J Algebr Comb (2009) 29: 215–227
DOI 10.1007/s10801-008-0130-7

The case of equality in the Livingstone-Wagner Theorem

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Received: 12 January 2007 / Accepted: 21 February 2008 / Published online: 4 March 2008

Abstract Let G be a permutation group acting on a set Ω of size $n \in \mathbb{N}$ and let $1 \leq k < (n-1)/2$. Livingstone and Wagner proved that the number of orbits of G on k-subsets of Ω is less than or equal to the number of orbits on $(k+1)$-subsets. We investigate the cases when equality occurs.

Keywords Livingstone-Wagner Theorem · Permutation groups · Orbits · Partitions

1 Introduction

Throughout this article we let G be a permutation group acting on a set Ω of size $n \in \mathbb{N}$ and let $1 \leq k < (n-1)/2$. In [7] Livingstone and Wagner proved the following theorem.

Theorem 1.1 (Livingstone, Wagner) [7] *The number of orbits of G on k-subsets of Ω is less than or equal to the number of orbits on $(k+1)$-subsets.*

Alternative proofs were subsequently given by Robinson [8] and Cameron [1] who extended the result to Ω infinite. An investigation of the cases when equality occurs for Ω infinite was then made by Cameron [2], [3] and Cameron and Thomas [5]. The case of equality also follows from a stronger "interchange property" examined by Cameron, Neumann and Saxl [4]. In this article, we will prove some similar results about the case of equality when Ω is finite.

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In Section 2 we consider the case when G is intransitive. We show (see Lemma 2.1) that G must have one orbit of length at least $n-k$ and (see Proposition 2.2) that the action of G on this orbit satisfies a strong condition which in almost all cases forces G to be k-homogeneous on this orbit.

Transitive but imprimitive groups are then investigated in Section 3. In this case there are too many examples for a complete classification to be feasible, so we concentrate on finding a necessary condition for the sizes and number of blocks in a system of imprimitivity. This quickly reduces to a combinatorial problem of determining when the number of partitions of k into at most r parts of size at most s is the same as for $k+1$. This problem is also of independent interest in invariant theory, where such partitions can be used to count the number of linearly independent semi-invariants of degree r and weight k of a binary form of degree s. We are able to determine all the cases of equality for $r \le 4$ (see Theorem 3.1) and conjecture that for $s \ge r \ge 5$, there are only finitely many cases of equality (see Conjecture 3.2 for details). Theorem 3.7 shows that for $s \ge r \ge 5$, equality can only occur when $2k \ge r(s-1)-1$, that is k is close to half n. We have strong experimental evidence for believing Conjecture 3.2 to be true. We observe that for large enough fixed r and s the number of partitions of k into at most r parts of size at most s approximates to a Gaussian distribution whose peak becomes sharper for larger r and s.

In the final section we make some observations about the case when G is primitive. Aside from $(k+1)$-homogeneous groups the only examples we know are the affine general linear groups over a field of size 2 (see Proposition 4.2) and a list of 19 further examples of degree at most 24, many of which are subgroups of M_{24}. The absence in [4] of any examples of degree greater than 24 suggests that such examples may also be rare or non-existent in our situation.

Notation and preliminary results

For each $0 \le l \le n$, let $\sigma_l(G)$ be the number of orbits of G on the set of l-subsets of Ω. A permutation group is said to be *l-homogeneous* if it is transitive in its action on l-subsets, that is $\sigma_l(G) = 1$. Let Δ be a G-invariant subset of Ω. Then G^{Δ} will denote the permutation group induced by G in its action on Δ.

Let H be a subgroup of a group G, χ be a character of G and ψ a character of H. Then $\chi \downarrow H$ will denote the restriction of χ to H and $\psi \uparrow G$ will denote the character induced by ψ on G. Furthermore 1_G will denote the trivial character on G.

Lemma 1.2 *Let $G \le \mathrm{Sym}(n)$, $0 \le l \le n$ and ψ_l be the character of* Sym(n) *induced by the trivial character on* $\mathrm{Sym}(l) \times \mathrm{Sym}(n-l)$. *Then $\langle \psi_l \downarrow G, 1_G\rangle$ is the number of orbits of G on l-subsets of $\{1,\dots,n\}$ and if $0 \le l < (n-1)/2$, then $\psi_{l+1} - \psi_l$ is an irreducible character of* Sym(n).

Proof See [8]. □

Lemma 1.3 *Let $H \le G \le \mathrm{Sym}(n)$ and $1 \le k < (n-1)/2$. Then $\sigma_{k+1}(G) - \sigma_k(G) \le \sigma_{k+1}(H) - \sigma_k(H)$. In particular, if $\sigma_{k+1}(H) = \sigma_k(H)$, then $\sigma_{k+1}(G) = \sigma_k(G)$.*

Proof Let $\chi := \psi_{k+1} - \psi_k$ be the irreducible character in the conclusion of Lemma 1.2. Then

$$\sigma_{k+1}(G) - \sigma_k(G) = \langle \chi \downarrow G, 1_G \rangle \leq \langle \chi \downarrow H, 1_H \rangle = \sigma_{k+1}(H) - \sigma_k(H).$$

In particular, if $\sigma_{k+1}(H) = \sigma_k(H)$, then the right-hand side is zero and by Theorem 1.1 the left-hand side is non-negative, so must also be zero. □

2 Intransitive groups with equality

In this section we investigate intransitive permutation groups which achieve equality in the Livingstone-Wagner Theorem.

