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Biplanes with flag-transitive automorphism groups of almost simple type, with exceptional socle of Lie type

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Abstract In this paper we prove that there is no biplane admitting a flag-transitive automorphism group of almost simple type, with exceptional socle of Lie type. A biplane is a $(v,k,2)$-symmetric design, and a flag is an incident point-block pair. A group G is almost simple with socle X if X is the product of all the minimal normal subgroups of G, and $X \trianglelefteq G \leq \text{Aut}\,(G)$.

Throughout this work we use the classification of finite simple groups, as well as results from P.B. Kleidman's Ph.D. thesis which have not been published elsewhere.

Keywords Automorphism group · Flag-transitive · Primitive group · Symmetric design

1 Introduction

A *biplane* is a $(v,k,2)$-symmetric design, that is, an incidence structure of v points and v blocks such that every point is incident with exactly k blocks, and every pair of blocks is incident with exactly two points. Points and blocks are interchangeable in the previous definition, due to their dual role. A *nontrivial* biplane is one in which $1 < k < v - 1$. A *flag* of a biplane D is an ordered pair (p, B) where p is a point of D, B is a block of D, and they are incident. Hence if G is an automorphism group of D, then G is *flag-transitive* if it acts transitively on the flags of D.

The only values of k for which examples of biplanes are known are $k = 3$, 4, 5, 6, 9, 11, and 13 [7, p. 76]. Due to arithmetical restrictions on the parameters, there are no examples with $k = 7$, 8, 10, or 12.

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For $k = 3$, 4, and 5 the biplanes are unique up to isomorphism [5], for $k = 6$ there are exactly three non-isomorphic biplanes [11], for $k = 9$ there are exactly four non-isomorphic biplanes [31], for $k = 11$ there are five known biplanes [3, 9, 10], and for $k = 13$ there are two known biplanes [1], in this case, it is a biplane and its dual.

In [28] it is shown that if a biplane admits an imprimitive, flag-transitive automorphism group, then it has parameters (16,6,2). There are three non-isomorphic biplanes with these parameters [4], two of which admit flag-transitive automorphism groups which are imprimitive on points, (namely $2^4 S_4$ and $(\mathbb{Z}_2 \times \mathbb{Z}_8)S_4$ [28]). Therefore, if any other biplane admits a flag-transitive automorphism group G, then G must be primitive. The O'Nan-Scott Theorem classifies primitive groups into five types [17]. It is shown in [28] that if a biplane admits a flag-transitive, primitive, automorphism group, it can only be of affine or almost simple type. The affine case was treated in [28]. The almost simple case when the socle of G is an alternating or a sporadic group was treated in [29], in which it is shown that no such biplane exists. The almost simple case with classical socle was treated in [30] where it was shown that if such a biplane exists, it must have parameters (7,4,2) or (11,4,2) and is unique up to isomorphism. In this paper we treat the almost simple case when the socle X of G is an exceptional group of Lie type, and we prove that no such biplane exists, namely:

Theorem 1 (Main) *There is no biplane admitting a flag-transitive, primitive almost simple automorphism group with exceptional socle of Lie type.*

In [30] the proof for biplanes follows the proof given in [32] for linear spaces. The last section in [32] is an appendix on exceptional groups of Lie type, the presentation of which is also followed here.

2 Preliminary results

In this section we state some results that we will use in the proof of our Main Theorem.

Lemma 1 *If D is a $(v, k, 2)$-biplane, then $8v - 7$ is a square.*

Proof The result follows from [28, Lemma 3]. □

Corollary 2 *If D is a flag-transitive $(v, k, 2)$-biplane, then $2v < k^2$, and hence $2|G| < |G_x|^3$.*

Proof Fix a point x in the biplane. Now count flags (p, B) where $p \neq x$ and x is incident with the block B. On one hand, there are $(v - 1)$ points different from x and each of them is, together with x, in 2 blocks, so there are $2(v - 1)$ such flags. On the other hand, there are k blocks through x, and each of them has $k - 1$ points different from x, that yields $k(k - 1)$ flags. So $k(k - 1) = 2(v - 1)$. The equality $k(k - 1) = 2(v - 1)$ implies $k^2 = 2v - 2 + k$, so clearly $2v < k^2$. Since $v = |G : G_x|$, and $k \leq |G_x|$, the result follows. □

Lemma 3 (Tits Lemma) [33, 1.6] *If X is a simple group of Lie type in characteristic p, then any proper subgroup of index prime to p is contained in a parabolic subgroup of X.*

Lemma 4 *If X is a simple group of Lie type in characteristic 2, ($X \not\cong A_5$ or A_6), then any proper subgroup H such that $[X:H]_2 \le 2$ is contained in a parabolic subgroup of X.*

Proof First assume that $X = Cl_n(q)$ is classical (q a power of 2), and take H maximal in X. By a theorem of Aschbacher [2], H is contained in a member of the collection $\mathcal{C}$ of subgroups of $\Gamma L_n(q)$, or in $\mathcal{S}$, that is, $H^{(\infty)}$ is quasisimple, absolutely irreducible, not realizable over any proper subfield of $\mathbb{F}_q$. (For a more precise description of this collection of subgroups, see [14].)

