Journal of Algebraic Combinatorics 2 (1993), 73–103


# The Subconstituent Algebra of an Association Scheme (Part II)*

PAUL TERWILLIGER
*Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706*

*Received July 1, 1991; Revised November 5, 1992*

**Abstract.** This is a continuation of an article from the previous issue. In this section, we determine the structure of a thin, irreducible module for the subconstituent algebra of a $P$- and $Q$- polynomial association scheme. Such a module is naturally associated with a Leonard system. The isomorphism class of the module is determined by this Leonard system, which in turn is determined by four parameters: the endpoint, the dual endpoint, the diameter, and an additional parameter $f$. If the module has sufficiently large dimension, the parameter $f$ takes one of a certain set of values indexed by a bounded integer parameter $e$.

**Keywords:** association scheme, $P$-polynomial, $Q$-polynomial, distance-regular graph

## 4. The subconstituent algebra of a $P$- and $Q$-polynomial scheme

In this section, we determine the structure of a thin, irreducible module for a subconstituent algebra in a $P$- and $Q$-polynomial scheme.

THEOREM 4.1. *Let $Y = (X, \{R_i\}_{0\leq i\leq D})$ denote a commutative association scheme with $D \geq 3$. Suppose $Y$ is $P$-polynomial with respect to the ordering $A_0, A_1, \ldots, A_D$ of it's associate matrices, and $Q$-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of it's primitive idempotents. Then*

*(i)*

$$LS(Y) := ((p^i_{1j})_{0\leq i,j\leq D},\ \mathrm{diag}(\theta_0, \theta_1, \ldots, \theta_D), (q^i_{1j})_{0\leq i,j\leq D},\ \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_D))$$

*is a Leonard system over $\mathbb{R}$, where $(p^i_{1j})_{0\leq i,j\leq D}$ has $i$, $j$ entry the intersection number $p^i_{1j}$ from Definition 3.1, $(q^i_{1j})_{0\leq i,j\leq D}$ has $i$, $j$ entry the Krein parameter $q^i_{1j}$ from (38), $\theta_i := p_1(i)$ is from (40), and $\theta^*_i := q_1(i)$ is from (41).*

*Fix any $x \in X$, and let $E^*_i = E^*_i(x)$, $A^*_i = A^*_i(x)\,(0 \leq i \leq D)$, $T = T(x)$ be as*

* Part I of this paper appears in *Journal of Algebraic Combinatorics*, Vol.1, No.4, December 1992. References for part II appear in part I.

*in (51), (56), and Definition 3.3. Let $W$ denote a thin, irreducible $T$-module, with endpoint $\mu$, dual-endpoint $\nu$, and diameter $d$, as defined in (79), (72), and Definition 3.5. Then (ii)–(viii) hold:*

*(ii) Pick a nonzero $u \in E_\mu W$, and a nonzero $v \in E^*_\nu W$. Then*

$$S := (E_\mu v, E_{\mu+1}v, \ldots, E_{\mu+d}v),$$
$$S^* := (E^*_\nu u, E^*_{\nu+1}u, \ldots, E^*_{\nu+d}u),$$

*are bases for $W$.*
*We call $S$ (resp. $S^*$) a standard basis* (resp. *dual basis) for $W$.*

*(iii)*

$$LS(W) := ([A]_{S^*}, [A]_S, [A^*]_S, [A^*]_{S^*})$$

*is a Leonard system over $\mathbb{R}$, where $A = A_1$, $A^* = A^*_1$, and where $[\alpha]_\beta$ denotes the matrix representing $\alpha$ with respect to the basis $\beta$. $LS(W)$ is uniquely determined by $W$ (once the orderings of the associate matrices and the primitive idempotents are fixed).*

*(iv)*

$$[A]_S = \text{diag}\,(\theta_\mu, \theta_{\mu+1}, \ldots, \theta_{\mu+d}),$$
$$[A^*]_{S^*} = \text{diag}\,(\theta^*_\nu, \theta^*_{\nu+1}, \ldots, \theta^*_{\nu+d}).$$

*Consequently, $LS(W)$, $LS(Y)$ are related as follows if $d \geq 1$:*

*Case I*

$$\begin{aligned} LS(Y) &= LS(I, q, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta^*_0, D) \\ &\qquad (q \succ q^{-1} \text{ if } ss^* \neq 0,\ r_1 \succeq r_2,\ r_1 r_2 = ss^* q^{D+1}), \\ LS(W) &= LS(I, q, hq^{-\mu}, h^* q^{-\nu}, f_1 q^{\mu+\nu}, f_2 q^{\mu+\nu}, sq^{2\mu}, s^* q^{2\nu}, \theta_\mu, \theta^*_\nu, d), \\ &\qquad (f_1 f_2 = ss^* q^{d+1}). \end{aligned} \tag{82}$$

*Case IA*

$$\begin{aligned} LS(Y) &= LS(IA, q, h^*, r, s, \theta_0, \theta^*_0, D), \\ LS(W) &= LS(IA, q, h^* q^{-\nu}, fq^\nu, sq^\mu, \theta_\mu, \theta^*_\nu, d). \end{aligned} \tag{83}$$

*Case II*

$$\begin{aligned} LS(Y) &= LS(II, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta^*_0, D) \\ &\qquad (r_1 \succeq r_2,\ r_1 + r_2 = s + s^* + D + 1), \\ LS(W) &= LS(II, h, h^*, f_1 + \mu + \nu, f_2 + \mu + \nu, s + 2\mu, s^* + 2\nu, \theta_\mu, \theta^*_\nu, d), \\ &\qquad (f_1 + f_2 = s + s^* + d + 1). \end{aligned} \tag{84}$$

*Case IIA*

$$LS(Y) = LS(IIA,\ h,\ r,\ s,\ s^*,\ \theta_0,\ \theta_0^*,\ D),$$
$$LS(W) = LS(IIA,\ h,\ f+\mu+\nu,\ s+2\mu,\ s^*,\ \theta_\mu,\ \theta_\nu^*,\ d). \tag{85}$$

*Case IIB*

$$LS(Y) = LS(IIB,\ h^*,\ r,\ s,\ s^*,\ \theta_0,\ \theta_0^*,\ D),$$
$$LS(W) = LS(IIB,\ h^*,\ f+\mu+\nu,\ s,\ s^*+2\nu,\ \theta_\mu,\ \theta_\nu^*,\ d). \tag{86}$$

*Case IIC*

$$LS(Y) = LS(IIC,\ r,\ s,\ s^*,\ \theta_0,\ \theta_0^*,\ D),$$
$$LS(W) = LS(IIC,\ f,\ s,\ s^*,\ \theta_\mu,\ \theta_\nu^*,\ d). \tag{87}$$

*Case III*

$$LS(Y) = LS(III,\ h,\ h^*,\ r_1,\ r_2,\ s,\ s^*,\ \theta_0,\ \theta_0^*,\ D)$$
$$(r_1 \succeq r_2 \text{ if } D \text{ is odd},\ r_1 + r_2 = -s - s^* + D + 1),$$
$$LS(W) = LS(III,\ h(-1)^\mu,\ h^*(-1)^\nu,\ f_1+\mu+\nu,\ f_2+\mu+\nu,\ s-2\mu,$$
$$s^*-2\nu,\ \theta_\mu,\ \theta_\nu^*,\ d), \quad (f_1 + f_2 = -s - s^* + d + 1). \tag{88}$$

(v) *If $d = 0$ set $f = \infty$ and if $d \geq 1$ let $f$ be as in part (iv) above, where we interpret $f = (f_1, f_2)$ (unordered pair) in Cases I, II, and Case III($d$ odd), and $f = (f_1, f_2)$ (ordered pair) in Case III($d$ even). Then $f$ is uniquely determined by $LS(W)$. We refer to the 4-tuple $(\mu, \nu, d, f)$ as the data sequence of $W$ (with respect to the given orderings of the associate matrices and primitive idempotents).*

(vi) *The statements*

$$d = D,\ \mu = 0,\ \nu = 0,\ W = M\hat{x},\ LS(W) = LS(Y)$$

*are all equivalent, where $M$ is the Bose-Mesner algebra of $Y$.*
*If $p$ is some object associated with $LS(W)$, we will occasionally write $p(W)$ to distinguish it from the corresponding object associated with $LS(Y)$.*

(vii)

$$\|E_i v\|^2 = \frac{b_0^*(W)b_1^*(W)\cdots b_{i-\mu-1}^*(W)}{c_1^*(W)c_2^*(W)\cdots c_{i-\mu}^*(W)}\|E_\mu v\|^2 \qquad (v \in E_\nu^* W,\quad \mu \leq i \leq \mu + d). \tag{89}$$

(viii)

$$\|E_i^* u\|^2 = \frac{b_0(W)b_1(W)\cdots b_{i-\nu-1}(W)}{c_1(W)c_2(W)\cdots c_{i-\nu}(W)}\|E_\nu^* u\|^2 \qquad (u \in E_\mu W,\quad \nu \leq i \leq \nu + d). \tag{90}$$

*Note* 4.2. $p^1_{10} = 1$, $q^1_{10} = 1$, $p^0_{10} = 0$, $q^0_{10} = 0$ by (31), (39), and these equations give relationships among the constants $q$, $h$, $h^*$, $r_1$, $r_2$, ... that appear in part (iv) above. However, we make no use of these relationships until Corollary 4.12.

*Proof of Theorem* 4.1. It is convenient to prove the parts in the order (ii), (iii), (vi), (i), (iv), (v), (vii), (viii).

*Proof of (ii)*. This is immediate from parts (ii) and (v) of Lemmas 3.9 and 3.12.