Lemma 2.1 *Let* $G \leq \mathrm{Sym}(n)$ *and suppose* $\sigma_k(G) = \sigma_{k+1}(G)$ *for some* $1 \leq k < (n-1)/2$. *Then* G *has an orbit of length at least* $n-k$.

Proof Suppose G has no orbit of length at least $n-k$. If $G \leq \mathrm{Sym}(n-l) \times \mathrm{Sym}(l) =: M$, for some $k < l < n-k$, then $\sigma_{k+1}(M) = k+2 > k+1 = \sigma_k(M)$, which contradicts Lemma 1.3. So the sum of the lengths of any set of orbits of G is either at most k or at least $n-k$. In particular, each orbit of G has length at most k and G has at least three orbits.

We claim that there exists a subgroup M of $\mathrm{Sym}(n)$ containing G isomorphic to $\mathrm{Sym}(l_1) \times \mathrm{Sym}(l_2) \times \mathrm{Sym}(l_3)$, where $n = l_1 + l_2 + l_3$ and $l_i \leq k$ for each $1 \leq i \leq 3$. Let g_1 and g_2 be the lengths of two distinct orbits of G. If $g_1 + g_2 \geq n-k$, then the total length of the remaining orbits is at most k. Then there exists M isomorphic to $\mathrm{Sym}(g_1) \times \mathrm{Sym}(g_2) \times \mathrm{Sym}(n-g_1-g_2)$, which fulfils the claim. Otherwise $g_1 + g_2 \leq k$ and there exists a group G' containing G which operates as $\mathrm{Sym}(g_1+g_2)$ on the union of these two orbits and operates in the same way as G on the other orbits. We then replace G by G' and repeat the argument to find M. We may assume without loss that $l_1 \geq l_2 \geq l_3$. It remains to show that $\sigma_k(M) < \sigma_{k+1}(M)$, which yields a contradiction to Lemma 1.3.

Note that $\sigma_k(M)$ is the number of tuples (a_1, a_2, a_3) such that $k = a_1 + a_2 + a_3$ and $a_i \leq l_i$, for each $1 \leq i \leq 3$. For each such tuple, (a_1+1, a_2, a_3) corresponds to a suitable partition of $k+1$, except in those cases when $a_1 = l_1$. The number of such exceptional tuples is $k - l_1 + 1$, because $l_1 + l_3 \geq n-k > k$ implies that $k - l_1 < l_3$. The tuples of the form $(0, a_2, a_3)$, where $a_2 + a_3 = k+1$ do not correspond to any partition of k as above. The number of such tuples is $l_2 + l_3 - k$, because a_2 can range between $k+1-l_3$ and l_2. Since $(l_2 + l_3 - k) - (k - l_1 + 1) = n - (2k+1) > 0$, this shows that $\sigma_{k+1}(M) > \sigma_k(M)$ as required. □

Proposition 2.2 *Let* $G \leq \mathrm{Sym}(n)$ *and* $1 \leq k < (n-1)/2$ *with* $\sigma_k(G) = \sigma_{k+1}(G)$. *Let* Δ *be an orbit of* G *of length at least* $n-k$. *Then* $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$, *for all* $k - (n - |\Delta|) \leq l \leq \min(k, |\Delta| - k - 2)$.

Proof Note that an orbit of length at least $n-k$ exists by Lemma 2.1. Let $M := G^\Delta \times \mathrm{Sym}(\Omega \setminus \Delta) \geq G$ and let $m := |\Delta|$. For $t \in \mathbb{N}$, two t-subsets of Ω are in the same M-orbit if and only if their intersections with Δ are in the same G^Δ-orbit. In particular, these intersections must be of the same size. Hence

$$\sigma_t(M) = \sum_{l=\max(0,t-(n-m))}^{min(t,m)} \sigma_l(G^\Delta).$$

Now $m \geq n-k \geq (2k+1)-k = k+1$. Also $k-(n-m) \geq k+(n-k)-n = 0$. Therefore

$$0 = \sigma_{k+1}(M) - \sigma_k(M) = \sum_{l=k+1-(n-m)}^{k+1} \sigma_l(G^\Delta) - \sum_{l=k-(n-m)}^{k} \sigma_l(G^\Delta)$$
$$= \sigma_{k+1}(G^\Delta) - \sigma_{k-(n-m)}(G^\Delta).$$

That is, $\sigma_{k+1}(G^\Delta) = \sigma_{k-(n-m)}(G^\Delta)$. If $2k < m-1$ then the Livingstone-Wagner Theorem forces $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$, for each $k-(n-m) \leq l \leq k$.