We check for every family $\mathcal{C}_i$ that if H is contained in C_i, then $2|H|_2 < |X|_2$, except when H is parabolic.

Now we take $H \in \mathcal{S}$. Then by [15, Theorem 4.2], $|H| < q^{2n+4}$, or H and X are as in [15, Table 4]. If $|X|_2 \le 2|H|_2 \le q^{2n+4}$, then if $X = L_n^{\epsilon}(q)$ we have $n \le 6$, and if $X = SP_n(q)$ or $P\Omega_n^{\epsilon}(q)$ then $n \le 10$. We check the list of maximal subgroups of X for $n \le 10$ in [12, Chapter 5], and we see that no group H satisfies $2|H|_2 \le |X|_2$. We then check the list of groups in [15, Table 4], and again, none of them satisfy this bound.

Finally, assume X to be an exceptional group of Lie type in characteristic 2. Then by [20], if $2|H| \ge |X|_2$, H is either contained in a parabolic subgroup, or H and X are as in [20, Table 1]. Again, we check all the groups in [20, Table 1], and in all cases $2|H|_2 < |X|_2$. □

As a consequence, we have a strengthening of Corollary 2:

Corollary 5 *Suppose D is a biplane with a primitive, flag-transitive almost simple automorphism group G with simple socle X of Lie type in characteristic p, and the stabilizer G_x is not a parabolic subgroup of G. If p is odd then p does not divide k; and if $p = 2$ then 4 does not divide k. Hence $|G| < 2|G_x||G_x|_{p'}^2$.*

Proof We know from Corollary 2 that $|G| < |G_x|^3$. Now, by Lemma 3, p divides $v = [G : G_x]$. Since k divides $2(v-1)$, if p is odd then $(k, p) = 1$, and if $p = 2$ then $(k, p) \le 2$. Hence k divides $2|G_x|_{p'}$, and since $2v < k^2$, we have $|G| < 2|G_x||G_x|_{p'}^2$. □

From the previous results we have the following lemma, which will be quite useful throughout this paper:

Lemma 6 *Suppose p divides v, and G_x contains a normal subgroup H of Lie type in characteristic p which is quasisimple and $p \nmid |Z(H)|$; then k is divisible by $[H : P]$, for some parabolic subgroup P of H.*

Proof As p divides v, and k divides $2(v-1)$ we have $(k, p) \le (2, p)$. Also, we have $k = [G_x : G_{x,B}]$ (where B is a block incident with x), so $[H : H_B]$ divides k, and therefore $([H : H_B], p) \le (2, p)$. This, and Lemmas 3 and 4 imply H_B is contained in a parabolic subgroup P of G_x, and since P is maximal, we have $G_{x,B}$ is contained in P, so k is divisible by $[G_x : P]$. □

We will also use the following two lemmas:

Lemma 7 [18] *If X is a simple group of Lie type in odd characteristic, and X is neither $PSL_d(q)$ nor $E_6(q)$, then the index of any parabolic subgroup is even.*

Lemma 8 [22, 3.9] *If X is a group of Lie type in characteristic p, acting on the set of cosets of a maximal parabolic subgroup, and X is not $PSL_d(q)$, $P\Omega^+_{2m}(q)$ (with m odd), nor $E_6(q)$, then there is a unique subdegree which is a power of p.*

Before stating the next result, we give the following [21]:

Definition 9 Let H be a simple adjoint algebraic group over an algebraically closed field of characteristic $p > 0$, and let σ be an endomorphism of H such that $X = (H_\sigma)'$ is a finite simple exceptional group of Lie type over $\mathbb{F}_q$, where $(q = p^a)$. Let G be a group such that $\mathrm{Soc}(G) = X$. The group $\mathrm{Aut}(X)$ is generated by H_σ, together with field and graph automorphisms. If D is a σ-stable closed connected reductive subgroup of H containing a maximal torus T of H, and $M = N_G(D)$, then we call M a *subgroup of maximal rank* in G.