*Proof of (iii)*. First, we show the 4-tuple $LS(W)$ is a Leonard system over $\mathbb{C}$. Certainly the matrices $B := [A]_{S^*}$, $B^* := [A^*]_S$ are tridiagonal, and have nonzero entries directly above and below the main diagonal, by parts (i)–(iii) of Lemmas 3.9, 3.12. The matrices $H := [A]_S$, $H^* := [A^*]_{S^*}$ are diagonal, for indeed $H = \mathrm{diag}(\theta_\mu, \theta_{\mu+1}, \ldots, \theta_{\mu+d})$ and $H^* = \mathrm{diag}(\theta^*_\nu, \theta^*_{\nu+1}, \ldots, \theta^*_{\nu+d})$ by construction. Also, $H$, $H^*$ each has distinct entries on the main diagonal by part (iii) of Lemmas 3.8, 3.11. So far we have (4)–(7). Now let $Q$ denote the transition matrix from the basis $S$ to the basis $S^*$, that is, the matrix whose columns represent the elements of $S$ with respect to $S^*$. Then by linear algebra

$$Q^{-1}BQ = H,$$
$$Q^{-1}H^*Q = B^*.$$

Note by (53) and part (ii) of the present theorem that the sum of the elements of $S^*$ is a scalar multiple of the first element in $S$. It follows that the entries in the leftmost column of $Q$ are all equal. Replacing $(Q, S, S^*)$ by $(Q^{-1}, S^*, S)$ in the above argument, we find the entries in the leftmost column of $Q^{-1}$ are all equal. Now conditions (8)–(11) of Theorem 2.1 are satisfied, so $LS(W)$ is a Leonard system over $\mathbb{C}$. In fact $LS(W)$ is over $\mathbb{R}$. Certainly $H \in \mathrm{Mat}_{d+1}(\mathbb{R})$ by part (iii) of Lemma 3.8, so consider the entries $a_i(W)$, $b_i(W)$, and $c_i(W)$ of $B$. We have

$$a_i(W) \in \mathbb{R} \qquad (0 \le i \le d), \tag{91}$$

since this is an eigenvalue of the real symmetric matrix $E^*_{i+\nu}AE^*_{i+\nu}$. From (12), we find

$$\begin{aligned} b_i(W) + c_i(W) &= \theta_\mu - a_i(W) \\ &\in \mathbb{R} \qquad (0 \le i \le d), \end{aligned}$$

where $c_0(W) = b_d(W) = 0$. In particular $b_0(W) \in \mathbb{R}$, and

$$c_i(W) \in \mathbb{R} \rightarrow b_i(W) \in \mathbb{R} \qquad (1 \le i \le d-1).$$

But also

$$b_i(W)c_{i+1}(W) \in \mathbb{R} \qquad (0 \le i \le d-1), \tag{92}$$

for this is an eigenvalue of the real symmetric matrix $E^*_{i+\nu}AE^*_{i+\nu+1}AE^*_{i+\nu}$. Now

$$b_i(W) \in \mathbb{R} \rightarrow c_{i+1}(W) \in \mathbb{R} \qquad (0 \le i \le d-1),$$

since the product in (92) is never 0. Combining the above implications we find $b_{i-1}(W)$, $c_i(W) \in \mathbb{R}$ $(1 \le i \le d)$, so $B \in \mathrm{Mat}_{d+1}(\mathbb{R})$ in view of (91). A similar argument shows $H^*$, $B^* \in \mathrm{Mat}_{d+1}(\mathbb{R})$, so $LS(W)$ is over $\mathbb{R}$.

*Proof of (vi).* It is immediate from (72), (79), and Definition 3.5 that

$$0 \le \mu,\ \nu \le D-d \le D. \tag{93}$$

Combining this with Lemma 3.6, we find the first four statements of (vi) are equivalent. Certainly the last statement implies the first and, hence, the first four, so now suppose $W = M\hat{x}$. Observe by (68), (69), and part (ii) of the present theorem, that

$$S := (E_0\hat{x}, E_1\hat{x}, \ldots, E_D\hat{x}) \tag{94}$$

$$= |X|^{-1}(A^*_0\delta, A^*_1\delta, \ldots, A^*_D\delta) \tag{95}$$

is a standard basis for $W$, and that

$$S^* := (E^*_0\delta, E^*_1\delta, \ldots, E^*_D\delta) \tag{96}$$

$$= (A_0\hat{x}, A_1\hat{x}, \ldots, A_D\hat{x}) \tag{97}$$

is a dual basis for $W$. Now $[A]_{S^*} = (p^i_{1j})_{0 \le i,j \le D}$ by (30), (97), $[A]_S = \mathrm{diag}\,(\theta_0, \theta_1, \ldots, \theta_D)$ by (46), (94), $[A^*]_S = (q^i_{1j})_{0 \le i,j \le D}$ by (61), (95), and $[A^*]_{S^*} = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_D)$ by (60), (96), so the 4-tuples $LS(W)$, $LS(Y)$ are indentical.

*Proof of (i).* The two 4-tuples $LS(Y)$, $LS(M\hat{x})$ are identical by part (vi) of the present theorem, and the 4-tuple $LS(M\hat{x})$ is a Leonard system over $\mathbb{R}$ by part (iii) of the present theorem.

*Proof of (iv).* The first statement is immediate from part (iii) of the present theorem. Now let $LS'$ denote the Leonard system on the right side of (82)–(88). Then one may readily verify using the data in Theorem 2.1 that $LS'$ has eigenvalue sequence $\theta_\mu, \theta_{\mu+1}, \ldots, \theta_{\mu+d}$ and dual eigenvalue sequence $\theta^*_\nu, \theta^*_{\nu+1}, \ldots, \theta^*_{\nu+d}$. It follows from Lemma 2.4 that $LS(W) = LS'$ for a suitable choice of the $f$ parameters.

*Proof of (v).* This is immediate from Lemma 2.4.

*Proof of (vii).* We may assume $v \ne 0$. Then since $A^*$ is real symmetric, and since the basis $S := (E_\mu v, E_{\mu+1}v, \ldots, E_{\mu+d}v)$ of $W$ is orthogonal, it follows from linear algebra that

$$\mathrm{diag}(\|E_\mu v\|^2, \|E_{\mu+1}v\|^2, \ldots, \|E_{\mu+d}v\|^2)\overline{[A^*]}_S$$
$$= [A^*]^t_S\ \mathrm{diag}\,(\|E_\mu v\|^2, \|E_{\mu+1}v\|^2, \ldots, \|E_{\mu+d}v\|^2).$$

$[A^*]_S$ is real by part (iii) of the present theorem, so we may eliminate the complex conjugate. Now computing the entries just above the main diagonal in the above products we find

$$\|E_i v\|^2 b^*_{i-\mu}(W) = \|E_{i+1} v\|^2 c^*_{i-\mu+1}(W) \qquad (\mu \le i < \mu + d).$$

The result is immediate from this and induction.

*Proof of (viii).* Similar to the proof of (vii). □

LEMMA 4.3. *Let $Y$ be as in Theorem 4.1, pick any $x \in X$, and let $W$, $W'$ denote any thin irreducible $T(x)$-modules. Then the following are equivalent.*

*(i) $W$, $W'$ are isomorphic as $T(x)$-modules.*
*(ii) $LS(W) = LS(W')$.*
*(iii) $W$, $W'$ have the same data sequence.*

*Proof.* Write $E^*_i = E^*_i(x)$ $(0 \le i \le D)$, $A^* = A^*_1(x)$, $T = T(x)$.

(i) → (ii). Let $\sigma : W \to W'$ denote an isomorphism of $T$-modules, and let $S$ (resp. $S^*$) denote a standard basis (resp. dual basis) for $W$. Since $\sigma E_i = E_i \sigma$, $\sigma E^*_i = E^*_i \sigma$ $(0 \le i \le D)$ by (3), we find $\sigma S$ (resp. $\sigma S^*$) is a standard basis (resp. dual basis) for $W'$. But now

$$\begin{aligned} LS(W) &= ([A]_{S^*}, [A]_S, [A^*]_S, [A^*]_{S^*}) \\ &= ([\sigma A \sigma^{-1}]_{\sigma S^*}, [\sigma A \sigma^{-1}]_{\sigma S}, [\sigma A^* \sigma^{-1}]_{\sigma S}, [\sigma A^* \sigma^{-1}]_{\sigma S^*}) \\ &= ([A]_{\sigma S^*}, [A]_{\sigma S}, [A^*]_{\sigma S}, [A^*]_{\sigma S^*}) \\ &= LS(W'). \end{aligned}$$

(ii) → (i). Let $S$, $S'$ denote standard bases for $W$, $W'$, respectively, and define the linear transformation $\sigma : W \to W'$ so that $\sigma S = S'$. Then for $B \in \{A, A^*\}$,

$$\begin{aligned} [B]_S &= [B]_{S'} \\ &= [B]_{\sigma S} \\ &= [\sigma^{-1} B \sigma]_S, \end{aligned}$$

so $\sigma A - A\sigma$, $\sigma A^* - A^* \sigma$ vanish on $W$. But $A$, $A^*$ generate $T$ by part (ii) of Lemmas 3.8, 3.11, so $\sigma a - a\sigma$ vanishes on $W$ for every $a \in T$. Now $\sigma$ is an isomorphism of $T$-modules by (3).

(ii) → (iii). The diameter of $W$ is determined by the sizes of the matrices in $LS(W)$. The endpoint of $W$ is determined by the eigenvalue sequence of $LS(W)$, and the dual endpoint of $W$ is determined by the dual eigenvalue sequence of $LS(W)$. The parameter $f$ in the data sequence of $W$ is determined by $LS(W)$ according to part (v) of Theorem 4.1.

(iii) $\rightarrow$ (ii). This is immediate from part (iv) of Theorem 4.1. □

In Theorem 4.10 we will show that if the parameters $\mu$, $\nu$, $d$ in a data sequence satisfy certain general inequalities, then the parameter $f$ in the data sequence takes the following special form.

*Definition 4.4.* Let the scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ be as in Theorem 4.1. Pick any $x \in X$, and let $W$ denote a thin irreducible $T(x)$-module, with data sequence $(\mu, \nu, d, f)$. Then $W$ is said to be *strong* whenever $d \geq 1$, and there exists an integer $e$ satisfying

$$e + d + D \text{ even}, \quad |e| \leq 2\mu - D + d, \quad |e| \leq 2\nu - D + d, \tag{98}$$

such that, [referring to part (iv) of Theorem 4.1],

*Case I* $\quad f_1, f_2$ is a permutation of

$$r_1 q^{\frac{d-D+e}{2}}, r_2 q^{\frac{d-D-e}{2}} \tag{99}$$

*Case IA*

$$f = r q^{\frac{d-D-e}{2}} \tag{100}$$

*Case II* $\quad f_1, f_2$ is a permutation of

$$r_1 + \frac{d-D+e}{2}, r_2 + \frac{d-D-e}{2} \tag{101}$$

*Case IIA, IIB*

$$f = r + \frac{d-D-e}{2} \tag{102}$$

*Case IIC*

$$f = r \tag{103}$$

*Case III* $\quad f_1, f_2$ is a permutation of
($d$ odd)

$$r_1 + \frac{d-D+e}{2}, r_2 + \frac{d-D-e}{2}, \quad \text{with } \mu + \nu + \frac{d-D-e}{2} \text{ even} \tag{104}$$

*Case III*
($d$ even)

$$f_1 = r_1 + \frac{d-D+e}{2}, \quad f_2 = r_2 + \frac{d-D-e}{2} \quad \left(\text{if } \mu + \nu + \frac{d-D-e}{2} \text{ is even}\right), \tag{105}$$

$$f_1 = r_2 + \frac{d-D-e}{2}, \quad f_2 = r_1 + \frac{d-D+e}{2}$$
$$\left(\text{if } \mu + \nu + \frac{d-D-e}{2} \text{ is odd}\right). \tag{106}$$

The parameter $e$ may not be unique.

