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Crystal graphs of irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules of level two and Uglov bipartitions

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Abstract We give a simple description of the natural bijection between the set of FLOTW bipartitions and the set of Uglov bipartitions (which generalizes the set of Kleshchev bipartitions). These bipartitions, which label the crystal graphs of irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules of level two, naturally appear in the context of the modular representation theory of Hecke algebras of type B_n.

Keywords Hecke algebras · Modular representations · Canonical basis · Crystal graph

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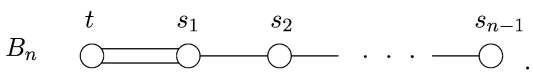
1 Introduction

Let $n > 0$, and let W_n be a Weyl group of type B_n with a set of simple reflections $S := \{t, s_1, \dots, s_{n-1}\}$ and relations symbolized by the following braid diagram:



Let k be a field and $Q, q \in k^\times$. We denote by $H_n := H_k(W_n, Q, q)$ the corresponding Iwahori–Hecke algebra. This is the associative unitary k-algebra generated by the elements T_s for $s \in S$, subject to the braid relations symbolized by the above diagram and the relations $(T_t - Q)(T_t + 1) = 0$, $(T_{s_j} - q)(T_{s_j} + 1) = 0$ for $1 \leq j \leq n - 1$. When H_n is semisimple, the Tits deformation theorem shows that the simple modules of this algebra are in natural bijection with the simple modules of the group algebra

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kW_n. In the nonsemisimple case, the classification of the simple H_n-modules was obtained by Dipper and James [7], Ariki [1], [2], and Ariki and Mathas [4], using the theory of canonical bases and crystal graphs for quantum groups.

Let $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ be the quantum group of type $A^{(1)}_{e-1}$. Ariki and Ariki-Mathas showed that the set of simple H_n-modules $\mathrm{Irr}(H_n)$ is in natural bijection with the Kashiwara crystal basis of the irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module with highest weight sum of two fundamental weights $\Lambda_{v_0} + \Lambda_{v_1}$ $(0 \leq v_0, v_1 < e)$. There are several natural ways to obtain a parametrization of this basis, depending on a choice of integers s_0 and s_1 in the classes of v_0 and v_1 modulo e. Hence we obtain several possibilities for labeling the same set $\mathrm{Irr}(H_n)$; they are given by a certain class of bipartitions $\Phi^{(s_0,s_1)}_{e,n}$ named "Uglov bipartitions." This kind of bipartitions both generalizes the set of FLOTW bipartitions (which correspond to the case $0 \leq s_0, s_1 \leq e$, see [12]), and the set of Kleshchev bipartitions (corresponding to the case where $s_0 - s_1 > n - 1 - e$, see [2]).

In [10], M. Geck and the author have given an interpretation of this fact in the context of the representation theory of Hecke algebras. We showed that each of the parameterizations by $\Phi^{(s_0,s_1)}_{e,n}$ corresponds to a natural indexation of the Geck–Rouquier's canonical basic set. As a consequence, these sets give natural indexations of the simple modules for Hecke algebras of type B_n in the nonsemisimple case.

In general we only know a recursive definition of the sets of Uglov bipartitions, and a natural problem is to obtain a nonrecursive (and simple) characterization of these sets. In the case where $s_0 - s_1 > n - 1 - e$ (known as the "asymptotic case"), this problem has been recently solved by Ariki, Kreiman, and Tsuchioka [5], using results of Littelmann. Our purpose is to obtain a new characterization of all Uglov bipartitions using the following facts:

- In the case where $0 \leq s_0 \leq s_1 < e$, we know a simple nonrecursive characterization of the set $\Phi^{(s_0,s_1)}_{e,n}$, the FLOTW bipartitions [6].
- If $s'_0 \equiv s_0 (\mathrm{mod}\, e)$ and $s'_1 \equiv s_1 (\mathrm{mod}\, e)$ or if $s'_0 \equiv s_1 (\mathrm{mod}\, e)$ and $s'_1 \equiv s_0 (\mathrm{mod}\, e)$, we have a bijection between $\Phi^{(s_0,s_1)}_{e,n}$ and $\Phi^{(s'_0,s'_1)}_{e,n}$.

Hence, if we know a simple (and nonrecursive) description of the above bijection, the desired characterizations of all Uglov bipartitions will follow. Quite remarkably, the main result of this paper, Theorem 4.6, together with the work of Leclerc and Miyachi, shows that this bijection is controlled by the canonical bases of the irreducible $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules. As a special case, we obtain a quite simple and new characterization of the set of Kleshchev bipartitions (but which remains recursive…) by using the notion of symbols.

The paper is organized as follows. The first section gives a brief exposition of the theory of crystal graphs and connections with the representation theory of Hecke algebras. In the second and third sections, our main results are stated and proved: we study the combinatoric of Uglov bipartitions and give a description of the above bijection. This description is largely inspired by the works of Leclerc and Miyachi. In the last section, we describe the relations of our results with these works.

2 Crystal graphs of v-deformed Fock spaces of level 2

2.1 Fock spaces

Let v be an indeterminate, and let e be a positive integer. Let $\mathfrak{h}$ be a free $\mathbb{Z}$-module with basis $\{h_i, \mathfrak{d} \mid 0 \leq i < e\}$, and let $\{\Lambda_i, \delta \mid 0 \leq i < e\}$ be the dual basis with respect to the pairing

$$\langle , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{Z}$$

such that $\langle \Lambda_i, h_j \rangle = \delta_{ij}$, $\langle \delta, \mathfrak{d} \rangle = 1$, and $\langle \Lambda_i, \mathfrak{d} \rangle = \langle \delta, h_j \rangle = 0$ for $0 \leq i, j < e$. The Λ_k $(1 \leq k \leq e)$ are called the *fundamental weights*. The quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ of type $A^{(1)}_{e-1}$ is the unital associative algebra over $\mathbb{C}(v)$ generated by the elements $\{e_i, f_i \mid i \in \{0, \ldots, e-1\}\}$ and $\{k_h \mid h \in \mathfrak{h}\}$ subject to the relations described, for example, in [16, Chap. 6].

In this paper, we want to study the irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-modules with highest weight Λ, where Λ is the sum of two fundamental weights $\Lambda_{v_0} + \Lambda_{v_1}$ with $0 \leq v_0, v_1 < e$. These modules can be constructed by using the *Fock space representation* which we now define. Let $\Pi_{2,n}$ be the set of bipartitions of rank n, that is, the set of 2-tuples $(\lambda^{(0)}, \lambda^{(1)})$ such that $\lambda^{(0)}$ (resp. $\lambda^{(1)}$) is a partition or rank a_1 (resp. a_2) with $a_1 + a_2 = n$. Let $\boldsymbol{s} = (s_0, s_1) \in \mathbb{Z}^2$ be such that $s_0 \equiv v_0 \pmod{e}$ and $s_1 \equiv v_1 \pmod{e}$ or such that $s_0 \equiv v_1 \pmod{e}$ and $s_1 \equiv v_0 \pmod{e}$. The *Fock space* (of level 2) is defined to be the $\mathbb{C}(v)$-vector space generated by the symbols $|\boldsymbol{\lambda}, \boldsymbol{s}\rangle$ with $\boldsymbol{\lambda} \in \Pi_{2,n}$:

$$\mathfrak{F}^{\boldsymbol{s}} := \bigoplus_{n \geq 0} \bigoplus_{\boldsymbol{\lambda} \in \Pi_{2,n}} \mathbb{C}(v) |\boldsymbol{\lambda}, \boldsymbol{s}\rangle.$$

Let us introduce some additional notation concerning the combinatorics of bipartitions. Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$ be a bipartition of rank n. The diagram of $\boldsymbol{\lambda}$ is the following set:

$$[\boldsymbol{\lambda}] = \left\{(a, b, c) \mid 0 \leq c \leq 1,\ 1 \leq b \leq \lambda_a^{(c)}\right\}.$$

The elements of this diagram are called the *nodes* of $\boldsymbol{\lambda}$. Let $\gamma = (a, b, c)$ be a node of $\boldsymbol{\lambda}$. The *residue* of γ associated to e and (s_0, s_1) is the element of $\mathbb{Z}/e\mathbb{Z}$ defined by

$$\operatorname{res}(\gamma) \equiv (b - a + s_c) (\operatorname{mod}\ e).$$

If γ is a node with residue i, we say that γ is an i-node. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two bipartitions of ranks n and $n+1$ such that $[\boldsymbol{\lambda}] \subset [\boldsymbol{\mu}]$. There exists a node γ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$. Then, we denote $[\boldsymbol{\mu}]/[\boldsymbol{\lambda}] = \gamma$ and if $\operatorname{res}(\gamma) = i$, we say that γ is an *addable* i-node for $\boldsymbol{\lambda}$ and a *removable* i-node for $\boldsymbol{\mu}$. Let $i \in \{0, \ldots, e-1\}$. We introduce a total order on the set of addable or removable i-nodes of a bipartition. Let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be two i-nodes of a bipartition. We denote $\gamma <_{(s_0, s_1)} \gamma'$ if:

$$b - a + s_c < b' - a' + s_{c'} \quad \text{or if} \quad b - a + s_c = b' - a' + s_{c'} \quad \text{and} \quad c' < c.$$

Note that this order strongly depends on the choice of s_0 and s_1 in the classes of v_0 and v_1 modulo e. Note also that this order coincides with that of [10].

Using this order, it is possible to define an action of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ on the Fock space $\mathfrak{F}^s$ such that $\mathfrak{F}^s$ becomes an integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module. Moreover, it is known that the submodule M_s generated by the empty bipartition is a highest-weight module with weight $\Lambda_{v_0}+\Lambda_{v_1}$ (see [14] for details). Hence, if $\mathbf{s}'=(s'_0,s'_1)\in\mathbb{Z}^2$ is such that $s_0\equiv s'_0(\operatorname{mod} e)$ and $s_1\equiv s'_1(\operatorname{mod} e)$ or such that $s_0\equiv s'_1(\operatorname{mod} e)$ and $s_1\equiv s'_0(\operatorname{mod} e)$, then the modules $M_{\mathbf{s}}$ and $M_{\mathbf{s}'}$ are isomorphic. However, it is important to note that the actions of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ on the elements of the standard basis $|\boldsymbol{\lambda},\mathbf{s}\rangle$ and $|\boldsymbol{\lambda},\mathbf{s}'\rangle$ are different in general.

