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Arithmetical rank of squarefree monomial ideals of small arithmetic degree

Kyouko Kimura · Naoki Terai · Ken-ichi Yoshida

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Abstract In this paper, we prove that the arithmetical rank of a squarefree monomial ideal I is equal to the projective dimension of R/I in the following cases: (a) I is an almost complete intersection; (b) $\operatorname{arithdeg} I = \operatorname{reg} I$; (c) $\operatorname{arithdeg} I = \operatorname{indeg} I + 1$.

We also classify all almost complete intersection squarefree monomial ideals in terms of hypergraphs, and use this classification in the proof in case (c).

Keywords Arithmetical rank · Almost complete intersection · Alexander duality · Regularity · Arithmetic degree · Initial degree

1 Introduction

Throughout this paper, let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k with the unique homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)R$, and let I be a homogeneous ideal of R, unless otherwise specified. Then the *arithmetical rank* of I, denoted by $\operatorname{ara} I$, is defined as follows:

$$\operatorname{ara} I := \min\left\{r \in \mathbb{N} : \text{there exist } a_1, \dots, a_r \in I \text{ such that } \sqrt{(a_1, \dots, a_r)} = \sqrt{I}\right\}.$$

K. Kimura · K. Yoshida (✉)
Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan
e-mail: yoshida@math.nagoya-u.ac.jp

K. Kimura
e-mail: m04012w@math.nagoya-u.ac.jp

N. Terai
Department of Mathematics, Faculty of Culture and Education, Saga University, Saga 840-8502, Japan
e-mail: terai@cc.saga-u.ac.jp

This paper deals with the problem of computing the arithmetical rank of a monomial ideal (that is, the minimal number of equations needed to define the variety associated to a monomial ideal).

A trivial upper bound on $\operatorname{ara} I$ is the minimal number of generators of I, denoted by $\mu(I)$. On the other hand, it is well known that $\operatorname{height} I$ gives a lower bound for $\operatorname{ara} I$. An ideal I satisfying $\operatorname{ara} I = \operatorname{height} I$ is said to be a *set-theoretic complete intersection*. Let $H_I^i(R)$ denote the ith local cohomology module of R with support at $V(I)$. Then the *cohomological dimension* of I is defined by $\operatorname{cd}(I) = \max\{i \in \mathbb{Z} : H_I^i(R) \neq 0\}$. From the expression of the local cohomology modules in terms of Čech complex, one can easily see that $\operatorname{cd}(I) \le \operatorname{ara} I$.

Now assume that I is a squarefree monomial ideal of R. Then Lyubeznik [9] showed that $\operatorname{cd}(I) = \operatorname{pd}_R R/I$, the *projective dimension* of R/I. We also note that $\operatorname{height} I \le \operatorname{pd}_R R/I$ always holds, and that equality holds if and only if R/I is Cohen–Macaulay. Combining all inequalities stated above, we have

$$\operatorname{height} I \le \operatorname{pd}_R R/I = \operatorname{cd}(I) \le \operatorname{ara} I \le \mu(I). \tag{1.1}$$

In particular, if I is a set-theoretic complete intersection, then R/I is Cohen–Macaulay. So, we consider the following fundamental question:

Question *Let I be a squarefree monomial ideal of R. When does $\operatorname{ara} I = \operatorname{pd}_R R/I$ hold? In particular, suppose that R/I is Cohen–Macaulay. When is I a set-theoretic complete intersection?*

Barile proved the equality for certain classes of squarefree monomial ideals in [1–6]. We remark that it does not always hold as was shown by Yan [15]. He showed that $\operatorname{ara} I = 4$ for the squarefree monomial ideal I generated by monomials

$$x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6,$$

which is the Stanley–Reisner ideal of the triangulation of $\mathbb{P}^2(\mathbb{R})$ with six vertices. However, when $\operatorname{char} k \neq 2$, R/I is Cohen–Macaulay and $\operatorname{pd}_R R/I = \operatorname{height} I = 3 < 4 = \operatorname{ara} I$. In particular, I is not a set-theoretic complete intersection. In this example, the deviation $d(I) = \mu(I) - \operatorname{height} I = 7$ is rather big. So, in this paper, we focus our attention on ideals with "small deviation" (e.g., almost complete intersection ideals) and on the Alexander dual of such ideals.

Before stating our results, we recall several definitions. Let M be an arbitrary noetherian graded R-module, and let

$$0 \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{tj}(M)} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{1j}(M)} \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \to M \to 0$$

be a graded minimal free resolution of M over R, where $R(-j)$ is a graded free R-module whose nth graded piece is given by R_{n-j}, and $t = \operatorname{pd}_R M$, the projective dimension of M over R. The *regularity* and the *initial degree* of M are defined as follows:

$$\operatorname{reg} M = \max\{j - i \in \mathbb{Z} : \beta_{ij}(M) \neq 0\};$$

$$\operatorname{indeg} M = \min\{j \in \mathbb{Z} : \beta_{0j}(M) \neq 0\},$$

i.e., $\operatorname{indeg} M$ is equal to the minimal degree of the generators of M. Note that $\operatorname{indeg} M \leq \operatorname{reg} M$.

For a squarefree monomial ideal I of R, the *arithmetic degree*, denoted by $\operatorname{arithdeg} I$, is coincident with the number of prime components of I. It is known that $\operatorname{reg} I \leq \operatorname{arithdeg} I$. See [7, 8].

Schenzel–Vogel [11] and Schmitt–Vogel [12] showed that $\operatorname{ara} I = \operatorname{pd}_R R/I$ for the squarefree monomial ideal I with $\operatorname{indeg} I = \operatorname{arithdeg} I$. One of the motivation for our study is to generalize this result.

Theorem 1.1 (See also Theorems 2.1, 5.1 and 6.1) *Let R be a polynomial ring over a field k, and let I be a squarefree monomial ideal of R which satisfies one of the following conditions*:

(1) $\mu(I) \leq \operatorname{pd}_R R/I + 1$ (*e.g., I is an almost complete intersection*).
(2) $\operatorname{arithdeg} I = \operatorname{reg} I$.
(3) $\operatorname{arithdeg} I = \operatorname{indeg} I + 1$.

Then we have that $\operatorname{ara} I = \operatorname{pd}_R R/I$.

Let us explain the organization of this paper. In Section 2, we consider the question in the case of almost complete intersection ideals (i.e., $\mu(I) = \operatorname{height} I + 1$); see Theorem 2.1.

In Section 3, we introduce the notion of hypergraphs associated to squarefree monomial ideals. In the next section, we classify almost complete intersection squarefree monomial ideals in terms of hypergraphs; see Theorem 4.4. As an application, we compute some invariants (the regularity, analytic spread etc.) for such ideals.

In Section 5, we consider the question in the case of $\operatorname{arithdeg} I = \operatorname{reg} I$ (those ideals satisfying this condition are obtained as the Alexander dual ideals of squarefree monomial ideals with $\mu(I) = \operatorname{pd}_R R/I$); see Theorem 5.1. The main tool in our argument is the Schmitt–Vogel method in [12].

