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The Subconstituent Algebra of an Association Scheme (Part III)*

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Abstract. This is the continuation of an article from the previous issue. In this part, we focus on the thin P- and Q-polynomial association schemes. We provide some combinatorial characterizations of these objects and exhibit the known examples with diameter at least 6. For each example, we give the irreducible modules for the subconstituent algebra. We close with some conjectures and open problems.

Keywords: association scheme, P-polynomial, Q-polynomial, distance-regular graph

5. The thin P- and Q-polynomial schemes

In this section we give some ways to determine whether a given scheme is thin or dual-thin with respect to a given vertex. We are mainly interested in the P-polynomial and/or Q-polynomial cases, but some results are proved under weaker conditions. Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote any commutative scheme, and let g denote a permutation matrix in $\mathrm{Mat}_X(\mathbb{C})$. Recall g is an *automorphism* of Y whenever (y, z), (gy, gz) are in the same associate class of Y for all $y, z \in X$. Equivalently, g is an automorphism of Y whenever g commutes with each element of the Bose-Mesner algebra of Y. Let $\mathrm{Aut}(Y)$ denote the set of automorphisms of Y. Recall the commutator $[\alpha, \beta] = \alpha\beta - \beta\alpha$.

THEOREM 5.1. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote any commutative association scheme with $D \ge 3$, associate matrices $A_0, A_1, \ldots, A_D$ and primitive idempotents $E_0, E_1, \ldots, E_D$. Fix any $x \in X$, and let $E_i^* = E_i^*(x)$, $A_i^* = A_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$ be as in (51), (56), and Definition 3.3. Then the following statements (i)–(v) hold.*

(i) *The following (a), (b) are equivalent for all integers $h, i, j, k\,(0 \le h, i, j, k \le D)$:*

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(a) $E_i^* A_h E_j^* A_k E_i^* = E_i^* A_k E_j^* A_h E_i^*$

(b) *For all* $y, z \in X$ *with* $(x, y), (x, z) \in R_i$, *the number of* $w \in X$ *with* $(x, w) \in R_j$, $(y, w) \in R_h$, $(w, z) \in R_k$ *equals the number of* $w' \in X$ *with* $(x, w') \in R_j$, $(y, w') \in R_k$, $(w', z) \in R_h$.

(Part (i) provides a way to interpret parts (iii), (v) below).

(ii) *We have the implications*

$$TH \leftrightarrow C \leftarrow S \leftarrow G, \tag{212}$$

$$TH^* \leftrightarrow C^* \leftarrow S^*, \tag{213}$$

where

- TH: Y *is thin with respect to* x.
- C: $E_i^* T E_i^*$ *is commutative for all integers* i $(0 \leq i \leq D)$.
- S: $E_i^* T E_i^*$ *is symmetric for all integers* i $(0 \leq i \leq D)$.
- G: *For all* $y, z \in X$ *such that* $(x, y), (x, z)$ *are in the same associate class of* Y, *there exists* $g \in Aut(Y)$ *such that*

$$gx = x, \quad gy = z, \quad gz = y.$$

To get TH^*, C^*, *and* S^*, *replace "thin" by "dual-thin" and* E_i^* *by* E_i *in* TH, C, *and* S.

(iii) *Suppose* Y *is P-polynomial with respect to the ordering* $A_0, A_1, \ldots, A_D$. *Then*

$$TH \leftrightarrow C \leftrightarrow S \leftrightarrow WS \rightarrow TH^*,$$

where

- WS: $E_i^* A E_j^* A_k E_i^* = E_i^* A_k E_j^* A E_i^*$ *for all integers* i, j, k $(0 \leq i, j, k \leq D)$.

(iv) *Suppose* Y *is Q-polynomial with respect to the ordering* $E_0, E_1, \ldots, E_D$. *Then*

$$TH^* \leftrightarrow C^* \leftrightarrow S^* \leftrightarrow WS^* \rightarrow TH,$$

where WS^* *denotes the statement obtained from* WS *by replacing* E_ψ^*, A_ψ, *by* E_ψ, A_ψ^*, *for all integers* ψ $(0 \leq \psi \leq D)$.

(v) *Suppose* Y *is P-polynomial with respect to the ordering* $A_0, A_1, \ldots, A_D$, *and Q-polynomial with respect to the ordering* $E_0, E_1, \ldots, E_D$. *Then for each integer* i $(1 \leq i \leq D)$, *the statements*

$$E_i^* A E_i^* A_2 E_i^* = E_i^* A_2 E_i^* A E_i^*, \tag{214}$$

$$E_i^* A E_{i+1}^* A_2 E_i^* = E_i^* A_2 E_{i+1}^* A E_i^*, \tag{215}$$

$$E_i^* A E_{i-1}^* A_2 E_i^* = E_i^* A_2 E_{i-1}^* A E_i^*, \tag{216}$$

$$[E_i^* A E_i^*, E_i^* A E_{i+1}^* A E_i^*] = 0, \tag{217}$$

$$[E_i^* A E_i^*, E_i^* A E_{i-1}^* A E_i^*] = 0, \tag{218}$$

all hold if $i \in \{1, D\}$, and are all equivalent if $2 \leq i \leq D-1$. For each integer i $(1 \leq i \leq D)$, the statements

$$E_iA^*E_iA_2^*E_i = E_iA_2^*E_iA^*E_i, \tag{219}$$
$$E_iA^*E_{i+1}A_2^*E_i = E_iA_2^*E_{i+1}A^*E_i, \tag{220}$$
$$E_iA^*E_{i-1}A_2^*E_i = E_iA_2^*E_{i-1}A^*E_i, \tag{221}$$
$$[E_iA^*E_i, E_iA^*E_{i+1}A^*E_i] = 0, \tag{222}$$
$$[E_iA^*E_i, E_iA^*E_{i-1}A^*E_i] = 0, \tag{223}$$

all hold if $i \in \{1, D\}$, and all are equivalent if $2 \leq i \leq D-1$. Now consider the statements

- VWS: *Lines (214)–(218) hold for all integers i $(2 \leq i \leq D-1)$.*
- VWS^*: *Lines (219)–(223) hold for all integers i $(2 \leq i \leq D-1)$.*

Then $TH, C, S, WS, VWS, TH^*, C^*, S^*, WS^*, VWS^*$, are all equivalent.

The bulk of this section is devoted to proving Theorem 5.1. We will prove parts (i), (ii), then a technical lemma, then parts (iii), (iv), then three more technical lemmas, and finally part (v).

Proof of (i). The numbers being equated in (b) are the corresponding entries of the two sides in (a).

Proof of (ii).