On the other hand, suppose $2k \geq m-1$. Then $\sigma_{k+1}(G^\Delta) = \sigma_{m-(k+1)}(G^\Delta)$ and $m-(k+1)$ is within the range to which the Livingstone-Wagner Theorem applies. We also have that

$$(m-(k+1)) - (k-(n-m)) = (n-1) - 2k > 0.$$

Hence, by the Livingstone-Wagner Theorem, $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$, for each $k-(n-m) \leq l \leq m-k-2$. Note that $\min(k, m-k-2)$ is k precisely when $2k < m-1$ and $m-k-2$ otherwise, so the proof is complete. □

Proposition 2.2 provides the means to reduce the case of equality for an intransitive group to that of equality for a transitive group. Indeed if G is intransitive with an orbit Δ satisfying the condition of Proposition 2.2, then we nearly always have equality $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$ for several consecutive values of l. (If there is just one value of l then either G is already transitive or $n = 2k+2$.) This almost forces G^Δ to be k-homogeneous. The only known exceptions with $k < (n-1)/2$ are where $G^\Delta \cong M_{24}$ or M_{23}.

3 Imprimitive groups with equality

There is an abundance of imprimitive groups which achieve equality in the Livingstone-Wagner Theorem and a complete classification of them seems intractable. Nevertheless, we are able to give a condition on the block sizes which is necessary if equality in the Livingstone-Wagner Theorem holds. Observe that by Lemma 1.3, if $\sigma_k(H) = \sigma_{k+1}(H)$ holds for an imprimitive group H with r blocks of size s, then $\sigma_k(G) = \sigma_{k+1}(G)$, where $G \cong \mathrm{Sym}(s) \wr \mathrm{Sym}(r)$ is the full stabiliser in $\mathrm{Sym}(rs)$ of the blocks of H. Note also that the number of orbits of G on k-subsets is equal to the number of ways, $P(r,s,k)$, to partition k into at most r parts of size at most s. We require $P(r,s,k) = P(r,s,k+1)$. The following result is established by Lemma 3.5, Proposition 3.6 and Proposition 3.9.

Theorem 3.1 *Let $r \in \{2,3,4\}$ with $r \le s$ and $1 \le k < (rs-1)/2$. Then $P(r,s,k) = P(r,s,k+1)$ if and only if one of the following holds.*

(a) $r = 2$ *and* k *is even.*
(b) $r = 3$ *and*

$$k = \begin{cases} \frac{3s-3}{2}, & \text{if } s \text{ is odd,} \\ \frac{3s-4}{2}, & \text{if } s \equiv 0 \mod 4, \\ \frac{3s-2}{2} \text{ or } \frac{3s-6}{2}, & \text{if } s \equiv 2 \mod 4. \end{cases}$$

(c) $r = 4$ *and* $k = 2s-2$ *or* $r = s = k = 4$.

We also make the following conjecture.

Conjecture 3.2 *Let $1 < r \le s$, $1 \le k < (rs-1)/2$ and suppose $P(r,s,k) = P(r,s,k+1)$. Then one of the following holds*:

(a) $r \in \{2,3,4\}$ *and the possibilities for s and k are as in Theorem 3.1; or*
(b) r, s *and* k *have the values given by a column of the following table*

r	5	5	5	6	6	6	6	6	7
s	6	10	14	6	7	9	11	13	10
k	14	24	34	16	20	26	32	38	34

Remark 3.3 The quantity $P(r,s,k) - P(r,s,k-1)$ is of interest in invariant theory. By a theorem of Cayley and Sylvester (see Satz 2.21 of [9]) it is equal to the number of linearly independent semi-invariants of degree r and weight k of a binary form of degree s. Conjecture 3.2, if proven, would then give the values of r, s and k for which no such semi-invariant exists.

We now define some more notation which we will use in this section. Let $\mathcal{P}(r,s,k)$ be the set of partitions of k into at most r parts of size at most s, so $P(r,s,k) = |\mathcal{P}(r,s,k)|$. We will use the convention that $P(r,s,k) = 0$ if $k < 0$ or $k > rs$. By considering dual partitions we observe that $P(r,s,k) = P(s,r,k)$, so without loss we will assume that $r \le s$. Elements of $\mathcal{P}(r,s,k)$ will be written $(a_1, a_2, \ldots, a_r)$ where $\sum_{i=1}^{r} a_i = k$ and $s \ge a_1 \ge \cdots \ge a_r \ge 0$. Let $\mathcal{A}(r,s,k)$ be the subset of $\mathcal{P}(r,s,k)$ consisting of all partitions of the form $(s, a_2, \ldots, a_r)$ and let $\mathcal{B}(r,s,k+1)$ be the subset of $\mathcal{P}(r,s,k+1)$ consisting of all partitions of the form $(x, x, a_3, \ldots, a_r)$, for some $x \le s$. Furthermore, let $A(r,s,k) = |\mathcal{A}(r,s,k)|$ and $B(r,s,k) = |\mathcal{B}(r,s,k)|$. Note that $A(r,s,k) = P(r-1,s,k-s)$. We will define a bijection from a subset of $\mathcal{P}(r,s,k)$ to a subset of $\mathcal{P}(r,s,k+1)$. Let $(a_1, a_2, \ldots, a_r) \in \mathcal{P}(r,s,k)$ with $s > a_1 \ge a_2 \ge \ldots \ge a_r \ge 0$, and define

$$f(a_1, a_2, \ldots, a_r) = (a_1 + 1, a_2, \ldots, a_r).$$

Then f is a bijection from $\mathcal{P}(r,s,k) \setminus \mathcal{A}(r,s,k)$ to $\mathcal{P}(r,s,k+1) \setminus \mathcal{B}(r,s,k+1)$. In particular we have the following result.

