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Sets of reflections defining twisted Bruhat orders

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Abstract Twisted Bruhat orders are certain partial orders on a Coxeter system (W, S) associated to initial sections of reflection orders, which are certain subsets of the set of reflections T of a Coxeter system. We determine which subsets of T give rise to a partial order on W in the same way.

1 Introduction and preliminaries

In a Coxeter system, (W, S), reflection orders are certain total orders of the set of reflections T. Initial sections of these reflection orders, which are certain subsets of T, lead to partial orders (twisted Bruhat orders) on W that are similar to Bruhat order. In fact, using the initial section $\emptyset \subseteq T$ we get the Bruhat order on W. In this paper, we determine all subsets of T that give rise to partial orders on W in the same manner. We see that subsets of T that have this property are closely related to initial sections of reflection orders and are conjecturally the same.

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2 Statement of main result

We begin with the same setup as [3]. We let (W, S) be a Coxeter system with $\ell : W \to \mathbb{N}$ the corresponding length function. Then, let $T = \cup_{w\in W} wSw^{-1}$ be the set of reflections of W, and regard $\mathcal{P}(T)$ as an Abelian group under symmetric difference. We define $N : W \to \mathcal{P}(T)$ by $N(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$. By [2], N can be characterized by $N(s) = \{s\}$ $(\forall s \in S)$ and $N(xy) = N(x) + xN(y)x^{-1}$ $(\forall x, y \in W)$.

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This implies there is a W-action on $\mathcal{P}(T)$ given by $w \cdot A = N(w) + wAw^{-1}$ (where $w \in W$ and $A \subseteq T$). For a reflection subgroup W' of W, we write $\chi(W')$ for the canonical set of generators for W', where $\chi(W') = \{t \in T | N(t) \cap W' = \{t\}\}$. Due to [2], $N_{(W',\chi(W'))}(x) = N(x) \cap W'$ for all reflection subgroups W' where $N_{(W',\chi(W'))}$ is the N function for $(W', \chi(W'))$. Additionally, let Φ be the root system for (W, S) with positive root system Φ^+. For general reference regarding Coxeter groups, see [5].

Now, for any $A \subseteq T$, we can define a directed graph $\Omega_{(W,A)}$ with the vertex set of $\Omega_{(W,A)}$ equal to W. We define the edge set $E_{(W,A)} = \{(tw, w) | t \in w \cdot A\}$. Conjugating by w, we get the equivalent statement $E_{(W,A)} = \{(wt, w) | t \in N(w^{-1}) + A\}$ (see [3]). For $x \in W$, the map $w \mapsto wx$ defines an isomorphism $\Omega_{(W,A)} \cong \Omega_{(W,x\cdot A)}$. In addition, for $A \subseteq T$ we can define a length function $\ell_A : W \to \mathbb{Z}$ in the following way:

$$\ell_A(v, w) = \ell(wv^{-1}) - 2\#[N(vw^{-1}) \cap v \cdot A] \in \mathbb{Z}$$

and then set $\ell_A(w) = \ell_A(1, w)$. We can define a pre-order $\leq_A$ for any $A \subseteq T$ given by the following: $v \leq_A w$ if and only if there exist $t_1, ..., t_n \in T$ with $w = vt_1...t_n$ such that $t_i \notin [N((vt_1...t_{i-1})^{-1}) + A]$ for all $i = 1, ..., n$.

Definition 2.1 *Following* [4], *we call a total order $\prec$ on T a* reflection order *if for any dihedral reflection subgroup W' of W either $r \prec rsr \prec \cdots \prec srs \prec s$ or $s \prec' srs \prec' \cdots \prec' rsr \prec' r$ where $\chi(W') = \{r, s\}$.*

Remark 2.2 We can also define a reflection order in terms of the root system Φ associated to the Coxeter system. This definition can be found in [1].

Recall from [4] that an initial section of a reflection order is a subset $A \subseteq T$ such that there is a reflection order $\prec$ with the property that $a \prec b$ for all $a \in A$ and $b \in T \setminus A$. It is shown in [3] that $\leq_A$ is a partial order of W if A is an initial section of a reflection order. Our main result, Theorem 2.3 below, describes all subsets A of T for which $\leq_A$ is a partial order.

