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Rees algebras and polyhedral cones of ideals of vertex covers of perfect graphs

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Abstract Let G be a perfect graph and let J be its ideal of vertex covers. We show that the Rees algebra of J is normal and that this algebra is Gorenstein if G is unmixed. Then we give a description–in terms of cliques–of the symbolic Rees algebra and the Simis cone of the edge ideal of G.

Keywords Perfect graphs · Normality · Edge ideals · Symbolic Rees algebras · Standard Gorenstein algebras · Max-flow min-cut · Clutters · Simis cone · Hilbert basis · Totally dual integral

Mathematics Subject Classification (2000) Primary 13H10 · Secondary 13F20 · 13B22 · 52B20

1 Introduction

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and let I be an ideal of R of height $g \geq 2$ minimally generated by a finite set $F = \{x^{v_1}, \dots, x^{v_q}\}$ of square-free monomials of degree at least two. As usual we use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \dots, a_n) \in \mathbb{N}^n$. A *clutter* $\mathcal{C}$ with vertex set X is a family of subsets of X, called edges, none of which is included in another. The set of vertices and edges of $\mathcal{C}$ are denoted by $V(\mathcal{C})$ and $E(\mathcal{C})$ respectively. We can associate to the ideal I a *clutter* $\mathcal{C}$ by taking the set of indeterminates $X = \{x_1, \dots, x_n\}$ as vertex set and $E = \{S_1, \dots, S_q\}$ as edge set, where S_k is the support of x^{v_k}, i.e., S_k is the set of variables that occur in x^{v_k}. For this reason I is called the *edge ideal* of $\mathcal{C}$. To stress

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the relationship between I and $\mathcal{C}$ we will use the notation $I = I(\mathcal{C})$. The $n \times q$ matrix with column vectors $v_1, \dots, v_q$ will be denoted by A. It is called the *incidence matrix* of $\mathcal{C}$. It is usual to call v_i the *incidence vector* or *characteristic vector* of S_i.

The *blowup algebras* studied here are: (a) the *Rees algebra*

$$R[It] = R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t],$$

where t is a new variable, and (b) the *symbolic Rees algebra*

$$R_s(I) = R \oplus I^{(1)}t \oplus \cdots \oplus I^{(i)}t^i \oplus \cdots \subset R[t],$$

where $I^{(i)}$ is the ith symbolic power of I.

The *Rees cone* of I, denoted by $\mathbb{R}_+(I)$, is the polyhedral cone consisting of the non-negative linear combinations of the set

$$\mathcal{A}' = \{e_1, \dots, e_n, (v_1, 1), \dots, (v_q, 1)\} \subset \mathbb{R}^{n+1},$$

where e_i is the ith unit vector. It is well documented [9–11] that Rees cones are an effective device to study algebraic and combinatorial properties of blowup algebras of square-free monomial ideals and clutters. They will play an important role here (Lemma 2.3). The normalization of $R[It]$ can be expressed in terms of Rees cones as we now explain. Let $\mathbb{N}\mathcal{A}'$ be the subsemigroup of $\mathbb{N}^{n+1}$ generated by $\mathcal{A}'$, consisting of the linear combinations of $\mathcal{A}'$ with non-negative integer coefficients. The Rees algebra of I can be written as

$$R[It] = K[\{x^a t^b \mid (a, b) \in \mathbb{N}\mathcal{A}'\}]. \tag{1}$$

According to [20, Theorem 7.2.28] the *integral closure* of $R[It]$ in its field of fractions can be expressed as

$$\overline{R[It]} = K[\{x^a t^b \mid (a, b) \in \mathbb{Z}^{n+1} \cap \mathbb{R}_+(I)\}]. \tag{2}$$

Hence, by Eqs. (1) and (2), we get that $R[It]$ is a normal domain if and only if the following equality holds:

$$\mathbb{N}\mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathbb{R}_+(I).$$

In geometric terms this means that $R[It] = \overline{R[It]}$ if and only if $\mathcal{A}'$ is an integral Hilbert basis, that is, a Hilbert basis for the cone it generates. Rees algebras and their integral closures are important objects of study in commutative algebra and geometry [19].

A subset $C \subset X$ is a *minimal vertex cover* of the clutter $\mathcal{C}$ if: (i) every edge of $\mathcal{C}$ contains at least one vertex of C, and (ii) there is no proper subset of C with the first property. If C satisfies condition (i) only, then C is called a *vertex cover* of $\mathcal{C}$. Let $C_1, \dots, C_s$ be the minimal vertex covers of $\mathcal{C}$. The ideal of *vertex covers* of $\mathcal{C}$ is the square-free monomial ideal

$$I_c(\mathcal{C}) = (x^{u_1}, \dots, x^{u_s}) \subset R,$$

where $x^{u_k} = \prod_{x_i \in C_k} x_i$. The clutter associated to $I_c(\mathcal{C})$ is the *blocker* of $\mathcal{C}$, see [6]. Notice that the edges of the blocker are the minimal vertex covers of $\mathcal{C}$.