Lemma 3.4 *Let $r, s, k \ge 1$. Then*

$$P(r,s,k+1) - P(r,s,k) = B(r,s,k+1) - A(r,s,k).$$

So the problem of determining when $P(r,s,k)=P(r,s,k+1)$ reduces to that of determining when $B(r,s,k+1)=A(r,s,k)$. We now consider in turn the cases when $r=2,3$ and 4.

Lemma 3.5 *Let $s\geq 0$. Then*

$$P(2,s,k)=\begin{cases}0, & \text{if } k>2s,\ \text{or } k<0,\\ s-\lceil \frac{k}{2}\rceil+1, & \text{if } s\leq k\leq 2s,\\ \lfloor \frac{k}{2}\rfloor+1, & \text{if } 0\leq k\leq s.\end{cases}$$

In particular, if $1\leq k<s$, then $P(2,s,k)=P(2,s,k+1)$ if and only if k is even.

Proof Elementary. □

Proposition 3.6 *Let $s\geq 3$ and $1\leq k<(3s-1)/2$. Then $P(3,s,k)=P(3,s,k+1)$ if and only if one of the following holds*:

(a) s is odd and $k=(3s-3)/2$,
(b) $s\equiv 0 \mod 4$ and $k=(3s-4)/2$,
(c) $s\equiv 2 \mod 4$ and $k=(3s-2)/2$ or $(3s-6)/2$.

Proof Let $d_k=P(3,s,k+1)-P(3,s,k)=B(3,s,k+1)-A(3,s,k)$. By Lemma 3.5,

$$A(3,s,k)=P(2,s,k-s)=\begin{cases}\lfloor \frac{k-s}{2}\rfloor+1 & \text{if } s\leq k<(3s-1)/2,\\ 0 & \text{if } k<s.\end{cases}$$

Moreover,

$$B(3,s,k+1)=|\{(a,a,b)\mid s\geq a\geq b, 2a+b=k+1\}|=\left\lfloor\frac{k+1}{2}\right\rfloor-\left\lceil\frac{k+1}{3}\right\rceil+1.$$

Hence

$$B(3,s,k+1)\geq\frac{k}{2}-\frac{k+3}{3}+1=\frac{k}{6}>0.$$

So if $A(3,s,k)=0$, then $d_k\geq k/6>0$. We may therefore assume that

$$s\leq k<(3s-1)/2 \ \text{and}\ A(3,s,k)=\left\lfloor\frac{k-s}{2}\right\rfloor+1.$$

Thus

$$d_k=\left\lfloor\frac{k+1}{2}\right\rfloor-\left\lceil\frac{k+1}{3}\right\rceil-\left\lfloor\frac{k-s}{2}\right\rfloor.$$

Suppose s is odd. Then $k+1\equiv k-s \mod 2$. Hence

$$d_k=\frac{k+1-(k-s)}{2}-\left\lceil\frac{k+1}{3}\right\rceil=\frac{s+1}{2}-\left\lceil\frac{k+1}{3}\right\rceil.$$

Therefore

$$d_k = 0 \Leftrightarrow k \in \left\{\frac{3s+1}{2}, \frac{3s-1}{2}, \frac{3s-3}{2}\right\}.$$

Since $k < (3s-1)/2$, this forces $k = \frac{3s-3}{2}$.

Suppose s is even. Then

$$d_k \geq \frac{k}{2} - \frac{k+3}{3} - \frac{k-s}{2} = \frac{1}{6}(3s - 2k - 6).$$

Assume $d_k = 0$. Then $2k \geq 3s - 6$. Thus $3s/2 - 3 \leq k \leq 3s/2 - 1$ and so $\left\lceil \frac{k+1}{3} \right\rceil = \frac{s}{2}$. Therefore

$$d_k = \begin{cases} \frac{k}{2} - \frac{s}{2} - \frac{k-s}{2} = 0, & \text{if } k \text{ is even} \\ \frac{k+1}{2} - \frac{s}{2} - \frac{k-s-1}{2} = 1, & \text{if } k \text{ is odd, a contradiction.} \end{cases}$$

Thus k is even, $\frac{3s-6}{2} \leq k \leq \frac{3s-2}{2}$ and hence

$$k = \begin{cases} \frac{3s-4}{2} & \text{if } s \equiv 0 \mod 4 \\ \frac{3s-2}{2} \text{ or } \frac{3s-6}{2} & \text{if } s \equiv 2 \mod 4. \end{cases}$$

□

Theorem 3.7 *Let* $4 \leq r \leq s$ *and* $1 \leq k < (rs-1)/2$. *If* $P(r,s,k) = P(r,s,k+1)$, *then* $k \geq (r(s-1)-1)/2$ *or* $r = s = k = 4$.