$TH \to C$: By part (ii) of Lemma 3.4, we may express the standard module V as an orthogonal direct sum of irreducible T-modules. Now fix any integer i $(0 \leq i \leq D)$, and apply E_i^* to each module in this sum. In each case the image is an $E_i^*TE_i^*$-module, with dimension at most 1 by TH. Now E_i^*V is a direct sum of one-dimensional $E_i^*TE_i^*$-modules. But E_i^*V is a faithful $E_i^*TE_i^*$-module, so $E_i^*TE_i^*$ is commutative by our comments at the end of Section 1.

$C \to TH$: Suppose $\dim E_i^*W \geq 2$ for some irreducible T-module W and some integer i $(0 \leq i \leq D)$. Then on the one hand, E_i^*W is an irreducible $E_i^*TE_i^*$-module, for if E_i^*W properly contains a nonzero $E_i^*TE_i^*$-module U then $TU = W$ by the irreducibility of W, and

$$\begin{aligned} E_i^*TE_i^*U &= E_i^*TU \\ &= E_i^*W, \end{aligned}$$

a contradiction. On the other hand, $E_i^*TE_i^*$ is commutative, so each irreducible $E_i^*TE_i^*$-module has dimension 1 by our comments at the end of Section 1. This gives a contradiction, so TH holds.

$S \to C$: Fix an integer i $(0 \le i \le D)$, and pick any $a, b \in E_i^* T E_i^*$. Then a, b, and ab are symmetric, so

$$\begin{aligned} ab &= (ab)^t \\ &= b^t a^t \\ &= ba. \end{aligned}$$

Now $E_i^* T E_i^*$ is commutative, so C holds.
$G \to S$: Fix an integer i $(0 \le i \le D)$, pick any $a \in E_i^* T E_i^*$, and pick any $y, z \in X$. It suffices to show $a_{yz} = a_{zy}$. Assume $(x, y), (x, z) \in R_i$, otherwise a_{yz}, a_{zy} are both 0. By G, there exists some $g \in Aut(Y)$ such that $gx = x$, $gy = z$, and $gz = y$. But g commutes with everything in the Bose-Mesner algebra M, and everything in the dual Bose-Mesner algebra $M^*(x)$, hence everything in T, so g commutes with a. Now

$$\begin{aligned} a_{yz} &= (g^{-1} a g)_{yz} \\ &= a_{gygz} \\ &= a_{zy}, \end{aligned}$$

and we are done.

We have now proved (212). The proof of (213) is similar, so we have proved part (ii) of Theorem 5.1. □

Definition 5.2. With the notation of Theorem 5.1, assume Y is P-polynomial with respect to the ordering $A_0, A_1, \ldots, A_D$. For each $\eta \in \frac{1}{2}\mathbb{Z}$, $(0 \le \eta \le D)$, let T_η denote the subalgebra of T generated by matrices

$$E_i^* \qquad (\lfloor \eta \rfloor \le i \le D), \tag{224}$$

$$E_i^* A E_j^*, \qquad (\lfloor \eta \rfloor \le i, j \le D, \quad |i - j| \le 1, \quad 2\eta \le i + j). \tag{225}$$

Also, for all integers i, n $(0 \le i \le D, 0 \le n)$, set

$$(E_i^* A^n E_i^*)^{\pm} = \begin{cases} E_i^* A^{\frac{n}{2}} E_{i \pm \frac{n}{2}}^* A^{\frac{n}{2}} E_i^* & \text{if } n \text{ is even} \\ E_i^* A^{\frac{n-1}{2}} E_{i \pm \frac{n-2}{2}}^* A E_{i \pm \frac{n-1}{2}}^* A^{\frac{n-1}{2}} E_i^* & \text{if } n \text{ is odd.} \end{cases}$$

LEMMA 5.3. *With the notation of Theorem 5.1, assume Y is P-polynomial with respect to the ordering $A_0, A_1, \ldots, A_D$. Then the following six statements are equivalent for all $\eta \in \frac{1}{2}\mathbb{Z}$ $(0 \le \eta \le D)$.*

- WS'_η: $[(E_i^* A^\xi E_i^*)^+, (E_i^* A^\zeta E_i^*)^-] \in E_i^* T_{i - \frac{\xi}{2} + 1} E_i^*$ *for all integers* i, ξ, ζ $(\xi \in \{1, 2\}, \eta \le i \le D + 1 - \xi, 3 - \xi \le \zeta \le 2i - 2\eta + 1)$.

- WS''_η: $[(E_i^* A^\xi E_i^*)^+, (E_i^* A^\zeta E_i^*)^-] = 0$ *for all integers* i, ξ, ζ $(\xi \in \{1, 2\}, \eta \leq i \leq D + 1 - \xi, \quad 3 - \xi \leq \zeta \leq 2i - 2\eta + 1)$.
- WS'''_η: $(E_i^* A^2 E_i^*)^+, \quad (E_i^* A^\zeta E_i^*)^-$ $(1 \leq \zeta \leq 2i - 2\eta + 1)$ *mutually commute, for each integer* i $(\lceil \eta \rceil \leq i \leq D)$.
- TH_η: $\dim E_i^* W \leq 1$ *for all irreducible* T_η*-modules* W *and all integers* i $(\lfloor \eta \rfloor \leq i \leq D)$.
- C_η: $E_i^* T_\eta E_i^*$ *is commutative for all integers* i $(\lfloor \eta \rfloor \leq i \leq D)$.
- S_η: $E_i^* T_\eta E_i^*$ *is symmetric for all integers* i $(\lfloor \eta \rfloor \leq i \leq D)$.

Proof of Lemma 5.3. The proof is by induction on $\eta = D, D - \frac{1}{2}, D - 1 \ldots$. Observe each of the six statements is trivially true if $\eta = D$, so assume $\eta < D$. Since we have the implication $\chi_\eta \to \chi_{\eta + \frac{1}{2}}$ for each $\chi \in \{WS', WS'', WS''', TH, C, S\}$, it suffices to prove $WS'_\eta, WS''_\eta, WS'''_\eta, TH_\eta, C_\eta, S_\eta$ are equivalent under the assumption that $WS'_{\eta+\frac{1}{2}}, WS''_{\eta+\frac{1}{2}}, WS'''_{\eta+\frac{1}{2}}, TH_{\eta+\frac{1}{2}}, C_{\eta+\frac{1}{2}}$, and $S_{\eta+\frac{1}{2}}$ all hold.

