times P : u \lhd v\}$ and let L be a labeling of the Hasse diagram of P with labels $+1$ and -1, that is a mapping $L : CovP \to \{+1, -1\}$.

Definition 5.1 *Let $x, y \in P$, $x \leq y$, $\rho(y) - \rho(x) = 2$, and let $x \lhd m \lhd y$ and $x \lhd n \lhd y$, $m \neq n$, be the two maximal chains in $[x, y]$. The labeling L is* balanced on $[x, y]$ *if*

$$L(x \lhd m)L(m \lhd y) + L(x \lhd n)L(n \lhd y) = 0,$$

(*or, equivalently, if* $L(x \lhd m)L(m \lhd y)L(x \lhd n)L(n \lhd y) = -1$).
Moreover, we say that L is a balanced labeling on the poset P (*or just a balanced labeling if the poset P is clear from the context*) *if it is balanced on all intervals of length* 2.

Suppose we have any labeling $L : CovP \to \{+1, -1\}$ and let $C_i(P)$ be the free Abelian group generated by the set $\{v \in P : \rho(v) = i\}$. Define a differential map $d_i(L) : C_i(P) \to C_{i-1}(P)$ (that we denote just by d_i if the labeling L is clear from the context) by linear extension of

$$d_i(v) = \textstyle\sum_{x:x \lhd v} L(x \lhd v)x \quad (\forall v \in P, \rho(v) = i).$$

It is easy to see that $d_{i-1} \circ d_i = 0$ for all i if and only if L is a balanced labeling. Hence, if L is a balanced labeling, we have the following differential complex $\mathcal{C}(L)$

$$\cdots \to C_n(P) \to C_{n-1}(P) \to \cdots \to C_1(P) \to C_0(P) \to 0.$$

Definition 5.2 *We say that a balanced labeling L is* exact *if $\mathcal{C}(L)$ is an exact sequence.*

Let L be any labeling on P and let $v \in P$. Then we can define a new labeling $\Phi_v(L)$ by

$$\Phi_v(L)(x \lhd y) := \begin{cases} L(x \lhd y) & \text{if } v \notin \{x, y\}, \\ -L(x \lhd y) & \text{if } v \in \{x, y\}, \end{cases}$$

for all $x \lhd y$.
Note that, for all $u, v \in P$, $\Phi_u \circ \Phi_v = \Phi_v \circ \Phi_u$ and $\Phi_v^2 = Id$. We can extend this definition to any subset S of P by setting

$$\Phi_S(L)(x \lhd y) := (-1)^{|S \cap \{x,y\}|} L(x \lhd y)$$

for all $x \lhd y$. Note that Φ_S is the composition of all Φ_v with $v \in S$ and that $\Phi_S^2 = Id$.

Proposition 5.3 *Let S and T be two subsets of P. Then $\Phi_S(L) = \Phi_T(L)$ if and only if $T \in \{S, P \setminus S\}$.*

Proof Suppose $\Phi_S(L) = \Phi_T(L)$. If either $v \in S \cap T$ or $v \notin S \cup T$, then $S \cap \{x : x \lhd v \text{ or } x \rhd v\} = T \cap \{x : x \lhd v \text{ or } x \rhd v\}$. As P is connected, we get the assertion. The converse follows from the definition. □

Theorem 5.4 *Let S be any subset of P. The labeling L is balanced if and only if $\Phi_S(L)$ is balanced.*

Proof Since $\Phi_S^2 = Id$, we need to prove only one implication, and since Φ_S is the composition of all Φ_v with $v \in S$, we may assume $S = \{v\}$. So we suppose that L is balanced and we show that $\Phi_v(L)$ is also balanced. Let $x, y \in P$, $x \leq y$, $\rho(y) - \rho(x) = 2$, and let $x \lhd m \lhd y$ and $x \lhd n \lhd y$, $m \neq n$, be the two maximal chains in $[x, y]$. Then $L(x \lhd m)L(m \lhd y)L(x \lhd n)L(n \lhd y) = \Phi_v(L)(x \lhd m)\Phi_v(L)(m \lhd y)\Phi_v(L)(x \lhd n)\Phi_v(L)(n \lhd y)$ since we have to change sign of the labels of exactly two edges if $v \in \{x, m, n, y\}$, of none otherwise. □