Proof Suppose first that $k < s$. Then $A(r,s,k) = 0$ but $B(r,s,k) > 0$, since $r \geq 4$. Therefore by Lemma 3.4 $P(r,s,k) < P(r,s,k+1)$. Now suppose that $k = s \geq 5$. Then

$$P(r,s,k) = P(r,k,k) = 2 + P(r,k-2,k)$$

and

$$P(r,s,k+1) = P(r,k,k+1) = 3 + P(r,k-2,k+1).$$

Since $(r(k-2)-1)/2 \geq (4(k-2)-1)/2 = 2k - 9/2 > k$, applying Theorem 1.1 yields $P(r,k-2,k) \leq P(r,k-2,k+1)$ and so $P(r,s,k) < P(r,s,k+1)$ in this case.

It remains to show for $s < k < (r(s-1)-1)/2$ that $P(r,s,k) < P(r,s,k+1)$. So we assume for a contradiction that $P(r,s,k) = P(r,s,k+1)$ in this case. Observe that

$$P(r,s,k) = P(r,s-1,k) + P(r-1,s,k-s).$$

Since $k < (r(s-1)-1)/2$ and $k - s < (r(s-1)-1-2s)/2 < ((r-1)s-1)/2$, by Theorem 1.1, $P(r,s-1,k) \leq P(r,s-1,k+1)$ and $P(r-1,s,k-s) \leq P(r-1, s,k-s+1)$. So under our assumption we have $P(r-1,s,k-s) = P(r-1,s, k-s+1)$. We now proceed by induction on r.

Suppose first that $r = 4$. Then by Proposition 3.6, $P(3, s, k - s) = P(3, s, k - s + 1)$ implies $3s/2 - 3 \leq k - s \leq 3s/2 - 1$. However $k < (4(s-1)-1)/2 = 2s - 5/2$, so $k - s \leq s - 3 < 3s/2 - 3$, a contradiction.

Now suppose $r > 4$ and the result holds for $r - 1$ in place of r. Since $P(r-1, s, k-s) = P(r-1, s, k-s+1)$, we obtain by induction that

$$k - s \geq \tfrac{(r-1)(s-1)-1}{2} = \tfrac{rs-r-s}{2}.$$

Hence $k \geq (rs - r + s)/2 > (rs-1)/2$, a contradiction. Therefore by induction the result holds for all $r \geq 4$. □

Proposition 3.8 *Let $s \geq 4$ and $2s - 2 \leq k \leq 2s - 1$. Then $P(4, s, k) = P(4, s, k+1)$ if and only if $k = 2s - 2$.*

Proof Since $r = 4$ is fixed, for this proof we will abbreviate $A(r, s, k)$ by $A(s, k)$ and $B(r, s, k)$ by $B(s, k)$. We first show that for all $s \geq 4$, $P(4, s, 2s-2) = P(4, s, 2s-1)$. We need to evaluate $B(s, k)$ more precisely. Now

$$\mathcal{B}(s, k) = \{(a, a, b, c) : s \geq a \geq b \geq c \geq 0, 2a + b + c = k\}.$$

Now $0 \leq b + c \leq 2a$ implies $2a \leq k \leq 4a$. Hence $\lceil \frac{k}{4} \rceil \leq a \leq \lfloor \frac{k}{2} \rfloor$. Thus

$$B(s, k) = \sum_{a=\lceil \frac{k}{4} \rceil}^{\lfloor \frac{k}{2} \rfloor} P(2, a, k - 2a).$$

By Lemma 3.5, the value of $P(2, a, k-2a)$ depends on whether $0 \leq k - 2a \leq a$ or $a \leq k - 2a \leq 2a$. Now $2a - (k - 2a) = 4a - k \geq 0$. Also $k - 2a \geq a$ whenever $a \leq \lfloor \frac{k}{3} \rfloor$. Therefore by Lemma 3.5

$$B(s, k) = \sum_{a=\lceil \frac{k}{4} \rceil}^{\lfloor \frac{k}{3} \rfloor} \left(a - \lceil \tfrac{k-2a}{2} \rceil + 1\right) + \sum_{a=\lfloor \frac{k}{3} \rfloor + 1}^{\lfloor \frac{k}{2} \rfloor} \left(\lfloor \tfrac{k-2a}{2} \rfloor + 1\right). \tag{1}$$

It follows that

$$\begin{aligned}
B(s, 2s-1) &= \sum_{a=\lceil \frac{2s-1}{4} \rceil}^{\lfloor \frac{2s-1}{3} \rfloor} \left(a - \lceil \tfrac{2s-1-2a}{2} \rceil + 1\right) + \sum_{a=\lfloor \frac{2s-1}{3} \rfloor + 1}^{\lfloor \frac{2s-1}{2} \rfloor} \left(\lfloor \tfrac{2s-1-2a}{2} \rfloor + 1\right) \\
&= \sum_{a=\lceil \frac{s}{2} \rceil}^{\lfloor \frac{2s-1}{3} \rfloor} (2a - s + 1) + \sum_{a=\lfloor \frac{2s-1}{3} \rfloor + 1}^{s-1} (s - a) \\
&= (1-s)\left(\lfloor \tfrac{2s-1}{3} \rfloor - \lceil \tfrac{s}{2} \rceil + 1\right) + \lfloor \tfrac{2s-1}{3} \rfloor \left(\lfloor \tfrac{2s-1}{3} \rfloor + 1\right) - \lceil \tfrac{s}{2} \rceil \left(\lceil \tfrac{s}{2} \rceil - 1\right) \\
&\quad + s\left(s - 1 - \lfloor \tfrac{2s-1}{3} \rfloor\right) - \tfrac{1}{2}(s-1)s + \tfrac{1}{2} \lfloor \tfrac{2s-1}{3} \rfloor \left(\lfloor \tfrac{2s-1}{3} \rfloor + 1\right)
\end{aligned}$$