Finally in Section 6, we consider the question in the Alexander dual case of almost complete intersection squarefree monomial ideals; see Theorem 6.1. We use Theorems 4.4, 5.1 in the proof of Theorem 6.1.

2 Arithmetical rank of almost complete intersection squarefree monomial ideals

A homogeneous ideal I of a polynomial ring R is said to be *an almost complete intersection* (resp. *a complete intersection*) if $\mu(I) = \operatorname{height} I + 1$ (resp. $\mu(I) = \operatorname{height} I$).

Let I be a squarefree monomial ideal of R. Then

$$\operatorname{height} I \leq \operatorname{pd}_R R/I \leq \operatorname{ara} I \leq \mu(I) \tag{2.1}$$

holds as stated in the introduction. In particular, if I is a complete intersection, then $\operatorname{ara} I = \operatorname{height} I$, and so there is nothing to do any more. On the other hand, if I is an almost complete intersection, then we have

$$0 \le \mu(I) - \operatorname{pd}_R R/I \le 1.$$

The purpose of this section is to determine the arithmetical rank in this situation. Before stating our result, let us recall the definition of a Taylor resolution. Let $I = (m_1, \dots, m_\mu)$ be a monomial ideal with the minimal set of monomial generators $G(I) = \{m_1, \dots, m_\mu\}$. Then the *Taylor resolution* $F_\bullet$ of I is a finite graded free complex of the following shape:

$$F_\bullet: \quad 0 \longrightarrow F_\mu \xrightarrow{d_\mu} F_{\mu-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow R/I \longrightarrow 0,$$

where

$$F_p = \bigoplus_{1 \le \ell_1 < \cdots < \ell_p \le \mu} R\, e_{\ell_1 \cdots \ell_p},$$

$$d_p(e_{\ell_1 \cdots \ell_p}) = \sum_{i=1}^{p} (-1)^i \frac{\operatorname{lcm}(m_{\ell_1}, \dots, m_{\ell_p})}{\operatorname{lcm}(m_{\ell_1}, \dots, \widehat{m_{\ell_i}}, \dots, m_{\ell_p})} e_{\ell_1 \cdots \widehat{\ell_i} \cdots \ell_p}.$$

That is, the free basis of F_p is $\{e_{\ell_1 \cdots \ell_p}\}$ with $\deg e_{\ell_1 \cdots \ell_p} = \deg \operatorname{lcm}(m_{\ell_1}, \dots, m_{\ell_p})$. It is known that $F_\bullet$ is not necessarily the minimal graded free resolution of R/I. This implies that $\operatorname{pd}_R R/I \le \mu(I)$, and that if $F_\bullet$ is minimal, the equality holds. Note that the converse is also true.

Theorem 2.1 (See also [2, Corollary 1]) *If I is a squarefree monomial ideal of R with $\mu(I) \le \operatorname{pd}_R R/I + 1$, then we have $\operatorname{ara} I = \operatorname{pd}_R R/I$. In particular, if I is an almost complete intersection, then the same formula holds.*

In order to prove the theorem, we need the following lemma.

Lemma 2.2 *Let I be a squarefree monomial ideal of R. Then $\operatorname{pd}_R R/I \le \mu(I) - 1$ if and only if $\operatorname{ara} I \le \mu(I) - 1$. In other words, $\operatorname{pd}_R R/I = \mu(I)$ if and only if $\operatorname{ara} I = \mu(I)$.*

Proof It is enough to show that $\operatorname{ara} I \le \mu(I) - 1$ holds whenever $\operatorname{pd}_R R/I \le \mu(I) - 1$. To do that, let $G(I) = \{m_1, \dots, m_\mu\}$ be the minimal set of monomial generators of I, where $\mu = \mu(I)$. Now suppose that $\operatorname{pd}_R R/I \le \mu - 1$. Then since the Taylor resolution of I is *not* minimal, we may assume that $m_1 m_2 \cdots m_{\mu-1}$ is divisible by m_μ. For each $i = 1, \dots, \mu - 1$, let s_i be the ith elementary symmetric polynomial in $m_1, m_2, \dots, m_{\mu-1}$. Since every m_j $(j = 1, \dots, \mu - 1)$ is a root of the polynomial

$$(X - m_1)(X - m_2) \cdots (X - m_{\mu-1}) = X^{\mu-1} - s_1 X^{\mu-2} + \cdots + (-1)^{\mu-1} s_{\mu-1},$$

we get

$$m_j^{\mu-1} = s_1 m_j^{\mu-2} - \cdots + (-1)^\mu s_{\mu-1} \in (s_1, \dots, s_{\mu-2}, s_{\mu-1}).$$

Hence $m_j^{\mu-1} \in (s_1, \cdots, s_{\mu-2}, m_\mu)$ because $m_\mu | s_{\mu-1} = m_1 \cdots m_{\mu-1}$. It follows that $I = \sqrt{(s_1, \ldots, s_{\mu-2}, m_\mu)}$, and thus $\operatorname{ara} I \le \mu - 1$, as required. □

Proof of Theorem 2.1 If $\operatorname{pd}_R R/I = \mu(I)$, then Lemma 2.2 implies that $\operatorname{ara} I = \mu(I) = \operatorname{pd}_R R/I$. Otherwise, $\operatorname{pd}_R R/I = \mu(I) - 1$ by assumption. Then Lemma 2.2 implies that $\operatorname{ara} I \le \mu(I) - 1 = \operatorname{pd}_R R/I$. But the converse is always true. Hence $\operatorname{ara} I = \operatorname{pd}_R R/I$. □

Example 2.3 Let $I = (x_1x_2x_3, x_2x_4x_6, x_3x_5x_6, x_2x_3x_4x_5)$, then $\mu(I) = 4$, height $I = 2$ and $\operatorname{pd}_R R/I = 3$. In particular, $\mu(I) - \operatorname{pd}_R R/I = 1$ holds, but I is *not* an almost complete intersection. The proof of the lemma above shows that $I = \sqrt{(s_1, s_2, x_2x_3x_4x_5)}$, where

$$s_1 = x_1x_2x_3 + x_2x_4x_6 + x_3x_5x_6,$$
$$s_2 = x_1x_2^2x_3x_4x_6 + x_1x_2x_3^2x_5x_6 + x_2x_3x_4x_5x_6^2.$$

3 Hypergraphs

In this section, we introduce the construction of a particular hypergraph for any given squarefree monomial ideal. In the next section, we will classify all almost complete intersection squarefree monomial ideals using this notion. Furthermore, in Section 6, we will use this classification in order to determine the arithmetical rank for the Alexander dual ideals of those ideals.

Let us begin with the definition of hypergraphs associated to squarefree monomial ideals. Let $[\mu]$ denote the subset $\{1, \ldots, \mu\}$ of $\mathbb{N}$.