We now describe the content of the paper. A characterization of perfect graphs–in terms of Rees cones–is given (Proposition 2.2). We are able to prove that $R[I_c(G)t]$ is normal if G is a perfect graph (Theorem 2.10) and that $R[I_c(G)t]$ is Gorenstein if G is a perfect and unmixed graph (Corollary 2.12). To show the normality of $R[I_c(G)t]$, we study when the system $x \geq 0; xA \leq \mathbf{1}$ is TDI (Proposition 2.5), where TDI stands for Totally Dual Integral (see Section 2). If this system is TDI and the monomials in F have the same degree, it is shown that $K[Ft]$ is an Ehrhart ring (Proposition 2.7). This is one of the results that will be used in the proof of Theorem 2.10.

If A is a balanced matrix, i.e., A has no square submatrix of odd order with exactly two 1's in each row and column, and $J = I_c(\mathcal{C})$, then $R[It] = R_s(I)$ and $R[Jt] = R_s(J)$, see [10]. We complement these results by showing that the Rees algebra of the dual I^* of I is normal if A is balanced (Proposition 2.14).

By a result of Lyubeznik [16], $R_s(I(\mathcal{C}))$ is a K-algebra of finite type. Let G be a graph. It is known that $R_s(I_c(G))$ is generated as a K-algebra by monomials whose degree in t is at most two [12, Theorem 5.1], and one may even give an explicit graph theoretical description of its minimal generators. Thus $R_s(I_c(G))$ is well understood for graphs. In contrast, the minimal set of generators of $R_s(I(G))$ is very hard to describe in terms of G (see [1]). If G is a perfect graph we compute the integral Hilbert basis $\mathcal{H}$ of the Simis cone of $I(G)$ (see Definition 3.1 and Theorem 3.2). Then, using that $R_s(I(G))$ is the semigroup ring of $\mathbb{N}\mathcal{H}$ over K, we are able to prove that $R_s(I(G))$ is generated as a K-algebra by monomials associated to cliques of G (Corollary 3.3).

Along the paper we introduce most of the notions that are relevant for our purposes. For unexplained terminology and notation we refer to [7, 14] and [3, 19]. See [6] for additional information about clutters and perfect graphs.

2 Perfect graphs, cones, and Rees algebras

We continue to use the notation and definitions used in the introduction. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of $I(\mathcal{C})$ and let $C_k = \{x_i \mid x_i \in \mathfrak{p}_k\}$ be the minimal vertex cover of $\mathcal{C}$ that corresponds to $\mathfrak{p}_k$, see [20, Proposition 6.1.16]. There is a unique irreducible representation

$$\mathbb{R}_+(I) = H_{e_1}^+ \cap H_{e_2}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap H_{\ell_2}^+ \cap \cdots \cap H_{\ell_r}^+$$

such that each ℓ_k is in $\mathbb{Z}^{n+1}$, the non-zero entries of each ℓ_k are relatively prime, and none of the closed halfspaces $H_{e_1}^+, \ldots, H_{e_{n+1}}^+, H_{\ell_1}^+, \ldots, H_{\ell_r}^+$ can be omitted from the intersection. Here H_a^+ denotes the closed halfspace $H_a^+ = \{x \mid \langle x, a\rangle \geq 0\}$ and H_a stands for the hyperplane through the origin with normal vector a, where $\langle\ ,\ \rangle$ denotes the standard inner product. According to [9, Lemma 3.1] we may always assume that $\ell_k = -e_{n+1} + \sum_{x_i \in C_k} e_i$ for $1 \leq k \leq s$. We shall be interested in the irreducible representation of the Rees cone of the ideal of vertex covers of a perfect graph G (see for instance Proposition 2.2).

Let G be a simple graph with vertex set $X=\{x_1,\dots,x_n\}$. In what follows we shall always assume that G has no isolated vertices. A *colouring* of the vertices of G is an assignment of colours to the vertices of G in such a way that adjacent vertices have distinct colours. The *chromatic number* of G is the minimal number of colours in a colouring of G. A graph is *perfect* if for every induced subgraph H, the chromatic number of H equals the size of the largest complete subgraph of H. The *complement* of G is denoted by G'. Recall that two vertices are adjacent in the graph G if and only if they are not adjacent in the graph G'.

Let S be a subset of the vertices of G. The set S is called *independent* if no two vertices of S are adjacent. Notice the following duality: S is a maximal independent set of G (with respect to inclusion) if and only if $X\setminus S$ is a minimal vertex cover of G. We denote a complete subgraph of G with r vertices by $\mathcal{K}_r$. The empty set is regarded as an independent set whose incidence vector is the zero vector.

Theorem 2.1 ([14, Theorem 16.14]) *The following statements are equivalent*:

(a) *G is a perfect graph.*
(b) *The complement of G is perfect.*
(c) *The independence polytope of G, i.e., the convex hull of the incidence vectors of the independent sets of G, is given by*:

$$\left\{(a_i)\in\mathbb{R}^n_+ \mid \textstyle\sum_{x_i\in\mathcal{K}_r} a_i\le 1;\ \forall\,\mathcal{K}_r\subset G\right\}.$$

Below we express the perfection of G in terms of a Rees cone. The next result is just a dual reinterpretation of part (c) above, which is adequate to examine the normality and Gorensteiness of Rees algebras. We regard $\mathcal{K}_0$ as the empty set with zero elements. A sum over an empty set is defined to be 0.

Proposition 2.2 *Let $J=I_c(G)$ be the ideal of vertex covers of G. Then G is perfect if and only if the following equality holds*

$$\mathbb{R}_+(J)=\left\{(a_i)\in\mathbb{R}^{n+1} \mid \textstyle\sum_{x_i\in\mathcal{K}_r} a_i\ge (r-1)a_{n+1};\ \forall\,\mathcal{K}_r\subset G\right\}. \tag{3}$$

Moreover this is the irreducible representation of $\mathbb{R}_+(J)$ if G is perfect.