$$= \lfloor \tfrac{2s-1}{3} \rfloor \left(\tfrac{3}{2} \lfloor \tfrac{2s-1}{3} \rfloor + 1 + 1 - s - s + \tfrac{1}{2} \right) + \lceil \tfrac{s}{2} \rceil \left(-\lceil \tfrac{s}{2} \rceil + s - 1 + 1 \right)$$
$$+1 - s + s^2 - s - \tfrac{1}{2}s^2 + \tfrac{1}{2}s$$
$$B(s, 2s-1) = \underbrace{\tfrac{1}{2} \lfloor \tfrac{2s-1}{3} \rfloor \left(3 \lfloor \tfrac{2s-1}{3} \rfloor + 5 - 4s \right)}_{X_3(B)} + \underbrace{\lceil \tfrac{s}{2} \rceil \left(s - \lceil \tfrac{s}{2} \rceil \right) + \tfrac{1}{2} \left(s^2 - 3s + 2 \right)}_{X_2(B)}$$

We now work out $A(s, 2s-2)$ in a similar fashion. Firstly note that $A(s, 2s-2) = P(3, s, s-2) = P(3, s-2, s-2)$, and

$$P(3, s-2, s-2) = \#\{a, b, c : a \geq b \geq c \geq 0, a + b + c = s - 2\}.$$

This implies that $\lceil \frac{s-2}{3} \rceil \leq a \leq s-2$. Thus $A(s, 2s-2) = \sum_{a=\lceil \frac{s-2}{3} \rceil}^{s-2} P(2, a, s-2-a)$. From Lemma 3.5, and noting that $s-2-a \geq a$ when $a \leq \lfloor \frac{s-2}{2} \rfloor$, we make the following calculation.

$$A(s, 2s-2) = \sum_{a=\lceil \frac{s-2}{3} \rceil}^{\lfloor \frac{s-2}{2} \rfloor} \left(a - \lceil \tfrac{s-2-a}{2} \rceil + 1 \right) + \sum_{a=\lfloor \frac{s-2}{2} \rfloor + 1}^{s-2} \left(\lfloor \tfrac{s-2-a}{2} \rfloor + 1 \right)$$
$$= \sum_{a=\lceil \frac{s-2}{3} \rceil}^{\lfloor \frac{s}{2} \rfloor - 1} (a - (s-2-a)) + \sum_{a=\lceil \frac{s-2}{3} \rceil}^{s-2} \left(\lfloor \tfrac{s-a}{2} \rfloor \right)$$
$$= \sum_{a=\lceil \frac{s-2}{3} \rceil}^{\lfloor \frac{s}{2} \rfloor - 1} (2a - s + 2)$$
$$+ \sum_{a=\lceil \frac{s-2}{3} \rceil}^{s-2} \left(\tfrac{s-a}{2} \right) - \tfrac{1}{2} \# \left\{ i \in \left\{ 2, \ldots, s - \lceil \tfrac{s-2}{3} \rceil \right\} : i \text{ odd} \right\}$$

Now the number of odd numbers in the range $\{2, \ldots, x\}$ is $\left\lfloor \frac{x-1}{2} \right\rfloor$, so the number of odd numbers in $\left\{ 2, \ldots, s - \lceil \frac{s-2}{3} \rceil \right\}$ is $\left\lfloor \frac{\left\lfloor \frac{2s+2}{3} \right\rfloor - 1}{2} \right\rfloor = \left\lfloor \frac{2s-1}{6} \right\rfloor = \left\lfloor \frac{s-1}{3} \right\rfloor$. Therefore

$$A(s, 2s-2) = (2-s) \left(\left\lfloor \tfrac{s}{2} \right\rfloor - \left\lceil \tfrac{s-2}{3} \right\rceil \right) + \left\lfloor \tfrac{s}{2} \right\rfloor \left(\left\lfloor \tfrac{s}{2} \right\rfloor - 1 \right) - \left\lceil \tfrac{s-2}{3} \right\rceil \left(\left\lceil \tfrac{s-2}{3} \right\rceil - 1 \right)$$
$$+ \tfrac{s}{2} \left(s - 2 - \left\lceil \tfrac{s-2}{3} \right\rceil + 1 \right) - \tfrac{1}{4}(s-2)(s-1)$$
$$+ \tfrac{1}{4} \left\lceil \tfrac{s-2}{3} \right\rceil \left(\left\lceil \tfrac{s-2}{3} \right\rceil - 1 \right) - \tfrac{1}{2} \left\lfloor \tfrac{s-1}{3} \right\rfloor$$
$$= \left\lfloor \tfrac{s}{2} \right\rfloor \left(2 - s + \left\lfloor \tfrac{s}{2} \right\rfloor - 1 \right) + \tfrac{s}{2}(s-1) - \tfrac{1}{4}(s-2)(s-1)$$
$$+ \left\lceil \tfrac{s-2}{3} \right\rceil \left(s - 2 - \tfrac{3}{4} \left\lceil \tfrac{s-2}{3} \right\rceil + \tfrac{3}{4} - \tfrac{s}{2} \right) - \tfrac{1}{2} \left\lfloor \tfrac{s-1}{3} \right\rfloor$$