Remark 2.1 Let $(v_0,v_1)\in\{0,1,\dots,e-1\}^2$. Then it is possible to define another order on the set of i-nodes of a bipartition as follows: we write $\gamma=(a,b,c)<_{(v_0,v_1)_+}\gamma'=(a',b',c')$ if:

$$c'<c \quad \text{or if} \quad c=c' \quad \text{and} \quad a'<a.$$

Note that if we fix a bipartition $\boldsymbol{\lambda}$ of rank n, then the above order on the i-nodes of $\boldsymbol{\lambda}$ coincides with $<_{(s_0,s_1)}$ in the case where $s_0\equiv v_0(\operatorname{mod} e)$ and $s_1\equiv v_1(\operatorname{mod} e)$ and $s_0\gg s_1$. This order will be referred to the *positive asymptotic order*, and this is the one used by Ariki [2] in its determination of the simple modules for Hecke algebras of type B_n.

Similarly, we can define another order on the set of i-nodes of a bipartition as follows. We write $\gamma=(a,b,c)<_{(v_0,v_1)_-}\gamma'=(a',b',c')$ if:

$$c'>c \quad \text{or if} \quad c=c' \quad \text{and} \quad a'<a.$$

If we fix a bipartition $\boldsymbol{\lambda}$ of rank n, then the above order on the i-nodes of $\boldsymbol{\lambda}$ coincides with $<_{(s_0,s_1)}$ in the case where $s_0\equiv v_0(\operatorname{mod} e)$ and $s_1\equiv v_1(\operatorname{mod} e)$ and $s_0\ll s_1$. This order will be referred to the *negative asymptotic order*.

In the two cases, we obtain an action of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ on the space $\mathfrak{F}^{(v_0,v_1)}$, and the submodules $M^+_{v_0,v_1}$ and $M^-_{v_0,v_1}$ generated by the empty bipartition are both irreducible highest-weight modules with weight $\Lambda_{v_0}+\Lambda_{v_1}$, and they are isomorphic.

2.2 Crystal graph of $M_{\mathbf{s}}$

As the modules $M_{\mathbf{s}}$ are integrable highest-weight modules, the general theory of Kashiwara and Lusztig provides us with a *canonical basis of* $M_{\mathbf{s}}$. We do not need in this paper the definition of this basis, but it is important to note that by the deep results of Ariki [1] one of the interest of this basis is that it provides a way to compute the decomposition matrices for Hecke algebras of type B_n over fields of characteristic zero (see [3, Theorem 14.49]). In order to make an efficient use of this, we need to determine a good parametrization of the canonical basis. This is given by studying the *Kashiwara crystal graph* which we now describe.

Let $\boldsymbol{\lambda}$ be a bipartition, and let γ be an i-node of $\boldsymbol{\lambda}$. We say that γ is a *normal* i-node of $\boldsymbol{\lambda}$ if, whenever η is an i-node of $\boldsymbol{\lambda}$ such that $\eta>_{(s_0,s_1)}\gamma$, there are more removable i-nodes between η and γ than addable i-nodes between η and γ. If γ is the minimal normal i-node of $\boldsymbol{\lambda}$ with respect to $<_{(s_0,s_1)}$, we say that γ is a *good* i-node.

Following [2, §2], the normal i-nodes of a bipartition $\boldsymbol{\lambda}$ can be easily obtained using the following process. We first read addable and removable i-nodes of $\boldsymbol{\lambda}$ in increasing order with respect to $<_{(s_0,s_1)}$. If we write A for an addable i-node and R for a removable one, we get a sequence of A and R. Then we delete RA as many as possible. The remaining removable i-nodes in the sequence are the normal i-nodes, and the node corresponding to the leftmost R is a good i-node.

Example 2.2 Let $e = 4$, $\mathbf{s} = (0, 6)$, and $\boldsymbol{\lambda} = ((4, 3, 1, 1), (4))$. The Young diagram of $\boldsymbol{\lambda}$ with residues is the following one:

$$\underline{\lambda} = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 3 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|c|} \hline 2 & 3 & 0 & 1 \\ \hline \end{array} \right).$$

We have one addable 1-node $(2, 1, 1)$ and three removable 1-nodes $(4, 1, 0)$, $(2, 3, 0)$, and $(1, 4, 1)$. We have:

$$(4, 1, 0) <_{(0,6)} (2, 3, 0) <_{(0,6)} (2, 1, 1) <_{(0,6)} (1, 4, 1),$$

and the associated sequence of removable and addable 1-nodes is $RRAR$. Hence, $(4, 1, 0)$ and $(1, 4, 1)$ are normal 1-nodes of $\boldsymbol{\lambda}$, and $(4, 1, 0)$ is a good 1-node for $\boldsymbol{\lambda}$.

Note that this notion depends on the order $<_{(s_0,s_1)}$ and, thus, on the choice of $\mathbf{s}$. To define the crystal graph of $M_{\mathbf{s}}$, we need to introduce the one of the Fock space $\mathcal{F}^{\mathbf{s}}$. This graph has been studied by Jimbo et al. [14], Foda et al. [6], and Uglov [17]. It is given by:

- Vertices: the bipartitions.
- Edges: $\boldsymbol{\lambda} \xrightarrow{i} \boldsymbol{\mu}$ if and only if $[\boldsymbol{\mu}]/[\boldsymbol{\lambda}]$ is a good i-node.

Then the crystal graph of $M_{\mathbf{s}}$ is the connected components of that of $\mathcal{F}^{\mathbf{s}}$ which contains the empty bipartition $\boldsymbol{\emptyset} := (\emptyset, \emptyset)$. The vertices of this graph that are in natural bijection with the canonical basis elements of $M_{\mathbf{s}}$ are given by the following class of bipartitions.

Definition 2.3 Let $\mathbf{s} \in \mathbb{Z}^2$. The set of *Uglov bipartitions* $\Phi^{\mathbf{s}}_{e,n}$ is defined recursively as follows.

- We have $\boldsymbol{\emptyset} := (\emptyset, \emptyset) \in \Phi^{\mathbf{s}}_{e,0}$.
- Let $\boldsymbol{\lambda} \in \Pi_{2,n}$ for $n > 0$. Then $\boldsymbol{\lambda} \in \Phi^{\mathbf{s}}_{e,n}$ if and only if there exist $i \in \{0, \dots, e-1\}$ and a good i-node γ such that if we remove γ from $\boldsymbol{\lambda}$, the resulting bipartition is in $\Phi^{\mathbf{s}}_{e,n-1}$.

It is known that the set $\coprod_{n\geq 0} \Phi^{\mathbf{s}}_{e,n}$ naturally label the canonical basis of the module $M_{\mathbf{s}}$. In the special case where $0 \leq s_0 \leq s_1 < e$, Foda, Leclerc, Okado, Thibon, and Welsh have given a nonrecursive parametrization of the set of Uglov bipartitions.

Proposition 2.4 (Foda et al. [6, Prop. 2.11]) *Assume that* $\mathbf{s} := (s_0, s_1) \in \mathbb{Z}^2$ *is such that* $0 \leq s_0 \leq s_1 < e$. *Then* $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$ *is in* $\Phi^{\mathbf{s}}_{e,n}$ *if and only if:*

(1) *for all* $i = 1, 2, \ldots,$ *we have*:

$$\lambda_i^{(0)} \geq \lambda_{i+s_1-s_0}^{(1)},$$
$$\lambda_i^{(1)} \geq \lambda_{i+e+s_0-s_1}^{(0)};$$

(2) *for all* $k > 0$, *among the residues appearing at the right ends of the length* k *rows of* $\boldsymbol{\lambda}$, *at least one element of* $\{0, 1, \ldots, e-1\}$ *does not occur.*

Such bipartitions are called FLOTW bipartitions.

When the condition $0 \leq s_0 \leq s_1 < e$ is not satisfied, the above characterization of Uglov bipartitions is no longer true. Hence, an important problem would be to obtain a simple description of $\Phi_{e,n}^{\mathbf{s}}$ in all cases.

Assume that $\mathbf{s} := (s_0, s_1) \in \mathbb{Z}^2$ and $\mathbf{s}' := (s_0', s_1') \in \mathbb{Z}^2$ are such that $s_0' \equiv s_0 (\operatorname{mod} e)$ and $s_1' \equiv s_1 (\operatorname{mod} e)$ or such that $s_0' \equiv s_1 (\operatorname{mod} e)$ and $s_1' \equiv s_0 (\operatorname{mod} e)$. Then the irreducible highest-weight modules $M_{\mathbf{s}}$ and $M_{\mathbf{s}'}$ are isomorphic, and this implies that the associated Kashiwara crystal graphs are also isomorphic: only the labeling of the vertices by the sets of Uglov bipartitions changes. Hence, in these cases, there exists a bijection

$$\Psi_{(s_0,s_1)}^{(s_0',s_1')} : \Phi_{e,n}^{(s_0,s_1)} \to \Phi_{e,n}^{(s_0',s_1')}.$$

This bijection can be obtained by following a sequence of arrows back to the empty bipartition in the crystal graph of $M_{\mathbf{s}}$ and then applying the reversed sequence to the empty bipartition of $M_{\mathbf{s}'}$. In other words, the bijection is obtained recursively as follows. We put $\Psi_{(s_0,s_1)}^{(s_0',s_1')}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$. Assume that we know $\Psi_{(s_0,s_1)}^{(s_0',s_1')} : \Phi_{e,n-1}^{(s_0,s_1)} \to \Phi_{e,n-1}^{(s_0',s_1')}$. Let $\boldsymbol{\lambda} \in \Phi_{e,n}^{(s_0,s_1)}$. Then there exist $i \in \{0, \ldots, e-1\}$ and a good i-node γ with respect to $<_{(s_0,s_1)}$ such that if we remove γ from $\boldsymbol{\lambda}$, the resulting bipartition $\boldsymbol{\lambda}'$ is in $\Phi_{e,n-1}^{\mathbf{s}}$. Let $\boldsymbol{\mu}' := \Psi_{(s_0,s_1)}^{(s_0',s_1')}(\boldsymbol{\lambda}')$. Then there exist an i-node γ' and a bipartition $\boldsymbol{\mu}$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\mu}'] \cup \{\gamma'\}$ and such that γ' is a good i-node for $\boldsymbol{\mu}$ with respect to $<_{(s_0',s_1')}$. Then we put $\Psi_{(s_0,s_1)}^{(s_0',s_1')}(\boldsymbol{\lambda}) = \boldsymbol{\mu}$.