If $W$ is strong, the *auxiliary parameter* of $W$ is the integer $e$ with

$$|e + 1/2| \text{ minimal} \tag{107}$$

subject to (98)–(106). (The auxiliary parameter is unique by the first condition of (98).)

On our way toward Theorem 4.10, our next task is to consider how the data sequences of the various modules are related. Theorems 4.6, 4.9 are our main results on this subject. They are preceeded by the technical lemmas 4.5, 4.8. Recall nonempty subsets $W, W'$ of the standard module $V$ are said to be *orthogonal* whenever $\langle w, w' \rangle = 0$ for all $w \in W$ and all $w' \in W'$.

LEMMA 4.5. *Let the scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be as in Theorem 4.1. Pick any $x, y \in X$, any thin irreducible $T(x)$-module $W$, and any thin irreducible $T(y)$-module $W'$, such that $W, W'$ are not orthogonal. Let $\nu, \nu'$ denote the dual endpoints of $W, W'$, respectively, and pick nonzero $v \in E^*_\nu(x)W$, $v' \in E^*_{\nu'}(y)W'$. Then there exist nonzero polynomials $\psi, \psi' \in \mathbb{C}[\lambda]$ such that*

$$\langle E_i v, E_i v' \rangle = \|E_i v\|^2 \psi(\theta_i) \prod_{\xi \in W_s \setminus W'_s} (\theta_i - \theta_\xi) \qquad (0 \leq i \leq D), \tag{108}$$

$$= \|E_i v'\|^2 \psi'(\theta_i) \prod_{\zeta \in W'_s \setminus W_s} (\theta_i - \theta_\zeta) \qquad (0 \leq i \leq D), \tag{109}$$

*and*

$$\deg \psi \leq \rho - \nu + \nu' - |W_s \setminus W'_s|, \tag{110}$$
$$\deg \psi' \leq \rho - \nu' + \nu - |W'_s \setminus W_s|, \tag{111}$$

*where $(x, y) \in R_\rho$. (The supports $W_s, W'_s$ are from Definition 3.5.)*

*Proof.* This will consist of two claims.

*Claim 1.* There exist nonzero polynomials $\varphi, \varphi' \in \mathbb{C}[\lambda]$ such that

$$\deg \varphi \leq \rho - \nu + \nu', \tag{112}$$
$$\deg \varphi' \leq \rho - \nu' + \nu,$$

and

$$\text{proj}_W v' = \overline{\varphi(A)}v, \qquad (113)$$
$$\text{proj}_{W'} v = \varphi'(A)v',$$

where $A = A_1$ is the first associate matrix of $Y$ and $\text{proj}_\beta \alpha$ denotes the orthogonal projection of $\alpha$ onto $\beta$.

*Proof of Claim 1.* By symmetry, it suffices to show there exists a nonzero polynomial $\varphi \in \mathbb{C}[\lambda]$ satisfying (112) and (113). To do this, it suffices to show $\text{proj}_W v'$ is nonzero and contained in $\text{Span}\{v, Av, \ldots, A^{\rho-\nu+\nu'}v\}$. Now by assumption, there exist $w \in W$, $w' \in W'$ with $\langle w, w'\rangle \neq 0$, and by (74) we may write $w' = av'$ for some element $a$ of the Bose-Mesner algebra $M$. Since $a$ is symmetric we obtain

$$\begin{aligned}\langle \bar{a}w, v'\rangle &= \langle w, av'\rangle \\ &= \langle w, w'\rangle \\ &\neq 0,\end{aligned}$$

and since $\bar{a}w \in W$, we observe $\text{proj}_W v' \neq 0$. Now write

$$\text{proj}_W v' = v_\nu + v_{\nu+1} + \cdots + v_{\nu+d}, \qquad (114)$$

where $d$ denotes the diameter of $W$, and where $v_i \in E_i^*(x)W$ $(\nu \leq i \leq \nu + d)$. We will now show

$$v_i = 0 \qquad (\rho + \nu' < i \leq \nu + d). \qquad (115)$$

To see this, note

$$\begin{aligned}\|v_i\|^2 &= \langle v_i, v'\rangle \\ &= 0 \qquad (\rho + \nu' < i \leq \nu + d),\end{aligned}$$

since

$$\begin{aligned}&v_i \in E_i^*(x)V \qquad (\nu \leq i \leq \nu + d), \\ &v' \in E_{\nu'}^*(y)V,\end{aligned}$$

($V$ = standard module), and $E_i^*(x)V$, $E_{\nu'}^*(y)V$ are orthogonal whenever $p_{i\nu'}^\rho = 0$ $(0 \leq i \leq D)$. Now by (74), (114), and (115) we have

$$\begin{aligned}\text{proj}_W v' &= v_\nu + v_{\nu+1} + \cdots + v_{\rho+\nu'} \\ &\in \text{Span}\{v, Av, \ldots, A^{\rho-\nu+\nu'}v\},\end{aligned}$$

as desired. This proves Claim 1.

*Claim 2.*

$$\begin{aligned}\langle E_i v, E_i v'\rangle &= \|E_i v\|^2 \varphi(\theta_i) \qquad (0 \leq i \leq D) \qquad (116) \\ &= \|E_i v'\|^2 \varphi'(\theta_i) \qquad (0 \leq i \leq D). \qquad (117)\end{aligned}$$

In particular, the polynomial

$$\lambda - \theta_\xi \text{ divides } \varphi \text{ for each } \xi \in W_s \backslash W_s', \tag{118}$$

$$\lambda - \theta_\zeta \text{ divides } \varphi' \text{ for each } \zeta \in W_s' \backslash W_s. \tag{119}$$

*Proof of Claim 2.* By symmetry it suffices to prove (116) and (118). But since $v' - \operatorname{proj}_W v'$ is orthogonal to $W$, we have, for each integer $i$ $(0 \leq i \leq D)$,

$$\begin{aligned} 0 &= \langle E_i v, v' - \operatorname{proj}_W v' \rangle \\ &= \langle E_i v, v' - \overline{\varphi(A)} v \rangle \\ &= \langle E_i v, E_i(v' - \overline{\varphi(A)} v) \rangle \\ &= \langle E_i v, E_i v' \rangle - \varphi(\theta_i) \|E_i v\|^2, \end{aligned}$$

which gives (116). Now pick any $\xi \in W_s \backslash W_s'$ , so that $E_\xi v \neq 0$, $E_\xi v' = 0$. Then from (116) and (117) we find

$$\begin{aligned} \varphi(\theta_\xi) &= \|E_\xi v\|^{-2} \|E_\xi v'\|^2 \varphi'(\theta_\xi) \\ &= 0, \end{aligned}$$

so $\lambda - \theta_\xi$ divides $\varphi$. Thus (118) holds, and we have proved Claim 2.

Now set

$$\psi := \varphi \prod_{\xi \in W_s \backslash W_s'} (\lambda - \theta_\xi)^{-1}, \tag{120}$$

$$\psi' := \varphi' \prod_{\zeta \in W_s' \backslash W_s} (\lambda - \theta_\zeta)^{-1}. \tag{121}$$

Observe $\psi$, $\psi'$ are nonzero by Claim 1, and contained in $\mathbb{C}[\lambda]$ by (118) and (119). They satisfy (108) and (109) by (116) and (117), and (110) and (111) by Claim 1. This proves Lemma 4.5. □

THEOREM 4.6. *Let the scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be as in Theorem 4.1. Pick any $x$, $y \in X$, any thin irreducible $T(x)$-module $W$, and any thin irreducible $T(y)$-module $W'$, such that $W$, $W'$ are not orthogonal. Let $(\mu, \nu, d, f)$, $(\mu', \nu', d', f')$ denote the data sequences of $W$, $W'$, respectively, and suppose $(x, y) \in R_\rho$ for some $\rho\,(0 \leq \rho \leq D)$. Then the following statements (i)–(v) hold.*

*(i) $W_s \cap W_s' \neq \emptyset$, where we recall $W_s = \{\mu, \mu + 1, \ldots, \mu + d\}$, $W_s' = \{\mu', \mu' + 1, \ldots, \mu' + d'\}$.*

*(ii)*

$$|\mu - \mu'| + |\mu - \mu' + d - d'| + |2(\nu - \nu') + d - d'| \leq 2\rho. \tag{122}$$

*(iii) Assume $|W_s \cup W_s'| \geq 2\rho + 2$. Then $d$, $d' \geq 1$. Furthermore, there exists an integer $\varepsilon$ satisfying*

$$\varepsilon + d + d' \text{ even}, \ |\varepsilon| \le 2\rho - |\mu - \mu'| - |\mu - \mu' + d - d'|, \tag{123}$$

*such that [referring to part (iv) of Theorem 4.1]*

*Case I* $f_1, f_2$ *is a permutation of*

$$f_1' q^{\frac{d-d'+\varepsilon}{2}}, \ f_2' q^{\frac{d-d'-\varepsilon}{2}}, \tag{124}$$

*Case IA*

$$f = f' q^{\frac{d-d'-\varepsilon}{2}} \tag{125}$$

*Case II* $f_1, f_2$ *is a permutation of*

$$f_1' + \frac{d-d'+\varepsilon}{2}, \ f_2' + \frac{d-d'-\varepsilon}{2} \tag{126}$$

*Case IIA, IIB*

$$f = f' + \frac{d-d'-\varepsilon}{2} \tag{127}$$

*Case IIC*

$$f = f' \tag{128}$$

*Case III*

$$f_1 = f_1' + \frac{d-d'+\varepsilon}{2}, \ f_2 = f_2' + \frac{d-d'-\varepsilon}{2},$$

*and*

$$\mu - \mu' + \nu - \nu' + \frac{d-d'-\varepsilon}{2} \quad \text{is even}, \tag{129}$$

*or*

$$f_1 = f_2' + \frac{d-d'-\varepsilon}{2}, \ f_2 = f_1' + \frac{d-d'+\varepsilon}{2},$$

*and*

$$\mu - \mu' + \nu - \nu' + \frac{d-d'-\varepsilon}{2} + d \quad \text{is odd} \tag{130}$$

*(iv) Suppose*

$$\rho \le \nu - \nu'.$$

*Then*

$$0 \le \mu - \mu' \le d' - d \le 2(\nu - \nu') = 2\rho. \tag{131}$$

*(v) Suppose $W'$ is strong, and that*

$$\nu - \nu' = \rho < d'/2, \qquad \nu \leq \mu. \tag{132}$$

*Then $W$ is strong.*

*Proof of (i).* $W_s \cap W'_s \neq \emptyset$, for otherwise at least one of $E_iW$, $E_iW'$ is zero for each integer $i$ $(0 \leq i \leq D)$, contradicting the assumption that $W$, $W'$ are not orthogonal.