$WS'_\eta \to WS''_\eta$: Fix integers i, ξ, ζ satisfying the bounds in WS'_η. Then $i - \frac{\zeta}{2} + 1 \geq \eta + \frac{1}{2}$, so $E_i^* T_{i - \frac{\zeta}{2} + 1} E_i^* \subseteq E_i^* T_{\eta + \frac{1}{2}} E_i^*$, implying the commutator in WS'_η is symmetric by $S_{\eta+\frac{1}{2}}$. But this commutator is antisymmetric by construction , so it equals 0.
$WS''_\eta \to WS'''_\eta$: Fix an integer i $(\lceil \eta \rceil \leq i \leq D)$. Then we must show $(E_i^* A^\vartheta E_i^*)^-, (E_i^* A^\zeta E_i^*)^-$ commute for all integers ϑ, ζ $(2 \leq \vartheta, \zeta \leq 2i - 2\eta + 1)$. We will do this by induction on $i = \lceil \eta \rceil, \lceil \eta \rceil + 1, \ldots$. Observe the commutator of these two matrices equals

$$E_i^* A E_{i-1}^* ((E_{i-1}^* A^{\vartheta-2} E_{i-1}^*)^- \ (E_{i-1}^* A^2 E_{i-1}^*)^+ \ (E_{i-1}^* A^{\zeta-2} E_{i-1}^*)^-$$
$$- (E_{i-1}^* A^{\zeta-2} E_{i-1}^*)^- \ (E_{i-1}^* A^2 E_{i-1}^*)^+ \ (E_{i-1}^* A^{\vartheta-2} E_{i-1}^*)^-) E_{i-1}^* A E_i^*.$$

This will be 0 if the middle terms

$$(E_{i-1}^* A^{\vartheta-2} E_{i-1}^*)^-, \qquad (E_{i-1}^* A^2 E_{i-1}^*)^+, \qquad (E_{i-1}^* A^{\zeta-2} E_{i-1}^*)^- \tag{226}$$

mutually commute. But this is the case, since the first and third terms in (226) commute by induction, and the second term in (226) commutes with the other two by WS''_η.
$WS'''_\eta \to TH_\eta$: The proof will involve two claims.

Claim 1. Pick any integer i $(\lceil \eta \rceil < i \leq D)$, and suppose w is a common eigenvector for

$$(E_i^* A^\zeta E_i^*)^- \ (1 \leq \zeta \leq 2i - 2\eta + 1). \tag{227}$$

Then

$$E^*_{i-1}AE^*_i w \tag{228}$$

is a common eigenvector for

$$(E^*_{i-1}A^\zeta E^*_{i-1})^- \quad (1 \le \zeta \le 2i - 2\eta - 1)$$

whenever (228) is not 0.

Proof of Claim 1. Assume (228) is not 0, otherwise there is nothing to prove. Observe $(E^*_i A^2 E^*_i)^-$ is included in (227), so

$$(E^*_i A^2 E^*_i)^- w = \lambda w$$

for some $\lambda \in \mathbb{C}$. In fact $\lambda \neq 0$, for otherwise

$$\begin{aligned} 0 &= \langle w, (E^*_i A^2 E^*_i)^- w\rangle \\ &= \langle w, E^*_i A E^*_{i-1} A E^*_i w\rangle \\ &= \langle w, \overline{(E^*_{i-1}AE^*_i)^t}(E^*_{i-1}AE^*_i)w\rangle \\ &= \|E^*_{i-1}AE^*_i w\|^2, \end{aligned}$$

contradicting the assumption that (228) is not 0. Now pick any integer ζ $(1 \le \zeta \le 2i - 2\eta - 1)$. Observe $(E^*_i A^{\zeta+2} E^*_i)^-$ is included in (227), so

$$(E^*_i A^{\zeta+2} E^*_i)^- w \in \text{Span } w.$$

Also observe $(E^*_{i-1}A^2E^*_{i-1})^+$, $(E^*_{i-1}A^\zeta E^*_{i-1})^-$ commute by WS'''_η, so

$$\begin{aligned} (E^*_{i-1}A^\zeta E^*_{i-1})^- E^*_{i-1}AE^*_i w &= \lambda^{-1}(E^*_{i-1}A^\zeta E^*_{i-1})^- E^*_{i-1}AE^*_i(E^*_i A^2 E^*_i)^- w \\ &= \lambda^{-1}(E^*_{i-1}A^\zeta E^*_{i-1})^-(E^*_{i-1}A^2E^*_{i-1})^+ E^*_{i-1}AE^*_i w \\ &= \lambda^{-1}(E^*_{i-1}A^2E^*_{i-1})^+(E^*_{i-1}A^\zeta E^*_{i-1})^- E^*_{i-1}AE^*_i w \\ &= \lambda^{-1}E^*_{i-1}AE^*_i(E^*_i A^{\zeta+2}E^*_i)^- w \\ &\in \text{Span } E^*_{i-1}AE^*_i w, \end{aligned}$$

and $E^*_{i-1}AE^*_i w$ is an eigenvector for $(E^*_{i-1}A^\zeta E^*_{i-1})^-$. This proves Claim 1.

Now let W denote an irreducible T_η-module such that $E^*_i W \neq 0$ for some integer i $(\lfloor \eta \rfloor \le i \le D)$.

Claim 2. There exist integers j, k $(\lfloor \eta \rfloor \le j \le k \le D)$ and nonzero vectors $w_i \in E^*_i W$ $(j \le i \le k)$ such that

$$E^*_{i-1}AE^*_i w_i \in \text{Span } w_{i-1} \qquad (j+1 \le i \le k), \tag{229}$$

$$E^*_{j-1}AE^*_j w_j = 0 \quad \text{if} \quad \lfloor \eta \rfloor < j, \tag{230}$$

$$E^*_i AE^*_i w_i \in \text{Span } w_i \qquad (j, \lceil \eta \rceil \le i \le k) \tag{231}$$

$$E^*_{i+1}AE^*_i w_i \in \text{Span } w_{i+1} \qquad (j \le i \le k-1), \tag{232}$$

$$E^*_{k+1}AE^*_k w_k = 0. \tag{233}$$

Moreover, $W = \mathrm{Span}\{w_j, w_{j+1}, \ldots, w_k\}$.

Proof of Claim 2. Set

$$k = \max\{i \mid \lfloor\eta\rfloor \le i \le D, \quad E_i^* W \ne 0\}.$$

Suppose for the moment that $k < \lceil\eta\rceil$, so that $k = \eta - \frac{1}{2}$, and pick any nonzero $w_k \in E_k^* W$. Then (229)–(233) holds with $j = k$, and the claim is proved. Now assume $k \ge \lceil\eta\rceil$. Now the matrices $(E_k^* A^\zeta E_k^*)^-$ $(1 \le \zeta \le 2k - 2\eta + 1)$ mutually commute by VWS_η''', and we observe they are contained in T_η, so they have a common eigenvector $w := w_k \in E_k^* W$. Now (233) holds. Define the vectors w_i $(\lfloor\eta\rfloor \le i \le k-1)$ by

$$w_i = E_i^* A E_{i+1}^* w_{i+1} \qquad (\lfloor\eta\rfloor \le i \le k-1),$$

and set

$$j = \min\{i \mid \lfloor\eta\rfloor \le i \le k, \qquad w_i \ne 0\}.$$

Now (229), (230) hold. By Claim 1 and induction on $i = k, k-1, \ldots$, each w_i $(j, \lceil\eta\rceil \le i \le k)$ is a common eigenvector for the matrices

$$(E_i^* A^\zeta E_i^*)^- \qquad (1 \le \zeta \le 2i - 2\eta + 1). \tag{234}$$

Pick any integer i $(j, \lceil\eta\rceil \le i \le k)$. Then $E_i^* A E_i^* = (E_i^* A E_i^*)^-$ is included in (234), so (231) holds. Now assume $i \ge j + 1$, so in particular $i \ge \eta + \frac{1}{2}$. Then $(E_i^* A^2 E_i^*)^-$ is included in (234), so

$$\begin{aligned} E_i^* A E_{i-1}^* w_{i-1} &= E_i^* A E_{i-1}^* A E_i^* w_i \\ &= (E_i^* A^2 E_i^*)^- w_i \\ &\in \mathrm{Span}\, w_i \qquad (j + 1 \le i \le k). \end{aligned}$$