$$A(s,2s-2)=\underbrace{\left\lfloor\tfrac{s}{2}\right\rfloor\left(\left\lfloor\tfrac{s}{2}\right\rfloor+1-s\right)+\tfrac{1}{4}(s-1)(s+2)}_{X_2(A)}$$
$$+\underbrace{\tfrac{1}{4}\left\lceil\tfrac{s-2}{3}\right\rceil\left(2s-5-3\left\lceil\tfrac{s-2}{3}\right\rceil\right)-\tfrac{1}{2}\left\lfloor\tfrac{s-1}{3}\right\rfloor}_{X_3(A)}.$$

Now $P(4,s,2s-2)=P(4,s,2s-1)$ if and only if $B(s,2s-1)=A(s,2s-2)$, which is if and only if $X_2(B)-X_2(A)=X_3(A)-X_3(B)$. We have

$$X_2(B)-X_2(A)=\left(\lceil\tfrac{s}{2}\rceil\left(s-\lceil\tfrac{s}{2}\rceil\right)+\tfrac{1}{2}\left(s^2-3s+2\right)\right)$$
$$-\left(\left\lfloor\tfrac{s}{2}\right\rfloor\left(\left\lfloor\tfrac{s}{2}\right\rfloor+1-s\right)+\tfrac{1}{4}(s-1)(s+2)\right).$$

A simple calculation shows that regardless of whether s is odd or even, $X_2(B)-X_2(A)=\frac{1}{4}(3s^2-9s+6)$.

$$X_3(A)-X_3(B)=\left(\tfrac{1}{4}\left\lceil\tfrac{s-2}{3}\right\rceil\left(2s-5-3\left\lceil\tfrac{s-2}{3}\right\rceil\right)-\tfrac{1}{2}\left\lfloor\tfrac{s-1}{3}\right\rfloor\right)$$
$$-\left(\tfrac{1}{2}\lfloor\tfrac{2s-1}{3}\rfloor\left(3\lfloor\tfrac{2s-1}{3}\rfloor+5-4s\right)\right).$$

Calculating for each possible value of s modulo 3 shows that in each case, $X_3(A)-X_3(B)=\frac{1}{4}(3s^2-9s+6)=X_2(B)-X_2(A)$. Therefore, for all $s\geq 4$, $P(4,s,2s-2)=P(4,s,2s-1)$.

We now show that $P(4,s,2s-1)<P(4,s,2s)$ for all $s\geq 4$. Since $P(4,s,2s-2)=P(4,s,2s-1)$ for all $s\geq 4$, by substituting $s+1$ for s in Lemma 3.4 we have

$$A(s+1,2s)=B(s+1,2s+1).\tag{2}$$

Now $A(s,2s-1)=P(3,s,s-1)=P(3,s-1,s-1)$ as no part of a partition of $s-1$ can exceed $s-1$. Similarly $A(s+1,2s)=P(3,s+1,s-1)=P(3,s-1,s-1)$. Hence

$$A(s,2s-1)=A(s+1,2s).\tag{3}$$

Now we consider $B(s+1,2s+1)$ compared to $B(s,2s)$.

Setting $k+1=2s$ and $k+1=2s+1$ in (1) , respectively, gives:

$$B(s,2s)=\sum_{a=\lceil\frac{s}{2}\rceil}^{\lfloor\frac{2s}{3}\rfloor}(a-(s-a)+1)+\sum_{a=\lfloor\frac{2s}{3}\rfloor+1}^{s}(s-a+1)$$
$$=\sum_{a=\lceil\frac{s}{2}\rceil}^{\lfloor\frac{2s}{3}\rfloor}(2a-s+1)+\sum_{a=\lfloor\frac{2s}{3}\rfloor+1}^{s}(s-a+1);$$

$$B(s+1,2s+1) = \sum_{a=\lceil \frac{2s+1}{4}\rceil}^{\lfloor \frac{2s+1}{3}\rfloor} (a-(s-a+1)+1) + \sum_{a=\lfloor \frac{2s+1}{3}\rfloor+1}^{s} ((s-a)+1)$$

$$= \sum_{a=\lceil \frac{s+1}{2}\rceil}^{\lfloor \frac{2s+1}{3}\rfloor} (2a-s) + \sum_{a=\lfloor \frac{2s+1}{3}\rfloor+1}^{s} (s-a+1)\,.$$