Proof $\Rightarrow$) The left hand side is contained in the right hand side because any minimal vertex cover of G contains at least $r-1$ vertices of any $\mathcal{K}_r$. For the reverse inclusion take a vector $a=(a_i)$ satisfying $b=a_{n+1}\neq 0$ and

$$\textstyle\sum_{x_i\in\mathcal{K}_r} a_i\ge (r-1)b;\ \forall\,\mathcal{K}_r\subset G \implies \sum_{x_i\in\mathcal{K}_r}(a_i/b)\ge r-1;\ \forall\,\mathcal{K}_r\subset G.$$

This implication follows because by making $r=0$ we get $b>0$. We may assume that $a_i\le b$ for all i. Indeed if $a_i>b$ for some i, say $i=1$, then we can write $a=e_1+(a-e_1)$. From the inequality

$$\sum_{\substack{x_i\in\mathcal{K}_r\\ x_1\in\mathcal{K}_r}} a_i = a_1+\sum_{x_i\in\mathcal{K}_{r-1}} a_i\ge a_1+(r-2)b\ge 1+(r-1)b$$

it is seen that $a - e_1$ belongs to the right hand side of Eq. (3). Thus, if necessary, we may apply this observation again to $a - e_1$ and so on till we get that $a_i \le b$ for all i. Hence, by Theorem 2.1(c), the vector $\gamma = \mathbf{1} - (a_1/b, \dots, a_n/b)$ belongs to the independence polytope of G. Thus we can write

$$\gamma = \lambda_1 w_1 + \cdots + \lambda_s w_s; \quad (\lambda_i \ge 0; \ \textstyle\sum_i \lambda_i = 1),$$

where $w_1, \dots, w_s$ are incidence vectors of independent sets of G. Hence

$$\gamma = \lambda_1(\mathbf{1} - u_1') + \cdots + \lambda_s(\mathbf{1} - u_s'),$$

where $u_1', \dots, u_s'$ are incidence vectors of vertex covers of G. Since any vertex cover contains a minimal one, for each i we can write $u_i' = u_i + \epsilon_i$, where u_i is the incidence vector of a minimal vertex cover of G and $\epsilon_i \in \{0, 1\}^n$. Therefore

$$\begin{aligned} &\mathbf{1} - \gamma = \lambda_1 u_1' + \cdots + \lambda_s u_s' \Longrightarrow \\ &\quad a = b\lambda_1(u_1, 1) + \cdots + b\lambda_s(u_s, 1) + b\lambda_1\epsilon_1 + \cdots + b\lambda_s\epsilon_s. \end{aligned}$$

Thus $a \in \mathbb{R}_+(J)$. If $b = 0$, clearly $a \in \mathbb{R}_+(J)$. Hence we get equality in Eq. (3), as required. The converse follows using similar arguments.

To finish the proof it suffices to show that the set

$$F = \{(a_i) \in \mathbb{R}^{n+1} | \textstyle\sum_{x_i \in \mathcal{K}_r} a_i = (r-1)a_{n+1}\} \cap \mathbb{R}_+(J)$$

is a facet of $\mathbb{R}_+(J)$. If $\mathcal{K}_r = \emptyset$, then $r = 0$ and $F = H_{e_{n+1}} \cap \mathbb{R}_+(J)$, which is clearly a facet because $e_1, \dots, e_n \in F$. If $r = 1$, then $F = H_{e_i} \cap \mathbb{R}_+(J)$ for some $1 \le i \le n$, which is a facet because $e_j \in F$ for $j \notin \{i, n+1\}$ and there is at least one minimal vertex cover of G not containing x_i. We may assume that $X' = \{x_1, \dots, x_r\}$ is the vertex set of $\mathcal{K}_r$ and $r \ge 2$. For each $1 \le i \le r$ there is a minimal vertex cover C_i of G not containing x_i. Notice that C_i contains $X' \setminus \{x_i\}$. Let u_i be the incidence vector of C_i. Since the rank of $u_1, \dots, u_r$ is r, it follows that the set

$$\{(u_1, 1), \dots, (u_r, 1), e_{r+1}, \dots, e_n\}$$

is a linearly independent set contained in F, i.e., $\dim(F) = n$. Hence F is a facet of $\mathbb{R}_+(J)$ because the hyperplane that defines F is a supporting hyperplane. □

There are computer programs that determine the irreducible representation of a Rees cone [4]. Thus we may use Proposition 2.2 to determine whether a given graph is perfect, and in the process we may also determine its complete subgraphs. However this proposition is useful mainly for theoretical reasons. A direct consequence of this result (Lemma 2.3(b) below) will be used to prove one of our main results (Theorem 2.10).

Let S be a set of vertices of a graph G. The *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S. A *clique* of a graph G is a subset of the set of vertices that induces a complete subgraph. We will also call a complete subgraph of G a clique. The *support* of $x^a = x_1^{a_1} \cdots x_n^{a_n}$ is $\mathrm{supp}(x^a) = \{x_i \mid a_i > 0\}$. If $a_i \in \{0, 1\}$ for all i, x^a is called a *square-free* monomial. We regard the empty set as an independent set with zero elements.