Replacing i by $i + 1$ in the lines above, we obtain (232). Now

$$\mathrm{Span}\{w_j, w_{j+1}, \ldots, w_k\}$$

is a T_η-module by (229)–(233), and hence equals W by the irreducibility of W. This proves Claim 2, and TH_η is immediate.

$TH_\eta \to C_n$: Observe T_η is closed under conjugate-transpose by (224), (225), so by the discussion at the end of Section 1, we may express the standard module V as an orthogonal direct sum of irreducible T_η-modules. Now fix any integer i $(\lfloor\eta\rfloor \le i \le D)$, and apply E_i^* to each module in this sum. In each case the image is an $E_i^* T_\eta E_i^*$-module, and has dimension at most 1 by $TH\eta$. Now $E_i^* V$ is a direct sum of one dimensional $E_i^* T_\eta E_i^*$-modules. But $E_i^* V$ is a faithful $E_i^* T_\eta E_i^*$-module, so $E_i^* T_\eta E_i^*$ is commutative.

$C_\eta \to S_\eta$: By a *monomial* in T_η, we mean a matrix of the form

$$E^*_{i_0}AE^*_{i_1}AE^*_{i_2}\cdots E^*_{i_{n-1}}AE^*_{i_n},$$

where n, $i_0, i_1, \ldots, i_n$ are integers $(0 \le n,\ \lfloor\eta\rfloor \le i_0, i_1, \ldots, i_n \le D)$, with

$$i_j + i_{j+1} \ge 2\eta \qquad \text{for all } j \qquad (0 \le j \le n-1),$$

and

$$|i_j - i_{j+1}| \le 1 \qquad \text{for all } j \qquad (0 \le j \le n-1). \tag{235}$$

The monomial is *balanced* if $i_0 = i_n$. The integer n is the *length* of the monomial. It suffices to show that every balanced monomial in T_η is symmetric. Let

$$u = E^*_{i_0}AE^*_{i_1}AE^*_{i_2}\cdots E^*_{i_{n-1}}AE^*_{i_n} \qquad (i_o = i_n)$$

be any balanced monomial in T_η. Then u is immediately seen to be symmetric if $n \le 1$, so assume $n \ge 2$. By induction on n, we may assume each balanced monomial in T_η with length less than n is symmetric. First assume $i_j = i_0$ for some integer j $(1 \le j \le n-1)$, and set

$$\begin{aligned} u_1 &= E^*_{i_0}AE^*_{i_1}AE^*_{i_2}\cdots E^*_{i_{j-1}}AE^*_{i_j},\\ u_2 &= E^*_{i_j}AE^*_{i_{j+1}}AE^*_{i_{j+2}}\cdots E^*_{i_{n-1}}AE^*_{i_n}. \end{aligned}$$

Then u_1, u_2 are balanced, contained in T_η, and have length less than n, so u_1, u_2 are symmetric by induction. But u_1, u_2 commute by assumption, so

$$\begin{aligned} u &= u_1u_2\\ &= u_2u_1\\ &= u_2^tu_1^t\\ &= (u_1u_2)^t\\ &= u^t, \end{aligned}$$

and u is symmetric. Now assume $i_j \ne i_0$ for all integers j $(1 \le j \le n-1)$. Then

$$i_j > i_0 \qquad \text{for all } j \qquad (1 \le j \le n-1)$$

or

$$i_j < i_0 \qquad \text{for all } j \qquad (1 \le j \le n-1)$$

by (235), so

$$\begin{aligned} i_1 &= i_{n-1}\\ &\in \{i_0 - 1, i_0 + 1\}. \end{aligned}$$

Now set

$$\begin{aligned} u_3 &= E^*_{i_0} A E^*_{i_1}, \\ u_4 &= E^*_{i_1} A E^*_{i_2} A E^*_{i_3} \cdots E^*_{i_{n-2}} A E^*_{i_{n-1}}. \end{aligned}$$

Now u_4 is balanced, is contained in T_η, and has length less than n, so u_4 is symmetric by induction. But now

$$\begin{aligned} u &= u_3 u_4 u_3^t \\ &= u_3 u_4^t u_3^t \\ &= u^t \end{aligned}$$

so u is symmetric.

$S_\eta \to WS'_\eta$: The commutator in WS'_η is actually 0, since it is antisymmetric by construction, and contained in the symmetric space $E^*_i T_\eta E^*_i$. This proves Lemma 5.3.

Proof of part (iii) of Theorem 5.1. In view of part (ii) of Theorem 5.1, it suffices to show $TH \to TH^*$, $C \to S$, $S \to WS$, and $WS \to S$. The first implication is from part (v) in Lemma 3.9. The second implication is just $C_0 \to S_0$ in Lemma 5.3. The third implication holds, since $E^*_i A E^*_j A_k E^*_i$ is the transpose of $E^*_i A_k E^*_j A E^*_i$. The last implication will follow if we can show $WS \to WS'_0$, since $WS'_0 \to S_0 = S$ by Lemma 5.3. To do this, fix integers i, ξ, ζ such that

$$\xi \in \{1, 2\}, \quad 0 \le i \le D + 1 - \xi, \quad 3 - \xi \le \zeta \le 2i + 1.$$

Then a direct calculation yields

$$E^*_i A E^*_{i+\xi-1} A^{\zeta+\xi-1} E^*_i - E^*_i A^{\zeta+\xi-1} E^*_{i+\xi-1} A E^*_i \tag{236}$$

$$-[(E^*_i A^\xi E^*_i)^+, (E^*_i A^\zeta E^*_i)^-] \in E^*_i T_{i-\frac{\zeta}{2}+1} E^*_i. \tag{237}$$

But the expression in (236) is some linear combination of the matrices

$$E^*_i A E^*_{i+\xi-1} A_k E^*_i - E^*_i A_k E^*_{i+\xi-1} A E^*_i \qquad (0 \le k \le D)$$

by (71), and hence is 0 by WS. Now WS'_0 follows from (237). This proves part (iii) of Theorem 5.1.