We now have the following theorem and table [24, Theorem 2, Table III]:

Theorem 10 *If X is a finite simple exceptional group of Lie type such that $X \le G \le$ $\mathrm{Aut}(X)$, and G_x is a maximal subgroup of G such that $X_0 = \mathrm{Soc}(G_x)$ is not simple, then one of the following holds*:

(1) *G_x is parabolic.*
(2) *G_x is of maximal rank.*
(3) *$G_x = N_G(E)$, where E is an elementary Abelian group given in* [6, *Theorem* 1(*II*).].
(4) *$X = E_8(q)$, $(p > 5)$, and X_0 is either $A_5 \times A_6$ or $A_5 \times L_2(q)$.*
(5) *X_0 is as in Table 1.*

We will also use the following theorem [23, Theorem 3]:

Theorem 11 *Let X be a finite simple exceptional group of Lie type, with $X \le G \le$ $\mathrm{Aut}(X)$. Assume G_x is a maximal subgroup of G, and $\mathrm{Soc}(G_x) = X_0(q)$ is a simple group of Lie type over $\mathbb{F}_q$ $(q > 2)$ such that $\frac{1}{2}\mathrm{rk}(X) < \mathrm{rk}(X_0)$. Then one of the following holds*:

(1) *G_x is a subgroup of maximal rank.*
(2) *X_0 is a subfield or twisted subgroup.*
(3) *$X = E_6(q)$ and $X_0 = C_4(q)$ (q odd) or $F_4(q)$.*

Table 1

X	X_0
$F_4(q)$	$L_2(q) \times G_2(q)$ $(p > 2, q > 3)$
$E_6^\epsilon(q)$	$L_3(q) \times G_2(q)$, $U_3(q) \times G_2(q)$ $(q > 2)$
$E_7(q)$	$L_2(q) \times L_2(q)$ $(p > 3)$, $L_2(q) \times G_2(q)$ $(p > 2, q > 3)$
	$L_2(q) \times F_4(q)$ $(q > 3)$, $G_2(q) \times PSp_6(q)$
$E_8(q)$	$L_2(q) \times L_3^\epsilon(q)$ $(p > 3)$, $G_2(q) \times F_4(q)$
	$L_2(q) \times G_2(q) \times G_2(q)$ $(p > 2, q > 3)$, $L_2(q) \times G_2(q^2)$ $(p > 2, q > 3)$

Finally, we will use the following theorem [26, Theorem 1.2]:

Theorem 12 *Let X be a finite exceptional group of Lie type such that $X \le G \le \mathrm{Aut}(X)$, and G_x a maximal subgroup of G with socle $X_0 = X_0(q)$ a simple group of Lie type in characteristic p. Then if $\mathrm{rk}(X_0) \le \frac{1}{2}\mathrm{rk}(X)$, we have the following bounds*:

(1) *If $X = F_4(q)$ then $|G_x| < q^{20}.4\log_p(q)$,*

(2) *If $X = E_6^\epsilon$ then $|G_x| < q^{28}.4\log_p(q)$,*

(3) *If $X = E_7(q)$ then $|G_x| < q^{30}.4\log_p(q)$, and*

(4) *If $X = E_8(q)$ then $|G_x| < q^{56}.12\log_p(q)$.*

In all cases, $|G_x| < |G|^{\frac{5}{13}}.5\log_p(q)$.

3 Proof of our main theorem

Lemma 13 *The group X is not a Suzuki group ${}^2B_2(q)$, with $q = 2^{2e+1}$.*

Proof Suppose that the socle X is a Suzuki group ${}^2B_2(q)$, with $q = 2^{2e+1}$. Then $|G| = f|X| = f(q^2+1)q^2(q-1)$, where $f \mid (2e+1)$, and so the order of any point stabilizer G_x is one of the following [34]:

(1) $fq^2(q-1)$

(2) $4f(q+\sqrt{2q}+1)$

(3) $4f(q-\sqrt{2q}+1)$

(4) $f(q_0^2+1)q_0^2(q_0-1)$, where $8 \le q_0^m = q$, with $m \ge 3$.

Case (1) Here $v = (q^2+1)$, so from $k(k-1) = 2(v-1)$ we obtain $k(k-1) = 2q^2$, a power of 2, which is a contradiction.

Cases (2) *and* (3) From the inequality $|G| < |G_x|^3$, we have

$$f \cdot \frac{7}{8}q^5 < f(q^2+1)q^2(q-1) < 4^4 f^3(q \pm \sqrt{2q}+1)^3$$
$$< 4^4 f^3(2q+1)^3 \le 4^4\left(\frac{17}{8}fq\right)^3,$$

so

$$q^2 < \frac{4^4 \cdot (17)^3 \cdot f^2}{8^2 \cdot 7} < 2808 f^2,$$

hence $q \leq 128$.