Definition 3.1 Let $V = [\mu]$. We call $\mathcal{H} \subset 2^V$ a *hypergraph* with the vertex set V if

$$\bigcup_{F \in \mathcal{H}} F = V.$$

Let I be a squarefree monomial ideal, and let $G(I) = \{m_1, m_2, \ldots, m_\mu\}$ denote the minimal set of monomial generators of I. For such an ideal I, we construct a hypergraph $\mathcal{H}(I)$ with the vertex set $V = \{1, 2, \ldots, \mu\}$ as follows:

$$F \in \mathcal{H}(I) \iff \begin{array}{l}\text{there exists } i\ (1 \le i \le n) \text{ such that}\\ \text{for all } j \in V,\\ m_j \text{ is divisible by } x_i \text{ if } j \in F\\ \text{and } m_j \text{ is not divisible by } x_i \text{ if } j \in V \setminus F.\end{array}$$

That is,

$$\mathcal{H}(I) = \Big\{ \{ j \in V : m_j \text{ is divisible by } x_i \} : 1 \le i \le n \Big\}.$$

We call $\mathcal{H}(I)$ the *hypergraph associated to a squarefree monomial ideal I*.

Notice that the hypergraph $\mathcal{H} = \mathcal{H}(I)$ satisfies

$$\begin{aligned}&\text{for all } j_1, j_2 \in V \ (j_1 \neq j_2),\\ &\text{there exist } F_1, F_2 \in \mathcal{H} \text{ such that } j_1 \in F_1 \cap (V \setminus F_2),\ \ j_2 \in F_2 \cap (V \setminus F_1).\end{aligned} \tag{3.1}$$

Conversely, for any hypergraph $\mathcal{H}$ on $V = [\mu]$ satisfying condition (3.1), there exists a squarefree monomial ideal I in a polynomial ring with enough variables such that $\mathcal{H} = \mathcal{H}(I)$. For example, if we put $I_{\mathcal{H}} := (\prod_{F \ni j} x_F : 1 \le j \le \mu)$ in a polynomial ring $k[x_F : F \in \mathcal{H}]$, then $\mathcal{H} = \mathcal{H}(I_{\mathcal{H}})$ holds. Note that such a choice of I is *not* unique. In fact, one can obtain the same hypergraph as the original one if one replaces each variable by a squarefree monomial no two of which have common factors. For example, let $I_1 = (x_1x_5, x_2x_5, x_3x_6, x_4x_6)$ and $I_2 = (x_1x_5, x_2x_5, x_3x_6x_7, x_4x_6x_7)$. Then we have

$$\mathcal{H}(I_1) = \mathcal{H}(I_2) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{3,4\}\}.$$

Definition 3.2 A subset $\mathcal{C}$ of $\mathcal{H}$ is said to be a *cover of* $\mathcal{H}$ if

$$\bigcup_{F \in \mathcal{C}} F = V.$$

In particular, $\mathcal{C}$ is called a *minimal cover of* $\mathcal{H}$ if no proper subset of $\mathcal{C}$ is a cover of $\mathcal{H}$.

$\mathcal{H}$ itself is a cover of $\mathcal{H}$. Assume that $\mathcal{H}$ satisfies condition (3.1). Then $\mathcal{H}$ is a minimal cover of $\mathcal{H}$ if and only if $\mathcal{H}$ consists of isolated points.

Note that the cardinality of the minimal cover is *not* constant in general: for instance, for a hypergraph $\mathcal{H} = \{\{1,2\}, \{2,4\}, \{1,4\}, \{1,3\}, \{3\}\}$ on $V = \{1,2,3,4\}$, $\mathcal{C}_1 = \{\{1,2\}, \{2,4\}, \{3\}\}$ and $\mathcal{C}_2 = \{\{2,4\}, \{1,3\}\}$ are both minimal covers of $\mathcal{H}$. In general, we have

Proposition 3.3 *Let I be a squarefree monomial ideal. Then the following two conditions are equivalent*:

(1) *I has a prime component of height h.*
(2) *$\mathcal{H} = \mathcal{H}(I)$ has a minimal cover of cardinality h.*

In particular,

$$\operatorname{height} I = \min\{\sharp\mathcal{C} : \mathcal{C} \textit{ is a (minimal) cover of } \mathcal{H}\}.$$

Proof Set $G(I) = \{m_1, \dots, m_\mu\}$ to be the minimal set of monomial generators of I.

(1) $\Rightarrow$ (2): Let $P = (x_{i_1}, \dots, x_{i_h})$ be a prime component of I with height h. Set $F_\ell = \{j \in V : m_j \text{ is divisible by } x_{i_\ell}\}$ for $1 \le \ell \le h$. Then $\mathcal{C} = \{F_1, \dots, F_h\}$ is a minimal cover of $\mathcal{H}$.

(2) $\Rightarrow$ (1): Let $\mathcal{C} = \{F_i : 1 \le i \le h\}$ be a minimal cover of $\mathcal{H}$. By definition, we may assume that for each F_i $(1 \le i \le h)$,

$$j \in F_i \Longleftrightarrow m_j \text{ is divisible by } x_i.$$

Since $\bigcup_{\ell=1}^{h} F_\ell = V$, we have $I = (m_1, \dots, m_\mu) \subset (x_1, \dots, x_h) = P$. Thus there is a prime component P' of I such that $P' = (x_{i_1}, \dots, x_{i_s}) \subset P$. By the argument as in the proof of (1) $\Rightarrow$ (2), $\mathcal{C}' = \{F_{i_1}, \dots, F_{i_s}\} \subset \mathcal{C}$ is a minimal cover of $\mathcal{H}$. The minimality of $\mathcal{C}$ implies that $\mathcal{C} = \mathcal{C}'$ and $P = P'$. □

Let $\mathcal{H}$ be a hypergraph. An element of $\mathcal{H}$ is said to be a *face* in $\mathcal{H}$. The *dimension* of a face F in $\mathcal{H}$ is defined by $\dim F = \sharp F - 1$, and the *dimension* of $\mathcal{H}$, denoted by $\dim \mathcal{H}$, is defined as the maximal dimension of all faces in $\mathcal{H}$. A face F with $\dim F = 1$ is said to be an *edge*. Two edges are said to be *disjoint* if they do not intersect.