To simplify the notation for the rest of the proof, set

$$\tau = \max\{\mu, \mu'\}, \tag{133}$$

$$m = \rho - \nu + \nu' - |W_s \backslash W'_s|, \tag{134}$$

$$n = \rho - \nu' + \nu - |W'_s \backslash W_s|, \tag{135}$$

and note

$$m - n = 2(\nu' - \nu) + d' - d, \tag{136}$$

$$m + n = 2\rho - |\mu - \mu'| - |\mu - \mu' + d - d'|. \tag{137}$$

*Proof of (ii).* Observe $m$, $n$ are nonnegative by (110) and (111), so using (136) and (137) we find

$$\begin{aligned} 0 &\leq 2 \min\{m, n\} \\ &= m + n - |m - n| \\ &= 2\rho - |\mu - \mu'| - |\mu - \mu' + d - d'| - |2(\nu - \nu') + d - d'|. \end{aligned}$$

*Proof of (iii).* Combining (134), (135), and the assumption $|W_s \cup W'_s| \geq 2\rho + 2$, we have

$$\begin{aligned} |W_s \cap W'_s| &= |W_s \cup W'_s| - |W_s \backslash W'_s| - |W'_s \backslash W_s| \\ &\geq m + n + 2 \\ &\geq 2, \end{aligned} \tag{138}$$

so $d, d' \geq 1$. Let $v, v', \psi, \psi'$ be as in Lemma 4.5. Then, comparing the right sides of (108) and (109), we find

$$\phi_i \psi(\theta_i) = \psi'(\theta_i) \qquad (i \in W_s \cap W'_s), \tag{139}$$

where

$$\phi_i = \frac{\|E_i v\|^2}{\|E_i v'\|^2} \prod_{\xi \in W_s \backslash W'_s} (\theta_i - \theta_\xi) \prod_{\zeta \in W'_s \backslash W_s} (\theta_i - \theta_\zeta)^{-1} \qquad (i \in W_s \cap W'_s).$$

We observe $\phi_i \neq 0$ for all $i \in W_s \cap W'_s$.

*Claim 1.* Assume Case I $(ss^* \neq 0)$. Then

$$a^2 q^{2-m+n} = bcde \tag{140}$$

and

$$\phi_i = \phi_\tau q^{(m-n)(i-\tau)} \frac{(b,\, c,\, d,\, e;\, q)_{i-\tau}}{(aq/b,\, aq/c,\, aq/d,\, aq/e;\, q)_{i-\tau}} \qquad (i \in W_s \cap W_s'), \tag{141}$$

where

$$(a,\, b,\, c,\, d,\, e) = (sq^{2\tau+1},\, f_1 q^{\tau+\nu+1},\, f_2 q^{\tau+\nu+1},\, sq^{\tau-\nu'+1}/f_1',\, sq^{\tau-\nu'+1}/f_2'). \tag{142}$$

*Proof of Claim 1.* From (82) we find

$$f_1 f_2 = ss^* q^{d+1},$$
$$f_1' f_2' = ss^* q^{d'+1},$$

so

$$f_1 f_2 = f_1' f_2' q^{d-d'}. \tag{143}$$

Evaluating this using (142), we obtain (140). To obtain (141), it suffices to show

$$\phi_i/\phi_{i-1} = \frac{q^{m-n}(1-bq^{i-\tau-1})(1-cq^{i-\tau-1})(1-dq^{i-\tau-1})(1-eq^{i-\tau-1})}{(1-aq^{i-\tau}/b)(1-aq^{i-\tau}/c)(1-aq^{i-\tau}/d)(1-aq^{i-\tau}/e)}$$
$$(i-1,\, i \in W_s \cap W_s'). \tag{144}$$

By (139) and (89) and the definition of $W_s$, $W_s'$, we find

$$\begin{aligned}
\phi_i/\phi_{i-1} &= \frac{\|E_i v\|^2}{\|E_{i-1} v\|^2} \frac{\|E_{i-1} v'\|^2}{\|E_i v'\|^2} \prod_{\xi \in W_s \setminus W_s'} \frac{(\theta_i - \theta_\xi)}{(\theta_{i-1} - \theta_\xi)} \prod_{\zeta \in W_s' \setminus W_s} \frac{(\theta_{i-1} - \theta_\zeta)}{(\theta_i - \theta_\zeta)} \\
&= \frac{b^*_{i-\mu-1}(W)}{c^*_{i-\mu}(W)} \frac{c^*_{i-\mu'}(W')}{b^*_{i-\mu'-1}(W')} \prod_{\mu \le \xi < \mu'} \frac{(\theta_i - \theta_\xi)}{(\theta_{i-1} - \theta_\xi)} \prod_{\mu'+d' < \xi \le \mu+d} \frac{(\theta_i - \theta_\xi)}{(\theta_{i-1} - \theta_\xi)} \\
&\quad \times \prod_{\mu' \le \zeta < \mu} \frac{(\theta_{i-1} - \theta_\zeta)}{(\theta_i - \theta_\zeta)} \prod_{\mu+d < \zeta \le \mu'+d'} \frac{(\theta_{i-1} - \theta_\zeta)}{(\theta_i - \theta_\zeta)} \quad (i-1,\, i \in W_s \cap W_s').
\end{aligned}$$

To evaluate this, we use the following notation. Set

$$\left(\left(\frac{\alpha}{\beta}\right)\right) = \begin{cases} 1 & \text{if } \alpha = \beta = 0 \\ \alpha/\beta & \text{if } \beta \ne 0 \\ \text{undefined} & \text{if } \alpha \ne 0,\ \beta = 0 \end{cases} \qquad (\alpha,\, \beta \in \mathbb{C}).$$

Then

$$\left(\left(\frac{\alpha}{\beta}\right)\right)\left(\left(\frac{\beta}{\gamma}\right)\right) = \left(\left(\frac{\alpha}{\gamma}\right)\right) \qquad (\alpha,\, \beta,\, \gamma \in \mathbb{C}) \tag{145}$$

as long as $((\alpha/\beta))$, $((\beta/\gamma))$ are defined. Evaluating the data in Case I of Theorem 2.1 using (82), we find that for all $i$ ($i-1, i \in W_s \cap W'_s$):

$$b^*_{i-\mu-1}(W) = \left(\left(\frac{1-sq^{i+\mu}}{1-sq^{2i-1}}\right)\right) \times \frac{h^* q^{-\nu}(1-q^{i-\mu-d-1})(1-f_1q^{i+\nu})(1-f_2q^{i+\nu})}{1-sq^{2i}}, \tag{146}$$

$$b^*_{i-\mu'-1}(W')^{-1} = \left(\left(\frac{1-sq^{2i-1}}{1-sq^{i+\mu'}}\right)\right) \times \frac{1-sq^{2i}}{h^* q^{-\nu'}(1-q^{i-\mu'-d'-1})(1-f'_1q^{i+\nu'})(1-f'_2q^{i+\nu'})}, \tag{147}$$

$$c^*_{i-\mu'}(W') = \left(\left(\frac{1-sq^{i+\mu'+d'+1}}{1-sq^{2i+1}}\right)\right) \times \frac{h^* s^* q^{\nu'+1}(1-q^{i-\mu'})(1-sq^{i-\nu'}/f'_1)(1-sq^{i-\nu'}/f'_2)}{1-sq^{2i}}, \tag{148}$$

$$c^*_{i-\mu}(W)^{-1} = \left(\left(\frac{1-sq^{2i+1}}{1-sq^{i+\mu+d+1}}\right)\right) \times \frac{1-sq^{2i}}{h^* s^* q^{\nu+1}(1-q^{i-\mu})(1-sq^{i-\nu}/f_1)(1-sq^{i-\nu}/f_2)}, \tag{149}$$

where the (( )) expressions in (146)–(149) are all defined. From (15) we also have

$$\prod_{\mu\le\xi<\mu'} \frac{(\theta_i-\theta_\xi)}{(\theta_{i-1}-\theta_\xi)} \prod_{\mu'\le\zeta<\mu} \frac{(\theta_{i-1}-\theta_\zeta)}{(\theta_i-\theta_\zeta)} = \left(\left(\frac{1-sq^{i+\mu'}}{1-sq^{i+\mu}}\right)\right) \frac{q^{\mu-\mu'}(1-q^{i-\mu})}{1-q^{i-\mu'}}, \tag{150}$$

$$\prod_{\mu'+d'<\xi\le\mu+d} \frac{(\theta_i-\theta_\xi)}{(\theta_{i-1}-\theta_\xi)} \prod_{\mu+d<\zeta\le\mu'+d'} \frac{(\theta_{i-1}-\theta_\zeta)}{(\theta_i-\theta_\zeta)} = \left(\left(\frac{1-sq^{i+\mu+d+1}}{1-sq^{i+\mu'+d'+1}}\right)\right) \frac{q^{\mu'-\mu+d'-d}(1-q^{i-\mu'-d'-1})}{1-q^{i-\mu-d-1}}, \tag{151}$$

where the (( )) expressions in (150) and (151) are defined. Now using (145), we find the product of the (( )) expressions in (146)–(151) is 1. Multiplying together the remaining factors in (146)–(151), we find

$$\phi_i/\phi_{i-1} = \frac{q^{2(\nu'-\nu)+d'-d}(1-f_1q^{i+\nu})(1-f_2q^{i+\nu})}{(1-f'_1q^{i+\nu'})(1-f'_2q^{i+\nu'})} \times \frac{(1-sq^{i-\nu'}/f'_1)(1-sq^{i-\nu'}/f'_2)}{(1-sq^{i-\nu}/f_1)(1-sq^{i-\nu}/f_2)} \qquad (i-1, i \in W_s \cap W'_s). \tag{152}$$

But (152) equals (144) upon applying (136) and (142).

To complete the proof of Theorem 4.6, we will need the following identity. See the given reference for a proof.