Lemma 2.3 (a) $I_c(G') = (\{x^a \mid X \setminus \mathrm{supp}(x^a)$ *is a maximal clique of* $G\})$.
(b) *If* G *is perfect and* $J' = I_c(G')$, *then* $\mathbb{R}_+(J')$ *is equal to*

$$\left\{(a_i) \in \mathbb{R}^{n+1} \mid \textstyle\sum_{x_i \in S} a_i \geq (|S|-1)a_{n+1};\ \forall S \text{ independent set of } G\right\}.$$

Proof (a) Let $x^a \in R$ and let $S = \mathrm{supp}(x^a)$. Then x^a is a minimal generator of $I_c(G')$ if and only if S is a minimal vertex cover of G', which happens if and only if $X \setminus S$ is a maximal independent set of G', which in turn happens if and only if $\langle X \setminus S\rangle$ is a maximal complete subgraph of G. Thus the equality holds. (b) By Theorem 2.1 the graph G' is perfect. Hence the equality follows from Proposition 2.2. □

Let A be an integral matrix. The system $x \geq 0;\ xA \leq \mathbf{1}$ is called *totally dual integral* (TDI) if the minimum in the LP-duality equation

$$\max\{\langle \alpha, x\rangle \mid x \geq 0; xA \leq \mathbf{1}\} = \min\{\langle y, \mathbf{1}\rangle \mid y \geq 0; Ay \geq \alpha\} \tag{4}$$

has an integral optimum solution y for each integral vector α with finite minimum.

An incidence matrix A of a clutter is called *perfect* if the polytope defined by the system $x \geq 0;\ xA \leq \mathbf{1}$ is integral, i.e., it has only integral vertices. The *vertex-clique matrix* of a graph G is the $\{0,1\}$-matrix whose rows are indexed by the vertices of G and whose columns are the incidence vectors of the maximal cliques of G.

Theorem 2.4 ([5, 15]) *Let* A *be the incidence matrix of a clutter. Then the following are equivalent*:

(a) *The system* $x \geq 0;\ xA \leq \mathbf{1}$ *is* TDI.
(b) A *is perfect.*
(c) A *is the vertex-clique matrix of a perfect graph.*

Proposition 2.5 *Let* A *be an* $n \times q$ *matrix with entries in* $\mathbb{N}$ *and let* $v_1, \ldots, v_q$ *be its column vectors. Then the system* $x \geq 0;\ xA \leq \mathbf{1}$ *is* TDI *if and only if*

(i) *the polyhedron* $\{x \mid x \geq 0;\ xA \leq \mathbf{1}\}$ *is integral, and*
(ii) $\mathbb{R}_+\mathcal{B} \cap \mathbb{Z}^{n+1} = \mathbb{N}\mathcal{B}$, *where* $\mathcal{B} = \{(v_1, 1), \ldots, (v_q, 1), -e_1, \ldots, -e_n\}$.

Proof $\Rightarrow$) By [17, Corollary 22.1c] we get that (i) holds. To prove (ii) take $(\alpha, b) \in \mathbb{R}_+\mathcal{B} \cap \mathbb{Z}^{n+1}$, where $\alpha \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. By hypothesis the minimum in Eq. (4) has an integral optimum solution $y = (y_i)$ such that $|y| = \langle y, \mathbf{1}\rangle \leq b$. Since $y \geq 0$ and $\alpha \leq Ay$ we can write

$$\begin{aligned} \alpha &= y_1v_1 + \cdots + y_qv_q - \delta_1e_1 - \cdots - \delta_ne_n \quad (\delta_i \in \mathbb{N}) \implies \\ (\alpha, b) &= y_1(v_1, 1) + \cdots + y_{q-1}(v_{q-1}, 1) + (y_q + b - |y|)(v_q, 1) - (b - |y|)v_q - \delta, \end{aligned}$$

where $\delta = (\delta_i)$. As the entries of A are in $\mathbb{N}$, the vector $-v_q$ can be written as a non-negative integer combination of $-e_1, \ldots, -e_n$. Thus $(\alpha, b) \in \mathbb{N}\mathcal{B}$. This proves (ii).

$\Leftarrow$) Assume that the system $x \geq 0; xA \leq \mathbf{1}$ is not TDI. Then there exists an $\alpha_0 \in \mathbb{Z}^n$ such that if y_0 is an optimal solution of the linear program:

$$\min\{\langle y, \mathbf{1}\rangle \mid y \geq 0;\ Ay \geq \alpha_0\}, \tag{5}$$

then y_0 is not integral. We claim that also the optimal value $|y_0| = \langle y_0, \mathbf{1}\rangle$ of this linear program is not integral. If $|y_0|$ is integral, then $(\alpha_0, |y_0|)$ is in $\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{B}$. Hence by (ii), we get that $(\alpha_0, |y_0|)$ is in $\mathbb{N}\mathcal{B}$, but this readily yields that the linear program of Eq. (5) has an integral optimal solution, a contradiction. This completes the proof of the claim. Consider the dual linear program:

$$\max\{\langle x, \alpha_0\rangle \,|\, x \geq 0,\ xA \leq \mathbf{1}\}.$$

Its optimal value is attained at a vertex x_0 of $\{x \,|\, x \geq 0;\ xA \leq \mathbf{1}\}$. Then by LP duality we get $\langle x_0, \alpha_0\rangle = |y_0| \notin \mathbb{Z}$. Hence x_0 is not integral, a contradiction to the integrality of $\{x \,|\, x \geq 0;\ xA \leq \mathbf{1}\}$. □

Remark 2.6 If A is a matrix with entries in $\mathbb{Z}$ satisfying (i) and (ii), then the system $x \geq 0;\ xA \leq \mathbf{1}$ is TDI.