Proposition 3.4 *Let I be a squarefree monomial ideal of R. Then we have*

$$\dim \mathcal{H}(I) \le \mu(I) - \operatorname{height} I.$$

Proof Put $d(I) = \mu(I) - \operatorname{height} I$. Suppose $\mathcal{H}(I)$ has a face F with $\dim F > d(I)$. For each $j \in V \setminus F$, we choose $G_j \in \mathcal{H}$ such that $j \in G_j$. Then $\mathcal{C} = \{F\} \cup \{G_j : j \in V \setminus F\}$ is a cover of $\mathcal{H}$. Since $\sharp\mathcal{C} \le \sharp V - \sharp F + 1 < \mu(I) - d(I)$, this contradicts Proposition 3.3. □

Example 3.5 The equality $\dim \mathcal{H}(I) = \mu(I) - \operatorname{height} I$ does *not* necessarily hold. For example, if we put $I = (x_1x_5, x_2x_5, x_3x_6, x_4x_6)$, then $\mu(I) = 4$, $\operatorname{height} I = 2$ and $\mathcal{H}(I) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}\}$. In particular, $\dim \mathcal{H}(I) = 1 < 2 = \mu(I) - \operatorname{height} I$.

4 Classification of almost complete intersection squarefree monomial ideals

In this section, we classify almost complete intersection squarefree monomial ideals in terms of hypergraphs. Let us begin with studying hypergraphs of those ideals.

Lemma 4.1 *Assume that I is an almost complete intersection. Then*:

(1) $\dim \mathcal{H}(I) = 1$.
(2) *There are no two disjoint edges in $\mathcal{H}(I)$.*

Proof (1) Proposition 3.4 shows that $\dim \mathcal{H}(I) \le 1$. Moreover, it is easy to see that I is a complete intersection if and only if $\dim \mathcal{H}(I) = 0$. Hence $\dim \mathcal{H}(I) = 1$.

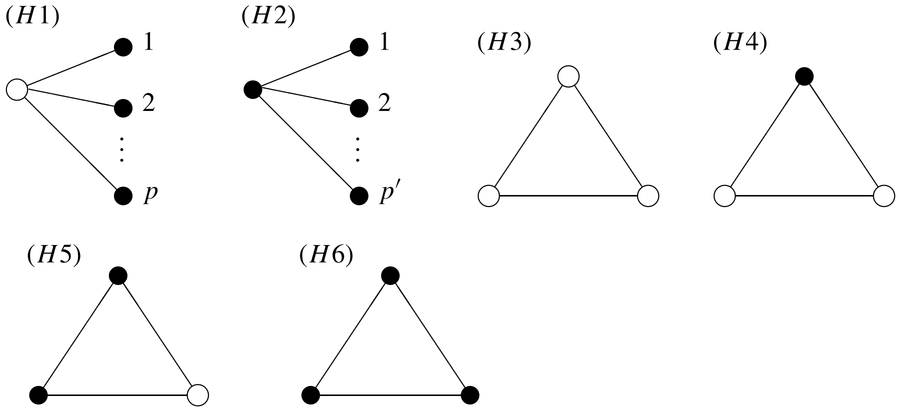
(2) Suppose $\mathcal{H} = \mathcal{H}(I)$ has two disjoint edges F_1, F_2. For each $j \in V \setminus (F_1 \cup F_2)$, we choose $G_j \in \mathcal{H}$ such that $j \in G_j$. Then $\mathcal{C} = \{F_1, F_2\} \cup \{G_j : j \in V \setminus (F_1 \cup F_2)\}$ is a cover of $\mathcal{H}$, and $\sharp\mathcal{C}$ is at most $\mu(I) - 2$. This contradicts Proposition 3.3 as $\operatorname{height} I = \mu(I) - 1$. □

By the lemma above, the hypergraph $\mathcal{H} = \mathcal{H}(I)$ associated to any almost complete intersection ideal I of height $h \ge 1$ can be represented as a simple graph H equipped with some weight function $w\colon V \longrightarrow \{0, 1\}$. In other words, $\mathcal{H}$ is the simple graph $H = (V, \mathcal{H}^1)$, where $\mathcal{H}^1 = \{F \in \mathcal{H} : \dim F = 1\}$ with the weight function $w\colon V \to \{0, 1\}$; $w(j) = 1$ if $\{j\} \in \mathcal{H}$ and $w(j) = 0$ otherwise.

In this paper, we will describe a vertex of the hypergraph (of dimension one) by the following rule: ● if $w(j) = 1$; ○ if $w(j) = 0$.

Proposition 4.2 *Assume that I is an almost complete intersection squarefree monomial ideal with $h = \operatorname{height} I \geq 2$. Then the hypergraph $\mathcal{H}(I)$ consists of one of the following one-dimensional hypergraphs with finitely many isolated points.*

In the picture below, p, p' are integers with $2 \leq p \leq h$ and $1 \leq p' \leq h$.



Remark 4.3 An almost complete intersection squarefree monomial ideal of height 1 is of the form (AB_1, AB_2), where A, B_1, B_2 are squarefree monomials no two of which have common factors. This ideal corresponds to $(H2)$ with $p' = 1$ in the proposition.

Proof of Proposition 4.2 Put $\mathcal{H} = \mathcal{H}(I)$. Since $\dim \mathcal{H} = 1$ by Lemma 4.1(1), $\mathcal{H}$ consists of vertices and 1-faces (edges). We may assume that $\mathcal{H}$ does not contain any isolated points. Then one can easily see that $\mathcal{H}$ is connected by Lemma 4.1(2).

Case 1: The case where $\mathcal{H}$ contains no cycles.
Since $\mathcal{H}$ is a connected graph without cycles, it is a tree. Moreover, $\mathcal{H}$ does not have two disjoint edges, thus it is isomorphic to either $(H1)$ or $(H2)$.

Case 2: The case where $\mathcal{H}$ contains a cycle C.
If the number of edges of C (say, m) is bigger than 3, then one can find two disjoint edges. Thus $m = 3$. Since $\mathcal{H}$ cannot have edges that do not belong to C, $\mathcal{H}$ is a triangle as a graph. Then, $\mathcal{H}$ is isomorphic to one of $(H3), \ldots, (H6)$. □

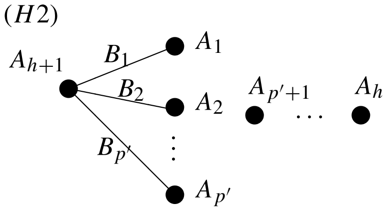
Using Proposition 4.2, we classify almost complete intersection squarefree monomial ideals. We say that I is isomorphic to J if I is obtained from J by renumbering the variables.

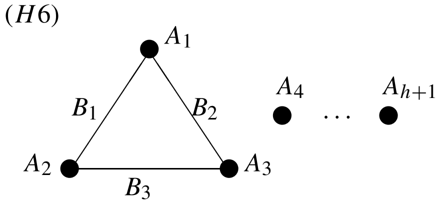
Theorem 4.4 *Let I be an almost complete intersection squarefree monomial ideal of $\operatorname{height} I = h \geq 2$. Then I can be written in one of the following forms, where $A_1, A_2, \ldots, B_1, B_2, \ldots$ are non-trivial squarefree monomials no two of which have common factors, and p, p' are integers with $2 \leq p \leq h$ and $1 \leq p' \leq h$.*

(1) $I_1 = (A_1B_1, A_2B_2, \dots, A_pB_p, A_{p+1}, \dots, A_h, B_1B_2\cdots B_p)$.
(2) $I_2 = (A_1B_1, A_2B_2, \dots, A_{p'}B_{p'}, A_{p'+1}, \dots, A_h, A_{h+1}B_1B_2\cdots B_{p'})$.
(3) $I_3 = (B_1B_2, B_1B_3, B_2B_3, A_4, \dots, A_{h+1})$.
(4) $I_4 = (A_1B_1B_2, B_1B_3, B_2B_3, A_4, \dots, A_{h+1})$.
(5) $I_5 = (A_1B_1B_2, A_2B_1B_3, B_2B_3, A_4, \dots, A_{h+1})$.
(6) $I_6 = (A_1B_1B_2, A_2B_1B_3, A_3B_2B_3, A_4, \dots, A_{h+1})$.