Remark 2.5 Let $(v_0, v_1) \in \{0, 1, \ldots, e-1\}^2$. Then the crystal associated to the modules M_{v_0,v_1}^{+} and M_{v_0,v_1}^{-} can be obtained in the same way as in Definition 2.3 by using the order $<_{(v_0,v_1)_+}$ and $<_{(v_0,v_1)_-}$. The bipartitions which label the vertices of the crystal graph are respectively called the *positive Kleshchev bipartitions* and *negative Kleshchev bipartitions*. They are denoted by $\Phi_{e,n}^{(v_0,v_1)+}$ and $\Phi_{e,n}^{(v_0,v_1)-}$.

Let $\mathbf{s} := (s_0, s_1) \in \mathbb{Z}^2$ be such that $s_0 \equiv v_0 (\operatorname{mod} e)$ and $s_1 \equiv v_1 (\operatorname{mod} e)$. Then the irreducible highest-weight modules $M_{\mathbf{s}}$, M_{v_0,v_1}^{+}, and M_{v_0,v_1}^{-} are isomorphic, and we also obtain the bijections:

$$\Psi_{(s_0,s_1)}^{(v_0,v_1)-} : \Phi_{e,n}^{(s_0,s_1)} \to \Phi_{e,n}^{(v_0,v_1)-},$$
$$\Psi_{(s_0,s_1)}^{(v_0,v_1)+} : \Phi_{e,n}^{(s_0,s_1)} \to \Phi_{e,n}^{(v_0,v_1)+},$$
$$\Psi_{(v_0,v_1)+}^{(v_0,v_1)-} : \Phi_{e,n}^{(v_0,v_1)+} \to \Phi_{e,n}^{(v_0,v_1)-}.$$

Note that we also have the bijections $\Psi^{(v_1,v_0)-}_{(s_0,s_1)}$, $\Psi^{(v_1,v_0)+}_{(s_0,s_1)}$, and $\Psi^{(v_1,v_0)-}_{(v_0,v_1)+}$. By the definitions of the order $<_{(v_0,v_1)+}$ and $<_{(v_1,v_0)-}$ and the definition of good nodes, it is clear that the last bijection is given by $\Psi^{(v_1,v_0)-}_{(v_0,v_1)+}(\lambda^{(0)},\lambda^{(1)}) = (\lambda^{(1)},\lambda^{(0)})$ for all $(\lambda^{(0)},\lambda^{(1)}) \in \Phi^{(v_0,v_1)+}_{e,n}$.

Now it is natural to try to obtain a more efficient description of these bijections. This is also motivated by the following results.

2.3 Hecke algebras of type B_n

One of the motivations for studying the class of Uglov bipartitions is provided by the study of the modular representations of Hecke algebras of type B_n. We briefly sketch this application in this subsection.

Let W_n be the Weyl group of type B_n, and let $(a,b) \in \mathbb{N}^2_{>0}$ and $\zeta_l := \exp(\frac{2i\pi}{l})$. Let $H_n := H_k(W_n, \zeta_l^b, \zeta_l^a)$ be the Hecke algebra with parameters $Q := \zeta_l^b$ and $q := \zeta_l^a$ defined over the field of complex numbers as it is defined in the introduction. In this case, the algebra H_n is nonsemisimple in general, and one of the main problem is to determine a parametrization of its simple modules and to compute the associated decomposition matrix. An approach to solve this problem has been given by Geck [8] and Geck–Rouquier [11]. This approach which is closely related to the existence of Kazhdan–Lusztig theory shows the existence of "canonical sets" of bipartitions which are in natural bijection with the set $\mathrm{Irr}(H_n)$. These sets are called "canonical basic sets," and they also show the unitriangularity of the decomposition matrix of H_n (for a good order on the rows provided by the Lusztig a-function). A complete survey of this theory can be found in [8] (see also [9] for further applications). Now, [10, Theorem 5.4] shows that these canonical basic sets are precisely given by the Uglov bipartitions.

Theorem 2.6 (Geck–Jacon [10]) *Let $H_n := H_k(W_n, \zeta_l^b, \zeta_l^a)$ be the Hecke algebra with parameters $Q := \zeta_l^b$ and $q := \zeta_l^a$, where $(a,b) \in \mathbb{N}^2_{>0}$. Let $d \in \mathbb{Z}$ be such that $\zeta_l^b = -\zeta_l^{a\cdot d}$. Let $e \geq 2$ be the multiplicative order of q, and let $p \in \mathbb{Z}$ be such that*:

$$d + pe < \frac{b}{a} < d + (p+1)e.$$

Then the set $\mathcal{B} = \Phi^{(d+pe,0)}_{e,n}$ is a canonical basic set in the sense of [10, *Def.* 2.4], *and it is in natural bijection with* $\mathrm{Irr}(H_n)$.

Thus it could be interesting to obtain another characterization of the set of Uglov bipartitions.

3 First results

In this section, we show that the characterization of the map $\Psi^{(s_0,s_1+e)}_{(s_0,s_1)}$ in the case where $s_0 \leq s_1$ is sufficient to obtain a characterization of the maps $\Psi^{(s_0',s_1')}_{(s_0,s_1)}$ in all cases.

3.1 Particular cases

The following proposition gives an explicit description of the map $\Psi^{(s_0,s_1)}_{(s'_0,s'_1)}$ in particular cases.

Proposition 3.1 *Let $(s_0,s_1)\in\mathbb{N}^2$, and let $e>1$ be a positive integer.*

(1) *Let $t\in\mathbb{Z}$. Then for all $\boldsymbol{\lambda}\in\Phi^{(s_0,s_1)}_{e,n}$, we have $\Psi^{(s_0+te,s_1+te)}_{(s_0,s_1)}(\boldsymbol{\lambda})=\boldsymbol{\lambda}$. Hence we have*

$$\Phi^{(s_0,s_1)}_{e,n}=\Phi^{(s_0+te,s_1+te)}_{e,n}.$$

(2) *For all $\boldsymbol{\lambda}=(\lambda^{(0)},\lambda^{(1)})\in\Phi^{(s_0,s_1)}_{e,n}$, we have $\Psi^{(s_1,s_0+e)}_{(s_0,s_1)}(\lambda^{(0)},\lambda^{(1)})=(\lambda^{(1)},\lambda^{(0)})$. Hence we have*

$$\Phi^{(s_1,s_0+e)}_{e,n}=\left\{\boldsymbol{\lambda}=\left(\lambda^{(0)},\lambda^{(1)}\right)\in\Pi_{2,n}\;\middle|\;\left(\lambda^{(1)},\lambda^{(0)}\right)\in\Phi^{(s_0,s_1)}_{e,n}\right\}.$$

Proof The first assertion is clear as the order associated to (s_0,s_1) and (s_0+te,s_1+te) on the set of i-nodes of a bipartition is the same in both cases.

We prove (2) by induction on the rank n. If $n=0$, then the result is clear. Assume that $n>0$. Let $\boldsymbol{\lambda}=(\lambda^{(0)},\lambda^{(1)})\in\Phi^{(s_0,s_1)}_{e,n}$, and let $\gamma=(a,b,c)$ be a good i-node of $\boldsymbol{\lambda}$. We must show that $\gamma'=(a,b,c+1(\mathrm{mod}\,2))$ is a good i-node for $(\lambda^{(1)},\lambda^{(0)})$ for the order induced by (s_1,s_0+e), and the result will follow by induction. To do this, by the definition of good nodes in Sect. 2.2, it is enough to show the following property: if $i\in\{0,1,\dots,e-1\}$, then $\gamma_1=(a_1,b_1,c_1)$ is an i-node in $(\lambda^{(0)},\lambda^{(1)})$ such that $\gamma>_{(s_0,s_1)}\gamma_1$ if and only if $\gamma'_1=(a_1,b_1,c_1+1(\mathrm{mod}\,2))$ is an i-node in $(\lambda^{(1)},\lambda^{(0)})$ such that $\gamma'>_{(s_1,s_0+e)}\gamma'_1$. We first assume that $\gamma>_{(s_0,s_1)}\gamma_1$ and we show that $\gamma'>_{(s_1,s_0+e)}\gamma'_1$. Note that as γ and γ_1 have the same residue modulo e, there exists $t\in\mathbb{Z}$ such that $b-a+s_c=b_1-a_1+s_{c_1}+te$.

- If $c=c_1$, then it is clear that $\gamma'>_{(s_1,s_0+e)}\gamma_1$.
- If $c=0$ and $c_1=1$, then we have $t\geq 0$. Hence $b-a+s_0\geq b_1-a_1+s_1$ and thus $b-a+s_0+e>b_1-a_1+s_1$ and $\gamma'>_{(s_1,s_0+e)}\gamma'_1$.
- If $c=1$ and $c_1=0$, then we have $t>0$. Hence we have $b-a+s_1\geq b_1-a_1+s_0+e$. If $t>1$, then we have $b-a+s_1>b_1-a_1+s_0+e$ and $\gamma'>_{(s_1,s_0+e)}\gamma'_1$. If $t=1$, then we have $b-a+s_1=b_1-a_1+s_0+e$ and $\gamma'>_{(s_1,s_0+e)}\gamma_1$, since γ' is in the first component of $(\lambda^{(1)},\lambda^{(0)})$.