Note that, for any $v \in P$, if $d_r(L)(\sum a_k x_k) = \sum b_k y_k$, we have

$$d_r(\Phi_v(L))\left(\sum a_k x_k\right) = \begin{cases} d_r(L)(-a_1 v + \sum_{k\neq 1} a_k x_k) & \text{if } \rho(v) = r \text{ and } v = x_1, \\ -b_1 v + \sum_{k\neq 1} b_k y_k & \text{if } \rho(v) = r-1 \text{ and } v = y_1, \\ d_r(L)(\sum a_k x_k) & \text{if } v \notin \{x_1, x_2, \dots, y_1, y_2, \dots\}. \end{cases} \tag{7}$$

Theorem 5.5 *Let S be any subset of P. The labeling L is exact if and only if $\Phi_S(L)$ is exact.*

Proof Since $\Phi_S^2 = Id$, we need to prove only one implication, and since Φ_S is the composition of all Φ_v with $v \in S$, we may assume $S = \{v\}$. So we suppose that L is exact and we show that $\Phi_v(L)$ is also exact. For notational convenience, we set $\Phi = \Phi_v$, $d_r = d_r(L)$, and $\Phi(d_r) = d_r(\Phi(L))$, for all possible r. Let $X = \sum a_k x_k \in \ker \Phi(d_i)$. We must show that $X \in \mathrm{Im}\,\Phi(d_{i+1})$. If $\rho(v) \notin \{i+1, i, i-1\}$, this is clear since $\Phi(d_i) = d_i$ and $\Phi(d_{i+1}) = d_{i+1}$ by (7).
Case $\rho(v) = i + 1$.
In this case, $\Phi(d_i) = d_i$ and so $X \in \ker d_i$. Thus, there exists $Y \in C_{i+1}(P)$ such that $d_{i+1}(Y) = X$, since L is exact. Suppose that $Y = \sum b_k y_k$ where $y_1, y_2, \dots \in P$ have rank $i+1$ and $b_1, b_2, \dots \in \mathbb{Z}$. If $v \notin \{y_1, y_2, \dots\}$, then $\Phi(d_{i+1})(\sum b_k y_k) = d_{i+1}(\sum b_k y_k) = X$. Otherwise, suppose $v = y_1$. Then $\Phi(d_{i+1})(-b_1 v + \sum_{k\neq 1} b_k y_k) = X$ by (7).
Case $\rho(v) = i$.
If $v \notin \{x_1, x_2, \dots\}$ then $d_i(X) = \Phi(d_i)(X)$, hence $X \in \ker d_i$. Thus, there exist $y_1, y_2, \dots \in P$ of rank $i+1$ and $b_1, b_2, \dots \in \mathbb{Z}$ such that $d_{i+1}(\sum b_k y_k) = X$, since L is exact. Then $\Phi(d_{i+1})(\sum b_k y_k) = X$ by (7). So we may assume that $v = x_1$. By (7), $d_i(-a_1 v + \sum_{k\neq 1} a_k x_k) = 0$ and there exist some $y_1, y_2, \dots \in P$ of rank $i+1$ and some $b_1, b_2, \dots \in \mathbb{Z}$ such that $d_{i+1}(\sum b_k y_k) = -a_1 v + \sum_{k\neq 1} a_k x_k$. Hence $\Phi(d_{i+1})(\sum b_k y_k) = a_1 v + \sum_{k\neq 1} a_k x_k = X$.
Case $\rho(v) = i - 1$.
By (7), $d_i(X) = \Phi(d_i)(X) = 0$, hence $X \in \ker d_i$. Thus, there exist some $y_1, y_2, \dots \in P$ of rank $i+1$ and some $b_1, b_2, \dots \in \mathbb{Z}$ such that $d_{i+1}(\sum b_k y_k) = X$, since L is exact. But by (7), $\Phi(d_{i+1}) = d_{i+1}$. □

Corollary 5.6 *Let P be finite. If P has a balanced (respectively, exact) labeling L, then it has exactly $2^{|P|-1}$ distinct balanced (respectively, exact) labelings of the form $\Phi_S(L)$, with $S \subseteq P$.*

Proof The assertion follows by Proposition 5.3 and Theorems 5.4 and 5.5. □

6 Labelings on zircons

In this section we prove that the concepts of balanced and exact labelings essentially coincide for zircons. We give an algorithm to construct all such labelings which implies that the number of balanced and exact labelings on a finite connected zircon Z is $2^{|Z|-1}$.