First assume $q = 128$. Then $v = 58781696$ and 75427840 and $|G_x| = 4060$ and 3164 in cases (2) and (3) respectively. We know k divides $2(|G_x|, v-1)$, but here $(|G_x|, v-1) = 1015$ in case (2), and 113 in case (3). In both cases $k^2 < v$, which is a contradiction.

Next assume $q = 32$. Then $v = 198400$ and 325376 in cases (2), and (3) respectively. In case (2), $(|G - x|, v-1) = 41$, and in case (3), $(|G_x|, v-1) = 25$ or 125, depending on whether $f = 1$ or 5. In all cases we see $k^2 < v$, a contradiction.

Finally assume $q = 8$. Then $v = 560$ and 1456, and $(|G_x|, v-1) = 13$ and $5f$ in cases (2) and (3) respectively, therefore k is again too small.

Case (4) Here $|G_x| = f(q_0^2+1)q_0^2(q_0-1)$, so q_0 divides v and hence q_0 and $v-1$ are relatively prime, so from $|G| < 2|G_x||G_x|_{p'}^2$ we obtain:

$$(q_0^{2m}+1)q_0^{2m}(q_0^m-1) < 4f^2(q_0^2+1)^3 q_0^2 (q_0-1)^3.$$

Now, $q_0^{5m-1} < (q_0^{2m}+1)q_0^{2m}(q_0^m-1)$, and also

$$4f^2(q_0^2+1)^3 q_0^2(q_0-1)^3 = 4f^2 q_0^2 (q_0^3 - q_0^2 + q_0 - 1)^3 < f^2 q_0^{13},$$

so

$$q_0^{5m-1} < f^2 q_0^{13} < q_0^{13+m}.$$

Therefore $5m - 1 < 13 + m$, which forces $m = 3$. Then

$$v = (q_0^4 - q_0^2 + 1)q_0^4(q_0^2 + q_0 + 1),$$

and so $k \leq 2(|G_x|, v-1) \leq 2fq_0^3 < 2q_0^{\frac{9}{2}}$. The inequality $v < k^2$ forces $q_0 = 2$, and so $q = 8$. Then $v = 1456$, and $|G_x| = 20f$, with $f = 1$ or 3. Hence $(|G_x|, v-1) = 5f$, and therefore $k^2 < v$, which is a contradiction. □

This completes the proof of Lemma 13.

Lemma 14 *The point stabilizer G_x is not a parabolic subgroup of G.*

Proof First assume $X \neq E_6(q)$. Then by Lemma 8 there is a unique subdegree which is a power of p. Therefore k divides twice a power of p, but it also divides $2(v-1)$, so it is too small.

Now assume $X = E_6(q)$. If G contains a graph automorphism or $G_x = P_i$ with $i = 2$ or 4, then there is a unique subdegree which is a power of p and again k is too small. If $G_x = P_3$, the A_1A_4 type parabolic, then

$$v = \frac{(q^3+1)(q^4+1)(q^{12}-1)(q^9-1)}{(q^2-1)(q-1)}.$$

Since k divides $2(|G_x|, v-1)$, then k divides $2q(q^5-1)(q-1)^5 \log_p q$, and hence $k^2 < v$, which is a contradiction. If $G_x = P_1$, then

$$v = \frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)},$$

and the nontrivial subdegrees are (see [19]): $\frac{q(q^8-1)(q^3+1)}{(q-1)}$, and $\frac{q^8(q^5-1)(q^4+1)}{(q-1)}$. The fact that k divides twice the highest common factor of these forces $k^2 < v$, again, a contradiction. □

This completes the proof of Lemma 14.

Lemma 15 *The group X is not a Chevalley group $G_2(q)$.*

Proof Assume $X = G_2(q)$, with $q > 2$ since $G_2(q)' = U_3(3)$. The list of maximal subgroups of $G_2(q)$ with q odd can be found in [13], and in [8] for q even.

First consider the case where $X \cap G_x = SL_3^{\epsilon}(q).2$. Here

$$v = \frac{q^3(q^3+\epsilon)}{2}.$$

From the factorization $\Omega_7(q) = G_2(q)N_1^{\epsilon}$, ([16]), it follows that the suborbits of $\Omega_7(q)$ are unions of G_2-suborbits, and so k divides each of the Ω_7-subdegrees. Now q cannot be odd, since this is ruled out by the first case with $i = 1$ in the section of orthogonal groups of odd dimension in [30]. For q even, the subdegrees for $Sp_6(q)$, given in the last case of the section on symplectic groups in [30] are $(q^3-\epsilon)(q^4+\epsilon)$ and $\frac{(q-2)q^2(q^3-\epsilon)}{2}$. This implies that k divides $2(q^3-\epsilon)(q-2, q^2+\epsilon)$, and since $v < k^2$ then $\epsilon = -1$, and so

$$v = \frac{q^3(q^3-1)}{2}.$$

So k divides $2(q^3+1)(q-2, q^2-1) \leq 6(q^3+1)$, and $k(k-1) = 2(v-1) = (q^3+1)(q^3-2)$. This is impossible.