Let $v_1, \ldots, v_q$ be a set of points in $\mathbb{N}^n$ and let $P = \mathrm{conv}(v_1, \ldots, v_q)$. The *Ehrhart ring* of the lattice polytope P is the K-subring of $R[t]$ given by

$$A(P) = K[\{x^a t^b \,|\, a \in bP \cap \mathbb{Z}^n\}].$$

Proposition 2.7 *Let A be a perfect matrix with column vectors $v_1, \ldots, v_q$. If there is $x_0 \in \mathbb{R}^n$ such that all the entries of x_0 are positive and $\langle v_i, x_0\rangle = 1$ for all i, then $A(P) = K[x^{v_1}t, \ldots, x^{v_q}t]$.*

Proof Let $x^a t^b \in A(P)$. Then we can write $(a, b) = \sum_{i=1}^{q} \lambda_i (v_i, 1)$, where $\lambda_i \geq 0$ for all i. Hence $\langle a, x_0\rangle = b$. By Theorem 2.4 the system $x \geq 0; xA \leq \mathbf{1}$ is TDI. Hence applying Proposition 2.5(ii) we have:

$$(a, b) = \eta_1(v_1, 1) + \cdots + \eta_q(v_q, 1) - \delta_1 e_1 - \cdots - \delta_n e_n \quad (\eta_i \in \mathbb{N};\ \delta_i \in \mathbb{N}).$$

Consequently $b = \langle a, x_0\rangle = b - \delta_1\langle x_0, e_1\rangle - \cdots - \delta_n\langle x_0, e_n\rangle$. Using that $\langle x_0, e_i\rangle > 0$ for all i, we conclude that $\delta_i = 0$ for all i, i.e., $x^a t^b \in K[x^{v_1}t, \ldots, x^{v_q}t]$. □

Recall that the clutter $\mathcal{C}$ (or the edge ideal $I(\mathcal{C})$) is called *unmixed* if all the minimal vertex covers of $\mathcal{C}$ have the same cardinality.

Corollary 2.8 *If G is a perfect unmixed graph and $v_1, \ldots, v_q$ are the incidence vectors of the maximal independent sets of G, then $K[x^{v_1}t, \ldots, x^{v_q}t]$ is normal.*

Proof The minimal vertex covers of G are exactly the complements of the maximal independent sets of G. Thus $|v_i| = d$ for all i, where $d = \dim(R/I(G))$. On the other hand the maximal independent sets of G are exactly the maximal cliques of G'. Thus, by Theorem 2.4 and Proposition 2.7, the subring $K[x^{v_1}t, \ldots, x^{v_q}t]$ is an Ehrhart ring, and consequently it is normal. □

Let $\mathcal{C}$ be a clutter and let A be its incidence matrix. The clutter $\mathcal{C}$ satisfies the *max-flow min-cut* (MFMC) property if both sides of the LP-duality equation

$$\min\{\langle \alpha, x\rangle \,|\, x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0; Ay \leq \alpha\}$$

have integral optimum solutions x and y for each non-negative integral vector α, see [6]. Let I be the edge ideal of $\mathcal{C}$. Closely related to $\mathbb{R}_+(I)$ is the *set covering polyhedron*:

$$Q(A)=\{x\in\mathbb{R}^n \mid x\geq 0,\ xA\geq \mathbf{1}\},$$

see [10, Theorem 3.1]. Its integral vertices are precisely the incidence vectors of the minimal vertex covers of $\mathcal{C}$ [10, Proposition 2.2].

Corollary 2.9 *Let $\mathcal{C}$ be a clutter and let A be its incidence matrix. If all the edges of $\mathcal{C}$ have the same cardinality and the polyhedra*

$$\{x\,|\,x\geq 0;\ xA\leq \mathbf{1}\}\ \textit{and}\ \{x\,|\,x\geq 0;\ xA\geq \mathbf{1}\}$$

are integral, then $\mathcal{C}$ has the max-flow min-cut property.

Proof By [10, Proposition 4.4 and Theorem 4.6] we have that $\mathcal{C}$ has the max-flow min-cut property if and only if $Q(A)$ is integral and $K[x^{v_1}t,\ldots,x^{v_q}t]=A(P)$, where $v_1,\ldots,v_q$ are the column vectors of A and $P=\mathrm{conv}(v_1,\ldots,v_q)$. Thus the result follows from Proposition 2.7. □

The *clique clutter* of a graph G, denoted by $\mathrm{cl}(G)$, is the clutter on $V(G)$ whose edges are the maximal cliques of G.

Theorem 2.10 *If G is a perfect graph, then $R[I_c(G)t]$ is normal.*

Proof Let G' be the complement of G and let $J'=I_c(G')$. Since G' is perfect it suffices to prove that $R[J't]$ is normal.