Let $\mathbf{A}_{(W,S)}$ be the set of initial sections of reflection orders of T. Now, we define $\hat{\mathbf{A}}_{(W,S)} = \{A \subseteq T | A \cap W' \in \mathbf{A}_{(W',\chi(W'))}\ \forall W' \subseteq W \text{ dihedral}\}$. It has been conjectured by Matthew Dyer that $\mathbf{A} = \hat{\mathbf{A}}$. We now come to the main result:

Theorem 2.3 *Let (W, S) be any Coxeter system. The following are equivalent*:

1. *$\Omega_{(W,A)}$ is acyclic.*
2. *$\leq_A$ is a partial order.*
3. *$\Omega_{(W,A)}$ has no cycle of length four.*
4. *$A \in \hat{\mathbf{A}}$.*
5. *$\ell_A(xt) < \ell_A(x)$ for all $x \in W$ and $t \in N(x^{-1}) + A$.*

3 Proof of main result

In the following proofs, for any positive root $\alpha \in \Phi^+$, let $t_\alpha \in T$ be the corresponding reflection, and for any reflection $t \in T$, let $\alpha_t \in \Phi^+$ be the corresponding positive root. To begin with we investigate the dihedral case. Suppose (W, S) is dihedral, i.e. $S = \{r, s\}$. There is a bijection between subsets $A \subseteq T$ and subsets $\Psi \subset \Phi$ such that $\Psi \cup -\Psi = \Phi$ and $\Psi \cap -\Psi = \emptyset$ given by $A = A_\Psi = \{t_\alpha \mid \alpha \in \Psi \cap \Phi^+\}$. We note that $A_{-\Psi} = A_\Psi + T$.

Lemma 3.1 *For any $A_\Psi \subseteq T$, $w \cdot A_\Psi = A_{w(\Psi)}$.*

Proof It is enough to show that this is true for $r \in S$. Now we know that $r \cdot A_\Psi = \{r\} + rA_\Psi r = \{r\} + \{rt_\alpha r \mid \alpha \in \Psi \cap \Phi^+\} = \{r\} + \{t_{r(\alpha)} \mid \alpha \in \Psi \cap \Phi^+\} = \{r\} + \{t_\alpha \mid \alpha \in r(\Psi) \cap r(\Phi^+)\}$. If $\alpha_r \in \Psi$ then $-\alpha_r \in r(\Psi) \cap r(\Phi^+)$ and so $r \in \{t_\alpha \mid \alpha \in r(\Psi) \cap r(\Phi^+)\}$. This implies that $r \cdot A_\Psi = \{t_\alpha \mid \alpha \in r(\Psi) \cap \Phi^+\}$. If $\alpha_r \notin \Psi$ then $r \notin \{t_\alpha \mid \alpha \in r(\Psi) \cap r(\Phi^+)\}$. But $\alpha_r \in r(\Psi)$ so $\{r\} + \{t_\alpha \mid \alpha \in r(\Psi) \cap r(\Phi^+)\} = \{t_\alpha \mid \alpha \in r(\Psi) \cap \Phi^+\}$. Thus, in both cases $r \cdot A_\Psi = \{t_\alpha \mid \alpha \in r(\Psi) \cap \Phi^+\}$. □

Since we are considering (W, S) dihedral, we choose an orientation of the plane spanned by Φ. Then for any root $\alpha \in \Phi$ we can define the **neighbor** of α, denoted $nbr(\alpha)$, to be the root lying directly next to α if we traverse the root system clockwise. Recall that $S = \{r, s\}$. Interchanging r and s if necessary, we assume without loss of generality that $\alpha_r = nbr(-\alpha_s)$. We note that this implies that $nbr(\alpha_s) \notin \Phi^+$.