Assume now that $\gamma'>_{(s_1,s_0+e)}\gamma'_1$. Then by the above argument $\gamma>_{(s_0+e,s_1+e)}\gamma_1$, and we conclude by using (1). □

The following proposition deals with the characterization of the maps $\Psi^{(v_0,v_1)-}_{(s_0,s_1)}$ and $\Psi^{(v_0,v_1)+}_{(s_0,s_1)}$.

Proposition 3.2 *Let $(s_0,s_1)\in\mathbb{Z}^2$, and let $(v_0,v_1)\in\{0,1,\dots,e-1\}^2$ be such that $v_0\equiv s_0(\mathrm{mod}\,e)$ and $v_1\equiv s_1(\mathrm{mod}\,e)$.*

(1) *if* $s_1 - s_0 > n-1$, *then for all* $\boldsymbol{\lambda} \in \Phi_{e,n}^{(s_0,s_1)}$, *we have* $\Psi_{(s_0,s_1)}^{(v_0,v_1)-}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$. *Hence we have*

$$\Phi_{e,n}^{(s_0,s_1)} = \Phi_{e,n}^{(v_0,v_1)-}.$$

(2) *If* $s_0 - s_1 > n-1-e$, *then for all* $\boldsymbol{\lambda} \in \Phi_{e,n}^{(s_0,s_1)}$, *we have* $\Psi_{(s_0,s_1)}^{(v_0,v_1)+}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$. *Hence we have*

$$\Phi_{e,n}^{(s_0,s_1)} = \Phi_{e,n}^{(v_0,v_1)+}.$$

Proof We prove (1). Let $(s_0,s_1) \in \mathbb{Z}^2$ be such that $s_1 - s_0 > n-1$ and let $\boldsymbol{\lambda} \in \Phi_{e,n}^{(s_0,s_1)}$. Let $\gamma = (a,b,c)$ be a removable i-node of $\boldsymbol{\lambda}$, and let $\gamma' = (a',b',c')$ be an addable or removable i-node of $\boldsymbol{\lambda}$. We show that $\gamma <_{(s_0,s_1)} \gamma'$ if and only if $\gamma <_{(v_0,v_1)_-} \gamma'$, and the result will follow by induction and by the definition of a good i-node as in the proof of the previous proposition. Assume first that $\gamma <_{(s_0,s_1)} \gamma'$. If $c = c'$, then the result is clear. So assume that $c \neq c'$. If $c' = 1$ and $c = 0$, then we have $\gamma <_{(v_0,v_1)_-} \gamma'$. Assume that $c = 1$ and $c' = 0$. As γ and γ' have the same residue modulo e, there exists $t \in \mathbb{Z}$ such that $b - a + s_1 = b' - a' + s_0 + te$. As $\gamma <_{(s_0,s_1)} \gamma'$, we have $t \leq 0$. Hence we have:

$$b - a - (b' - a') \leq (s_0 - s_1) < 1 - n.$$

This is impossible. Indeed, as $\boldsymbol{\lambda}$ is a bipartition of rank n, we must have:

$$\left| b' - a' - (b-a) \right| \leq n - 1.$$

Assume now that $\gamma <_{(v_0,v_1)_-} \gamma'$. If $c = c'$, then $\gamma <_{(s_0,s_1)} \gamma'$. If otherwise, we have $c' = 1$ and $c = 0$, then $b' - a' + s_1 - (b - a + s_0) \geq 1 - n + s_1 - s_0 > 0$, and we conclude that $\gamma <_{(s_0,s_1)} \gamma'$. Hence the first assertion is proved. (2) follows by using Proposition 3.1(2) and Remark 2.5. □

3.2 The map $\Psi_{(s_0,s_1)}^{(s_0,s_1+e)}$

In this subsection, we show that to characterize $\Psi_{(s_0,s_1)}^{(s'_0,s'_1)}$ in all cases, it is enough to characterize $\Psi_{(s_0,s_1)}^{(s_0,s_1+e)}$ in the case where $0 \leq s_0 \leq s_1$. First, let us assume that we know $\Psi_{(s_0,s_1)}^{(s_0,s_1+e)}$ and its inverse map if $s_0 \leq s_1$.

Let $(u_0,u_1) \in \mathbb{Z}^2$ be such that $0 \leq u_c < e$ and $s_c \equiv u_c (\mathrm{mod}\, e)$ for $c = 0,1$. By Proposition 3.1(2), we can assume that we have $0 \leq u_0 \leq u_1 < e$. Then, we have a characterization of all the following maps:

$$\Phi_{e,n}^{(u_0,u_1)} \xrightarrow[\Psi_{(u_0,u_1)}^{(u_0,u_1+e)}]{} \Phi_{e,n}^{(u_0,u_1+e)} \xrightarrow[\Psi_{(u_0,u_1+e)}^{(u_0,u_1+2e)}]{} \cdots$$

$$\cdots \xrightarrow[\Psi_{(u_0,u_1+(t-1)e)}^{(u_0,u_1+te)}]{} \Phi_{e,n}^{(u_0,u_1+te)} \xrightarrow[\Psi_{(u_0,u_1+te)}^{(u_0,u_1+(t+1)e)}]{} \Phi_{e,n}^{(u_0,u_1)-},$$

where t is such that $(t-1)e > n-1$. Now, by Proposition 3.1(1), for all $s \in \mathbb{N}$, we have that $\Phi_{e,n}^{(u_0,u_1+se)} = \Phi_{e,n}^{(u_0-se,u_1)}$ and $\Psi_{(u_0,u_1+se)}^{(u_0-se,u_1)}$ is the identity. Hence all the following maps are known:

$$\Phi_{e,n}^{(u_0,u_1)} \xrightarrow[\Psi_{(u_0,u_1)}^{(u_0-e,u_1)}]{} \Phi_{e,n}^{(u_0-e,u_1)} \xrightarrow[\Psi_{(u_0-e,u_1)}^{(u_0-2e,u_1)}]{} \cdots$$
$$\cdots \xrightarrow[\Psi_{(u_0-(t-1)e,u_1)}^{(u_0-te,u_1)}]{} \Phi_{e,n}^{(u_0-te,u_1)} \xrightarrow[\Psi_{(u_0-te,u_1)}^{(u_0-(t+1)e,u_1)}]{} \Phi_{e,n}^{(u_0,u_1)-}.$$

As we have $0 \leq u_0 \leq u_1 < e$, we have $u_1 \leq u_0 + e$. Hence, we have a characterization of the following maps:

$$\Phi_{e,n}^{(u_1,u_0+e)} \xrightarrow[\Psi_{(u_1,u_0+e)}^{(u_1,u_1+2e)}]{} \Phi_{e,n}^{(u_1,u_0+2e)} \xrightarrow[\Psi_{(u_1,u_0+2e)}^{(u_1,u_0+3e)}]{} \cdots$$
$$\cdots \xrightarrow[\Psi_{(u_1,u_0+te)}^{(u_1,u_0+(t+1)e)}]{} \Phi_{e,n}^{(u_1,u_0+(t+1)e)} \xrightarrow[\Psi_{(u_1,u_0+(t+1)e)}^{(u_1,u_0+(t+2)e)}]{} \Phi_{e,n}^{(u_1,u_0)-}.$$

Hence by Proposition 3.1(2), we have a characterization of the following maps:

$$\Phi_{e,n}^{(u_0,u_1)} \xrightarrow[\Psi_{(u_0,u_1)}^{(u_0+e,u_1)}]{} \Phi_{e,n}^{(u_0+e,u_1)} \xrightarrow[\Psi_{(u_0+e,u_1)}^{(u_0+2e,u_1)}]{} \cdots$$
$$\cdots \xrightarrow[\Psi_{(u_0+(t-1)e,u_1)}^{(u_0+te,u_1)}]{} \Phi_{e,n}^{(u_0+te,u_1)} \xrightarrow[\Psi_{(u_0+te,u_1)}^{(u_0+(t+1)e,u_1)}]{} \Phi_{e,n}^{(u_0,u_1)+}.$$

By Proposition 3.1(1), for all $s \in \mathbb{N}$, we have that $\Phi_{e,n}^{(u_0+se,u_1)} = \Phi_{e,n}^{(u_0,u_1-se)}$ and $\Psi_{(u_0,u_1+se)}^{(u_0-se,u_1)}$ is the identity. Hence all the following maps are known:

$$\Phi_{e,n}^{(u_0,u_1)} \xrightarrow[\Psi_{(u_0,u_1)}^{(u_0,u_1-e)}]{} \Phi_{e,n}^{(u_0,u_1-e)} \xrightarrow[\Psi_{(u_0,u_1-e)}^{(u_0,u_1-2e)}]{} \cdots$$
$$\cdots \xrightarrow[\Psi_{(u_0,u_1-(t-1)e)}^{(u_0,u_1-te)}]{} \Phi_{e,n}^{(u_0,u_1-te)} \xrightarrow[\Psi_{(u_0,u_1-te)}^{(u_0,u_1-(t+1)e)}]{} \Phi_{e,n}^{(u_0,u_1)+}.$$

In addition, note that by Proposition 3.1(1), we can also assume that s_0 and s_1 are both positive integers. Thus, we conclude that the characterization of $\Psi_{(s_0,s_1)}^{(s_0,s_1+e)}$ in the case where $0 \leq s_0 \leq s_1$ yields a characterization of $\Psi_{(s_0,s_1)}^{(s'_0,s'_1)}$ in all cases.

4 Characterization of the map $\Psi_{(s_0,s_1)}^{(s_0,s_1+e)}$

4.1 Properties of Uglov bipartitions

We begin with a general result on the set of Uglov bipartitions. This will be useful for the proof of the main result.