Moreover, R/I is unmixed if and only if I is isomorphic to I_i for some $i = 1, 3, 4, 5$. *When this is the case, R/I is Cohen–Macaulay.*

Proof We assign each vertex (resp. edge) in $\mathcal{H}$ to A_i (resp. B_j). We give pictures only for the cases (H2) and (H6).





Then I is isomorphic to one of I_i for $1 \le i \le 6$ by virtue of Proposition 4.2.

It is clear that R/I_i is unmixed if and only if $i = 1, 3, 4$ or 5. In I_1, if we put $m_{h+1} = B_1\cdots B_p$ and $m_j = A_jB_j$ for every $j = 1, \dots, p$, then $m_1\cdots m_p$ is divisible by m_{h+1}. In I_3, I_4 or I_5, if we put $m_1 = A_1B_1B_2$, $m_2 = A_2B_2B_3$ and $m_3 = B_2B_3$, then m_1m_2 is divisible by m_3, where we consider $A_1 = A_2 = 1$ in I_3 (resp. $A_2 = 1$ in I_4). In any case, using the Taylor resolution we obtain that $\mathrm{pd}_R\, R/I_i \le \mu(I_i) - 1 = \mathrm{height}\, I_i$, that is, R/I_i is Cohen–Macaulay. □

The following corollary gives an answer to the question stated in the introduction in the case of almost complete intersection squarefree monomial ideals. Let $r(R/I)$ denote the *Cohen–Macaulay type* of R/I.

Corollary 4.5 *Let $I = (m_1, \dots, m_{h+1})$ be an almost complete intersection squarefree monomial ideal with $h \ge 2$. Then the following conditions are equivalent*:

(1) *R/I is Cohen–Macaulay.*
(2) *R/I is unmixed.*
(3) *There exists m_ℓ such that $m_\ell \,|\, m_1\cdots\widehat{m_\ell}\cdots m_{h+1}$.*
(4) *I is a set-theoretic complete intersection.*

When this is the case, under the same notation as in Theorem 4.4, we have

$$r(R/I_1) = p; \quad r(R/I_i) = 2 \quad (i = 3, 4, 5).$$

Moreover, the regularity is obtained by the following formula:

$$\mathrm{reg}\, I_1 = \sum_{i=1}^{h} \deg A_i + \sum_{j=1}^{p} \deg B_j - \min\{\deg A_i \,:\, 1 \le i \le p\} - h + 1;$$

$$\operatorname{reg} I_i = \sum_{i=4}^{h+1} \deg A_i + \sum_{j=1}^{3} \deg B_j + \max\{\deg A_1, \deg A_2\} - h + 1$$

for each $i = 3, 4, 5$, *where we consider* $A_1 = A_2 = 1$ *in* I_3 (*resp.* $A_2 = 1$ *in* I_4).

Proof We first show that the above four conditions are equivalent. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear. By Lemma 2.2, we have (3) $\Rightarrow$ (4). (4) $\Rightarrow$ (1): from (4), height $I = \operatorname{pd}_R R/I$. Thus R/I is Cohen–Macaulay.

Secondly, let us determine the Cohen–Macaulay type and the regularity in the case $I = I_1$. We may assume $p = h$. Set $m_j = A_j B_j$ for $1 \le j \le h$ and $m_{h+1} = B_1 \cdots B_h$. Considering the Taylor resolution of I_1, by [10, Theorem 5.2], we have

$$\begin{aligned}\operatorname{reg} I_1 &= \max\{j \in \mathbb{Z} : \beta_{h,j}(R/I) \neq 0\} - h + 1\\ &= \max\{\deg \operatorname{lcm}(m_1, \ldots, \widehat{m_i}, \ldots, m_h, m_{h+1}) : 1 \le i \le h\} - h + 1\\ &= \sum_{i=1}^{h} \deg A_i + \sum_{j=1}^{h} \deg B_j - \min\{\deg A_i : 1 \le i \le h\} - h + 1,\end{aligned}$$

as required. Moreover, $r(R/I) = \sum_{j\in\mathbb{Z}} \beta_{h,j}(R/I) = p$.

In the case $I = I_3$, I_4 or I_5, one can also prove the formula by a similar argument as above. So we omit the proof here. □

As an application of our classification, we consider the analytic spread. The *analytic spread* of I is defined by $\ell(I) := \ell(I_{\mathfrak{m}}) = \dim \bigoplus_{n\ge 0} I_{\mathfrak{m}}^n/\mathfrak{m}I_{\mathfrak{m}}^n$, and satisfies $\operatorname{ara} I \le \ell(I) \le \mu(I)$.

Corollary 4.6 *Let I be an almost complete intersection squarefree monomial ideal. Then it is of linear type, that is,* $\ell(I) = \mu(I)$.

Proof Set $I = (m_1, \ldots, m_{h+1})$. It suffices to show that the kernel of the natural map $R[Y_1, \ldots, Y_{h+1}] \to R[m_1 t, \ldots, m_{h+1} t]$ is generated by $m_i Y_j - m_j Y_i$, $1 \le i < j \le h + 1$. One can easily reduce the proof to the case $p = h$ in I_1 (resp. $h = 2$ in I_3, I_4 or I_5). Then it is easy to check it. □

From Theorem 4.4, we can also classify almost complete intersection (not necessarily squarefree) monomial ideals. For $A = x_{i_1}^{j_1} \cdots x_{i_m}^{j_m}$ $(j_1, \ldots, j_m > 0)$, we set $\sqrt{A} := x_{i_1} \cdots x_{i_m}$.

Corollary 4.7 *Let I be an almost complete intersection monomial ideal. Then I is one of the following types, where $C_1, C_2, \ldots, D_1, D_2, \ldots$ are monomials no two of which have common factor*:

(1) $(D_1, \ldots, D_p, C_{p+1}D_{p+1}, \ldots, C_q D_q, C_{q+1}, \ldots, C_h, C_{h+1}D_1' \cdots D_q')$, *where* $p \ge 0$, $q \ge 1$, $p \le q$, $\sqrt{D_i} = \sqrt{D_i'}$ *for each of* $i = 1, \ldots, q$ *and* D_i' *is not divisible by* D_i *for each of* $i = 1, \ldots, p$, $C_{p+1}, \ldots, C_h, D_1, \ldots, D_p \neq 1$. *Moreover, if* $p = 0$, *then* $q \ge 2$ *or* $C_{h+1} \neq 1$.