Proof of part (iv) of Theorem 5.1. Similar to part (iii).

To prove part (v) of Theorem 5.1, we need some lemmas concerning a P- and Q-polynomial scheme Y with diameter at least 3. For notational convenience, we will take Y to be the scheme given in Theorem 4.1. The cases I, IA, II, IIA, IIB, IIC, III refer to part (iv) of Theorem 4.1.

LEMMA 5.4. *Let the scheme* $Y = (X, \{R_i\}_{0 \le i \le D})$ *be as in Theorem 4.1. Pick any* $x \in X$, *and write* $A^* = A_1^*(x)$. *Then*

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] \tag{238}$$

$$= A^3A^* - (\beta + 1)A^2A^*A + (\beta + 1)AA^*A^2 - A^*A^3 - \gamma(A^2A^* - A^*A^2) - \varrho(AA^* - A^*A), \tag{239}$$

$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A] \tag{240}$$

$$= A^{*3}A - (\beta + 1)A^{*2}AA^* + (\beta + 1)A^*AA^{*2} - AA^{*3} - \gamma^*(A^{*2}A - AA^{*2}) - \varrho^*(A^*A - AA^*), \tag{241}$$

where

$$\beta = \frac{\theta_i - \theta_{i+1} + \theta_{i+2} - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}} \qquad (0 \le i \le D - 3) \tag{242}$$

$$= \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} \qquad (0 \le i \le D - 3), \tag{243}$$

$$\gamma = \theta_i - \beta\theta_{i+1} + \theta_{i+2} \qquad (0 \le i \le D - 2), \tag{244}$$

$$\gamma^* = \theta_i^* - \beta\theta_{i+1}^* + \theta_{i+2}^* \qquad (0 \le i \le D - 2), \tag{245}$$

$$\varrho = \theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) \qquad (0 \le i \le D - 1), \tag{246}$$

$$\varrho^* = \theta_i^{*2} - \beta\theta_i^*\theta_{i+1}^* + \theta_{i+1}^{*2} - \gamma^*(\theta_i^* + \theta_{i+1}^*) \qquad (0 \le i \le D - 1). \tag{247}$$

Proof of (i). One may routinely verify (242)–(247) for each of Case I, IA, II, IIA, IIB, IIC, III (or see [71]). Indeed $\beta = q + q^{-1}$ (in Case I, IA), $\beta = 2$ (in Case II, IIA, IIB, IIB, IIC), and $\beta = -2$ (in Case III). Now let C denote the expression on the right side of (239). Expanding the commutator in (238), one observes this commutator equals C. To show $C = 0$, we observe

$$C = \sum_{i=0}^{D}\sum_{j=0}^{D} E_iCE_j, \tag{248}$$

and show

$$E_iCE_j = 0 \quad \text{for all integers } i, j \quad (0 \le i, j \le D). \tag{249}$$

Fix the integers i, j $(0 \le i, j \le D)$, and recall by (46) that

$$E_\eta A = AE_\eta = \theta_\eta E_\eta \qquad (0 \le \eta \le D).$$

Evaluating E_iCE_j using this, we find

$$\begin{aligned} E_iCE_j &= E_iA^*E_j(\theta_i^3 - (\beta+1)\theta_i^2\theta_j + (\beta+1)\theta_i\theta_j^2 - \theta_j^3 \\ &\quad -\gamma(\theta_i^2 - \theta_j^2) - \varrho(\theta_i - \theta_j)) \\ &= E_iA^*E_j(\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \varrho)(\theta_i - \theta_j). \end{aligned} \tag{250}$$

But

$$E_iA^*E_j = 0 \qquad \text{if} \quad |i-j| > 1$$

by Lemma 3.2 and Definition 3.10,

$$\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \varrho = 0 \quad \text{if} \quad |i-j| = 1$$

by (246), and certainly

$$\theta_i - \theta_j = 0 \qquad \text{if} \qquad i = j,$$

so $E_iCE_j = 0$ by (250). We now have (249), and hence (238), (239). Lines (240), (241) hold by a dual argument, so Lemma 5.4 is proved.

LEMMA 5.5. *Let the scheme $Y = (X, \{R_i\}_{0\le i\le D})$ be as in Theorem 4.1, and let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ be as in Lemma 5.4. Fix any $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \le i \le D)$. Then*

(i)

$$[E_i^*AE_i^*, E_i^*AE_{i+1}^*AE_i^*] = h_i[E_i^*AE_i^*, E_i^*AE_{i-1}^*AE_i^*] \ (0 \le i \le D), \tag{251}$$

where

$$h_i = \frac{\theta_{i-1}^* - \theta_i^*}{\theta_i^* - \theta_{i+1}^*} \qquad (1 \le i \le D-1), \tag{252}$$

and h_0, h_D are indeterminates.

(ii)

$$\begin{aligned} &e_i^- E_{i-1}^*AE_{i-2}^*A^2E_i^* + (\beta+2)E_{i-1}^*AE_i^*AE_{i-1}^*AE_i^* \\ &\quad + e_i^+ E_{i-1}^*A^2E_{i+1}^*AE_i^* + E_{i-1}^*AE_i^*AE_i^*AE_i^* - \beta E_{i-1}^*AE_{i-1}^*AE_i^*AE_i^* \\ &\quad + E_{i-1}^*AE_{i-1}^*AE_{i-1}^*AE_i^* \\ &= \gamma(E_{i-1}^*AE_i^*AE_i^* + E_{i-1}^*AE_{i-1}^*AE_i^*) + \varrho E_{i-1}^*AE_i^* \quad (1 \le i \le D), \end{aligned} \tag{253}$$

where

$$e_i^+ = \frac{\theta_{i-1}^* - (\beta+2)\theta_i^* + (\beta+1)\theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \qquad (1 \le i \le D-1), \tag{254}$$

$$e_i^- = \frac{-(\beta+1)\theta_{i-2}^* + (\beta+2)\theta_{i-1}^* - \theta_i^*}{\theta_{i-1}^* - \theta_i^*} \qquad (2 \le i \le D), \tag{255}$$

and e^+_D, e^-_1 *are indeterminants.*

(iii)

$$g_i^- E^*_{i-2} A E^*_{i-2} A^2 E^*_i + E^*_{i-2} A E^*_{i-1} A E^*_{i-1} A E^*_i + g_i^+ E^*_{i-2} A^2 E^*_i A E^*_i$$
$$= \gamma E^*_{i-2} A^2 E^*_i \qquad (2 \le i \le D), \tag{256}$$

where

$$g_i^+ = \frac{\theta^*_{i-2} - (\beta+1)\theta^*_{i-1} + \beta\theta^*_i}{\theta^*_{i-2} - \theta^*_i} \qquad (2 \le i \le D), \tag{257}$$

$$g_i^- = \frac{-\beta\theta^*_{i-2} + (\beta+1)\theta^*_{i-1} - \theta^*_i}{\theta^*_{i-2} - \theta^*_i} \qquad (2 \le i \le D). \tag{258}$$