LEMMA 4.7. (Terwilliger [68]). *Let $m$, $n$ denote any nonnegative integers with $m \le n$, and pick any nonzero scalars $a, b, c, d, e, q \in \mathbb{C}$ such that*

$$a^2 q^{2-m+n} = bcde. \tag{153}$$

*Then*

$$\det\begin{pmatrix} \phi_0^+ & \phi_1^+ & \cdots & \phi_{m+n+1}^+ \\ \phi_0^+\vartheta_0 & \phi_1^+\vartheta_1 & \cdots & \phi_{m+n+1}^+\vartheta_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0^+\vartheta_0^m & \phi_1^+\vartheta_1^m & \cdots & \phi_{m+n+1}^+\vartheta_{m+n+1}^m \\ \phi_0^- & \phi_1^- & \cdots & \phi_{m+n+1}^- \\ \phi_0^-\vartheta_0 & \phi_1^-\vartheta_1 & \cdots & \phi_{m+n+1}^-\vartheta_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0^-\vartheta_0^n & \phi_1^-\vartheta_1^n & \cdots & \phi_{m+n+1}^-\vartheta_{m+n+1}^n \end{pmatrix} \tag{154}$$

$$\begin{aligned} = {} & \phi_0^+\phi_1^+\cdots\phi_m^+\phi_0^-\phi_1^-\cdots\phi_n^- \det(\vartheta_i^j)_{0\le i,j\le m} \det(\vartheta_{m+i+1}^j)_{0\le i,j\le n} \\ & \times \prod_{k=0}^{m}\left(aq^{m+k+1}, \frac{aq^{k-m+1}}{bc}, \frac{aq^{k-m+1}}{bd}, \frac{aq^{k-m+1}}{be}; q\right)_{m+n+1-2k} \\ & \times b^{(m+1)(n+1)} q^{(m+n)(m+1)(n+1)}, \end{aligned}$$

*where*

$$\begin{aligned} \vartheta_i &= q^{-i} + aq^i && (0 \le i \le m+n+1), \\ \phi_i^+ &= q^{mi}(b, c, d, e; q)_i && (0 \le i \le m+n+1), \\ \phi_i^- &= q^{ni}(aq/b, aq/c, aq/d, aq/e; q)_i && (0 \le i \le m+n+1). \end{aligned}$$

*We now return to the proof of Theorem 4.6.*

*Claim 2.* Suppose $m \le n$. Then

$$\det\begin{pmatrix} \phi_\tau & \phi_{\tau+1} & \cdots & \phi_{\tau+m+n+1} \\ \phi_\tau\theta_\tau & \phi_{\tau+1}\theta_{\tau+1} & \cdots & \phi_{\tau+m+n+1}\theta_{\tau+m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_\tau\theta_\tau^m & \phi_{\tau+1}\theta_{\tau+1}^m & \cdots & \phi_{\tau+m+n+1}\theta_{\tau+m+n+1}^m \\ 1 & 1 & \cdots & 1 \\ \theta_\tau & \theta_{\tau+1} & \cdots & \theta_{\tau+m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_\tau^n & \theta_{\tau+1}^n & \cdots & \theta_{\tau+m+n+1}^n \end{pmatrix} \tag{155}$$

$$= \phi_\tau \phi_{\tau+1} \cdots \phi_{\tau+m} \det (\theta^j_{i+\tau})_{0 \le i,j \le m} \det (\theta^j_{i+\tau+m+1})_{0 \le i,j \le n} \Delta_0 \Delta_1 \cdots \Delta_m, \quad (156)$$

where $\Delta_k$ $(0 \le k \le m)$ is given as follows:

*Case I* $(ss^* \neq 0)$

$$\Delta_k = \frac{(sq^{\tau-\nu'+m+1})^{m+n+1-2k}(sq^{2\tau+m+2+k},\ s^*q^{2\nu+d+1-n+k};\ q)_{m+n+1-2k}}{(f_2')^{m+n+1-2k}(sq^{\tau-\nu+1+k}/f_1,\ sq^{\tau-\nu+1+k}/f_2;\ q)_{m+n+1-2k}} \times \frac{(f_2'q^{\nu'-\nu-m+k}/f_1,\ f_2'q^{\nu'-\nu-m+k}/f_2;\ q)_{m+n+1-2k}}{(f_1'q^{\tau+\nu'+1+k},\ f_2'q^{\tau+\nu'+1+k};\ q)_{m+n+1-2k}} \quad (157)$$

*Case I* $(s = f_1 = f_1' = 0)$

$$\Delta_k = \frac{(s^*q^{2\nu+d+1-n+k};\ q)_{m+n+1-2k} \prod_{\eta=\nu'-\nu-m+k}^{\nu'-\nu+n-k}(f_2 - f_2'q^\eta)}{(f_2'q^{\tau+\nu'+1+k};\ q)_{m+n+1-2k} \prod_{\eta=d+\nu+k-\tau-m-n}^{\nu+d-\tau-k}(f_2 - s^*q^\eta)}$$

*Case I* $(s^* = f_1 = f_1' = 0)$

$$\Delta_k = \frac{(sq^{2\tau+m+2+k};\ q)_{m+n+1-2k} \prod_{\eta=\nu'-\nu-m+k}^{\nu'-\nu+n-k}(f_2 - f_2'q^\eta)}{(f_2'q^{\tau+\nu'+1+k};\ q)_{m+n+1-2k} \prod_{\eta=\tau-\nu+1+k}^{\tau-\nu+m+n+1-k}(f_2 - sq^\eta)}$$

*Case IA*

$$\Delta_k = \frac{(sq^{\tau-\nu'+m+1})^{m+n+1-2k}(f'q^{\nu'-\nu-m+k}/f;\ q)_{m+n+1-2k}}{(f')^{m+n+1-2k}(sq^{\tau-\nu+1+k}/f;\ q)_{m+n+1-2k}}$$

*Case II*

$$\Delta_k = \frac{(-2\tau - m - s - 2 - k,\ n - 2\nu - d - 1 - s^* - k)_{m+n+1-2k}}{(f_1 - s - \tau + \nu - 1 - k,\ f_2 - s - \tau + \nu - 1 - k)_{m+n+1-2k}} \times \frac{(f_1 - f_2' + \nu - \nu' + m - k,\ f_2 - f_2' + \nu - \nu' + m - k)_{m+n+1-2k}}{(-\nu' - 1 - \tau - f_1' - k,\ -\nu' - 1 - \tau - f_2' - k)_{m+n+1-2k}}$$

*Case IIA*

$$\Delta_k = \frac{(-2\tau - m - s - 2 - k,\ f - f' + \nu - \nu' + m - k)_{m+n+1-2k}}{(f - s - \tau + \nu - 1 - k,\ -\nu' - 1 - \tau - f' - k)_{m+n+1-2k}}$$

*Case IIB*

$$\Delta_k = \frac{(s^* + 2\nu + d + 1 + m - k,\ f - f' + \nu - \nu' + m - k)_{m+n+1-2k}}{(s^* - f + \nu + d - \tau - k,\ -\nu' - 1 - \tau - f' - k)_{m+n+1-2k}}$$

*Case IIC*

$$\Delta_k = \frac{(f - f')^{m+n+1-2k}(ss^*)^{m+n+1-2k}}{(f - ss^*)^{m+n+1-2k}(f')^{m+n+1-2k}}$$

*Case III*

$$\Delta_k = \frac{\prod_{\substack{\eta=2\tau+m+2+k \\ \eta\, even}}^{2\tau+2m+n+2-k}(\eta - s)\prod_{\substack{\eta=2\nu+d+1-n+k \\ \eta\, even}}^{2\nu+d+1+m-k}(\eta - s^*)}{\prod_{\substack{\eta=\tau-\nu+1+k \\ \mu+\nu+\eta+d\, even}}^{\tau-\nu+m+n+1-k}(\eta - f_1 - s)\prod_{\substack{\eta=\tau-\nu+1+k \\ \mu+\nu+\eta\, odd}}^{\tau-\nu+m+n+1-k}(\eta - f_2 - s)} \times \frac{\prod_{\substack{\eta=\nu'-\nu-m+k \\ \mu-\mu'+\nu-\nu'+\eta+d\, odd}}^{\nu'-\nu+n-k}(f_2' - f_1 + \eta)\prod_{\substack{\eta=\nu'-\nu-m+k \\ \mu-\mu'+\nu-\nu'+\eta\, even}}^{\nu'-\nu+n-k}(f_2' - f_2 + \eta)}{\prod_{\substack{\eta=\tau+\nu'+1+k \\ \eta-\mu'-\nu+d'\, even}}^{\tau+\nu'+m+n+1-k}(\eta + f_1')\prod_{\substack{\eta=\tau+\nu'+1+k \\ \eta-\mu'-\nu'\, odd}}^{\tau+\nu'+m+n+1-k}(\eta + f_2')}$$

*Proof of Claim 2.* First assume Case I ($ss^* \neq 0$), and consider the matrix in Lemma 4.7, where $(a, b, c, d, e, q)$ are from (142). Note the determinant formula in that lemma remains valid if we replace $\vartheta_i$ by $\alpha\vartheta_i + \beta$, $(0 \le i \le m + n + 1)$, where $\alpha$, $\beta$ are any complex numbers. Choose

$$\alpha = hq^{-\tau}, \qquad \beta = \theta_0 - h(1 + sq),$$

and observe by (14), (142) that

$$\begin{aligned} \alpha\vartheta_i + \beta &= hq^{-\tau}(q^{-i} + aq^i) + \theta_0 - h(1 + sq) \\ &= hq^{-\tau}(q^{-i} + sq^{2\tau+1+i}) + \theta_0 - h(1 + sq) \\ &= \theta_0 + h(1 - q^{i+\tau})(1 - sq^{i+\tau+1})q^{-i-\tau} \\ &= \theta_{i+\tau} \qquad (0 \le i \le m + n + 1). \end{aligned}$$

Thus the determinant formula in Lemma 4.7 remains valid if we replace $\vartheta_i$ by $\theta_{i+\tau}$, $(0 \le i \le m+n+1)$. But after this replacement, the matrix (155) is obtained from the matrix (154) by dividing column $i$ by $\phi_i^-(0 \le i \le m + n + 1)$. Now the determinant (155) can be readily determined from Lemma 4.7. In Case I ($ss^* = 0$), IA, II, IIA, IIB, IIC, III, the determinant (155) is obtained by taking limits as indicated in Note 2.6.