Let Z be a zircon, which we may assume to be connected. By the definition of a zircon, we can fix a family $\mathcal{M} = \{M_v\}_{v\in Z\setminus\hat{0}}$ of special matchings such that $M_v \in SM_v$ for all $v \in Z \setminus \hat{0}$ (namely, the pair $(Z, \mathcal{M})$ is a refined zircon, see Sect. 4). Our algorithm will depend on $\mathcal{M}$. We construct a labeling L of the edges of the Hasse diagram of Z step by step. We start from Step 1) and we go on. At the i-th step, the edges connecting an element of rank k to an element of rank $k-1$ are already labeled for all $k \in [i-1]$. The i-th step is as follows.

Step i).

Part 1: If there are no elements in Z of rank i, the labeling is complete. Otherwise, for all $v \in Z$ of rank i, label the edge $\{v, M_v(v)\}$ at random.

Part 2: If there are edges with no label connecting an element v of rank i to an element u of rank $i-1$, go to Part 3. Otherwise go to Step $i+1$).

Part 3: Choose at random an edge $E = \{v, u\}$ with no label connecting an element v of rank i to an element u of rank $i-1$. By construction, $u \neq M_v(v)$. By the definition of a special matching, the elements $v, u, M_v(v), M_v(u)$ form a square Q (see the Remark of Sect. 2). All edges of Q have already been labeled except E. Then label the edge E as to obtain a balanced labeling on Q and go to Part 2.

Theorem 6.1 *Any labeling $L : CovP \to \{+1, -1\}$ given by the previous algorithm is a balanced labeling.*

Proof By contradiction, suppose that L is not a balanced labeling. Let $Q = [u, v] = \{v, m, n, u\}$ be a square of minimal rank such that L is not a balanced labeling on Q, i.e.

$$L(u \lhd m)L(m \lhd v)L(u \lhd n)L(n \lhd v) = 1. \tag{8}$$

We distinguish two cases, according to as whether $M_v(v) \in \{m, n\}$ or not. For notational convenience, we let $M = M_v$ in the sequel of the proof.

Case 1: $M(v) \notin \{m, n\}$.

By the definition of a special matching, $M(m) \lhd M(v)$, $M(n) \lhd M(v)$ and, since Z is thin, $M(m) \neq u$, $M(n) \neq u$. Hence $M(u) \lhd u$, $M(u) \lhd M(m)$ and $M(u) \lhd M(n)$, and we are in the situation of Fig. 4.

By the minimality of the square Q and by the definition of the algorithm, we have

$$-1 = L(M(u) \lhd M(m))\ L(M(u) \lhd u)\ L(M(m) \lhd m)\ L(u \lhd m),$$

$$-1 = L(M(u) \lhd M(m))\ L(M(u) \lhd M(n))\ L(M(m) \lhd M(v))\ L(M(n) \lhd M(v)),$$