If $X \cap G_x = G_2(q_0) < G_2(q)$ or ${}^2G_2(q) < G_2(q)$ then p does not divide $[G_x : G_{xB}]$, so by Lemma 6, k is divisible by the index of a parabolic subgroup of G_x which is $\frac{q_0^6-1}{q_0-1}$ in the case of $G_2(q_0)$, or q^3+1 in the case of ${}^2G_2(q)$. But this is not so since k also divides $2(v-1, |G_x|)$.

If $G_x = N_G(SL_2(q) \circ SL_2(q))$, then

$$v = \frac{q^4(q^6-1)}{q^2-1}.$$

Now k divides $2(q^2-1)^2 \log_p q$ but $(q^2-1, v-1) \leq 2$, so k is too small.

If $X \cap G_x = J_2 < G_2(4)$ then $v = 416$. But k divides $2(|G_x|, 415)$, which is too small.

Now suppose $X \cap G_x = G_2(2)$, with $p = q \geq 5$. Then the inequality $v < k^2$ forces $q = 5$ or 7. In both cases $(v-1, |G_x|)$ is too small.

If $X \cap G_x = PGL_2(q)$, or $L_2(8)$, then the inequality $|G| < |G_x|^3$ is not satisfied.

Next consider $X \cap G_x = L_2(13)$. Then the inequality $|G| < |G_x|^3$ forces $q \leq 5$. If $q = 5$ then $v = 2^3 \cdot 3^2 \cdot 5^6 \cdot 13 \cdot 31$, so $(v-1, |G_x|) \leq 7$, hence k is too small. If $q = 3$ then $v = 2^3 \cdot 3^5$, and k divides $2(v-1, |G_x|) \leq 2 \cdot 7 \cdot 13$, this does not satisfy the equation $k(k-1) = 2(v-1)$.

Finally, if $X \cap G_x = J_1$ with $q = 11$ then the inequality $v < k^2$ cannot be satisfied.

There is no other maximal subgroup G_x satisfying the inequality $|G| < |G_x|$. □

This completes the proof of Lemma 15.

Lemma 16 *The group X is not a Ree group* ${}^2G_2(q)$, $(q > 3)$.

Proof Suppose $X = {}^2G_2(q)$, with $q = 3^{2e+1} > 3$. A complete list of maximal subgroups of G can be found in [13, p. 61]. First suppose $G_x \cap X = 2 \times SL_2(q)$. Then

$$v = \frac{q^2(q^2 - q + 1)}{2},$$

so $2(v-1) = q^4 - q^3 + q^2 - 2$, and k divides $2(|G_x|, v-1)$. But $(q(q^2-1), q^4 - q^3 + q^2 - 1) = q - 1$, which is too small.

The groups $X \cap G_x = N_X(S_2)$, (where S_2 is a Sylow 2-subgroup of X of order 8), of order $2^3 \cdot 3 \cdot 7$ and $L_2(8)$ are not allowed since $|G| < |G_x|^3$ forces $q = 3$.

If $X \cap G_x = {}^2G_2(q_0)$, with $q_0^m = q$ and m prime, then

$$\begin{aligned} v = {} & q_0^{3(m-1)}(q_0^{3(m-1)} - q_0^{3(m-2)} + \ldots + (-1)^m q_0^3 + (-1)^{m-1}) \\ & \times (q_0^{m-1} + q_0^{m-2} + \ldots + 1). \end{aligned}$$

Now k divides $2mq_0^3(q_0^3+1)(q_0-1)$, but since q_0 and $v-1$ are relatively prime, q_0 does not divide k, so in fact $k \leq 2m(q_0^3+1)(q_0-1)$, and the inequality $v < k^2$ forces $m = 2$, which is a contradiction.

If $X \cap G_x = \mathbb{Z}_{q \pm \sqrt{3q}+1} : \mathbb{Z}_6$, since $q \geq 27$ the inequality $|G| < |G_x|^3$ is not satisfied.

Finally, if $X \cap G_x = (2^2 \times D_{(\frac{1}{2})(q+1)}) : 3$, since $q \geq 27$ then the inequality $|G| < |G_x|^3$ is not satisfied. □

This completes the proof of Lemma 16.

Lemma 17 *The group X is not a Ree group* ${}^2F_4(q)$.