Lemma 3.2 *Let (W, S) be dihedral. Suppose $\alpha \in \Phi^+$ and $\gamma := nbr(\alpha) \in \Phi^+$. Then $t_\alpha t_\gamma = sr$.*

Proof Let $(rsr...)_n$ denote an element with n simple reflections listed (i.e. $\ell((rsr...)_n)$ is not necessarily n). With this terminology, $t \in T$ can be written $t = (rsr...)_{2i+1}$ or $t = (srs...)_{2i+1}$ for some $i \geq 0$. Under the chosen orientation for Φ^+, if $t_\alpha = (rsr...)_{2i+1}$ then we can write $t_\gamma = (rsr...)_{2i+3}$ so that $t_\alpha t_\gamma = (rsr...)_{2i+1}(rsr...)_{2i+3} = sr$. Otherwise, if $t_\alpha = (srs...)_{2i+1}$ $(i \geq 1)$ then $t_\gamma = (srs...)_{2i-1}$ and so $t_\alpha t_\gamma = sr$. □

Now, given $A = A_\Psi$, we introduce two conditions that a positive system, Γ^+, of Φ can have:

C1: There are roots $\alpha, nbr(\alpha) \in \Gamma^+$ such that $\alpha \notin \Psi$ and $nbr(\alpha) \in \Psi$
C2: There are roots $\beta, nbr(\beta) \in \Gamma^+$ such that $\beta \in \Psi$ with $nbr(\beta) \notin \Psi$

Lemma 3.3 *Let (W, S) be dihedral and let $A = A_\Psi \subseteq T$.*

1. *If the positive system $r(\Phi^+)$ has condition* **C1**, *then there exists a path $1 \to x \to sr$ in $\Omega_{(W,A)}$.*
2. *If the positive system $r(\Phi^+)$ has condition* **C2**, *then there exists a path $sr \to x \to 1$ in $\Omega_{(W,A)}$.*

Proof Replacing A by $A + T$ reverses the orientation of edges in $\Omega_{(W,A)}$ and so 2 follows from 1.

We now prove 1. There are two cases to consider. If $\alpha \neq \alpha_s$, then both α and $\gamma := nbr(\alpha) \in \Phi^+$. So $t_\alpha \notin A$ and $t_\gamma \in A$. Also, since $t_\alpha \notin \{r, s\}$, it follows that $t_\gamma \in N(t_\alpha)$. So we have that $t_\gamma \notin N(t_\alpha) + A$. Thus, we have a path $1 \to t_\alpha \to t_\alpha t_\gamma = sr$, where the last equality follows from Lemma 3.2. Now, if $\alpha = \alpha_s$ then $nbr(\alpha) = -\alpha_r$. Since $-\alpha_r \in \Psi$ we know that $\alpha_r \notin \Psi$ which implies $r \notin A$. Also, $s \notin A$ and clearly $r \notin N(s)$. Together, we see that there is a path $1 \to s \to sr$ in $\Omega_{(W,A)}$. □

Lemma 3.4 *For (W, S) dihedral, let $A = A_\Psi \subseteq T$. If there are no 4-cycles in $\Omega_{(W,A)}$ then there is no positive system, $\Gamma^+ \subset \Phi$ satisfying* **C1** *and* **C2**.

Proof Suppose there is a positive system Γ^+ satisfying **C1** and **C2**. It is clear that $-\Gamma^+$ also satisfies **C1** and **C2**. Since Γ^+ satisfies both **C1** and **C2**, then $w(\Gamma^+)$ will also satisfy both conditions (w with even length respects both conditions and w with odd length interchanges the conditions). Thus, we can find a $w \in W$ such that $r(\Phi^+) = w(\Gamma^+)$ or $r(\Phi^+) = w(-\Gamma^+)$. So we have that $r(\Phi^+)$ satisfies **C1** and **C2** with respect to $A_{w(\Psi)} = w \cdot A_\Psi$. Since $\Omega_{(W,A)}$ is isomorphic to $\Omega_{(W,w\cdot A)}$, we can assume without loss of generality that $r(\Phi^+)$ satisfies **C1** and **C2** with respect to A_Ψ. Thus, Lemma 3.3 implies that we have a path $sr \to u \to 1 \to v \to sr$ in $\Omega_{(W,A)}$. □

Lemma 3.5 *Let (W, S) be dihedral and $A = A_\Psi \subseteq T$. If there is no positive system, Γ^+, of Φ satisfying* **C1** *and* **C2**, *then $A \in \mathbf{A}_{(W,S)}$.*

Proof Let $A \notin \mathbf{A}_{(W,S)}$. Recall that the only two reflection orders on (W, S) are $\prec$ and $\prec'$ described in Definition 2.1. Since A is not an initial section of either of these orders, without loss of generality (replacing A with $T \setminus A$ if necessary) we can find $t_0 \in A$ and $t_1, t_2 \in T \setminus A$ such that $t_1 \prec t_0$ and $t_2 \prec' t_0$, i.e. $t_2 \prec t_0 \prec t_1$.