Proposition 4.1 *Let* $\mathbf{s} := (s_0, s_1) \in \mathbb{N}^2$, *and assume that* $s_1 \geq s_0$. *Let* $\boldsymbol{\lambda} \in \Phi_{e,n}^{\mathbf{s}}$, *Then* $\boldsymbol{\lambda} \in \Phi_{f,n}^{\mathbf{s}}$, *where* $f > s_1 + n$. *Hence, for all* $i = 1, 2, \ldots,$ *we have*:

$$\lambda_i^{(0)} \geq \lambda_{i+s_1-s_0}^{(1)}.$$

Proof This is proved by induction on n. If $n=0$, the result is trivial. Let $n>0$, and let $\boldsymbol{\lambda}\in\Phi^{\mathbf{s}}_{e,n}$. Then by the definition of Uglov bipartitions, there exists a good i-node $\eta=(a,b,c)$ such that if we remove η from $\boldsymbol{\lambda}$, the resulting bipartition is in $\Phi^{\mathbf{s}}_{e,n-1}$. We have $\lambda^{(c)}_a-a+s_c\equiv i\,(\mathrm{mod}\,e)$. Now, we have two cases to consider:

- If there is no addable node $\eta'=(a',b',c')$ such that $\lambda^{(c')}_{a'}-a'+s_{c'}=\lambda^{(c)}_a-a+s_c$. Then, as $f>s_1+n$, there is no addable node such that $\lambda^{(c')}_{a'}-a'+s_{c'}\equiv\lambda^{(c)}_a-a+s_c(\mathrm{mod}\,f)$. This implies that η is a normal node for the order induced by $\mathbf{s}$ and f. If there is no removable node $\eta'=(a',b',c')$ such that $\lambda^{(c')}_{a'}-a'+s_{c'}=\lambda^{(c)}_a-a+s_c$ then this is a good node for the order induced by $\mathbf{s}$ and f. If otherwise, as η is a good node for the order induced by $\mathbf{s}$ and e, we must have $c'<c$. We conclude that η is a good i-node for the order induced by $\mathbf{s}$ and f.
- If there is an addable node $\eta'=(a',b',c')$ such that $\lambda^{(c')}_{a'}-a'+s_{c'}=\lambda^{(c)}_a-a+s_c$, then, as η is a good i-node for the order induced by $\mathbf{s}$ and e, we must have $c'>c$ (if otherwise, we have $\eta'>_{(s_0,s_1)}\eta$ and no removable i-node between these two i-nodes). η' is the only addable node which has the same residue as η' modulo f. Moreover, in this case, there is no removable node $\eta'=(a',b',c')$ such that $\lambda^{(c')}_{a'}-a'+s_{c'}=\lambda^{(c)}_a-a+s_c$ and thus such that $\lambda^{(c')}_{a'}-a'+s_{c'}\equiv\lambda^{(c)}_a-a+s_c(\mathrm{mod}\,f)$. Hence η must be a good i-node for the order induced by $\mathbf{s}$ and f.

Thus, the first part of the proposition follows by induction.

Now, as $f>s_1+n\geq s_1$, the elements of $\Phi^{\mathbf{s}}_{f,n}$ are FLOTW bipartitions. Hence, we can use the characterization of Proposition 2.4 to get the second part of the proposition. □

4.2 Symbol of a bipartition

Let $\boldsymbol{s}:=(s_0,s_1)\in\mathbb{N}^2$ be such that $s_0\leq s_1$, and let $\boldsymbol{\lambda}:=(\lambda^{(0)},\lambda^{(1)})$ be a bipartition of rank $n\geq 0$. Assume that $\lambda^{(0)}=(\lambda^{(0)}_1,\lambda^{(0)}_2,\dots,\lambda^{(0)}_{r_0})$ and $\lambda^{(1)}=(\lambda^{(1)}_1,\lambda^{(1)}_2,\dots,\lambda^{(1)}_{r_1})$ (where $\lambda^{(0)}_1\geq\lambda^{(0)}_2\geq\cdots\geq\lambda^{(0)}_{r_0}$ and $\lambda^{(1)}_1\geq\lambda^{(1)}_2\geq\cdots\geq\lambda^{(1)}_{r_1}$). Let $m\in\mathbb{N}$ be such that $m>\mathrm{Max}(r_0-s_0,r_1-s_1)$. We define the following numbers which depend on $\boldsymbol{\lambda}$, $\mathbf{s}$, and m:

- For $i=1,\dots,m+s_0$, we put $\beta^{(0)}_i=\lambda^{(0)}_i-i+s_0+m$
- For $j=1,\dots,m+s_1$, we put $\beta^{(1)}_j=\lambda^{(1)}_j-j+s_1+m$

where we put $\lambda^{(0)}_k:=0$ (resp. $\lambda^{(1)}_k:=0$) if $k>r_0$ (resp. $k>r_1$). We have $\beta^{(1)}_1>\beta^{(1)}_2>\cdots>\beta^{(1)}_{m+s_1}\geq 0$ and $\beta^{(0)}_1>\beta^{(0)}_2>\cdots>\beta^{(0)}_{m+s_0}\geq 0$. In this paper, we sometimes identified $\beta^{(c)}$ ($c=0,1$) with the set $\{\beta^{(c)}_{m+s_c},\dots,\beta^{(c)}_1\}$. Then, the $\mathbf{s}$-symbol $S_{\mathbf{s}}(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$ is defined to be the pair of these two partitions. This is written as follows:

$$\begin{pmatrix}\beta^{(1)}_{m+s_1} & \beta^{(1)}_{m+s_1-1} & \cdots & \cdots & \beta^{(1)}_1\\ \beta^{(0)}_{m+s_0} & \beta^{(0)}_{m+s_0-1} & \cdots & \beta^{(0)}_1 & \end{pmatrix}.$$

On the other hand, given an $\mathbf{s}$-symbol $S_{\mathbf{s}}$, it is easy to get the bipartition $\boldsymbol{\lambda}$ such that $S_{\mathbf{s}}=S_{\mathbf{s}}(\boldsymbol{\lambda})$. By Proposition 4.1, note that if $\boldsymbol{\lambda}$ is in $\Phi^{\mathbf{s}}_{e,n}$, the $\mathbf{s}$-symbol $S_{\mathbf{s}}(\boldsymbol{\lambda})$ has the property that $\beta^{(1)}_{i+s_1-s_0} \leq \beta^{(0)}_i$ for $i = 1, \dots, m+s_0$. Such symbols are called *standard* in [15].

We will now define a map from the set of Uglov bipartitions $\Phi^{\mathbf{s}}_{e,n}$ to the set of bipartitions of rank n using this notion of $\mathbf{s}$-symbol. Let $\boldsymbol{\lambda} := (\lambda^{(0)}, \lambda^{(1)}) \in \Phi^{\mathbf{s}}_{e,n}$, and let $S_{\mathbf{s}}(\boldsymbol{\lambda}) = \binom{\beta^{(1)}}{\beta^{(0)}}$ be its s-symbol. Following [15, §2.5], we first define an injective map $\theta : \beta^{(0)} \to \beta^{(1)}$ such that $\theta(\beta^{(0)}_j) \leq \beta^{(0)}_j$ for all $j \in \{1, \dots, m+s_0\}$ as follows.

- Let $\beta^{(1)}_i$ be the maximal element of $\beta^{(1)}$ such that $\beta^{(0)}_{m+s_0} \geq \beta^{(1)}_i$. Then we put $\theta(\beta^{(0)}_{m+s_0}) = \beta^{(1)}_i$.
- Assume that we have defined $\theta(\beta^{(0)}_j)$ for $j = p+1, p+2, \dots, m+s_0$. Let $\beta^{(1)}_k$ be the maximal element of $\beta^{(1)} \setminus \{\theta(\beta^{(0)}_{m+s_0} \cup \cdots \cup \beta^{(0)}_{p+2} \cup \beta^{(0)}_{p+1})\}$ such that $\beta^{(0)}_p \geq \beta^{(1)}_k$. Then we put $\theta(\beta^{(0)}_p) = \beta^{(1)}_k$.

Observe that the standardness of $S_{\mathbf{s}}(\boldsymbol{\lambda})$ implies that θ is well defined. The 2-tuples $(j, \theta(j))$ such that $\theta(j) \neq j$ are called the *pairs* of $S_{\mathbf{s}}(\boldsymbol{\lambda})$.

Example 4.2 Let $e = 4$ and $\mathbf{s} = (0, 2)$. Then by Proposition 2.4, the bipartition $\boldsymbol{\lambda} := ((2, 2, 1), (3, 2))$ is in $\Phi^{(0,2)}_{4,10}$. The $\mathbf{s}$-symbol of this bipartition is the following one (where we put $m = 4$):

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 6 & 8 \\ 0 & 2 & 4 & 5 & & \end{pmatrix}.$$

We have $\theta(0) = 0$, $\theta(2) = 2$, $\theta(4) = 3$, $\theta(5) = 1$.

Definition 4.3 Let e be a positive integer such that $e > 1$, and let $\boldsymbol{s} := (s_0, s_1) \in \mathbb{N}^2$ be such that $s_0 \leq s_1$. We define a map

$$\Upsilon_{(s_0,s_1)} : \Phi^{(s_0,s_1)}_{e,n} \to \Pi_{2,n}$$

as follows. Let $\boldsymbol{\lambda} \in \Phi^{(s_0,s_1)}_{e,n}$, and let $S_{\mathbf{s}}(\boldsymbol{\lambda})$ be the associated $\mathbf{s}$-symbol. Let $S'_{\mathbf{s}}$ be the symbol obtained from $S_{\mathbf{s}}(\boldsymbol{\lambda})$ by permuting the pairs in $S_{\mathbf{s}}(\boldsymbol{\lambda})$ and reordering the rows (so that $S'_{\mathbf{s}}$ is a well-defined $\mathbf{s}$-symbol, see the example below). Let $\boldsymbol{\mu}$ be the bipartition such that $S'_{\mathbf{s}} = S_{\mathbf{s}}(\boldsymbol{\mu})$. Observe that $\boldsymbol{\mu} \in \Pi_{2,n}$. Then we put:

$$\Upsilon_{(s_0,s_1)}(\boldsymbol{\lambda}) = \boldsymbol{\mu}.$$

Example 4.4 Keeping the above example, the symbol $S'_{\mathbf{s}}$ is given by

$$\begin{pmatrix} 0 & 2 & 4 & 5 & 6 & 8 \\ 0 & 1 & 2 & 3 & & \end{pmatrix}.$$

This is the $\mathbf{s}$-symbol of the bipartition $(\emptyset, (3, 2, 2, 2, 1))$.