Proof Suppose $X = {}^2F_4(q)$. Then from [27] we see there are no maximal subgroups G_x that are not parabolic satisfying the inequality $|G| < 2|G_x||G_x|^2_{2'}$, except for the case $q = 2$. In this case $G_x \cap X = L_3(3).2$ or $L_2(25)$. In both cases, since k must divide $2(v-1, |G_x|)$ it is too small. □

Lemma 18 *The group X is not ${}^3D_4(q)$.*

Proof Suppose $X = {}^3D_4(q)$. If $X \cap G_x = G_2(q)$ or $SL_2(q^3) \circ SL_2(q).(2, q-1)$ then $v = q^e(q^8 + q^4 + 1)$, where $e = 6$ or 8 respectively. By Lemma 6, k is divisible by $q+1$, which forces $q = 3$ (since $q+1$ also divides $2(v-1)$), but then in neither case is $8v - 7$ a square.

If $X \cap G_x = PGL_3^{\epsilon}(q)$ then the inequality $|G| < |G_x|^3$ is not satisfied. □

Lemma 19 *The group X is not $F_4(q)$.*

Proof Suppose $X = F_4(q)$. First assume that $X_0 = \mathrm{Soc}(X \cap G_x)$ is not simple. Then by Theorem 10 and Table 1, $G_x \cap X$ is one of the following,

(1) Parabolic.
(2) Of maximal rank.
(3) $3^3.SL_3(3)$.

or $X_0 = L_2(q) \times G_2(q) (p > 2, q > 3)$.

The parabolic subgroups have been ruled out by Lemma 14.

The possibilities for the second case are given in [21, Table 5.1]. We check that in every case there is a large power of q dividing v, and since $(k, v) \le 2$, then $q \ne 2$ does not divide k. If $q = 2$, then 4 does not divide k. Therefore k divides $2(|G_x|, v-1)$, and in each case $(|G_x|_{p'}, v-1)$ is too small for k to satisfy $k^2 > v$.

The local subgroup is too small to satisfy the bound $|G_x|^3 > |G|$.

Finally, $|L_2(q) \times G_2(q)| \le q^7(q^2-1)^2(q^6-1) < |F_4(q)|^{\frac{1}{3}}$. Therefore X_0 is simple.

First suppose $X_0 \notin \mathrm{Lie}(p)$. Then by [25, Table 1], it is one of the following: $A_7, A_8, A_9, A_{10}, L_2(17), L_2(25), L_2(27), L_3(3), U_4(2), Sp_6(2), \Omega_8^+(2), {}^3D_4(2), J_2, A_{11}(p = 11), L_3(4)(p = 3), L_4(3)(p = 2), {}^2B_2(8)(p = 5), M_{11}(p = 11)$.

The only possibilities for X_0 that could satisfy the bound $|G_x|^3 > |G|$ are $A_9, A_{10}(q = 2)$, $Sp_6(2)(q = 2)$, $\Omega_8^+(2)(q = 2, 3)$, ${}^3D_4(2)(q = 3)$, $J_2(q = 2)$, and $L_4(3)(q = 2)$. However, since k divides $2(|G_x|, v-1)$, in all these cases $k^2 < v$.

Now assume $X_0 \in \mathrm{Lie}(p)$. First consider the case $\mathrm{rk}(X_0) > \frac{1}{2}\mathrm{rk}(G)$, where $X_0 = X_0(r)$. If $r > 2$, then by Theorem 11 it is a subfield subgroup. We have seen earlier that the only subgroups which could satisfy the bound $|G_x|^3 > |G|$ are $F_4(q^{\frac{1}{2}})$ and $F_4(q^{\frac{1}{3}})$. If $q_0 = q^{\frac{1}{2}}$, then

$$v = q^{12}(q^6+1)(q^4+1)(q^3+1)(q+1) > q^{26}.$$

Now k divides $2F_4(q^{\frac{1}{2}})$, and $(k, v) \le 2$. Since $(q, k) \le 2$, then k divides

$$2(2(q^6-1)(q^4-1)(q^3-1)(q-1), v-1) < q^{13},$$

so $k^2 < v$, a contradiction.

If $q_0 = q^{\frac{1}{3}}$, then

$$v = \frac{q^{16}(q^{12}-1)(q^4+1)(q^6-1)}{(q^{\frac{8}{3}}-1)(q^{\frac{2}{3}}-1)},$$

but $k < q^{10}$, so $k^2 < v$, which is a contradiction.

If $r = 2$, then the subgroups $X_0(2)$ with $\mathrm{rk}(X_0) > \frac{1}{2}\mathrm{rk}(G)$ that satisfy the bound $|G_x|^3 > |G|$ are $A_4^\epsilon(2)$, $B_3(2)$, $B_4(2)$, $C_3(2)$, $C_4(2)$, and $D_4^\epsilon(2)$. Again, in all cases the fact that k divides $2(|G_x|, v-1)$ forces $k^2 < v$, a contradiction.