$$-1 = L(M(u) \lhd u)\ L(M(u) \lhd M(n))\ L(u \lhd n)\ L(M(n) \lhd n),$$

$$-1 = L(M(m) \lhd m)\ L(M(m) \lhd M(v))\ L(m \lhd v)\ L(M(v) \lhd v),$$

$$-1 = L(M(n) \lhd n)\ L(M(n) \lhd M(v))\ L(n \lhd v)\ L(M(v) \lhd v).$$

By multiplying right hand sides and left hand sides we get

$$-1 = L(m \lhd v)\ L(u \lhd m)\ L(u \lhd n)\ L(n \lhd v),$$

Fig. 4 $M(v) \notin \{m, n\}$

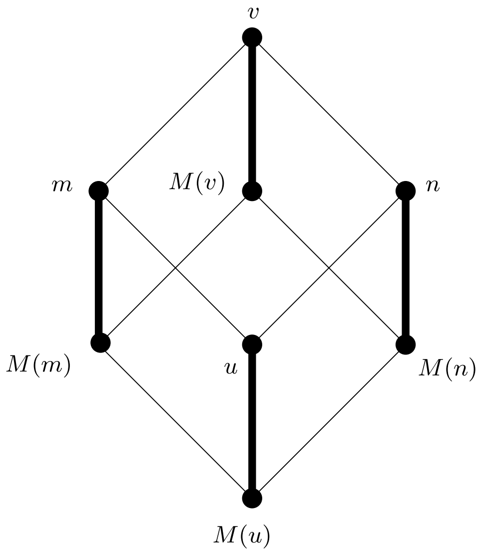
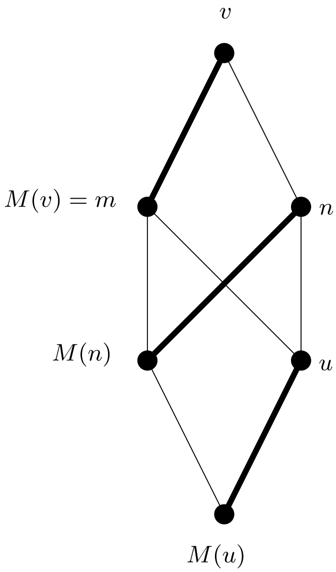


Fig. 5 $M(v) = m$



which contradicts (8).
Case 2: $M(v) \in \{m, n\}$.
We may assume that $M(v) = m$. By the definition of the algorithm, L is a balanced labeling on the square $\{v, n, M(v) = m, M(n)\}$, that is

$$-1 = L(M(n) \lhd m)L(m \lhd v)L(M(n) \lhd n)L(n \lhd v) \tag{9}$$

and hence $M(n) \neq u$ by (8). Then, by the definition of a special matching, $M(u) \lhd u$, $M(u) \lhd M(n)$ and we are in the situation of Fig. 5.

By the minimality of the square Q we have

$$-1 = L(M(u) \lhd M(n))\ L(M(n) \lhd m)\ L(M(u) \lhd u)\ L(u \lhd m),$$
$$-1 = L(M(u) \lhd M(n))\ L(M(n) \lhd n)\ L(M(u) \lhd u)\ L(u \lhd n).$$

By multiplying right hand sides and left hand sides of the two previous equalities and of (9), we get

$$-1 = L(u \lhd m)L(u \lhd n)L(m \lhd v)L(n \lhd v),$$

which contradicts (8). □

Corollary 6.2 *To any family $\mathcal{M} = \{M_v\}_{v \in Z \setminus \hat{0}}$ of special matchings with $M_v \in SM_v$ for all $v \in Z \setminus \hat{0}$ we can associate a bijection $\Phi_{\mathcal{M}}$ between the set of balanced labelings and the set of subsets of $Z \setminus \{\hat{0}\}$. The bijection $\Phi_{\mathcal{M}}$ sends a balanced labeling L to the subset $\{v \in Z : L(M_v(v) \lhd v) = 1\}$.*
In particular, if Z is finite, the number of balanced labelings on Z is $2^{|Z|-1}$.

Proof By Theorem 6.1, any mapping $L : \{M_v(v) \lhd v : v \in Z\} \to \{+1, -1\}$ can be uniquely extended to a balanced labeling on Z. □

Corollary 6.3 *Let L be a balanced labeling on Z. Given any other balanced labeling L' on Z, there exists $S \subseteq Z$ such that $L' = \Phi_S(L)$.*