Case (A): Assume that all the maximal cliques of G have the same number of elements. Let $F=\{x^{v_1},\ldots,x^{v_q}\}$ be the set of monomials of R whose support is a maximal clique of G. We set $F'=\{x^{w_1},\ldots,x^{w_q}\}$, where $x^{w_i}=x_1\cdots x_n/x^{v_i}$. By Lemma 2.3(a) we have $J'=(F')$. Consider the matrices

$$B=\begin{pmatrix} v_1 & \cdots & v_q\\ 1 & \cdots & 1\end{pmatrix}\quad\text{and}\quad B'=\begin{pmatrix} w_1 & \cdots & w_q\\ 1 & \cdots & 1\end{pmatrix},$$

where the v_i's and w_j's are regarded as column vectors. Using the last row of B as a pivot it is seen that B is equivalent over $\mathbb{Z}$ to B'. Let A be the incidence matrix of $\mathrm{cl}(G)$, the clique clutter of G, whose columns are $v_1,\ldots,v_q$. As the matrix A is perfect, by Proposition 2.7, we obtain that $K[Ft]=A(P)$, where $A(P)$ is the Ehrhart ring of $P=\mathrm{conv}(v_1,\ldots,v_q)$. In particular $K[Ft]$ is normal because Ehrhart rings are normal. According to [8, Theorem 3.9] we have that $K[Ft]=A(P)$ if and only if $K[Ft]$ is normal and B diagonalizes over $\mathbb{Z}$ to an "identity" matrix. Consequently the matrix B' diagonalizes to an identity matrix along with B. Since the rings $K[F't]$ and $K[Ft]$ are isomorphic, we get that $K[F't]$ is normal. Thus, again by [8, Theorem 3.9], we obtain the equality $K[F't]=A(P')$, where $A(P')$ is the Ehrhart ring of $P'=\mathrm{conv}(w_1,\ldots,w_q)$. Let H_a^+ be any of the halfspaces that occur in the irreducible representation of the Rees cone $\mathbb{R}_+(J')$. By Lemma 2.3(b) the first n entries of a are either 0 or 1. Hence by [10, Proposition 4.2] we get the equality

$$A(P')[x_1,\dots,x_n]=\overline{R[J't]}.$$

Therefore $R[J't]=K[F't][x_1,\dots,x_n]=A(P')[x_1,\dots,x_n]=\overline{R[J't]}$, that is, $R[J't]$ is normal.

Case (B): Assume that not all the maximal cliques of G have the same number of elements. Let C be a maximal clique of G of lowest size and let w be its incidence vector. For simplicity of notation assume that $C=\{x_1,\dots,x_r\}$. Let $z=x_{n+1}\notin V(G)$ be a new vertex. We construct a new graph H as follows. Its vertex set is $V(H)=V(G)\cup\{z\}$ and its edge set is

$$E(H)=E(G)\cup\{\{z,x_1\},\dots,\{z,x_r\}\}.$$

Notice that $C\cup\{z\}$ is the only maximal clique of H containing z. Thus it is seen that the edges of the clique clutter of H are related to those of the clique clutter of G as follows:

$$E(\mathrm{cl}(H))=(E(\mathrm{cl}(G))\setminus\{C\})\cup\{C\cup\{z\}\}.$$

From the proof of [7, Proposition 5.5.2] it follows that if we paste together G and the complete subgraph induced by $C\cup\{z\}$ along the complete subgraph induced by C we obtain a perfect graph, i.e., H is perfect. This construction is different from the famous Lovász replication of a vertex, as explained in [6, Lemma 3.3]. The contraction of $\mathrm{cl}(H)$ at z, denoted by $\mathrm{cl}(H)/z$, is the clutter of minimal elements of $\{S\setminus\{z\}\,|\,S\in\mathrm{cl}(H)\}$. In our case we have $\mathrm{cl}(H)/z=\mathrm{cl}(G)$, i.e., $\mathrm{cl}(G)$ is a minor of $\mathrm{cl}(H)$ obtained by contraction. By successively adding new vertices $z_1=z,z_2,\dots,z_r$, following the construction above, we obtain a perfect graph H whose maximal cliques have the same size and such that $\mathrm{cl}(G)$ is a minor of $\mathrm{cl}(H)$ obtained by contraction of the vertices $z_1,\dots,z_s$. By case (A) we obtain that the ideal $L=I_c(H')$ of minimal vertex covers of H' is normal. Since L is generated by all the square-free monomials m of $R[z_1,\dots,z_s]$ such that $V(H)\setminus\mathrm{supp}(m)$ is a maximal clique of H, it follows that J' is obtained from L by making $z_i=1$ for all i. Hence $R[J't]$ is normal because the normality property of Rees algebras of edge ideals is closed under taking minors [9, Proposition 4.3]. □

Example 2.11 If G is a pentagon, then the Rees algebra of $I_c(G)$ is normal and G is not perfect.