Now consider the case $\mathrm{rk}(X_0) \le \frac{1}{2}\mathrm{rk}(G)$. Theorem 12 implies $|G_x| < q^{20}.4\log_p q$. Looking at the orders of groups of Lie type, we see that if $|G_x| < q^{20}.4\log_p q$, then $|G_x|_{p'} < q^{12}$, so $2|G_x||G_x|^2_{p'} < |G|$, contrary to Corollary 5. □

This completes the proof of Lemma 19.

Lemma 20 *The group X is not $E_6^\epsilon(q)$.*

Proof Suppose $X = E_6^\epsilon(q)$. As in the previous lemma, assume first that X_0 is not simple. Then Theorem 10 implies $G_x \cap X$ is one of the following,

(1) Parabolic.
(2) Of maximal rank.
(3) $3^6.SL_3(3)$.

or $X_0 = L_3(q) \times G_2(q)$, $U_3(q) \times G_2(q) (q > 2)$.

The first case was ruled out in Lemma 14.

The possibilities for the second case are given in [21, Table 5.1]. In some cases $|G_x|^3 < |G|$, and in each of the remaining cases, calculating $2(|G_x|, v-1)$ we obtain $k^2 < v$.

The local subgroup for the third case is too small.

Finally, the orders of the groups in the last case are less than $q^{17} < |E_6^\epsilon|^{\frac{1}{3}}$.

Now assume X_0 is simple. If $X_0 \notin \mathrm{Lie}(p)$, then we find the possibilities in [25, Table 1]. However, the only two cases which satisfy Corollary 2 have order that does not divide $|E_6^\epsilon|$. Hence $X_0 = X_0(r) \in \mathrm{Lie}(p)$.

If $\mathrm{rk}(X_0) > \frac{1}{2}\mathrm{rk}(G)$, then when $r > 2$ by Theorem 11 the only possibilities are $E_6^\epsilon(q^{\frac{1}{s}})$ with $s = 2$ or 3, $C_4(q)$, and $F_4(q)$. In all cases k is too small. When $q = 2$ then the possibilities satisfying $|G_x|^3 > |G|$ with order dividing $E_6^\epsilon(2)$ are $A_5^\epsilon(2)$, $B_4(2)$, $C_4(2)$, $D_4^\epsilon(2)$, and $D_5^\epsilon(2)$. However since k divides $2(|G_x|, v-1)$, in all cases $k^2 < v$, a contradiction.

If $\mathrm{rk}(X_0) \le \frac{1}{2}\mathrm{rk}(G)$, then Theorem 12 implies $|G_x| < q^{28}.4\log_p q$. Looking at the $p-$ and $p'-$ parts of the orders of the possible subgroups, we see that the p'-part is always less than q^{17}. Hence $|G_x|_{p'} < q^{17}$, so $2|G_x||G_x|^2_{p'} < |G|$, contradicting Corollary 5. □

This completes the proof of Lemma 20.

Lemma 21 *The group X is not $E_7(q)$.*

Proof Suppose $X = E_7(q)$. First assume X_0 is not simple. Then by Theorem 10, $G_x \cap X$ is one of the following,

(1) Parabolic.

(2) Of maximal rank.
(3) $2^2.S_3$.

or $X_0 = L_2(q) \times L_2(q)(p > 3)$, $L_2(q) \times G_2(q)(p > 2, q > 3)$, $L_2(q) \times F_4(q)(q > 3)$, or $G_2(q) \times PSp_6(q)$.

The parabolic subgroups have been ruled out in Lemma 14. The subgroups of maximal rank can be found in [21, Table 5.1]. Of these, the only ones with order greater than $|E_7(q)|^{\frac{1}{3}}$ are $d.(L_2(q) \times P\Omega^+_{12}(q)).d$ and $f.L^\epsilon_8(q).g.(2 \times (\frac{2}{f}))$, where $d = (2, q-1)$, $f = (4, \frac{q-\epsilon}{d})$, and $g = (8, \frac{q-\epsilon}{d})$. However in both cases the fact that $(k, v) \leq 2$ forces $k^2 < v$, a contradiction.

The local subgroup is too small to satisfy $|G_x|^3 > |G|$.

In the last case, the only group that is not too small to satisfy $|G_x|^3 > |G|$ is $L_2(q) \times F_4(q)$, but here q^{38} divides v, and since $(v, k) \leq 2$, then $k^2 < v$. So X_0 is simple.

First assume $X_0 \notin \mathrm{Lie}(p)$. Then by [25, Table 1], the possibilities are $A_{14}(p = 7)$, $M_{22}(p = 5)$, $Ru(p = 5)$, and $HS(p = 5)$. None of these groups satisfy Corollary 2.