Proof If Z is finite, then by Corollary 5.6 there are exactly $2^{|Z|-1}$ distinct balanced labelings on Z of the form $\Phi_S(L)$ with $S \subseteq Z$. By Corollary 6.2, this is also the number of balanced labelings and so the assertion follows. To find the subset S we can proceed step by step. At the i-th step, we already know $S \cap \{z \in Z : \rho(z) < i\}$ and we find $S \cap \{z \in Z : \rho(z) = i\}$ considering the edges connecting elements of rank i to elements of rank $i-1$. We start from the first step and we go on till the maximal rank. Note that, for all $z \in Z$, the restrictions of L and L' to $[\hat{0}, z]$ determine whether z is in S or not.
Now suppose that Z is infinite. For all $T \subseteq Z$, $|T| < \infty$, let $Z_T := \cup_{z \in T}[\hat{0}, z]$ (Z has $\hat{0}$ by Corollary 3.3). Clearly $Z = \cup_{|T|<\infty} Z_T$. Since every order ideal of a zircon is itself a zircon and a zircon is locally finite, Z_T is a finite zircon for all $T \subseteq Z$, $|T| < \infty$. Then by what we have already proved, for all $T \subseteq Z$, $|T| < \infty$, there exists $S_T \subseteq Z_T$ such that the restriction of L' to Z_T is equal to the labeling we obtain by applying Φ_{S_T} to the restriction of L to Z_T. Note that, for all finite subsets $T, T' \subseteq Z$, $T \subseteq T'$, we have that $Z_T \subseteq Z_{T'}$, $S_T \subseteq S_{T'}$ and $S_{T'} \cap Z_T = S_T$. Let $S = \cup_{|T|<\infty} S_T$. Let us show that $S \cap Z_T = S_T$ for all $T \subseteq Z$, $|T| < \infty$. Clearly $S \cap Z_T \supseteq S_T$. Let us prove that $S \cap Z_T \subseteq S_T$ by contradiction. So assume that $z \in (S \cap Z_T) \setminus S_T$. This means that there exist $T' \subseteq Z$, $|T'| < \infty$, such that $z \in S_{T'}$. Consider $U = T \cup T'$. Then $S_U \cap Z_T = S_T$ and $S_U \cap Z_{T'} = S_{T'}$. This is a contradiction since $z \in Z_T$, $z \in Z_{T'}$, $z \in S_{T'}$, but $z \notin S_T$.
So $S \cap Z_T = S_T$ for all $T \subseteq Z$, $|T| < \infty$. Then the restriction of L' to Z_T is equal to the restriction of $\Phi_S(L)$ to Z_T. Since $Z = \cup_{|T|<\infty} Z_T$, we have $L' = \Phi_S(L)$. □

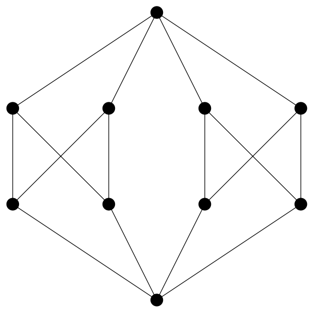


Fig. 6 A poset with no exact labelings

We now show that all balanced labelings on a zircon Z are exact if we assume Z to be directed (a poset P is directed if for every $z_1, z_2 \in P$, there is some $z \in P$ with $z \geq z_1$ and $z \geq z_2$). This follows by the existence of at least one exact labeling, whose proof is based on the fact that the reduced cellular homology of the ball vanishes in all dimensions.

Corollary 6.4 *All balanced labelings on a directed zircon Z are exact labelings.*

Proof First we claim that, for all $u, v \in Z$, $u \leq v$, there exists an exact labeling on $[u, v]$. After Corollary 3.5, we can proceed as in the case of Bruhat intervals (see Corollary 2.7.14 of [2] and references cited there). So we omit the proof of our claim and we just note that it is based on the fact that the reduced cellular homology of the ball vanishes in all dimensions, and on the existence of the incidence numbers. These are numbers given by a mapping from pairs of balls (c, c') of a regular CW complex with $c \subset c'$ and $\dim c' = \dim c + 1$ to numbers $[c : c'] \in \{+1, -1\}$ such that the boundary maps are given by

$$d_i(c') = \sum_c [c : c']c.$$

So, in particular, if Z has a top element $\hat{1}$, there exists an exact labeling on Z. In this case, by Corollary 5.6, Z has at least $2^{|Z|-1}$ exact labelings. But this is also the number of its balanced labelings by Corollary 6.2. So the assertion is proved for zircons with top element.
Now suppose that Z is an arbitrary directed zircon and that L is a balanced labeling on Z. Then, for all $z \in Z$, the restriction of L to $[\hat{0}, z]$ is exact. Hence we get the assertion because, given $X = \sum a_k x_k \in \ker d_i$ (where $x_1, x_2, \ldots \in P$ are finite in number and have rank $i + 1$, and $a_1, a_2, \ldots \in \mathbb{Z}$), there is some $z \in Z$ such that $x_k \leq z$ for all k. $\square$

Remarks

1. The hypothesis that Z is directed is essential. For example, suppose that the zircon Z consists of the bottom element $\hat{0}$ and two atoms. The trivial labeling with two +1 is a balanced labeling which is not exact. However, this condition is not

particularly restrictive and all Coxeter groups are directed posets (see Lemma 6.4 of [4]).

2. The hypothesis that Z is a zircon is essential. For example, the directed Eulerian poset P in Fig. 6 has $2^{|P|}$ balanced labelings and none of them is exact.

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