If $\lfloor \frac{2s}{3}\rfloor = \lfloor \frac{2s+1}{3}\rfloor$, then $B(s,2s)-B(s+1,2s+1) \geq \sum_{a=\lceil s/2\rceil}^{\lfloor (2s+1)/3\rfloor} 1 \geq \frac{2s-1}{3} - \frac{s-1}{2} > 0$. If $\lfloor \frac{2s}{3}\rfloor < \lfloor \frac{2s+1}{3}\rfloor$ then $\lfloor \frac{2s}{3}\rfloor = \frac{2s-2}{3}$, $\lfloor \frac{2s+1}{3}\rfloor = \frac{2s+1}{3}$ and

$$\begin{aligned} B(s,2s)-B(s+1,2s+1) &\geq \left(\sum_{a=\lceil \frac{s}{2}\rceil}^{\lfloor \frac{2s}{3}\rfloor} 1\right) - \left(2\lfloor \tfrac{2s+1}{3}\rfloor - s\right) \\ &\quad + \left(s - \left(\lfloor \tfrac{2s}{3}\rfloor + 1\right) + 1\right) \\ &\geq \tfrac{2s-2}{3} - \tfrac{s-1}{2} - \tfrac{4s+2}{3} + 2s - \tfrac{2s-2}{3} \\ &= \tfrac{1}{6}(-3s+3-8s-4+12s) = \tfrac{1}{6}(s-1) > 0. \end{aligned}$$

Thus in any case $B(s,2s) > B(s+1,2s+1)$. Therefore by equations (2) and (3),

$$B(s,2s) - A(s,2s-1) > B(s+1,2s+1) - A(s+1,2s) = 0.$$

Hence by Lemma 3.4 $P(4,s,2s-1) < P(4,s,2s)$. □

Proposition 3.9 *Let* $s \geq 4$ *and* $1 \leq k \leq 2s-1$. *Then* $P(4,s,k) = P(4,s,k+1)$ *if and only if* $k = 2s-2$ *or* $s = k = 4$.

Proof In the case $s = k = 4$ it can be easily computed that $P(4,4,4) = P(4,4,5) = 5$. Otherwise, by Theorem 3.7, if $P(4,s,k) = P(4,s,k+1)$, then $4(s-1)-1 \leq 2k$ and, since k is an integer, $2s-2 \leq k$. We may now apply Proposition 3.8 to get the result. □

Theorem 3.1 now follows immediately from Lemma 3.5, Proposition 3.6 and Proposition 3.9.

4 Primitive groups with equality

Primitive groups which are not $(k+1)$-homogeneous but achieve equality in the Livingstone-Wagner Theorem for some $k < (n-1)/2$ are fairly rare. Searching the database of primitive groups in GAP [6] for degrees up to 28 produced the list given below. The lack of any primitive examples in the more special situation in [4] for degrees greater than 24 suggests very tentatively that this may be the complete list.

It might be possible to prove such a result using the O'Nan-Scott Theorem and the Classification of Finite Simple Groups, but such a proof would certainly be very labourious and we do not attempt this here. The difficulty in finding a more enlightening approach to this problem is that one would like to exploit the high degree of transitivity in the examples below. In particular it would be useful to have a result relating the number of orbits of a group to that of its point-stabilizers, but a simple relationship in general does not appear to exist. We therefore leave this problem open.

Remark 4.1 The known primitive but not $(k+1)$-homogeneous groups G such that $\sigma_k(G) = \sigma_{k+1}(G)$, for some $k < (n-1)/2$, are:

(a) $AGL(m, 2)$, for $m \geq 4, n = 2^m, k = 4$,
(b) $ASL(2, 3)$ or $AGL(2, 3)$, for $n = 9, k = 3$,
(c) Sym(5), Sym(6), $PGL(2, 9)$ or $P\Gamma L(2, 9)$, for $n = 10, k = 4$,
(d) M_{11}, $PSL(2, 11)$, $PGL(2, 11)$, for $n = 12, k = 4$,
(e) $PSL(3, 3)$, for $n = 13, k = 4$,
(f) $PGL(2, 13)$, for $n = 14, k = 4$,
(g) 2^4 : Alt(6), 2^4 : Sym(6), 2^4 : Alt(7), for $n = 16, k = 6$,
(h) $PGL(2, 17)$, for $n = 18, k = 6$ or 8,
(i) M_{22} or Aut(M_{22}), for $n = 22, k = 8$,
(j) M_{23}, for $n = 23, k = 8, 9$,
(k) M_{24}, for $n = 24, k = 6, 8, 9$ or 10.

Observe that many of these groups are subgroups of M_{24}.

Regarding case (a), we prove the following.

Proposition 4.2 *Let $G = AGL(m, 2)$, for $m \geq 4$, acting naturally on an m-dimensional vector space V over $GF(2)$. Then $\sigma_4(G) = \sigma_5(G) = 2$.*

Proof Observe that the stabiliser in G of any three points of V fixes the fourth point in the unique affine plane containing these three points and is transitive on the remaining points of V. It follows that $\sigma_4(G) = 2$ and also G has a single orbit on the set of 5-subsets which contain affine planes. Let Δ be any set of five distinct points in V which does not contain any affine plane. Then Δ is not contained in an affine 3-dimensional subspace of V. Furthermore the stabiliser in G of an affine 3-dimensional subspace W is transitive on pairs (α, Λ), where α is a point not in W and Λ is any set of four points in W which is not an affine plane. Therefore G has a single orbit on 5-subsets which do not contain any affine plane. Thus $\sigma_5(G) = 2$. □

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