(2) $(C_1D_1D_2, C_2{D_1}'D_3, C_3{D_2}'{D_3}', C_4, \dots, C_{h+1})$, *where* $\sqrt{D_i} = \sqrt{{D_i}'}$ *for each of* $i = 1, 2, 3$, $C_4, \dots, C_{h+1}, D_1, D_2, D_3 \neq 1$.

When this is the case, $\sqrt{I}$ *is a complete intersection if and only if* $p \geq 1$ *in* (1).

Proof As $h = \operatorname{height} I = \operatorname{height} \sqrt{I} \leq \mu(\sqrt{I}) \leq \mu(I) = h + 1$, $\sqrt{I}$ is a complete intersection or an almost complete intersection.

Case 1: $\sqrt{I}$ is a complete intersection

By the assumption, the ideal I can be written as $I = (M_1, \dots, M_{h+1})$, where M_i are monomials such that $\sqrt{M_{h+1}} \in (\sqrt{M_1}, \dots, \sqrt{M_h})$. Put $B_i = \gcd(\sqrt{M_i}, \sqrt{M_{h+1}})$. By renumbering the monomials, we may assume that $p \geq 1$, and that $B_i \neq 1$ if and only if $1 \leq i \leq q$ and

$$\sqrt{M_i} = \begin{cases} B_i & \text{if } 1 \leq i \leq p; \\ A_i B_i & \text{if } p+1 \leq i \leq q, \end{cases}$$

where $A_i \neq 1$ is a squarefree monomial for $i = p+1, \dots, q$. Thus we can write

$$I = (D_1, \dots, D_p, C_{p+1}D_{p+1}, \dots, C_qD_q, C_{q+1}, \dots, C_h, C_{h+1}D'_1 \cdots D'_p),$$

where C_i, D_j (resp. C_i, D'_j) are coprime monomials and $\sqrt{C_i} = A_i$ and $\sqrt{D_j} = B_j$ (resp. $\sqrt{D'_j} = B_j$). Conversely, if I can be written in the above form, then

$$\sqrt{I} = (B_1, \dots, B_p, A_{p+1}B_{p+1}, \dots, A_qB_q, A_{q+1}, \dots, A_h)$$

is a complete intersection.

Case 2: $\sqrt{I}$ is an almost complete intersection

By the assumption, the ideal I can be written as $I = (M_1, \dots, M_{h+1})$, where M_i are monomials such that $\sqrt{I} = (\sqrt{M_1}, \dots, \sqrt{M_{h+1}})$.

Suppose that $\sqrt{I} = (A_1B_1, \dots, A_qB_q, A_{h+1}, \dots, A_h, A_{h+1}B_1 \cdots B_q)$, where A_i, B_j are non-trivial monomials for $i = 1, \dots, h$, $j = 1, \dots, q$, and A_{h+1} is a monomial; see Theorem 4.4(1),(2). Then I can be written as:

$$(C_1D_1, \dots, C_qD_q, C_{q+1}, \dots, C_h, C_{h+1}{D_1}' \cdots {D_q}'),$$

where $\sqrt{C_i} = A_i$, $\sqrt{D_i} = B_i$, and $q \geq 2$ when $C_{h+1} = 1$. This is the form described in (1) with $p = 0$.

Next suppose that $\sqrt{I} = (A_1B_1B_2, A_2B_1B_3, A_3B_2B_3, A_4, \dots, A_{h+1})$, that is, it is isomorphic to I_i for some $i = 3, 4, 5, 6$. Then one can easily see that I can be written as the ideal in (2). ☐

Remark 4.8 One can prove this corollary using *polarization* (see [13, p. 107, Chapter II Section 1]).

5 Arithmetical rank of the case arithdeg $I = \operatorname{reg} I$

In this section, using Alexander duality, we consider the arithmetical rank of squarefree monomial ideals with $\operatorname{arithdeg} I = \operatorname{reg} I$. Before stating our result, we recall the definition and fundamental properties of Alexander duality.

Put $V = [n]$. For $\Delta \subseteq 2^V$, Δ is called a *simplicial complex* on the vertex set V if (a) $\{i\} \in \Delta$ for every $i \in V$ and (b) $F \in \Delta$, $G \subseteq F$ implies $G \in \Delta$. For a simplicial complex Δ on V, the *Alexander dual complex* of Δ is defined by $\Delta^* := \{F \subset V : V \setminus F \notin \Delta\}$.

Let I be a squarefree monomial ideal of R. Then there exists a simplicial complex Δ on V such that $I = I_\Delta$, where I_Δ is the *Stanley–Reisner ideal* of Δ: $I_\Delta = (x_{i_1} \cdots x_{i_p} \ : \ 1 \le i_1 < \cdots < i_p \le n,\ \{i_1, \ldots, i_p\} \notin \Delta)R$. Now suppose that $\operatorname{height} I \ge 2$. Then we set $I^* := I_{\Delta^*}$ and call it the *Alexander dual ideal* of I. Then it is easy to see that $I^{**} = I$. Let $I = Q_1 \cap Q_2 \cap \cdots \cap Q_q$ be the irredundant primary decomposition of I, and let m_ℓ be the product of all variables which appear in Q_ℓ for each $\ell = 1, \ldots, q$. Then $I^* = (m_1, m_2, \ldots, m_q)$. This implies that $\operatorname{height} I = \operatorname{indeg} I^*$ and $\mu(I^*) = \operatorname{arithdeg} I$. Moreover, it is known that $\operatorname{pd}_R R/I = \operatorname{reg} I^*$ (see, e.g., [14, Corollary 1.6]). Considering Alexander dual of the relation (2.1), we have

$$\operatorname{indeg} I \le \operatorname{reg} I \le \operatorname{arithdeg} I. \tag{5.1}$$

See [7, 8].

Schenzel–Vogel [11] and Schmitt–Vogel [12] showed that $\operatorname{ara} I = \operatorname{pd}_R R/I$ for the squarefree monomial ideal I with $\operatorname{indeg} I = \operatorname{arithdeg} I$. We generalize it as follows:

Theorem 5.1 *Let I be a squarefree monomial ideal. If* $\operatorname{arithdeg} I = \operatorname{reg} I$, *then we have* $\operatorname{ara} I = \operatorname{pd}_R R/I$.