Now, if (W, S) is finite, we can list all of the reflections and thus can find $t', t'' \in T$ such that $t_2 \preceq t' \prec t_0 \preceq t'' \prec t_1$ with $\alpha_{t'} \notin \Psi$ and $nbr(\alpha_{t'}) \in \Psi$, and $\alpha_{t''} \in \Psi$ and $nbr(\alpha_{t''}) \notin \Psi$. Thus Φ^+ satisfies **C1** and **C2**.

If (W, S) is infinite, then $T = T_r \cup T_s$ (disjoint union) where $T_r = \{t \in T |\ r \in N(t)\}$ and $T_s = \{t \in T |\ s \in N(t)\}$. Suppose Φ^+ does not satisfy **C1** and **C2**. We cannot have $t_1, t_2, t_0 \in T_u$ for some $u \in \{r, s\}$ since this would imply, by the reasoning for (W, S) finite, that Φ^+ satisfies **C1** and **C2**. So, we can assume that $t_2, t_0 \in T_r$ and $t_1 \in T_s$ (the case $t_2 \in T_r$ and $t_0, t_1 \in T_s$ is exactly similar). Additionally, since **C1** and **C2** aren't both satisfied by Φ^+, we may assume the following conditions hold (see Figure 1 above):

1. $\alpha_{t_0} = nbr(\alpha_{t_2})$,
2. $t \in T \setminus A$ if $t \preceq t_2$,
3. $t \in T \setminus A$ if $t \in T_s$ and $t \preceq t_1$,
4. $t \in A$ if $t \in T_r$ and $t_0 \preceq t$,
5. $t \in A$ if $t_1 \prec t$.

Now, consider the positive system Γ^+ with simple roots α_{t_0} and $-nbr(\alpha_{t_0})$. Then Γ^+ satisfies **C1** using the roots α_{t_2} and $\alpha_{t_0} = nbr(\alpha_{t_2})$. Also, by above, $\alpha_{t_1} \notin \Psi$ and $nbr(\alpha_{t_1}) \in \Psi$ (note that even if $t_1 = s$ this is true since $nbr(\alpha_s) = -\alpha_r \in \Psi$ because by above $\alpha_r \notin \Psi$). Using this, we see that Γ^+ satisfies **C2** using the roots $-\alpha_{t_1}$ and $nbr(-\alpha_{t_1}) = -nbr(\alpha_{t_1})$ (again see Figure 1). □

With the dihedral case taken care of, we proceed to the general case. For the remainder of the paper, we assume that (W, S) is a general Coxeter system.

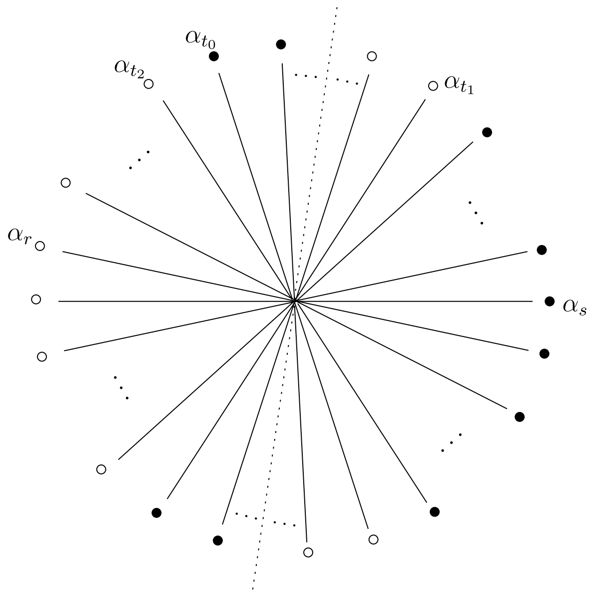

Fig. 1 This is a schematic diagram of the root system for (W, S) infinite. Φ^+ consists of the roots which are non-negative linear combinations of α_r and α_s. The dotted ray through the origin represents a limit line of roots. If **C1** and **C2** are not both satisfied by Φ^+ then Ψ is pictured above with the roots in Ψ labeled by $\bullet$, and those not in Ψ labeled by $\circ$

Proposition 3.6 *For all* $w \in W$ *and* $A \in \hat{\mathbf{A}}_{(W,S)}$, $w \cdot A \in \hat{\mathbf{A}}_{(W,S)}$.