Corollary 2.12 *If G is perfect and unmixed, then $R[I_c(G)t]$ is a Gorenstein standard graded K-algebra.*

Proof Let g be the height of the edge ideal $I(G)$ and let $J=I_c(G)$. By assigning $\deg(x_i)=1$ and $\deg(t)=-(g-1)$, the Rees algebra $R[Jt]$ becomes a graded K-algebra generated by monomials of degree 1. The Rees ring $R[Jt]$ is a normal domain by Theorem 2.10. Then according to a formula of Danilov-Stanley [3, Theorem 6.3.5] its canonical module is the ideal of $R[Jt]$ given by

$$\omega_{R[Jt]}=(\{x_1^{a_1}\cdots x_n^{a_n}t^{a_{n+1}}\,|\,a=(a_i)\in\mathbb{R}_+(J)^{\mathrm{o}}\cap\mathbb{Z}^{n+1}\}),$$

where $\mathbb{R}_+(J)^{\mathrm{o}}$ denotes the topological interior of the Rees cone of J. By a result of Hochster [13] the ring $R[Jt]$ is Cohen-Macaulay. Using Eq. (3) it is seen that the vector $(1,\dots,1)$ is in the interior of the Rees cone, i.e., $x_1\cdots x_nt$ belongs to $\omega_{R[Jt]}$. Take an arbitrary monomial $x^at^b=x_1^{a_1}\cdots x_n^{a_n}t^b$ in the ideal $\omega_{R[Jt]}$, that is $(a,b)\in\mathbb{R}_+(J)^{\mathrm{o}}$. Hence the vector (a,b) has positive integer entries and satisfies

$$\textstyle\sum_{x_i\in\mathcal{K}_r} a_i \geq (r-1)b+1 \tag{6}$$

for every complete subgraph $\mathcal{K}_r$ of G. If $b=1$, clearly x^at^b is a multiple of $x_1\cdots x_nt$. Now assume $b\geq 2$. Using the normality of $R[Jt]$ and Eqs. (3) and (6) it follows that the monomial $m=x_1^{a_1-1}\cdots x_n^{a_n-1}t^{b-1}$ belongs to $R[Jt]$. Since $x^at^b=mx_1\cdots x_nt$, we obtain that $\omega_{R[Jt]}$ is generated by $x_1\cdots x_nt$ and thus $R[Jt]$ is a Gorenstein ring. □

A graph G is *chordal* if every cycle of G of length $n\geq 4$ has a chord. A *chord* of a cycle is an edge joining two non adjacent vertices of the cycle.

Corollary 2.13 *If J is a Cohen-Macaulay square-free monomial ideal of height two, then $R[Jt]$ is normal.*

Proof Consider the graph G whose edges are the pairs $\{x_i,x_j\}$ such that (x_i,x_j) is a minimal prime of J. Notice that $J=I_c(G)$. By [20, Theorem 6.7.13], the ideal $I_c(G)$ is Cohen-Macaulay if and only if G' is a chordal graph. Since chordal graphs are perfect [7, Proposition 5.5.2], we obtain that G' is perfect. Thus G is a perfect graph by Theorem 2.1. Applying Theorem 2.10 we conclude that $R[Jt]$ is normal. □

Recall that a matrix with $\{0,1\}$-entries is called *balanced* if A has no square submatrix of odd order with exactly two 1's in each row and column,

Proposition 2.14 *Let A be a $\{0,1\}$-matrix with column vectors $v_1,\dots,v_q$ and let $w_i=\mathbf{1}-v_i$. If A is balanced, then the Rees algebra of $I^*=(x^{w_1},\dots,x^{w_q})$ is a normal domain.*

Proof According to [2], [18, Corollary 83.1a(vii), p. 1441] A is balanced if and only if every submatrix of A is perfect. By adjoining rows of unit vectors to A and since the normality property of edge ideals is closed under taking minors [9, Proposition 4.3] we may assume that $|v_i|=d$ for all i. By Theorem 2.4 there is a perfect graph G such that A is the vertex-clique matrix of G. Thus following the first part of the proof of Theorem 2.10, we obtain that $R[I^*t]$ is normal. □

Consider the ideals $I=(x^{v_1},\dots,x^{v_q})$ and $I^*=(x^{w_1},\dots,x^{w_q})$. Following the terminology of matroid theory we call I^* the *dual* of I. Notice the following duality. If A is the vertex-clique matrix of a graph G, then I^* is precisely the ideal of vertex covers of G'.

3 Symbolic Rees algebras of edge ideals

Let G be a graph with vertex set $X=\{x_1,\dots,x_n\}$ and let $I=I(G)$ be its edge ideal [20, Chapter 6]. The main purpose of this section is to study the symbolic Rees algebra of I and the Simis cone of I when G is a perfect graph. We show that the cliques of a perfect graph G completely determine both the Hilbert basis of the Simis cone and the symbolic Rees algebra of $I(G)$.

Definition 3.1 Let $C_1,\dots,C_s$ be the minimal vertex covers of G. The *symbolic Rees cone* or *Simis cone* of I is the rational polyhedral cone:

$$\mathrm{Cn}(I)=H_{e_1}^+\cap\cdots\cap H_{e_{n+1}}^+\cap H_{(u_1,-1)}^+\cap\cdots\cap H_{(u_s,-1)}^+,$$

where $u_k=\sum_{x_i\in C_k}e_i$ for $1\le k\le s$.

Simis cones were introduced in [9] to study symbolic Rees algebras of square-free monomial ideals. If $\mathcal{H}$ is an integral Hilbert basis of $\mathrm{Cn}(I)$, then $R_s(I(G))$ equals $K[\mathbb{N}\mathcal{H}]$, the semigroup ring of $\mathbb{N}\mathcal{H}$ (see [9]). This result is interesting because it allows us to compute the minimal generators of $R_s(I(G))$ using Hilbert basis. Next we describe $\mathcal{H}$ when G is perfect.