(iv) Let h^*_i, e^{*+}_i, e^{*-}_i, g^{*+}_i, g^{*-}_i *denote the constants obtained from (252), (254), (255), (257), (258) by replacing* θ^*_j *by* θ_j $(0 \le j \le D)$*. Then the equations (251), (253), (256) still hold after replacing* γ, ϱ, A, h_i, $e^\pm_i$, $g^\pm_i$ *and* E^*_j $(0 \le j \le D)$ *by* γ^*, ϱ^*, A^*, h^*_i, $e^{*\pm}_i$, $g^{*\pm}_i$, *and* E_j $(0 \le j \le D)$*, respectively.*

Proof of (i)–(iii). Pick any integers i, j $(0 \le i, j \le D, 0 \le i - j \le 2)$, where $i - j = 0$ (in part (i)), $i - j = 1$ (in part (ii)), and $i - j = 2$ (in part (iii)). Now multiply both sides of (239) on the right by E^*_i and on the left by E^*_j. Writing the result as a linear combination of monomials, we get (i)–(iii). In part (i), one finds one needs $\beta + 1 \neq 0$ to carry this out. But this holds, for otherwise we are in Case I or Case IA, with $q + q^{-1} + 1 = \beta + 1 = 0$, forcing $q^3 = 1$ and contradicting (16).

Part (iv) is obtained by a similar argument, so Lemma 5.5 is proved.

LEMMA 5.6. *Let the scheme* $Y = (X, \{R_i\}_{0 \le i \le D})$ *be as in Theorem 4.1, and let* h_i, h^*_i, $e^\pm_i$, $e^{*\pm}_i$, $g^\pm_i$, $g^{*\pm}_i$ *be as in Lemma 5.5. Then*

$$e_i^+ = \frac{\theta^*_i - \theta^*_{i+2}}{\theta^*_i - \theta^*_{i-1}} \qquad (1 \le i \le D-2),$$

$$e_i^- = \frac{\theta^*_{i-1} - \theta^*_{i-3}}{\theta^*_{i-1} - \theta^*_i} \qquad (3 \le i \le D),$$

$$g_i^+ = \frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_i - \theta^*_{i-2}} \qquad (2 \le i \le D-1),$$

$$g_i^- = \frac{\theta^*_{i-2} - \theta^*_{i-3}}{\theta^*_{i-2} - \theta^*_i} \qquad (3 \le i \le D).$$

To get e^{*+}_i, e^{*-}_i, g^{*+}_i, g^{*-}_i, *replace* θ^*_j *by* θ_j $(0 \le j \le D)$ *in the above formulae. In* *particular,*

$$h_i,\ h_i^* \neq 0 \qquad (1 \leq i \leq D-1), \tag{259}$$
$$e_i^+,\ e_i^{*+} \neq 0 \qquad (1 \leq i \leq D-2), \tag{260}$$
$$e_i^-,\ e_i^{*-} \neq 0 \qquad (3 \leq i \leq D), \tag{261}$$
$$g_i^+,\ g_i^{*+} \neq 0 \qquad (2 \leq i \leq D-1), \tag{262}$$
$$g_i^-,\ g_i^{*-} \neq 0 \qquad (3 \leq i \leq D). \tag{263}$$

Proof. In each of (254), (255), (257), (258), equate the expression on the right with the corresponding expression above. In each case, the resulting equation is equivalent to (242) or (243). The last assertion is immediate from (7).

We now return to the proof of Theorem 5.1.

Proof of part (v) of Theorem 5.1. Line (216) holds for $i = 1$, since both sides are 0, and (215) is certainly true for $i = D$. For the first part of (v), it now suffices to show (214)–(218) are equivalent for all integers i $(1 \leq i \leq D)$. Fix such an integer i. Now lines (217), (218) are equivalent by (251), (259). Using

$$p_{11}^2 A_2 = A^2 - p_{11}^1 A - p_{11}^0 I,$$

we find by (251) that

$$\begin{aligned}
&E_i^* A E_i^* A_2 E_i^* - E_i^* A_2 E_i^* A E_i^* \\
&= [E_i^* A E_i^*,\ E_i^* A_2 E_i^*] \\
&= (p_{11}^2)^{-1}[E_i^* A E_i^*,\ E_i^* A^2 E_i^*] \\
&= (p_{11}^2)^{-1}[E_i^* A E_i^*,\ E_i^* A E_{i-1}^* A E_i^* + E_i^* A E_{i+1}^* A E_i^*] \\
&= (p_{11}^2)^{-1}(1 + h_i)[E_i^* A E_i^*,\ E_i^* A E_{i-1}^* A E_i^*].
\end{aligned}$$

Now (214), (218) are equivalent, since

$$\begin{aligned}
1 + h_i &= \frac{\theta_{i-1}^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i+1}^*} \\
&\neq 0 \qquad (1 \leq i \leq D-1),
\end{aligned}$$

and $1 + h_D$ is indeterminant. Similarly

$$E_i^* A E_{i+1}^* A_2 E_i^* - E_i^* A_2 E_{i+1}^* A E_i^* = (p_{11}^2)^{-1}[E_i^* A E_i^*, E_i^* A E_{i+1}^* A E_i^*],$$

so (215), (217) are equivalent, and

$$E_i^* A E_{i-1}^* A_2 E_i^* - E_i^* A_2 E_{i-1}^* A E_i^* = (p_{11}^2)^{-1}[E_i^* A E_i^*, E_i^* A E_{i-1}^* A E_i^*],$$

so (216), (218) are equivalent. Now (214)–(218) are all equivalent. The lines (219)–(223) can be treated in a similar manner, so consider the last assertion of part (v). The statements TH, C, S, WS, TH^*, C^*, S^*, WS^* are equivalent by parts (i)–(iv) of Theorem 5.1, and the implications $S \to VWS$, $S^* \to VWS^*$ are immediate, so it suffices to show $VWS \to TH$ and $VWS^* \to TH^*$.

$VWS \to TH$: Assume VWS. Then it suffices to show WS_1' (in the language of Lemma 5.3), since $WS_1' \to TH_1$ by that Lemma, and since $TH_1 \to TH$ by Lemma 3.6. Thus it suffices to pick integers i, ξ, ζ satisfying

$$\xi \in \{1, 2\}, 1 \le i \le D + 1 - \xi, \qquad 3 - \xi \le \zeta \le 2i - 1, \tag{264}$$

and show

$$[(E_i^* A^\xi E_i^*)^+, (E_i^* A^\zeta E_i^*)^-] \in E_i^* T_{i-\frac{\xi}{2}+1} E_i^*. \tag{265}$$

Case $(\xi, \zeta) = (1, 2)$ *or* $(2, 1)$. Immediate from (217), (218).