*Claim 3.* There exists an integer $\eta$ such that

$$\nu' - \nu - m \le \eta \le \nu' - \nu + n, \tag{158}$$

and such that

*Case I* $\quad f_1, f_2$ is a permutation of

$$f_1'q^{d-d'-\eta},\ f_2'q^\eta \tag{159}$$

*Case IA*

$$f = f'q^\eta \tag{160}$$

*Case II* $\quad f_1, f_2$ is a permutation of

$$f_1' + d - d' - \eta,\ f_2' + \eta \tag{161}$$

*Case IIA, IIB*

$$f = f' + \eta \tag{162}$$

*Case IIC*

$$f = f' \tag{163}$$

*Case III*

$$\begin{aligned} &f_1 = f_1' + d - d' - \eta,\ f_2 = f_2' + \eta, \\ \text{and}\quad & \\ &\mu - \mu' + \nu - \nu' + \eta \text{ is even,} \end{aligned} \tag{164}$$

or

$$\begin{aligned} &f_1 = f_2' + \eta,\ f_2 = f_1' + d - d' - \eta, \\ \text{and}\quad & \\ &\mu - \mu' + \nu - \nu' + \eta + d \text{ is odd.} \end{aligned} \tag{165}$$

*Proof of Claim 3.* Interchanging the roles of $W$, $W'$, if necessary, we may assume $m \leq n$. Also observe by (133) and (138) that

$$\tau, \tau + 1, \ldots, \tau + m + n + 1 \in W_s \cap W_s'.$$

Now the rows of the matrix (155) are linearily dependent by (139), (110), and (111), so the determinant (156) of that matrix is 0. We have observed the constants $\phi_\tau, \phi_{\tau+1}, \ldots, \phi_{\tau+m}$ are nonzero, and the Vandermonde determinants in (156) are nonzero, so $\Delta_k = 0$ for some integer $k$ ($0 \leq k \leq m$). Now assume Case I ($ss^* \neq 0$), and consider the factors in the numerator of (157). The first two factors $s$, $q$ are assumed to be nonzero. The next factor is

$$\prod (1 - sq^\xi), \tag{166}$$

where the product is over all integers $\xi$ such that

$$2\tau + m + 2 + k \leq \xi \leq 2\tau + 2m + n + 2 - k. \tag{167}$$

Recall by (17) that $sq^\xi \neq 1$ ($2 \leq \xi \leq 2D$). Therefore, the product (166) is nonzero if we can show

$$2 \leq 2\tau + m + 2 + k \tag{168}$$

and

$$2\tau + 2m + n + 2 - k \leq 2D. \tag{169}$$

The bound (168) is immediate, since $\tau$, $m$, $k$ are nonnegative by (133) and (110), and the definition of $k$. To see (169), observe by (111) and (138), that

$$\begin{aligned} 2\tau + 2m + n + 2 - k &\leq 2(2\tau + m + n + 1) \\ &\leq 2(\tau + |W_s \cap W_s'| - 1) \\ &= 2\min\{\mu + d, \mu' + d'\} \\ &\leq 2D. \end{aligned}$$

The next factor in the numerator of (157) is

$$\prod (1 - s^* q^\zeta), \tag{170}$$

where the product is over all integers $\zeta$ such that

$$2\nu + d + 1 - n + k \leq \zeta \leq 2\nu + d + 1 + m - k.$$

Just as above, by (17) we have $s^* q^\zeta \neq 1$ $(2 \leq \zeta \leq 2D)$. Therefore, the product (170) is nonzero if we can show

$$2 \leq 2\nu + d + 1 - n + k \tag{171}$$

and

$$2\nu + d + 1 + m - k \leq 2D. \tag{172}$$

Line (171) holds, since $n \leq d - 1$ by (138) and $\nu$, $k$ are nonnegative. To see (172), observe

$$\begin{aligned} 2\nu + d + 1 + m - k &\leq 2\nu + d + m + n + 1 \\ &\leq 2\nu + d + |W_s \cap W_s'| - 1 \\ &\leq 2(\nu + d) \\ &\leq 2D. \end{aligned}$$

The remaining factor in the numerator of (157) is

$$\prod (1 - f_2' q^\eta / f_1)(1 - f_2' q^\eta / f_2), \tag{173}$$

where the product is over all integers $\eta$ such that

$$\nu' - \nu - m + k \leq \eta \leq \nu' - \nu + n - k. \tag{174}$$

The product (173) must be 0 since $\Delta_k$ is 0, so

$$f_1 = f_2' q^\eta \text{ or } f_2 = f_2' q^\eta \tag{175}$$

for some integer $\eta$ that satisfies (174). Now (158) holds since $k$ is nonnegative, and line (159) follows from (175) and (143). We have now proved Claim 3 for Case I ($ss^* \neq 0$). The remaining cases are very similar.

Now set

$$\varepsilon = d - d' - 2\eta, \tag{176}$$

where $\eta$ is from Claim 3. Let us check that $\varepsilon$ satisfies (123). Certainly $\varepsilon + d + d'$ is even. Also, by (136), (158)

$$\begin{aligned} \varepsilon &\leq d - d' + 2(\nu - \nu') + 2m \\ &= m + n, \end{aligned} \tag{177}$$

and

$$\begin{aligned} \varepsilon &\geq d - d' + 2(\nu - \nu') - 2n \\ &= -m - n, \end{aligned} \tag{178}$$

so

$$\begin{aligned} |\varepsilon| &\leq m + n \\ &= 2\rho - |\mu - \mu'| - |\mu - \mu' + d - d'| \end{aligned} \tag{179}$$

by (137), (177) and (178). Thus (123) holds. Solving (176) for $\eta$, we find

$$\eta = \frac{d - d' - \varepsilon}{2}. \tag{180}$$

Now (124)–(130) are obtained upon evaluating the data in Claim 3 using (180). This proves (iii).

*Proof of (iv).* Assuming $\rho \leq \nu - \nu'$ in (122), we find

$$\begin{aligned} 2(\nu - \nu') &\geq 2\rho && (181) \\ &\geq |\mu - \mu'| + |\mu - \mu' + d - d'| + |2(\nu - \nu') + d - d'| && (182) \\ &\geq (\mu - \mu') + (\mu' - \mu + d' - d) + (2\nu - 2\nu' + d - d') && (183) \\ &= 2(\nu - \nu'), \end{aligned}$$

so equality holds in (181)–(183). From (181) we find $\rho = \nu - \nu'$. Comparing (181) and (183), we see the three terms in parenthesis in (183) are equal to their absolute value, and are, hence, nonnegative. This implies (131).

*Proof of (v).* Note $W_s \subseteq W'_s$ by (131), so

$$\begin{aligned} |W_s \cup W'_s| &= |W'_s| \\ &= d' + 1 \\ &\geq 2\rho + 2, \end{aligned}$$

and part (iii) of the present theorem applies. Let the integer $\varepsilon$ be from that part, let $e'$ denote the auxiliary parameter of $W'$, and set

$$e = \begin{cases} e' - \varepsilon, & \text{if Case III, } \mu' + \nu' + \dfrac{d' - D - e'}{2} \text{ odd,} \\ e' + \varepsilon, & \text{otherwise.} \end{cases} \tag{184}$$

To show $W$ is strong, it suffices to show $e$ satisfies (98)–(106). First consider (98). Certainly $e + d + D$ is even by (184), since $e' + d' + D$ is even by Definition 4.4 and $\varepsilon + d + d'$ is even by (123). By Definition 4.4, (123), (131), (132), and (184) we also have

$$\begin{aligned}
|e| &\leq |e'| + |\varepsilon| \\
&\leq (2\nu' - D + d') + (2\rho - |\mu - \mu'| - |\mu - \mu' + d - d'|) \\
&= (2\nu - 2\rho - D + d') + (2\rho + \mu' - \mu + \mu - \mu' + d - d') \\
&= 2\nu - D + d \\
&\leq 2\mu - D + d,
\end{aligned}$$

so (98) holds. Now for the moment assume Case I. Then by (124), $f_1$, $f_2$ is a permutation of

$$f_1' q^{\frac{d-d'+\varepsilon}{2}}, \quad f_2' q^{\frac{d-d'-\varepsilon}{2}},$$

where we may assume

$$f_1' = r_1 q^{\frac{d'-D+e'}{2}}, \quad f_2' = r_2 q^{\frac{d'-D-e'}{2}}$$

by Definition 4.4. Combining this information, we obtain (99). Lines (100)–(106) are obtained in a similar manner. This proves (v), and the theorem. □

We now give "dual" versions of Lemma 4.5 and Theorem 4.6.

LEMMA 4.8. *Let the scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be as in Theorem 4.1. Pick any $x, y \in X$, any thin irreducible $T(x)$-modules $W$, $W'$, and suppose $W$, $A^*_\rho(y)W'$ are not orthogonal for some integer $\rho$ $(0 \leq \rho \leq D)$. Let $\mu$, $\mu'$ denote the endpoints of $W$, $W'$, respectively, and pick nonzero vectors $u \in E_\mu W$, $u' \in E_{\mu'} W'$. Write $E^*_i = E^*_i(x)$ $(0 \leq i \leq D)$. Then there exist nonzero polynomials $\psi^*$, $\psi^{*\prime} \in \mathbb{C}[\lambda]$ such that*

$$\begin{aligned}
\langle E^*_i u, A^*_\rho(y) E^*_i u' \rangle &= \|E^*_i u\|^2 \psi^*(\theta^*_i) \prod_{\xi \in W_\sigma \setminus W'_\sigma} (\theta^*_i - \theta^*_\xi) \qquad (0 \leq i \leq D), &(185)\\
&= \|E^*_i u'\|^2 \psi^{*\prime}(\theta^*_i) \prod_{\zeta \in W'_\sigma \setminus W_\sigma} (\theta^*_i - \theta^*_\zeta) \qquad (0 \leq i \leq D), &(186)
\end{aligned}$$

*and*

$$\deg \psi^* \leq \rho - \mu + \mu' - |W_\sigma \setminus W'_\sigma|, \qquad (187)$$

$$\deg \psi^{*\prime} \leq \rho - \mu' + \mu - |W'_\sigma \setminus W_\sigma|. \qquad (188)$$

*Proof.* Write $A^* = A^*_1(x)$, and observe $A^*$, $E^*_i$ $(0 \leq i \leq D)$ commute with $A^*_\rho(y)$.

*Claim 1.* There exist nonzero polynomials $\varphi^*$, $\varphi^{*\prime} \in \mathbb{C}[\lambda]$ such that

$$\begin{aligned} &\deg \varphi^* \le \rho - \mu + \mu', \\ &\deg \varphi^{*\prime} \le \rho - \mu' + \mu, \end{aligned} \tag{189}$$

and

$$\begin{aligned} &\mathrm{proj}_W A^*_\rho(y)u' = \overline{\varphi(A^*)}u, \\ &\mathrm{proj}_{W'} A^*_\rho(y)u = \varphi^{*\prime}(A^*)u'. \end{aligned} \tag{190}$$

*Proof of Claim 1.* By symmetry, it suffices to show (189) and (190). To do this, it suffices to show $\mathrm{proj}_W A^*_\rho(y)u'$ is a nonzero and contained in $\mathrm{Span}\{u, A^*u, \ldots, (A^*)^{\rho-\mu+\mu'}u\}$. Now by assumption, there exists $w \in W$, $w' \in W'$ such that $\langle w, A^*_\rho(y)w'\rangle \neq 0$, and by (81) we may write $w' = au'$ for some element $a$ of the dual Bose-Mesner algebra $M^*(x)$. Since $a$ is symmetric and commutes with $A^*_\rho(y)$, we obtain

$$\begin{aligned} \langle \bar{a}w, A^*_\rho(y)u'\rangle &= \langle w, aA^*_\rho(y)u'\rangle \\ &= \langle w, A^*_\rho(y)w'\rangle \\ &\neq 0, \end{aligned}$$

and since $\bar{a}w \in W$, we observe $\mathrm{proj}_W A^*_\rho(y)u' \neq 0$. Now write

$$\mathrm{proj}_W A^*_\rho(y)u' = u_\mu + u_{\mu+1} + \cdots + u_{\mu+d}, \tag{191}$$

where $d$ denotes the diameter of $W$, and where $u_i \in E_iW$ $(\mu \le i \le \mu + d)$. Then

$$u_i = 0 \qquad (\rho + \mu' < i \le \mu + d), \tag{192}$$

since

$$\begin{aligned} \|u_i\|^2 &= \langle u_i, A^*_\rho(y)u'\rangle \\ &= 0 \qquad (\rho + \mu' < i \le \mu + d) \end{aligned}$$

by (65). Now by (81), (191), and (192), we have

$$\begin{aligned} \mathrm{proj}_W A^*_\rho(y)u' &= u_\mu + u_{\mu+1} + \cdots + u_{\rho+\mu'} \\ &\in \mathrm{Span}\{u, A^*u, \ldots, (A^*)^{\rho-\mu+\mu'}u\}, \end{aligned}$$

as desired. This proves Claim 1.