Remark 4.5 Note that the inverse map $\Upsilon^{-1}_{(s_0,s_1)}$ can be easily obtained as follows. Let $\boldsymbol{\mu} := (\mu^{(0)}, \mu^{(1)}) \in \Upsilon_{(s_0,s_1)}(\Phi^{(s_0,s_1)}_{e,n})$, and let $S_{\mathbf{s}}(\boldsymbol{\mu}) = \binom{\beta^{(1)}}{\beta^{(0)}}$ be its s-symbol. We define an injective map $\tau : \beta^{(0)} \to \beta^{(1)}$ such that $\tau(\beta^{(0)}_j) \geq \beta^{(1)}_j$ for all $j \in \{1, \dots, m+s_1\}$ as follows.

- Let $\beta_i^{(1)}$ be the minimal element of $\beta^{(1)}$ such that $\beta_1^{(0)} \leq \beta_i^{(1)}$. Then we put $\tau(\beta_1^{(0)}) = \beta_i^{(1)}$.
- Assume that we have defined $\tau(\beta_j^{(0)})$ for $j = 1, 2, \dots, p-1$. Let $\beta_k^{(1)}$ be the minimal element of $\beta^{(1)} \setminus \{\tau(\beta_1^{(0)} \cup \beta_2^{(0)} \cup \cdots \cup \beta_{p-1}^{(0)})\}$ such that $\beta_p^{(0)} \leq \beta_k^{(1)}$. Then we put $\tau(\beta_p^{(0)}) = \beta_k^{(1)}$.

Let $\boldsymbol{\lambda}$ be the bipartition associated to the $\mathbf{s}$-symbol obtained from $S_{\mathbf{s}}(\boldsymbol{\mu})$ by permuting the pairs $(j, \tau(j))$ with $j \neq \tau(j)$ and reordering the rows. Then we have $\boldsymbol{\mu} = \Upsilon_{(s_0,s_1)}^{-1}(\boldsymbol{\lambda})$.

4.3 Main result

We can now state the main theorem of this paper which gives an explicit description of the bijection $\Psi_{(s_0,s_1)}^{(s_0,s_1+e)}$.

Theorem 4.6 *Let e be a positive integer such that $e > 1$, and let $\mathbf{s} := (s_0, s_1) \in \mathbb{N}^2$ be such that $s_0 \leq s_1$. Then*:

$$\Psi_{(s_0,s_1)}^{(s_0,s_1+e)} = \Upsilon_{(s_0,s_1)}.$$

To prove this theorem, we need combinatorial properties of the map $\Upsilon_{(s_0,s_1)}$. Recall that $m \in \mathbb{N}$ is such that $m > \mathrm{Max}(r_0 - s_0, r_1 - s_1)$. For a bipartition $\boldsymbol{\nu} \in \Pi_{2,n}$, let $S_{\mathbf{s}}(\boldsymbol{\nu}) = \begin{pmatrix} \beta^{(1)} \\ \beta^{(0)} \end{pmatrix}$ be its $\mathbf{s}$-symbol. Observe that each node γ on the *border* of $\boldsymbol{\nu}$ (that is at the right ends of the Young diagram of $\boldsymbol{\nu}$) corresponds to an element of $S_{\mathbf{s}}(\boldsymbol{\nu})$. Indeed, to each node $(a, \nu_a^{(c)}, c)$, we can associate the element $\beta_a^{(c)} = \nu_a^{(c)} - a + s_c + m$. Observe also that:

- If the number $\beta_a^{(c)} - 1$ does not occur in $\beta^{(c)}$, then γ is a removable node of ν.
- If the number $\beta_a^{(c)} + 1$ does not occur in $\beta^{(c)}$, then we have an addable node $\gamma' := (a, \nu_a^{(c)} + 1, c)$ in ν.
- The residue of the node γ associated to $\beta_a^{(c)}$ is $\beta_a^{(c)} - m (\mathrm{mod}\, e)$.

In addition, recall that if $\eta = (a, b, c)$ and $\eta' = (a', b', c')$ are two i-nodes of a bipartition, we have $\eta <_{(s_0,s_1)} \eta'$ if and only if:

$$b - a + s_c < b' - a' + s_{c'} \quad \text{or if} \quad b - a + s_c = b' - a' + s_{c'} \quad \text{and} \quad c > c'.$$

On the other hand, assume that $\eta = (a, b, c)$ and $\eta' = (a', b', c')$ are two i-nodes such that $\eta <_{(s_0,s_1+e)} \eta'$.

- If $c = c' = 0$, then we have $b - a + s_0 < b' - a' + s_0$.
- If $c = c' = 1$, then we have $b - a + s_1 < b' - a' + s_1$.
- If $c = 0$ and $c' = 1$, then we have $b - a + s_0 < b' - a' + s_1 + e$. Thus we have $b - a + s_0 < b' - a' + s_1$ or $b - a + s_0 = b' - a' + s_1$.
- If $c = 1$ and $c' = 0$, we have $b - a + s_1 + e \geq b' - a' + s_0$. Thus we have $b - a + s_1 < b' - a' + s_0$.

Hence, if $\eta=(a,b,c)$ and $\eta'=(a',b',c')$ are two i-nodes of a bipartition, we have $\eta <_{(s_0,s_1+e)} \eta'$ if and only if:

$$b-a+s_c<b'-a'+s_{c'} \quad \text{or if} \quad b-a+s_c=b'-a'+s_{c'} \quad \text{and} \quad c<c'.$$

4.4 Proof of Theorem 4.6

This is proved by induction on n. If $n=0$, then the result is trivial as

$$\Psi^{(s_0,s_1+e)}_{(s_0,s_1)}(\boldsymbol{\emptyset})=\Upsilon_{(s_0,s_1)}=(\boldsymbol{\emptyset})=\boldsymbol{\emptyset}.$$

Let $n>0$, let $\boldsymbol{\lambda}:=(\lambda^{(0)},\lambda^{(1)})\in\Phi^{\mathbf{s}}_{e,n}$, and let $S_{\mathbf{s}}(\boldsymbol{\lambda})=\binom{\beta^{(1)}}{\beta^{(0)}}$ be its $\boldsymbol{s}$-symbol. Let $\boldsymbol{\mu}=(\mu^{(0)},\mu^{(1)}):=\Upsilon_{(s_0,s_1)}(\lambda^{(0)},\lambda^{(1)})$, and let $S_{\mathbf{s}}(\boldsymbol{\mu})=\binom{\alpha^{(1)}}{\alpha^{(0)}}$ be its $\mathbf{s}$-symbol.

As in Sect. 2.2, we write the sequence of removable and addable i-nodes of $\boldsymbol{\lambda}$ in increasing order with respect to $<_{(s_0,s_1)}$:

$$A_1A_2R_3R_4A_5R_6\cdots A_s,$$

where we write R_j for a removable i-node and A_j for an addable i-node. We delete the occurrences R_jA_{j+1} in this sequence. Then, we obtain a sequence $\mathfrak{S}$ of removable i-nodes and addable i-nodes:

$$A_{j_1}\cdots A_{j_s}R_{i_1}R_{i_2}\cdots,$$

where $j_1<j_2<\cdots<i_1<i_2<\cdots$. The R_{i_k} correspond to the normal i-nodes of $\boldsymbol{\lambda}$, and the leftmost one, R_{i_1}, is a good i-node for $\boldsymbol{\lambda}$.

Let R_{i_l} be an element of $\mathfrak{S}$. As explained above, R_{i_l} corresponds to an element $\beta^{(c)}_a$ in $S_{\mathbf{s}}(\boldsymbol{\lambda})$. As R_{i_l} is removable, we have $\beta^{(c)}_{a-1}<\beta^{(c)}_a-1$. We will associate to this node a removable i-node R'_{i_l} in $\boldsymbol{\mu}$. To do this, we will distinguish several cases. We also have to consider the other removable and addable i-nodes of $\boldsymbol{\lambda}$ as it will be useful for the proof of the main result. In each case, we give an example of the symbols $S_{\mathbf{s}}(\boldsymbol{\mu})$ and $S_{\mathbf{s}}(\boldsymbol{\lambda})$ in which the elements corresponding to R_{i_l} and R'_{i_l} are written in bold.

(1) Assume that $c=0$ and that we have $\theta(\beta^{(0)}_a)=\beta^{(1)}_b<\beta^{(0)}_a$ for $b\in\{1,\dots,m+s_1\}$. Then to obtain $S_{\mathbf{s}}(\boldsymbol{\mu})$, we have to permute $\beta^{(0)}_a$ and $\beta^{(1)}_b$. As $\beta^{(1)}_b<\beta^{(0)}_a$, the node R'_{i_l} associated to $\beta^{(0)}_a$ in $\alpha^{(1)}$ is a removable i-node (because $\beta^{(0)}_a-1$ cannot occur in $\alpha^{(1)}$). Note that if we have $\beta^{(1)}_b=\beta^{(0)}_a-1$, then we have an addable i-node A on the part of $\boldsymbol{\lambda}$ associated to $\beta^{(1)}_b$ in $\beta^{(1)}$ such that $A<_{(s_0,s_1)}R_{i_l}$. In this case, we have an addable i-node A' on the part of $\boldsymbol{\mu}$ associated to $\beta^{(1)}_b$ in $\alpha^{(0)}$ such that $A'<_{(s_0,s_1+e)}R'_{i_l}$.