Case $(\xi, \zeta) = (1, 3)$. Assume for the moment that $\beta = 0$ in (242). Then by the statement below (247), we are in Case I or Case IA, with $q + q^{-1} = \beta = 0$. But then $q^2 = -1$ and then $q^4 = 1$, so $D = 3$ by (16). But now (265) can be seen to hold using $A^3 \in \text{Span}\{E_0, I, A, A^2\}$, so we may assume $\beta \ne 0$. Now multiply both sides of (253) on the left by $E_i^* A E_{i-1}^*$, and solve for $E_i^* A E_{i-1}^* A E_{i-1}^* A E_i^* A E_i^*$. We conclude this matrix is a linear combination of the symmetric matrices $E_i^* A E_{i-1}^* A E_{i-1}^* A E_{i-1}^* A E_i^*$, $E_i^* A^2 E_{i-2}^* A^2 E_i^*$, $E_i^* A E_{i-1}^* A E_{i-1}^* A E_i^*$, plus a matrix (which we will denote by K) contained in $E_i^* T_{i-\frac{1}{2}} E_i^*$. Now observe

$$\begin{aligned}
[E_i^* A E_i^*, (E_i^* A^3 E_i^*)^-] &= (E_i^* A E_{i-1}^* A E_{i-1}^* A E_i^* A E_i^*)^t \\
&\qquad - E_i^* A E_{i-1}^* A E_{i-1}^* A E_i^* A E_i^* \\
&= K^t - K \\
&\in E_i^* T_{i-\frac{1}{2}} E_i^*,
\end{aligned}$$

since $E_i^* T_{i-\frac{1}{2}} E_i^*$ is closed under transpose. Thus (265) holds, and we are done in the present case.

Case $\zeta = 2\eta$ *for some integer* $\eta \ge 3 - \xi$. This case will follow if we can show

$$E_{j-1}^* A E_{j-2}^* A^2 E_j^* \in T_{j-1} \qquad (3 \le j \le D). \tag{266}$$

Observe $e_j^- \ne 0$ by (261), so we may solve for $E_{j-1}^* A E_{j-2}^* A^2 E_j^*$ in (253). Inspecting the other terms in (253), we find (266).

Case $\zeta = 2\eta + 1$ *for some integer* $\eta \ge 3 - \xi$. This will follow if we can show

$$E_{j-2}^* A E_{j-2}^* A^2 E_j^* \in T_{j-\frac{3}{2}} \qquad (3 \le j \le D). \tag{267}$$

Observe $g_j^- \neq 0$ by (263), so we may solve for $E^*_{j-2}AE^*_{j-2}A^2E^*_j$ in (256). Inspecting the other terms in (256), we find (267). We are now done in the present case.

The above four cases exhaust the possibilities in (264), so we have proved $VWS \to TH$. The implication $VWS^* \to TH^*$ is obtained by a dual argument, so we have now proved part (v) of Theorem 5.1.

COROLLARY 5.7. *Suppose the scheme $Y = (X, \{R_i\}_{0\leq i\leq D})$ in Theorem 5.1 is P-polynomial with respect to the ordering $A_0, A_1, \ldots, A_D$ of the associate matrices, and Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Further assume the intersection numbers satisfy*

$$p^i_{1i} = 0 \qquad \text{for all integers } i \qquad (2 \leq i \leq D-1), \tag{268}$$

or the Krein parameters satisfy

$$q^i_{1i} = 0 \qquad \text{for all integers } i \qquad (2 \leq i \leq D-1). \tag{269}$$

Then Y is thin.

Proof. Pick any $x \in X$, and write $A^* = A^*(x)$, $E^*_i = E^*_i(x)$ $(0 \leq i \leq D)$. Now (268) implies VWS by (219), since $E^*_iAE^*_i = 0$ whenever $p^i_{1i} = 0$ $(0 \leq i \leq D)$ by (64). Similarly (269) implies VWS^*, since $E_iA^*E_i = 0$ whenever $q^i_{1i} = 0$ by (65). In either case, Y is thin by part (v) of Theorem 5.1. This proves Corollary 5.7. □

6. Examples of thin P- and Q-polynomial schemes

In this section we exhibit the known thin P- and Q-polynomial schemes with diameter at least 6. For each example, we give the irreducible $T(x)$-modules. If the scheme has more than one P- and Q-polynomial structure, we view each structure as a separate scheme. Information on the examples can be found in the books of Bannai and Ito [3] and Brouwer, et al. [11]. See also [23], [63], [74], and the references below. We suppress the details of our calculations. $\frac{1}{2}Y$ and $\tilde{Y}$ represent, respectively, the bipartite half and antipodal quotient of a P-polynomial scheme Y. (Bannai and Ito [3, p. 316]).

Example 6.1.

Let D denote an integer at least 3. Then for each example below, $Y = (X, \{R_i\}_{0\leq i\leq D})$ is a thin P- and Q-polynomial scheme. The constants q_1, r_1, r_2, s, s^*, and the case given in parenthesis, refer to $LS(Y)$, as indicated in (82)–(88).

(1) The Johnson scheme $J(D, N)(2D \leq N)$ [22], [52], [61], [71], [73], [75].

$$X = \text{ all subsets of } \{1,2,\ldots,N\} \text{ of order } D,$$
$$(x, y) \in R_i \text{ iff } |x \cap y| = D - i \qquad (0 \le i \le D), (x, y \in X),$$
$$p^i_{1\ i-1} = i^2 \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = (D-i)(N-D-i) \qquad (0 \le i \le D-1),$$
$$r = -N + D - 1, \qquad s = -N - 2, \qquad s^* = \frac{-N(N-1)}{D(N-D)}. \qquad \text{(IIA)}$$

(2) The Odd graph O_{D+1} [9], [50], [76].
$J(D, 2D+1)$ has another P-polynomial structure $R_0, R_D, R_1, R_{D-1}, \ldots$ in terms of the original one.

$$p^i_{1\ i-1} = \lfloor\frac{i+1}{2}\rfloor \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = D + 1 - \lfloor\frac{i+1}{2}\rfloor \qquad (0 \le i \le D-1),$$
$$r_1 = -D - 1, \qquad r_2 = -2D - 3,$$
$$s = 2D + 3, \qquad s^* = 2D + 2. \qquad \text{(III)}$$

(3) $\tilde{J}(2D, 4D)$ [67].

$$p^i_{1\ i-1} = i^2 \qquad (1 \le i \le D-1),$$
$$p^D_{1\ D-1} = 2D^2,$$
$$p^i_{1\ i+1} = (2D-i)^2 \qquad (0 \le i \le D-1),$$
$$r_1 = -D - \frac{1}{2}, \qquad r_2 = -2D - 1,$$
$$s = -2D - \frac{3}{2}, \qquad s^* = -2D - 1. \qquad \text{(II)}$$