*Claim 2.*

$$\langle E^*_i u, A^*_\rho(y)E^*_i u'\rangle = \|E^*_i u\|^2 \varphi^*(\theta^*_i) \qquad (0 \le i \le D) \tag{193}$$

$$= \|E^*_i u'\|^2 \varphi^{*\prime}(\theta^*_i) \qquad (0 \le i \le D). \tag{194}$$

In particular

$$\lambda - \theta^*_\xi \text{ divides } \varphi^* \text{ for each } \xi \in W_\sigma \backslash W'_\sigma, \tag{195}$$

$$\lambda - \theta^*_\zeta \text{ divides } \varphi^{*\prime} \text{ for each } \zeta \in W'_\sigma \backslash W_\sigma. \tag{196}$$

*Proof of Claim 2.* Since $A^*_\rho(y)u' - \mathrm{proj}_W A^*_\rho(y)u'$ is orthogonal to $W$, we have, for each integer $i$ $(0 \le i \le D)$,

$$\begin{aligned} 0 &= \langle E^*_i u,\ A^*_\rho(y)u' - \mathrm{proj}_W A^*_\rho(y)u'\rangle \\ &= \langle E^*_i u,\ A^*_\rho(y)u' - \overline{\varphi^*(A^*)u}\rangle \\ &= \langle E^*_i u,\ E^*_i(A^*_\rho(y)u' - \overline{\varphi^*(A^*)u})\rangle \\ &= \langle E^*_i u,\ A^*_\rho(y)E^*_i u'\rangle - \varphi^*(\theta^*_i)\|E^*_i u\|^2, \end{aligned}$$

which gives (193). The remaining assertions of the claim are obtained as in Claim 2 of Theorem 4.6.

Now set

$$\psi^* := \varphi^* \prod_{\xi \in W_\sigma \backslash W'_\sigma} (\lambda - \theta^*_\xi)^{-1}, \tag{197}$$

$$\psi^{*\prime} := \varphi^{*\prime} \prod_{\zeta \in W'_\sigma \backslash W_\sigma} (\lambda - \theta^*_\zeta)^{-1}.$$

Observe $\psi^*$, $\psi^{*\prime}$ are nonzero by Claim 1, and contained in $\mathbb{C}[\lambda]$ by (195) and (196). They satisfy (185) and (186) by (193) and (194), and satisfy (187) and (188) by Claim 1. This proves Lemma 4.8. □

THEOREM 4.9. *Let the scheme $Y = (X, \{R_i\}_{0\le i\le D})$ be as in Theorem 4.1. Pick any $x, y \in X$, any thin irreducible $T(x)$-modules $W$, $W'$, and suppose $A^*_\rho(y)W'$, $W$ are not orthogonal for some integer $\rho$ $(0 \le \rho \le D)$. Let $(\mu, \nu, d, f)$, $(\mu', \nu', d', f')$ denote the data sequences of $W$, $W'$, respectively. Then the following statements (i)–(v) hold.*

*(i) $W_\sigma \cap W'_\sigma \ne \emptyset$, where we recall $W_\sigma = \{\nu, \nu+1, \ldots, \nu+d\}$, $W'_\sigma = \{\nu', \nu'+1, \ldots, \nu'+d'\}$.*
*(ii)*

$$|\nu - \nu'| + |\nu - \nu' + d - d'| + |2(\mu - \mu') + d - d'| \le 2\rho. \tag{198}$$

*(iii) Assume $|W_\sigma \cup W'_\sigma| \ge 2\rho + 2$. Then $d, d' \ge 1$. Furthermore, there exists an integer $\varepsilon$ satisfying*

$$\varepsilon + d + d' \text{ even}, \quad |\varepsilon| \le 2\rho - |\nu - \nu'| - |\nu - \nu' + d - d'|, \tag{199}$$

*such that*

*Case I* $\quad$ *$f_1, f_2$ is a permutation of*

$$f'_1 q^{\frac{d-d'+\varepsilon}{2}},\ f'_2 q^{\frac{d-d'-\varepsilon}{2}} \tag{200}$$

*Case IA*

$$f = f' q^{\frac{d-d'-\varepsilon}{2}}$$

*Case II* $\quad f_1, f_2$ *is a permutation of*

$$f_1' + \frac{d-d'+\varepsilon}{2}, \quad f_2' + \frac{d-d'-\varepsilon}{2}$$

*Case IIA, IIB*

$$f = f' + \frac{d-d'-\varepsilon}{2}$$

*Case IIC*

$$f = f'$$

*Case III*

$$f_1 = f_1' + \frac{d-d'+\varepsilon}{2}, \quad f_2 = f_2' + \frac{d-d'-\varepsilon}{2},$$

*and*

$$\mu - \mu' + \nu - \nu' + \frac{d-d'-\varepsilon}{2} \text{ is even,}$$

*or*

$$f_1 = f_2' + \frac{d-d'-\varepsilon}{2}, \quad f_2 = f_1' + \frac{d-d'+\varepsilon}{2},$$

*and*

$$\mu - \mu' + \nu - \nu' + \frac{d-d'-\varepsilon}{2} + d \text{ is odd.}$$

*(iv) Suppose*

$$\rho \leq \mu - \mu'.$$

*Then*

$$0 \leq \nu - \nu' \leq d' - d \leq 2(\mu - \mu') = 2\rho. \tag{201}$$

*(v) Suppose $W'$ is strong, and that*

$$\mu - \mu' = \rho < d'/2, \qquad \mu \leq \nu. \tag{202}$$

*Then $W$ is strong.*

*Proof.* Similar to the proof of Theorem 4.6. □

THEOREM 4.10. *Let the scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be as in Theorem 4.1. Pick any $x \in X$, and let $W$ denote a thin irreducible $T(x)$-module, with some endpoint $\mu$, dual endpoint $\nu$, and diameter $d$ $(0 \leq \mu, \nu \leq D - d \leq D)$. Then*

*(i) $W$ is strong whenever*

$$\mu < D/2 \quad \text{or} \quad \nu < D/2. \tag{203}$$

*(ii)*

$$(D - d)/2 \leq \mu, \nu. \tag{204}$$

*(iii) Suppose $Y$ is thin. Then $W$ is strong whenever*

$$\mu < D/2 \quad \text{or} \quad \nu < D/2 \quad \text{or} \quad d \geq 3.$$

*Proof of (i).* Assume (203). Then one of the following (a)–(c) holds.

(a)

$$\mu = 0 \quad \text{or} \quad \nu = 0, \tag{205}$$

(b)

$$0 < \nu \leq \mu, \qquad \nu < D/2,$$

(c)

$$0 < \mu < \nu, \qquad \mu < D/2.$$

First suppose (a), so that $W = M\hat{x}$ by part (vi) of Theorem 4.1. Then, using Definition 4.4 one may readily check that $M\hat{x}$ is strong (with auxiliary parameter $e = 0$), so we are done in this case. Next suppose (b), and set $W' = M\hat{y}$, where $y$ is any element in $X$ with $(x, y) \in R_\nu$, such that $\hat{y}$ is not orthogonal to $W$ ($y$ exists by the definition of $\nu$). Then $W'$ is a thin irreducible $T(y)$-module by Lemma 3.6, $W'$ is strong by case (a) above, and $W$, $W'$ are not orthogonal by construction. Now $W$, $W'$ satisfy the conditions of part (v) of Theorem 4.6 (with $\rho = \nu$, $\nu' = 0$, $d' = D$), so $W$ is strong. Next assume (c), and set $W' = M\hat{x}$. Then $W'$ is a thin, irreducible $T(x)$-module, and strong by part (a) above. Also, there exists $y \in X$ such that $W$, $A^*_\mu(y)W'$ are not orthogonal, since the all 1s vector $\delta \in W'$, $A^*_\mu(y)\delta = |X|E_\mu\hat{y}$ by (69), and Span $\{E_\mu\hat{y} \mid y \in X\} = E_\mu V$ is not orthogonal to $W$ by the definition of $\mu$. Now $W$, $W'$, and $y$ satisfy the conditions of part (v) of Theorem 4.9 (with $\mu' = 0$, $d' = D$, and $\rho = \mu$), so $W$ is strong. Thus $W$ is strong in general, and we are done.

*Proof of (ii).* Suppose $\mu < (D-d)/2$ or $\nu < (D-d)/2$. Then (203) holds, so $W$ is strong. But then $2\mu - D + d$, $2\nu - D + d$ are nonnegative by (98), a contradiction. This proves (ii).

*Proof of (iii).* In view of part (i) above, it suffices to prove $W$ is strong under the assumption $d \geq 3$. The proof is by induction on $\mu + \nu$.