Proof It suffices to check the condition for $w = r \in S$. Let W' be a dihedral reflection subgroup. If $r \in W'$, then $(r \cdot A) \cap W' = N_{(W',\chi(W'))}(r) + r(A \cap W')r \in \mathbf{A}_{(W',\chi(W'))}$ by [4]. Now, if $r \notin W'$, then $(r \cdot A) \cap W' = r(A \cap rW'r)r$. However, conjugation by r defines an isomorphism $(W', \chi(W')) \cong (rW'r, r\chi(W')r)$ in this case, and by assumption $A \cap rW'r \in \mathbf{A}_{(rW'r, r\chi(W')r)}$, thus $(r \cdot A) \cap W' \in \mathbf{A}_{(W',\chi(W'))}$. □

Proposition 3.7 *Let* $A \in \hat{\mathbf{A}}_{(W,S)}$, $x \in W$, $t \in T$. *Then* $\ell_A(x, xt) > 0$ *iff* $t \notin N(x^{-1}) + A$.

Proof Because of Proposition 3.6, the argument will follow directly from the proof of [3] (Proposition 1.2) which only requires that $A \in \hat{\mathbf{A}}_{(W,S)}$. □

We now can turn to the proof of Theorem 2.3:

Proof (1)⇔(2)⇒(3) is clear.

(3)⇒(4): By [2] (Lemma 3.2), we know that for a dihedral subgroup W' of W, $N_{(W',S')}(w) = N_{(W,S)}(w) \cap W'$. This implies $\Omega_{(W',A\cap W')}$ is a subgraph of $\Omega_{(W,A)}$. So, if $A \notin \hat{\mathbf{A}}_{(W,S)}$, Lemma 3.4 and Lemma 3.5 imply that there exists some dihedral subgroup W' of W with $\Omega_{(W',A\cap W')}$ containing a cycle with four edges.

(4)⇒(5): Suppose that $A \in \hat{\mathbf{A}}_{(W,S)}$, $x \in W$ and $t \in N(x^{-1}) + A$. Then Proposition 3.7 implies that $\ell_A(x, xt) < 0$. But by [3] $\ell_A(1, x) + \ell_A(x, xt) = \ell_A(1, xt)$ and this implies $\ell_A(xt) - \ell_A(x) = \ell_A(x, xt) < 0$.

(5)⇒(1): Suppose $\Omega_{(W,A)}$ has a cycle. This means that $xt_1 \ldots t_n = x$ for some $n > 0$ (all $t_i \in T$ and with $t_i \notin N((xt_1 \ldots t_{i-1})^{-1}) + A$ for all i). By assumption, $\ell_A(x) < \ell_A(xt_1) < \ell_A(xt_1t_2) < \ldots < \ell_A(xt_1t_2 \ldots t_n) = \ell_A(x)$ which is a contradiction. □

Remark 3.8 In case (W, S) is finite, say of order m, we can give a much simpler proof of the equivalence of (1), (2), (4), and (5). By the proof above, we have (4)⇒(5)⇒(1)⇔(2). If (4) fails, then $w \cdot A \neq T$ for all $w \in W$ (or else $A = w^{-1} \cdot T$ which is an initial section). So we can choose $t \notin w \cdot A$. It follows that we can recursively choose $t_1, ..., t_m$ such that $t_1 \notin A$ and $t_i \notin t_{i-1} \cdots t_1 \cdot A$ for $i = 2, ..., m$. This gives us the following path in $\Omega_{(W,A)}$: $1 \to t_1 \to \ldots \to t_m \cdots t_1$. However, this path has $m + 1$ elements and W has m elements, so there must be two elements in the path that are the same thus creating a cycle.

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