Now assume $X_0 = X_0(r) \in \mathrm{Lie}(p)$. If $\mathrm{rk}(X_0) \leq \frac{1}{2}\mathrm{rk}(G)$, then by Theorem 12, $|G_x|^3 < |G|$, which is a contradiction.

If $\mathrm{rk}(X_0) > \frac{1}{2}\mathrm{rk}(G)$ then if $r > 2$ Theorem 11 implies $X \cap G_x = E_7(q^{\frac{1}{s}})$, with $s = 2$ or 3. However in both cases $(v, k) \leq 2$ forces $k^2 < v$, a contradiction. If $r = 2$ then the possible subgroups satisfying the bound $|G_x|^3 > |G|$ and having order dividing $|E_7(2)|$ are $A^\epsilon_6(2)$, $A^\epsilon_7(2)$, $B_5(2)$, $C_5(2)$, $D^\epsilon_5(2)$, and $D^\epsilon_6(2)$. However in all of these cases $(v, k) \leq 2$ forces $k^2 < v$.

□

Lemma 22 *The group X is not $E_8(q)$.*

Proof Suppose $X = E_8(q)$. First suppose that X_0 is not simple. Then by Theorem 10, $G_x \cap X$ is one of the following,

(1) Parabolic.
(2) Of maximal rank.
(3) $(2^{15}).L_5(2)$ (q odd) or $5^3.SL_3(5)$ $(5|q^2 - 1)$.
(4) $G_x \cap X = (A_5 \times A_6).2^2$.

or $X_0 = L_2(q) \times L^\epsilon_3(q)(p > 3)$, $G_2(q) \times F_4(q)$, $L_2(q) \times G_2(q) \times G_2(q)(p > 2, q > 3)$, or $L_2(q) \times G_2(q^2)(p > 2, q > 3)$.

We know from Lemma 14 that the first case does not hold.

From [21, Table 5.1] the only subgroups of maximal rank such that $|G_x|^3 \geq |G|$ are $d.P\Omega^+_{16}(q).d$, $d.(L_2(q) \times E_7(q)).d$, $f.L^\epsilon_9(q).e.2$, and $e.(L^\epsilon_3(q) \times E^\epsilon_6(q)).e.2$, (where $d = (2, q-1)$, $e = (3, q-\epsilon)$, and $f = \frac{(9,q-\epsilon)}{e}$). In all cases, $(k, v) \leq 2$ implies $k^2 < v$, which is a contradiction.

In all other cases, for all possible groups we have that $|G_x|^3 < |G|$, a contradiction. Hence X_0 is simple.

First consider the case $X_0 \notin \mathrm{Lie}(p)$. Then by [25, Table 1] the possibilities are Alt_{14}, Alt_{15}, Alt_{16}, Alt_{17}, $Alt_{18}(p = 3)$, $L_2(16)$, $L_2(31)$, $L_2(32)$, $L_2(41)$,

$L_2(49)$, $L_2(61)$, $L_3(5)$, $L_4(5)(p=2)$, $PSp_4(5)$, $G_2(3)$, ${}^2B_2(8)$, ${}^2B_2(32)(p=5)$, and $Th(p=3)$. In every case the inequality $|G_x|^3 > |G|$ is not satisfied.

Now consider the case $X_0 \in \mathrm{Lie}(p)$. If $\mathrm{rk}(X_0) \leq \frac{1}{2}\mathrm{rk}(G)$, then by Theorem 12 we have $|G_x|^3 \geq |G|$, which is a contradiction.

So $\mathrm{rk}(X_0) > \frac{1}{2}\mathrm{rk}(G)$. If $r > 2$, then by Theorem 11, $G_x \cap X$ is a subfield subgroup. The only cases in which $|G_x|^3 > |G|$ can be satisfied are when $q = q_0^2$ or $q = q_0^3$, but in all cases since $(v,k) \leq 2$ then k is too small.

If $r = 2$, then $\mathrm{rk}(X_0) \geq 5$. The groups for which $|G| < |G_x|^3$ are $A_8^\epsilon(2)$, $B_8(2)$, $B_7(2)$, $C_8(2)$, $C_7(2)$, $D_8^\epsilon(2)$, and $D_7^\epsilon(2)$. However, in all cases $(v,k) \leq 2$ forces $k^2 < v$, which is a contradiction. □

This completes the proof of Lemma 22, completing thus the proof of our Main Theorem. As a consequence of this and the results in [29, 30] we have the following:

Theorem 23 *If D is a biplane with a primitive, flag-transitive automorphism group of almost simple type, then D has parameters either* (7,4,2), *or* (11,5,2), *and is unique up to isomorphism.*

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