From now on, we prove this theorem. When $\operatorname{height} I = 1$, I can be written in the form uI_0, where u is a squarefree monomial and $\operatorname{height} I_0 \ge 2$. In order to prove Theorem 5.1, we may assume that $\operatorname{height} I \ge 2$ by replacing I with I_0. Then we have following:

Lemma 5.2 *Any squarefree monomial ideal I with* $\operatorname{arithdeg} I = \operatorname{reg} I$ *can be written (by renumbering the variables) in the form*

$$I = (y_1, x_{t_{11}}, \ldots, x_{t_{1j_1}}) \cap (y_2, x_{t_{21}}, \ldots, x_{t_{2j_2}}) \cap \cdots \cap (y_q, x_{t_{q1}}, \ldots, x_{t_{qj_q}}),$$

where $y_\ell, x_{t_{ij}}$ are variables in R with $y_\ell \ne x_{t_{ij}}$, $y_i \ne y_{i'}$ for $i \ne i'$, and $x_{t_{ij}} \ne x_{t_{ij'}}$ for $j \ne j'$.

Proof Since a squarefree monomial ideal has no embedded associated primes, the assertion follows from [8, Theorem 2.6]. □

Lemma 5.3 *Using the same notation as in Lemma 5.2, we have*

$$\operatorname{pd}_R R/I = \sharp\{x_{t_{11}}, x_{t_{12}}, \ldots, x_{t_{1j_1}}, x_{t_{21}}, x_{t_{22}}, \ldots, x_{t_{2j_2}}, \ldots, x_{t_{q1}}, x_{t_{q2}}, \ldots, x_{t_{qj_q}}\} + 1.$$

Proof Since the Taylor resolution of I^* is minimal, we have

$$\operatorname{reg} R/I^* = \deg \operatorname{lcm}\left(m_1/y_1, \ldots, m_q/y_q\right).$$

Since $\operatorname{pd}_R R/I = \operatorname{reg} I^* = \operatorname{reg} R/I^* + 1$, we obtain the required assertion. □

Proof of Theorem 5.1 By Lemma 5.2, I can be written as follows:

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_q, \quad Q_i = (y_i, x_{t_{i1}}, \ldots, x_{t_{ij_i}}),\ i = 1, \ldots, q,$$

where $q = \operatorname{arithdeg} I$. Set $\{x_1, x_2, \ldots, x_r\} = \{x_{t_{ij}}\}$. By Lemma 5.3, to prove the theorem, it suffices to find $r+1$ elements $g_0, \ldots, g_r \in I$ such that $\sqrt{I} = \sqrt{(g_0, \ldots, g_r)}$.

Note that any squarefree monomial with respect to $x_1, \ldots, x_r$ can be written as follows:

$$x_{\underline{i}} = x_{i_1, \ldots, i_\ell} = x_{i_1} \cdots x_{i_\ell} \quad (1 \le i_1 < \cdots < i_\ell \le r;\ \ell = 0, 1, \ldots, r).$$

For $\underline{i} = (i_1, \ldots, i_\ell)$, we put $\Sigma(\underline{i}) = \{j \,:\, 1 \le j \le q,\ \ x_{\underline{i}} \notin Q_j\}$. Set

$$\mathcal{P}_{r-\ell} = \left\{ x_{i_1} \cdots x_{i_\ell} \Big(\prod_{j \in \Sigma(\underline{i})} y_j \Big) \,:\, 1 \le i_1 < \cdots < i_\ell \le r \right\}$$

for every $\ell = 0, 1, \ldots, r$. In particular, $\mathcal{P}_0 = \{x_1 \cdots x_r\}$ and $\mathcal{P}_r = \{y_1 \cdots y_q\}$. Then the following are satisfied:

(SV-1) $\bigcup_{\ell=0}^{r} \mathcal{P}_\ell = \mathcal{P}$ contains all minimal monomial generators of I.
(SV-2) $\sharp \mathcal{P}_0 = 1$.
(SV-3) For any ℓ $(0 \le \ell < r)$ and any $a, a'' \in \mathcal{P}_{r-\ell}$ $(a \ne a'')$, there is ℓ' $(\ell < \ell' \le r)$ and $a' \in \mathcal{P}_{r-\ell'}$ such that $a \cdot a'' \in (a')$.

Let us check (SV-3) only. For $\underline{i} = (i_1, \ldots, i_\ell)$, $\underline{i''} = (i''_1, \ldots, i''_\ell)$ with $\underline{i} \ne \underline{i''}$, we have that $\sharp(\{\underline{i}\} \cup \{\underline{i''}\}) \ge \ell + 1$. We also see that $\Sigma(\{\underline{i}\} \cup \{\underline{i''}\}) \subseteq \Sigma(\{\underline{i}\})$. The assertion immediately follows from here.

If we set

$$g_\ell = \sum_{\underline{i} = (i_1, \ldots, i_\ell)} x_{i_1} \cdots x_{i_\ell} \Big(\prod_{j \in \Sigma(\underline{i})} y_j \Big) \quad \text{for every } \ell = 0, 1, \ldots, r,$$

then we have $\sqrt{I} = \sqrt{(a : a \in \mathcal{P})R} = \sqrt{(g_0, \ldots, g_r)R}$ by virtue of Schmitt–Vogel lemma (see [12, Lemma, p. 249]). □

Example 5.4 Let us consider

$$\begin{aligned} I &= (y_1, x_1, x_2) \cap (y_2, x_2, x_3) \cap (y_3, x_4) \\ &= (x_1x_3x_4, x_1x_3y_3, x_1x_4y_2, x_2x_4, x_3x_4y_1, x_1y_2y_3, x_2y_3, x_3y_1y_3, x_4y_1y_2, y_1y_2y_3). \end{aligned}$$

Then g_ℓ's in the proof of Theorem 5.1 are given by the following:

$$\begin{aligned} g_0 &= x_1x_2x_3x_4,\\ g_1 &= x_1x_2x_3y_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4,\\ g_2 &= x_1x_2y_3 + x_1x_3y_3 + x_1x_4y_2 + x_2x_3y_3 + x_2x_4 + x_3x_4y_1,\\ g_3 &= x_1y_2y_3 + x_2y_3 + x_3y_1y_3 + x_4y_1y_2,\\ g_4 &= y_1y_2y_3. \end{aligned}$$

6 Alexander dual of almost complete intersection squarefree monomial ideals

In this section, we consider squarefree monomial ideals with $\operatorname{arithdeg} I = \operatorname{indeg} I + 1$. For such an ideal I with $\operatorname{height} I \geq 2$, the Alexander dual J of I is an almost complete intersection. Utilizing this fact, we determine $\operatorname{ara} I$.

Theorem 6.1 *If I is a squarefree monomial ideal with* $\operatorname{arithdeg} I = \operatorname{indeg} I + 1$, *then* $\operatorname{ara} I = \operatorname{pd}_R R/I$.