Example 4.7 In the following example, we put $\beta^{(0)}_a=j$, $\beta^{(1)}_b=j-1$, $\beta^{(0)}_{a-1}=j-2$, and $\beta^{(1)}_{b-1}=j-2$:

$$S_{\mathbf{s}}(\boldsymbol{\lambda})=\begin{pmatrix}\cdots & j-2 & j-1 & \cdots & \cdots\\ \cdots & j-2 & \mathbf{j} & \cdots & \end{pmatrix}.$$

Then

$$S_{\mathbf{s}}(\boldsymbol{\mu}) = \begin{pmatrix} \cdots & j-2 & \mathbf{j} & \cdots & \cdots \\ \cdots & j-2 & j-1 & \cdots & \end{pmatrix}.$$

(2) Assume that $c=0$ and that we have $\theta(\beta_a^{(0)}) = \beta_b^{(1)} = \beta_a^{(0)}$ for $b \in \{1,\dots,m+s_1\}$ and that $\beta_{b-1}^{(1)} < \beta_b^{(1)} - 1$. In this case, we have a removable i-node R associated to $\beta_b^{(1)}$ in $\beta^{(1)}$. Observe that $R <_{(s_0,s_1)} R_{i_l}$. Then to obtain $S_{\mathbf{s}}(\boldsymbol{\mu})$, $\beta_a^{(0)}$ is not permuted with any elements of $\beta^{(1)}$. The node R'_{i_l} associated to $\beta_b^{(1)}$ in $\alpha^{(1)}$ is a removable i-node. Note that the removable i-node R' associated to $\beta_a^{(0)}$ in $\alpha^{(0)}$ is such that $R' <_{(s_0,s_1+e)} R'_{i_l}$.

Example 4.8 In the following example, we put $\beta_a^{(0)} = j = \beta_b^{(1)}$, $\beta_{a-1}^{(0)} = j-2$, and $\beta_{b-1}^{(1)} = j-3$:

$$S_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} \cdots & j-3 & j & \cdots & \cdots \\ \cdots & j-2 & \mathbf{j} & \cdots & \end{pmatrix}.$$

Then

$$S_{\mathbf{s}}(\boldsymbol{\mu}) = \begin{pmatrix} \cdots & j-2 & \mathbf{j} & \cdots & \cdots \\ \cdots & j-3 & j & \cdots & \end{pmatrix}.$$

(3) Assume that $c=0$ and that we have $\theta(\beta_a^{(0)}) = \beta_b^{(1)} = \beta_a^{(0)}$ for $b \in \{1,\dots,m+s_1\}$ and that $\beta_{b-1}^{(1)} = \beta_b^{(1)} - 1$. Then to obtain $S_{\mathbf{s}}(\boldsymbol{\mu})$, $\beta_a^{(0)}$ is not permuted with any elements of $\beta^{(1)}$. The node R'_{i_l} associated to $\beta_a^{(0)}$ in $\alpha^{(0)}$ is a removable i-node.

Example 4.9 In the following example, we put $\beta_a^{(0)} = j = \beta_b^{(1)}$, $\beta_{a-1}^{(0)} = j-2$, and $\beta_{b-1}^{(1)} = j-1$:

$$S_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} \cdots & \cdots & j-1 & j & \cdots & \cdots \\ \cdots & j-2 & \mathbf{j} & \cdots & \cdots & \end{pmatrix}.$$

Then

$$S_{\mathbf{s}}(\boldsymbol{\mu}) = \begin{pmatrix} \cdots & \cdots & j-1 & j & \cdots & \cdots \\ \cdots & j-2 & \mathbf{j} & \cdots & \cdots & \end{pmatrix}.$$

(4) Assume that $c=1$ and that we have $\theta(\beta_b^{(0)}) = \beta_a^{(1)} < \beta_b^{(0)}$ for a $b \in \{1,\dots,m+s_1\}$. Then to obtain $S_{\mathbf{s}}(\boldsymbol{\mu})$, $\beta_b^{(0)}$ must be permuted with $\beta_a^{(1)}$. The node R'_{i_l} associated to $\beta_a^{(1)}$ in $\alpha^{(0)}$ is a removable i-node. Note that if we have $\beta_{b-1}^{(0)} = \beta_a^{(1)} - 1$, then we have an addable i-node A on the part of $\boldsymbol{\lambda}$ associated to $\beta_{b-1}^{(0)}$ such that $A >_{(s_0,s_1)} R_{i_l}$. This cannot happen for R_{i_l} since we have assumed that σ is reduced.

However note that if R is such a removable i-node and if R' is the associated removable i-node of $\boldsymbol{\mu}$ as above, we have an addable i-node A' on the part of $\boldsymbol{\mu}$ associated to $\beta_{b-1}^{(0)}$ in $\alpha^{(1)}$ such that $A' >_{(s_0,s_1+e)} R'$.

Example 4.10 In the following example, we put $\beta_a^{(1)} = j$, $\beta_b^{(0)} = j+1$, $\beta_{a-1}^{(1)} = j-3$, and $\beta_{b-1}^{(0)} = j-1$:

$$S_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} \cdots & j-3 & \mathbf{j} & \cdots & \cdots \\ \cdots & j-1 & j+1 & \cdots & \end{pmatrix}.$$

Then

$$S_{\mathbf{s}}(\boldsymbol{\mu}) = \begin{pmatrix} \cdots & j-1 & j+1 & \cdots & \cdots \\ \cdots & j-3 & \mathbf{j} & \cdots & \end{pmatrix}.$$

(5) Assume that $c=1$ and that we have $\theta(\beta_b^{(0)}) = \beta_a^{(1)} = \beta_b^{(0)}$ for a $b \in \{1,\dots,m+s_1\}$. Then to obtain $S_{\mathbf{s}}(\boldsymbol{\mu})$, $\beta_a^{(1)}$ is not permuted with any elements of $\beta^{(0)}$. The node R'_{i_l} associated to $\beta_b^{(0)}$ in $\alpha^{(0)}$ must be a removable i-node. Note that if $\beta_{b-1}^{(0)} < \beta_b^{(0)} - 1$, then the node R associated to $\beta_b^{(0)}$ in $\beta^{(0)}$ is a removable i-node such that $R >_{(s_0,s_1)} R_{i_l}$. Then, the node R' associated to $\beta_a^{(1)}$ in $\alpha^{(1)}$ is a removable i-node such that $R' >_{(s_0,s_1+e)} R'_{i_l}$.

Example 4.11 In the following example, we put $\beta_a^{(1)} = j = \beta_b^{(0)}$, $\beta_{a-1}^{(1)} = j-2$, and $\beta_{b-1}^{(0)} = j-1$:

$$S_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} \cdots & j-2 & \mathbf{j} & \cdots & \cdots \\ \cdots & j-1 & j & \cdots & \end{pmatrix}.$$

Then

$$S_{\mathbf{s}}(\boldsymbol{\mu}) = \begin{pmatrix} \cdots & j-1 & j & \cdots & \cdots \\ \cdots & j-2 & \mathbf{j} & \cdots & \end{pmatrix}.$$

(6) Assume that $c=1$ and that we have $\theta(\beta_b^{(0)}) \neq \beta_a^{(1)}$ for all $b \in \{1,\dots,m+s_0\}$. Then the node R'_{i_l} associated to $\beta_a^{(1)}$ in $\beta^{(1)}$ is a removable i-node except possibly in the following case: there exists $d \in \{1,\dots,m+s_0\}$ such that $\beta_d^{(0)} = \beta_a^{(1)} - 1$. In this case, we have an addable i-node A in $\beta^{(0)}$ such that $A >_{(s_0,s_1)} R_{i_l}$ and there is no removable i-node between R_{i_l} and A in $\boldsymbol{\lambda}$, contradicting the fact that R_{i_l} is a normal i-node.

Example 4.12 In the following example, we put $\beta_a^{(1)} = j$:

$$S_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} \cdots & j-3 & \mathbf{j} & j+1 & \cdots \\ \cdots & j-2 & j+3 & \cdots & \end{pmatrix}.$$

Then

$$S_{\mathbf{s}}(\boldsymbol{\mu}) = \begin{pmatrix} \cdots & j-2 & \mathbf{j} & j+3 & \cdots \\ \cdots & j-3 & j+1 & \cdots & \end{pmatrix}.$$

Thus we have associated to each normal i-node R_{i_l} in $\boldsymbol{\lambda}$ a removable i-node R'_{i_l} in $\boldsymbol{\mu}$.

Similarly, one can easily check that if A_{i_t} is an addable i-node of $\boldsymbol{\lambda}$ in $\mathfrak{S}$, then we can associate an addable i-node A'_{i_t} in $\boldsymbol{\mu}$ as above. If otherwise, one can show that there exists a removable i-node R such that $R <_{(s_0,s_1)} A_{i_t}$ and that there is no addable or removable i-node between these two i-nodes for the order $>_{(s_0,s_1)}$. This contradicts the fact that we have deleted all the occurrences $R_j A_{j+1}$ in the sequence $\mathfrak{S}$.

Hence, we have associated to the sequence $\mathfrak{S}$, a sequence of removable and addable i-nodes of $\boldsymbol{\mu}$:

$$A'_{j_1}\cdots A'_{j_s} R'_{i_1} R'_{i_2}\cdots,$$

where $j_1 < j_2 < \cdots < i_1 < i_2 < \cdots$.

Now, by the above observations, it is easy to verify that this sequence corresponds to the sequence $\mathfrak{S}'$ of the removable and addable i-nodes of $\boldsymbol{\mu}$, written in increasing order with respect to $<_{(s_0,s_1+e)}$ and where the occurrences RA have been deleted.

The only problem may appear in the following situation. We have an i-node corresponding to an element $\beta_a^{(1)}$ which is not removable. There exists $d\in\{1,\dots,m+s_0\}$ such that $\theta(\beta_d^{(0)})=\beta_a^{(1)}<\beta_d^{(0)}$ and $\beta_{d-1}^{(0)}<\beta_a^{(1)}-1$. In this situation, to obtain $S_{\mathbf{s}}(\boldsymbol{\mu})$, we must permute $\beta_d^{(0)}$ and $\beta_a^{(1)}$. Moreover, $\beta_{a-1}^{(1)}$ is not permuted with any elements of $\beta^{(0)}$. Thus the i-node R' associated to $\beta_a^{(1)}$ in $\alpha^{(0)}$ must be removable for $\boldsymbol{\mu}$.