Theorem 3.2 *Let $\omega_1,\dots,\omega_p$ be the incidence vectors of the non-empty cliques of a perfect graph G and let*

$$\mathcal{H}=\{(\omega_1,|\omega_1|-1),\dots,(\omega_p,|\omega_p|-1)\}.$$

Then $\mathbb{N}\mathcal{H}=\mathrm{Cn}(I)\cap\mathbb{Z}^{n+1}$, where $\mathbb{N}\mathcal{H}$ is the subsemigroup of $\mathbb{N}^{n+1}$ generated by $\mathcal{H}$, that is, $\mathcal{H}$ is the integral Hilbert basis of $\mathrm{Cn}(I)$.

Proof The inclusion $\mathbb{N}\mathcal{H}\subset\mathrm{Cn}(I)\cap\mathbb{Z}^{n+1}$ is clear because each clique of size r intersects any minimal vertex cover in at least $r-1$ vertices. Let us show the reverse inclusion. Let (a,b) be a minimal generator of $\mathrm{Cn}(I)\cap\mathbb{Z}^{n+1}$, where $0\neq a=(a_i)\in\mathbb{N}^n$ and $b\in\mathbb{N}$. Then

$$\textstyle\sum_{x_i\in C_k}a_i=\langle a,u_k\rangle\ge b, \tag{7}$$

for all k. If $b=0$ or $b=1$, then $(a,b)=e_i$ for some $i\le n$ or $(a,b)=(e_i+e_j,1)$ for some edge $\{x_i,x_j\}$ respectively. In both cases $(a,b)\in\mathcal{H}$. Thus we may assume that $b\ge 2$ and $a_j\ge 1$ for some j. Using Eq. (7) we obtain

$$\sum_{x_i\in C_k}a_i+\sum_{x_i\in X\setminus C_k}a_i=|a|\ge b+\sum_{x_i\in X\setminus C_k}a_i=b+\langle\mathbf{1}-u_k,a\rangle, \tag{8}$$

for all k, where $X=\{x_1,\dots,x_n\}$ is the vertex set of G. Set $c=|a|-b$. Notice that $c\ge 1$ because $a\neq 0$. Indeed if $c=0$, from Eq. (8) we get $\sum_{x_i\in X\setminus C_k}a_i=0$ for all k, i.e., $a=0$, a contradiction. Consider the vertex-clique matrix of G':

$$A'=(\mathbf{1}-u_1\cdots\mathbf{1}-u_s),$$

where $\mathbf{1}-u_1,\ldots,\mathbf{1}-u_s$ are regarded as column vectors. From Eq. (8) we get $(a/c)A' \le \mathbf{1}$. Hence by Theorem 2.1(c) we obtain that a/c belongs to $\mathrm{conv}(\omega_0,\omega_1,\ldots,\omega_p)$, where $\omega_0=0$, i.e., we can write $a/c=\lambda_0\omega_0+\cdots+\lambda_p\omega_p$, where $\lambda_i\ge 0$ for all i and $\sum_i\lambda_i=1$. Thus we can write

$$(a,c)=c\lambda_0(\omega_0,1)+\cdots+c\lambda_p(\omega_p,1).$$

Using Theorem 2.4(a) it is not hard to see that the subring $K[\{x^{\omega_i}t|0\le i\le p\}]$ is normal. Hence there are $\eta_0,\ldots,\eta_p$ in $\mathbb{N}$ such that

$$(a,c)=\eta_0(\omega_0,1)+\cdots+\eta_p(\omega_p,1).$$

Thus $|a|=\eta_0|\omega_0|+\cdots+\eta_p|\omega_p|$ and $c=\eta_0+\cdots+\eta_p=|a|-b$, consequently:

$$(a,b)=\eta_0(\omega_0,|\omega_0|-1)+\eta_1(\omega_1,|\omega_1|-1)+\cdots+\eta_p(\omega_p,|\omega_p|-1).$$

Notice that there is u_ℓ such that $\langle a,u_\ell\rangle=b$; otherwise since $a_j\ge 1$, by Eq. (7) the vector $(a,b)-e_j$ would be in $\mathrm{Cn}(I)\cap\mathbb{Z}^{n+1}$, contradicting the minimality of (a,b). Therefore from the equality

$$0=\langle(a,b),(u_\ell,-1)\rangle=\eta_0+\textstyle\sum_{i=1}^p\eta_i\langle(\omega_i,|\omega_i|-1),(u_\ell,-1)\rangle$$

we conclude that $\eta_0=0$, i.e., $(a,b)\in\mathbb{N}\mathcal{H}$, as required. □

Corollary 3.3 *If G is a perfect graph, then*

$$R_s(I(G))=K[x^at^r\,|\,x^a \textit{ is square-free};\ \langle\mathrm{supp}(x^a)\rangle=\mathcal{K}_{r+1};\ 0\le r<n].$$

Proof Let $K[\mathbb{N}\mathcal{H}]$ be the semigroup ring with coefficients in K of the semigroup $\mathbb{N}\mathcal{H}$. By [9, Theorem 3.5] we have the equality $R_s(I(G))=K[\mathbb{N}\mathcal{H}]$, thus the formula follows from Theorem 3.2. □

Corollary 3.4 ([1]) *If G is a complete graph, then*

$$R_s(I(G))=K[x^at^r\,|\,x^a \textit{ is square-free};\ \deg(x^a)=r+1;\ r\ge 0].$$

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