Proof We may assume that $\operatorname{height} I \geq 2$. Put $h = \operatorname{indeg} I \geq 2$, and let $J = I^*$ denote the Alexander dual of I. Then since J is an almost complete intersection, it is isomorphic to one of $I_1, \ldots, I_6$; see Theorem 4.4. Noting that $I = J^*$, we get

Lemma 6.2 *Let I be a squarefree monomial ideal with* $\operatorname{arithdeg} I = \operatorname{indeg} I + 1$ *and* $\operatorname{height} I \geq 2$. *Then I is isomorphic to one of the following ideals, where x_{ij}, y_ℓ are variables that are different from each other*:

$$\begin{aligned} \bullet I_1' = {} & (x_{11}, x_{12}, \ldots, x_{1j_1}) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}) \cap \cdots \cap (x_{h1}, x_{h2}, \ldots, x_{hj_h})\\ & \cap (x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}, \ldots, x_{p1}, x_{p2}, \ldots, x_{pi_p}), \end{aligned}$$

where $2 \leq p \leq h$, $1 \leq i_\ell < j_\ell$ $(\ell = 1, 2, \ldots, p)$, $j_{p+1}, \ldots, j_h \geq 1$.

$$\begin{aligned} \bullet I_2' = {} & (x_{11}, x_{12}, \ldots, x_{1j_1}) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}) \cap \cdots \cap (x_{h1}, x_{h2}, \ldots, x_{hj_h})\\ & \cap (x_{h+11}, x_{h+12}, \ldots, x_{h+1j_{h+1}},\\ & \quad x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}, \ldots, x_{p1}, x_{p2}, \ldots, x_{pi_p}), \end{aligned}$$

where $1 \leq p \leq h$, $1 \leq i_\ell < j_\ell$ $(\ell = 1, 2, \ldots, p)$, $j_{p+1}, \ldots, j_h, j_{h+1} \geq 1$.

$$\begin{aligned} \bullet I_3' = {} & (x_{11}, x_{12}, \ldots, x_{1i_1}, y_1, y_2, \ldots, y_p) \cap (x_{21}, x_{22}, \ldots, x_{2i_2}, y_1, y_2, \ldots, y_p)\\ & \cap (x_{31}, x_{32}, \ldots, x_{3j_3}) \cap \cdots \cap (x_{h1}, x_{h2}, \ldots, x_{hj_h})\\ & \cap (x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}), \end{aligned}$$

where $h \geq 2$, $p, i_1, i_2, j_3, \ldots, j_h \geq 1$.

$$\bullet I'_4 = (x_{11}, x_{12}, \ldots, x_{1i_1}, y_1, y_2, \ldots, y_p) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}, y_1, y_2, \ldots, y_p)$$
$$\cap (x_{31}, x_{32}, \ldots, x_{3j_3}) \cap \cdots \cap (x_{h1}, x_{h2}, \ldots, x_{hj_h})$$
$$\cap (x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}),$$

where $h \geq 2$, $p, i_1, i_2, j_3, \ldots, j_h \geq 1$ *and* $j_2 > i_2$.

$$\bullet I'_5 = (x_{11}, x_{12}, \ldots, x_{1j_1}, y_1, y_2, \ldots, y_p) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}, y_1, y_2, \ldots, y_p)$$
$$\cap (x_{31}, x_{32}, \ldots, x_{3j_3}) \cap \cdots \cap (x_{h1}, x_{h2}, \ldots, x_{hj_h})$$
$$\cap (x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}),$$

where $h \geq 2$, $p \geq 1$, $1 \leq i_\ell < j_\ell$ $(\ell = 1, 2)$, $j_3, \ldots, j_h \geq 1$.

$$\bullet I'_6 = (x_{11}, x_{12}, \ldots, x_{1j_1}, y_1, y_2, \ldots, y_p) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}, y_1, y_2, \ldots, y_p)$$
$$\cap (x_{31}, x_{32}, \ldots, x_{3j_3}) \cap \cdots \cap (x_{h1}, x_{h2}, \ldots, x_{hj_h})$$
$$\cap (x_{h+11}, x_{h+12}, \ldots, x_{h+1j_{h+1}}, x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}),$$

where $h \geq 2$, $p \geq 1$, $1 \leq i_\ell < j_\ell$ $(\ell = 1, 2)$, $j_3, \ldots, j_h, j_{h+1} \geq 1$.

We now return to the proof.

Case 1. *The case* $I = I'_2$ *or* $I = I'_6$.
Then as R/J is not Cohen–Macaulay, we have that $\mathrm{pd}_R\, R/J = \mathrm{height}\, J + 1$. This means that $\mathrm{reg}\, I = \mathrm{indeg}\, I + 1$. Hence we can apply Theorem 5.1 by assumption.

Case 2. *The case* $I = I'_1$.
We first compute $\mathrm{pd}_R\, R/I$. We may assume that all variables appear in the minimal monomial generators of I, and that

$$j_1 - i_1 = \min\{j_\ell - i_\ell : \ell = 1, 2, \ldots, p\}.$$

By Corollary 4.5, we have

$$\mathrm{pd}_R\, R/I = \mathrm{reg}\, J = i_1 + j_2 + \cdots + j_h - h + 1.$$

Next, we will find $\mathrm{pd}_R\, R/I$ elements generating I up to radical. Indeed, it is enough to take

$$g_\ell = \sum_{\substack{\ell_1 + \cdots + \ell_h = \ell + h \\ \ell_1 \leq i_1 \text{ or } \ell_2 \leq i_2 \text{ or } \cdots \text{ or } \ell_p \leq i_p}} x_{1\ell_1} x_{2\ell_2} \cdots x_{h\ell_h}, \qquad \ell = 0, 1, \ldots, i_1 + \sum_{t=2}^{h} j_t - h.$$

Case 3. *The case* $I = I'_3$, I'_4 *or* I'_5.
Put $j_1 = i_1$ in the case of I'_3, I'_4, and $j_2 = i_2$ in the case of I'_3. We may assume that $j_1 - i_1 \leq j_2 - i_2$ in the case of I'_5. Then

$$\mathrm{pd}_R\, R/I = i_1 + j_2 + \cdots + j_h + p - h + 1.$$

It is enough to take

$$g_\ell = \sum_{\substack{\ell_1+\cdots+\ell_h=\ell+h \\ \ell_1 \le i_1 \text{ or } \ell_2 \le i_2}} x_{1\ell_1} x_{2\ell_2} \cdots x_{h\ell_h}, \qquad \ell = 0, 1, \ldots, i_1 + \sum_{t=2}^{h} j_t + p - h,$$

where $x_{1 j_1+i} = x_{2 j_2+i} = y_i$. This completes the proof. □

Remark 6.3 We define the homogeneous arithmetical degree of the monomial ideal I by

$$\operatorname{ara}_h I = \min \left\{ r \in \mathbb{N} : \begin{array}{l} \text{there are homogeneous } a_1, \ldots, a_r \in I \\ \text{such that } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I} \end{array} \right\}.$$

Then, obviously, one has $\operatorname{ara} I \le \operatorname{ara}_h I$. However, the converse is open in general. Note that one can easily obtain that $\operatorname{pd}_R R/I = \operatorname{ara}_h I$ in Theorems 2.1, 5.1 and 6.1.

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