Example 4.13 In the following example, we put $\beta_a^{(1)}=j$, $\beta_d^{(0)}=j+1$, and $\beta_{d-1}^{(0)}=j-2$:

$$S_{\mathbf{s}}(\boldsymbol{\lambda})=\begin{pmatrix}\cdots & j-3 & j-1 & \mathbf{j} & \cdots\\ \cdots & j-2 & j+1 & \cdots & \end{pmatrix}.$$

Then

$$S_{\mathbf{s}}(\boldsymbol{\mu})=\begin{pmatrix}\cdots & j-2 & j-1 & j+1 & \cdots\\ \cdots & j-3 & \mathbf{j} & \cdots & \end{pmatrix}.$$

Note that in this case, we have an addable i-node A' on the part of $\mu^{(1)}$ associated to $\beta_{a-1}^{(1)}$ such that $A' >_{(s_0,s_1+e)} R'$. Thus, to obtain $\mathfrak{S}'$, the occurrence $R'A'$ must be deleted.

Now the leftmost removable i-node R_{i_1} in $\mathfrak{S}$ is a good i-node for $\boldsymbol{\lambda}$ (with respect to $<_{(s_0,s_1)}$), and the above discussion shows that this corresponds to a removable i-node R'_{i_1} in $\mathfrak{S}'$ which must be a good i-node for $\boldsymbol{\mu}$ (with respect to $<_{(s_0,s_1+e)}$).

Similarly, one easily shows that if we have an i-node which is not addable for $\boldsymbol{\lambda}$ and if this node corresponds to an addable i-node A' for $\boldsymbol{\mu}$, then we have a removable i-node R' for $\boldsymbol{\mu}$ such that $A' >_{(s_0,s_1+e)} R'$ and to obtain $\mathfrak{S}'$, the occurrence $R'A'$ must be deleted.

Finally, let $\boldsymbol{\nu}$ be the bipartition obtained by removing R_{i_1} from $\boldsymbol{\lambda}$. Note that in case (2) above, the normal i-node R_{i_l} cannot be a good i-node. Indeed, we have a removable i-node R such that $R <_{(s_0,s_1)} R_{i_l}$ and no addable i-node between these two nodes. Hence R is a normal i-node such that $R <_{(s_0,s_1)} R_{i_l}$, and thus R_{i_l} is not a good i-node. Studying the other cases above, one can verify that $\Upsilon_{(s_0,s_1)}(\boldsymbol{\nu})$ is the bipartition obtained by removing R'_{i_1} from $\boldsymbol{\mu}$. This concludes the proof of the main theorem.

4.5 Example

In this subsection, we give an example for the computation of the bijection $\Psi_{(s_0,s_1)}^{(s_0,s_1+e)}$. We put $\mathbf{s}=(0,1)$ and $e=4$ and we consider the bipartition $\boldsymbol{\lambda}:=((8),(4))\in\Phi_{4,12}^{(0,1)}$ (this is a FLOTW bipartition, see Proposition 2.4). We compute the $(0,1)$-symbol of $((8),(4))$ (with $m=2$):

$$S_{(0,1)}((8),(4))=\begin{pmatrix}0 & 1 & 6\\ 0 & 9 & \end{pmatrix}.$$

Then, the injection $\theta:\{0,9\}\to\{0,1,6\}$ is such that $\theta(0)=0$ and $\theta(9)=1$. Thus, the $(0,1)$-symbol of $\Psi_{(0,1)}^{(0,5)}((8),(4))$ is:

$$\begin{pmatrix}0 & 1 & 9\\ 0 & 6 & \end{pmatrix}.$$

Thus, we have $\Psi_{(0,1)}^{(0,5)}((8),(4))=((5),(7))$. We now want to find $\Psi_{(0,5)}^{(0,9)}((5),(7))$. The $(0,9)$-symbol of $((5),(7))$ is:

$$S_{(0,5)}((5),(7)))=\begin{pmatrix}0 & 1 & 2 & 3 & 4 & 5 & 13\\ 0 & 6 & & & & & \end{pmatrix}.$$

Then, the injection $\theta:\{0,6\}\to\{0,1,2,3,4,5,13\}$ is such that $\theta(0)=0$ and $\theta(6)=5$. Thus, the $(0,5)$-symbol of $\Psi_{(0,5)}^{(0,9)}((8),(4))$ is:

$$\begin{pmatrix}0 & 1 & 2 & 3 & 4 & 6 & 13\\ 0 & 5 & & & & & \end{pmatrix}.$$

Thus, we have $\Psi_{(0,5)}^{(0,9)}((8),(4))=((4),(7,1))$. We want now to find the Uglov bipartition $\Psi_{(0,9)}^{(0,13)}((4),(7,1))$. The $(0,9)$-symbol of $((4),(7,1))$ is:

$$S_{(0,9)}((4),(7,1)))=\begin{pmatrix}0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 17\\ 0 & 6 & & & & & & & & & \end{pmatrix}.$$

Then, the injection $\theta:\{0,6\}\to\{0,1,2,3,4,5,6,7,8,9,17\}$ is such that $\theta(0)=0$ and $\theta(6)=6$.

Thus, the $(0,9)$-symbol of $\Psi_{(0,5)}^{(0,9)}((8),(4))$ is $S_{(0,9)}((4),(7,1))$. Hence we have $\Psi_{(0,9)}^{(0,13)}((4),(7,1))=((4),(7,1))$. Now, we have $\Phi_{4,12}^{(0,13)}=\Phi_{4,12}^{(0,1)+}$ because $13-0>n-1$.

5 Relation with results of Leclerc and Miyachi

Following the works of Leclerc and Miyachi and using the above results, it is possible to describe the bijection $\Psi_{(s_0,s_1)}^{(s_0,s_1+e)}$ by using the theory of canonical basis for $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules. We first recall the results of [15].

Let $\mathcal{U}_v(\mathfrak{sl}_\infty)$ be the quantum algebra associated to the doubly infinite diagram of type A_∞. The fundamental weights are denoted by Λ_i with $i \in \mathbb{Z}$. Let $\mathbf{s} := (s_0, s_1) \in \mathbb{N}^2$ with $s_0 \leq s_1$, and let $L_\mathbf{s}$ be the irreducible highest-weight module with highest weight $\Lambda_{s_0} + \Lambda_{s_1}$. Then the theory of Kashiwara and Lusztig provides us with a canonical basis for $L_\mathbf{s}$. This basis is naturally labeled by the vertices of the associated crystal graph which can be constructed as in Sect. 2.2. It is easy to see that the class of bipartitions which label this graph is given by:

$$\Phi^{\mathbf{s}}_{\infty,n} = \left\{ \boldsymbol{\lambda} = \left(\lambda^{(0)}, \lambda^{(1)}\right) \in \Pi_{2,n} \mid \lambda^{(0)}_i \geq \lambda^{(1)}_{i+s_1-s_0},\ i = 1, 2, 3, \ldots \right\}$$

with $n \geq 0$. Thus, if $\boldsymbol{\lambda} \in \Phi^{\mathbf{s}}_{\infty,n}$, the associated element of the canonical basis is given by:

$$b(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\mu} \in \Pi_{2,n}} c_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)\boldsymbol{\mu}$$

with $c_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(v) = 1$ and $c_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in v\mathbb{Z}[v]$ if $\boldsymbol{\mu} \neq \boldsymbol{\lambda}$.

Let $\boldsymbol{\lambda} := (\lambda^{(0)}, \lambda^{(1)}) \in \Phi^{\mathbf{s}}_{\infty,n}$, and let $S_\mathbf{s}(\boldsymbol{\lambda})$ be its associated symbol. By the above characterization of $\Phi^{\mathbf{s}}_{\infty,n}$, this symbol is standard. Let p be the number of pairs in this symbol (see Sect. 4.2), and let $C(\boldsymbol{\lambda})$ be the set of bipartitions $\boldsymbol{\mu}$ of n such that $S_\mathbf{s}(\boldsymbol{\mu})$ is obtained from $S_\mathbf{s}(\boldsymbol{\lambda})$ by permuting some pairs in $S_\mathbf{s}(\boldsymbol{\lambda})$ and reordering the rows. For $\boldsymbol{\mu} \in C(\boldsymbol{\lambda})$, we denote by $l(\boldsymbol{\mu})$ the number of pairs permuted in $S_\mathbf{s}(\boldsymbol{\lambda})$ to obtain $S_\mathbf{s}(\boldsymbol{\mu})$. In particular, we have $l(\boldsymbol{\lambda}) = 0$. Then, the following result gives an explicit description of the canonical basis.

Theorem 5.1 (Leclerc–Miyachi [15, Theorem 3]) *Let $\boldsymbol{\lambda} \in \Phi^{\mathbf{s}}_{\infty,n}$, and let $b(\boldsymbol{\lambda})$ be the associated element of the canonical basis of $L_\mathbf{s}$. Then we have*:

$$b(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\mu} \in C(\boldsymbol{\lambda})} v^{l(\boldsymbol{\mu})}\boldsymbol{\mu}.$$

Now, let e be a positive integer such that $e > 1$, and let $\mathbf{s} := (s_0, s_1) \in \mathbb{N}^2$ with $s_0 \leq s_1$. Let $\boldsymbol{\lambda} \in \Phi^{\mathbf{s}}_{e,n}$. By Proposition 4.1, we have $\boldsymbol{\lambda} \in \Phi^{\mathbf{s}}_{\infty,n}$. Thus $\boldsymbol{\lambda}$ is labeling the element of the canonical basis of the irreducible highest-weight module $L_\mathbf{s}$ with highest weight $\Lambda_{s_0} + \Lambda_{s_1}$. Hence Theorem 4.6 together with Theorem 5.1 yields the following remarkable property:

Theorem 5.2 *Let $\boldsymbol{\lambda} \in \Phi^{\mathbf{s}}_{e,n}$. Then we have $\Psi^{(s_0,s_1+e)}_{(s_0,s_1)}(\boldsymbol{\lambda}) = \boldsymbol{\mu}$ if and only if the degree of $c_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ is maximal in $b(\boldsymbol{\lambda})$.*

It could be interesting to obtain a noncombinatorial proof of the above theorem which shows why the bijections $\Psi^{(s'_0,s'_1)}_{(s_0,s_1)}$ are controlled by the canonical basis of irreducible $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules.

Another open problem is obtaining similar statements for the irreducible highest-weight $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules of level $l > 2$. In this case, relations between the sets of Uglov multipartitions and the representation theory of Ariki-Koike algebras have been established in [13].

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