First assume $\nu \leq \mu$. Pick any $y' \in X$ such that $(x, y') \in R_\nu$, and such that $\hat{y}'$ is not orthogonal to $W$. Now pick any $y \in X$ such that $(x, y) \in R_1$ and

$(y, y') \in R_{\nu-1}$. Now $E^*_{\nu-1}(y)V$ contains $\hat{y}'$, and is therefore not orthogonal to $W$. But now there exists a (thin) irreducible $T(y)$-module $W'$, with endpoint $\nu' \leq \nu - 1$, that is not orthogonal to $W$. Applying part (iv) of Theorem 4.6 to $W$, $W'$ with $\rho = 1$, we find by (131) that $\mu' \leq \mu$, $d' \geq d$, and $\nu' = \nu - 1$, where $\mu'$, $d'$ are, respectively, the endpoint and diameter of $W'$. In particular $d' \geq 3$ and $\mu' + \nu' < \mu + \nu$, so $W'$ is strong by induction. But now $W$, $W'$ satisfy the conditions of part (v) of Theorem 4.6, so $W$ is strong. Next assume $\mu \leq \nu$. Since $E_\mu V$ is contained in the column space of $E_1 \circ E_{\mu-1}$ by (38), there exists $y \in X$ such that $A^*_1(y)E_{\mu-1}\hat{y}$ is not orthogonal to $W$. But then there exists a (thin) irreducible $T(x)$-module $W'$, with endpoint $\mu' \leq \mu - 1$, such that $A^*_1(y)W'$ is not orthogonal to $W$. Applying part (iv) of Theorem 4.9 to $W$, $W'$ with $\rho = 1$, we find $\nu' \leq \nu$, $d' \geq d$, and $\mu' = \mu - 1$, where $\nu'$ and $d'$ are, respectively, the dual endpoint and diameter of $W'$. In particular $d' \geq 3$ and $\mu' + \nu' < \mu + \nu$, so $W'$ is strong by induction. But now $W$, $W'$ satisfy the conditions of part (v) of Theorem 4.9, so $W$ is strong. □

COROLLARY 4.11. *Let the scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be as in Theorem 4.1, and pick any $x \in X$. Pick any integers $\mu$, $\nu$, $d$ $(0 \leq \mu, \nu \leq D - d \leq D)$, and assume $\mu < D/2$ or $\nu < D/2$. Then the number of pairwise nonisomorphic, thin, irreducible $T(x)$-modules with*

*(i) endpoint $\mu$, dual endpoint $\nu$, diameter $d$*
*(ii) endpoint $\mu$, dual endpoint $\nu$*
*(iii) endpoint $\mu$*
*(iv) dual endpoint $\nu$*

*is at most*

*(i)*

$$\binom{2\mu - D + d + 1}{1} \qquad (\text{if } \mu \leq \nu), \tag{206}$$

$$\binom{2\nu - D + d + 1}{1} \qquad (\text{if } \nu \leq \mu). \tag{207}$$

*(ii)*

$$\binom{2\mu - \nu + 2}{2} \qquad (\text{if } \mu \leq \nu), \tag{208}$$

$$\binom{2\nu - \mu + 2}{2} \qquad (\text{if } \nu \leq \mu). \tag{209}$$

*(iii)*

$$(4\mu^3 + 18\mu^2 + 28\mu + 15 + (-1)^\mu)/16 \qquad (\text{if } \mu < D/2), \tag{210}$$

*(iv)*

$$(4\nu^3 + 18\nu^2 + 28\nu + 15 + (-1)^\nu)/16 \qquad (\text{if } \nu < D/2). \tag{211}$$

*(In (206)–(209), interpret $\binom{\alpha}{\beta} = 0$ if $\alpha < \beta$.)*

*Proof of (i).* Immediate from Lemma 4.3, Definition 4.4, and (203).

*Proof of (ii).* To obtain (208), sum (206) over all integers $d$ $(D-2\mu \le d \le D-\nu)$. To obtain (209), sum (207) over all integers $d$ $(D-2\nu \le d \le D-\mu)$.

*Proof of (iii).* Line (210) is the sum of (208) over all integers $\nu$ $(\mu/2 \le \nu < \mu)$, plus the sum of (209) over all integers $\nu$ $(\mu \le \nu \le 2\mu)$.

*Proof of (iv).* Similar to (iii). □

COROLLARY 4.12. *Let the scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ be as in Theorem 4.1. Pick any $x \in X$, pick any integer $i$ $(0 \le i \le D)$, and write $E_i^* = E_i^*(x)$, $T = T(x)$. (Observe $E_i^*AE_i^* : E_i^*V \to E_i^*V$ is the adjacency map for the undirected graph with vertex set $X \cap E_i^*V$, and edge set $\{(y, z) \mid (y, z) \in R_1, y, z \in X \cap E_i^*V\}$.) Let $W$ denote an irreducible $T$-module with $E_i^*W \ne 0$. Then the following statements (1)–(5) hold.*

*(1) $E_i^*W$ is an $E_i^*AE_i^*$-invariant subspace of $E_i^*V$.*
*(2) Suppose $W$ is thin. Then $E_i^*W$ is a (one-dimensional) eigenspace of $E_i^*AE_i^*$. The eigenvalue is $\lambda := a_{i-\nu}(W)$, where $\nu$ is the dual endpoint of $W$.*

*An $E_i^*AE_i^*$-eigenvalue of this form will be said to be of thin type.*

*(3) $E_i^*V$ is an orthogonal direct sum $E_i^*W_0 + E_i^*W_1 + \cdots + E_i^*W_n$, where $W_0, W_1, \ldots, W_n$ are irreducible $T$-modules that intersect $E_i^*V$ nontrivially. In particular, if $Y$ is thin with respect to $x$ then every eigenvalue of $E_i^*AE_i^* : E_i^*V \to E_i^*V$ is of thin type.*
*(4) Suppose $i < D/2$, and that $W$ is thin. Then there are at most $\sigma_i$ ways to choose $W$ up to isomorphism of $T(x)$-modules, where*

$$\sigma_i = (2i^4 + 16i^3 + 48i^2 + 64i + 31 + (-1)^i)/32.$$

*In particular, $E_i^*AE_i^*$ has at most $\sigma_i$ distinct eigenvalues of thin type.*

*We note $\sigma_0 = 1$, $\sigma_1 = 5$, $\sigma_2 = 16$, $\sigma_3 = 39, \ldots$*

*(5) Suppose $i = 1$, and that $W$ is thin, with some endpoint $\mu$, diameter $d$, and auxiliary parameter $e$. Then $(\mu, \nu, d, e)$, $\lambda$ is given in one of (i)–(v) below (here we use the notation of part (iv) of Theorem 4.1).*

*(i)* $(\mu, \nu, d, e) = (0, 0, D, 0)$, $\lambda = p^1_{11}$.
*(ii)* $(\mu, \nu, d, e) = (1, 1, D-1, 1)$, *$\lambda$ equals*

*Case* $I(s^* \neq 0)$

$$-1 - \frac{s^*(1 - r_2 q^2)(1 - q^{1-D})(1 - s^* q^2)}{(r_2 - s^* q)(q^{-1-D} - s^* q)(1 - s^* q^4)}$$

*Case* $I(s^* = r_1 = 0)$ $\quad -1$

*Case IA* $\quad -1$

*Case II*

$$-1 - \frac{(r_2 + 2)(D - 1)(s^* + 2)}{(r_2 - s^* - 1)(D + s^* + 2)(s^* + 4)}$$

*Case IIA* $\quad -1$

*Case IIB*

$$-1 - \frac{(r + 2)(D - 1)(s^* + 2)}{(r - s^* - 1)(D + s^* + 2)(s^* + 4)}$$

*Case IIC* *does not occur*

*Case III(D even)* *does not occur*

*Case III(D odd)*

$$-1 + \frac{(D - 1)(s^* - 2)}{(r_2 + s^* - 1)(s^* - 4)}.$$

*(iii)* $(\mu, \nu, d, e) = (1, 1, D-1, -1)$, *$\lambda$ equals*

*Case* $I(s^* \neq 0)$

$$-1 - \frac{s^*(1 - r_1 q^2)(1 - q^{1-D})(1 - s^* q^2)}{(r_1 - s^* q)(q^{-1-D} - s^* q)(1 - s^* q^4)}$$

*Case* $I(s^* = r_1 = 0)$

$$-1 - \frac{r_2(1 - q^{1-D})}{s - r_2 q^{-D}}$$

*Case IA*

$$-1 - \frac{r(1 - q^{1-D})}{q(1 - q)}$$

*Case II*

$$-1 - \frac{(r_1 + 2)(D - 1)(s^* + 2)}{(r_1 - s^* - 1)(D + s^* + 2)(s^* + 4)}$$

*Case IIA*

$$-1 + \frac{D - 1}{r - s - D}$$

*Case IIB*

$$-1 - \frac{(D - 1)(s^* + 2)}{(D + s^* + 2)(s^* + 4)}$$

*Case IIC* $-1$

*Case III(D even)*

$$-1 - \frac{(r_1 + 2)(s^* - 2)}{(s^* - D - 2)(s^* - 4)}$$

*Case III(D odd)*

$$-1 + \frac{(D - 1)(s^* - 2)}{(s^* + r_1 - 1)(s^* - 4)}.$$

*(iv)* $(\mu, \nu, d, e) = (1, 1, D - 2, 0)$, $\lambda$ *equals*

*Case* $I(s^* \neq 0)$

$$-1 - \frac{s^*(1 - r_1 q^2)(1 - r_2 q^2)(1 - s^* q^2)}{(r_1 - s^* q)(r_2 - s^* q)(1 - s^* q^4)}$$

*Case* $I(s^* = r_1 = 0)$

$$-1 - \frac{1 - r_2 q^2}{s q^{D+1} - r_2 q}$$

*Case IA*

$$-1 - r q^{2-D}(q - 1)$$

*Case II*

$$-1 - \frac{(r_1 + 2)(r_2 + 2)(s^* + 2)}{(r_1 - s^* - 1)(r_2 - s^* - 1)(s^* + 4)}$$

*Case IIA*

$$-1 - \frac{r+2}{r-s-D}$$

*Case IIB*

$$-1 - \frac{(r+2)(s^*+2)}{(r-s^*-1)(s^*+4)}$$

*Case IIC*

$$-s-2$$

*Case III(D even)*

$$-1 - \frac{(r_1+2)(s^*-2)}{(r_2+s^*-1)(s^*-4)}$$

*Case III(D odd)* *does not occur*

*(v)* $(\mu, \nu, d, e) = (2, 1, D-2, 0)$, $\lambda$ *equals*

*Case I*$(s^* \neq 0)$

$$-1 - \frac{q(1-s^*q^2)}{1-s^*q^4}$$

*Case I*$(s^* = r_1 = 0)$

$$-1-q$$

*Case IA*

$$-1-q$$

*Case II*

$$-1 - \frac{s^*+2}{s^*+4}$$

*Case IIA is* $-2$

*Case IIB*

$$-1 - \frac{s^*+2}{s^*+4}$$

*Case IIC is* $-2$

*Case III*

$$\frac{2}{s^* - 4}$$

*Proof of* (1). Immediate.

*Proof of* (2). Immediate from Theorem 2.1.

*Proof of* (3). Immediate from Lemma 3.4.

*Proof of* (4). $\sigma_i$ is the sum of (211) over $\nu = 0, 1, \ldots, i$.

*Proof of* (5). The given values for $(\mu, \nu, d, e)$ represent all the integer solutions to (93) and (98) that satisfy $\nu \leq 1$. In case (i), we have $\lambda = p^1_{11}$ by parts (i) and (vi) of Theorem 4.1. In case (ii)–(v), $\lambda = a_0(W)$ is computed using Theorem 2.1 and Note 4.2. □

**Acknowledgments**

This research was partially supported by NSF grant DMS-880-0764.