(4) $\tilde{J}(2D+1, 4D+2)$ [67].

$$p^i_{1\ i-1} = i^2 \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = (2D+1-i)^2 \qquad (0 \le i \le D-1),$$
$$r_1 = -D - \frac{3}{2}, \qquad r_2 = -2D - 2,$$
$$s = -2D - \frac{5}{2}, \qquad s^* = -2D - 2. \qquad \text{(II)}$$

(5) The generalized Johnson scheme $J_{p^n}(D, N)\,(2D \le N)$ [25].
X = all D dimensional subspaces of a fixed N dimensional vector space over the finite field $GF(p^n)$,

$$(x, y) \in R_i \text{ iff } \dim(x \cap y) = D - i \qquad (0 \le i \le D), \qquad (x, y \in X),$$
$$p^i_{1\ i-1} = \left(\frac{q^i - 1}{q - 1}\right)^2 \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = \frac{q^{2i+1}(q^{D-i} - 1)(q^{N-D-i} - 1)}{(q-1)^2} \qquad (0 \le i \le D - 1),$$

where

$$q = p^n, \qquad r_1 = 0, \qquad r_2 = q^{-N+D-1}, \qquad s = q^{-N-2}, \qquad s^* = 0.$$

(6) The dual polar spaces of rank D [16], [64].
Let U denote a finite vector space with one of the following nondegenerate forms:

name	**dim U**	**field**	**form**	ϵ
$B_D(p^n)$	$2D + 1$	$GF(p^n)$	quadratic	0
$C_D(p^n)$	$2D$	$GF(p^n)$	symplectic	0
$D_D(p^n)$	$2D$	$GF(p^n)$	quadratic (Witt index D)	-1
${}^2D_{D+1}(p^n)$	$2D + 2$	$GF(p^n)$	quadratic (Witt index D)	1
${}^2A_{2D}(p^n)$	$2D + 1$	$GF(p^{2n})$	Hermitean	$\frac{1}{2}$
${}^2A_{2D-1}(p^n)$	$2D$	$GF(p^{2n})$	Hermitean	$-\frac{1}{2}$.

A subspace of U is called *isotropic* whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is D.

$$X = \text{ set of all maximal isotropic subspaces of } U.$$
$$(x, y) \in R_i \text{ iff } \dim(x \cap y) = D - i \qquad (0 \le i \le D) \qquad (x, y \in X),$$
$$p^i_{1\ i-1} = \frac{q^i - 1}{q - 1} \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = \frac{q^{i+\epsilon+1}(q^{D-i} - 1)}{q - 1} \qquad (0 \le i \le D - 1),$$

where

$$q = p^n, p^n, p^n, p^n, p^{2n}, p^{2n},$$
$$r_1 = r_2 = s^* = 0, \quad s = -q^{-D-\epsilon-2}. \qquad \text{(I)}$$

(7) ${}^2A_{2D-1}(p^n)'$ [41].
${}^2A_{2D-1}(p^n)$ has another Q-polynomial structure $E_0, E_D, E_1, E_{D-1}, \ldots$ in terms of the original one above.

$$p^i_{1\ i-1} = \frac{q^{2i}-1}{q^2-1} \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = -\frac{q^{2i+1}(q^{2D-2i}-1)}{q^2-1} \qquad (0 \le i \le D-1),$$

where

$$q = -p^n, \qquad r_1 = s^* = 0, \qquad r_2 = -q^{-D-1}, \qquad s = q^{-2D-2}.$$

(8) $\frac{1}{2}D_{2D}(p^n)$.

$$p^i_{1\ i-1} = \frac{(q^i-1)(q^{i-\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = \frac{q^{2i+\frac{1}{2}}(q^{D-i}-1)(q^{D-i-\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (0 \le i \le D-1),$$

where

$$q = p^{2n}, \qquad r_1 = s^* = 0, \qquad r_2 = q^{-D-\frac{1}{2}}, \qquad s = q^{-2D-1}.$$

(9) $\frac{1}{2}D_{2D+1}(p^n)$.

$$p^i_{1\ i-1} = \frac{(q^i-1)(q^{i-\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = \frac{q^{2i+\frac{1}{2}}(q^{D-i}-1)(q^{D-i+\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (0 \le i \le D-1),$$

where

$$q = p^{2n}, \qquad r_1 = s^* = 0, \qquad r_2 = q^{-D-\frac{3}{2}}, \qquad s = q^{-2D-2}.$$

(10) Hemmeter's scheme $Hem_D(p^n)$ [12].
Let $\hat{X}$ denote the vertex set of the scheme $C_{D-1}(p^n)$ (p odd), and let $X^+ := \{x^+ | x \in \hat{X}\}, X^- := \{x^- | x \in \hat{X}\}$ denote two copies of $\hat{X}$.

$$X = X^+ \cup X^- \qquad \text{(disjoint union)},$$

For all integers i $(0 \le i \le D-1)$, and for all $x, y \in \hat{X}$ with $(x, y) \in R_i$:

$$(x^+, y^+),\ (x^-, y^-) \in \begin{cases} R_i & \text{if } i \text{ is even}, \\ R_{i+1} & \text{if } i \text{ is odd}, \end{cases}$$

$$(x^+, y^-),\ (x^-, y^+) \in \begin{cases} R_{i+1} & \text{if } i \text{ is even}, \\ R_i & \text{if } i \text{ is odd}. \end{cases}$$

$$p^i_{1\ i-1} = \frac{q^i - 1}{q - 1} \qquad (1 \le i \le D),$$

$$p^i_{1\ i+1} = \frac{q^i(q^{D-i} - 1)}{q - 1} \qquad (0 \le i \le D-1),$$

where

$$q = p^n, \qquad r_1 = r_2 = s^* = 0, \qquad s = -q^{-D-1}. \tag{I}$$

(11) $\frac{1}{2}Hem_{2D}(p^n)$. Same data as 8.

(12) $\frac{1}{2}Hem_{2D+1}(p^n)$. Same data as 9.
The bipartite half of Hemmeter's scheme is known as *Ustimenko's scheme* [40].

(13) The Hamming scheme $H(D, q)$ $(q \ge 2)$ [27], [52], [61], [71], [73].

$$X = \text{ all } D\text{-tuples of elements from the set } \{1, 2, \ldots, q\},$$

$$(x, y) \in R_i \text{ iff } x, y \text{ differ in exactly } i \text{ coordinates } (x, y \in X),$$

$$p^i_{1\ i-1} = i, \qquad p^i_{1\ i+1} = (q-1)(D-i) \qquad (0 \le i \le D),$$

$$r = q(q-1), \qquad s = s^* = -q. \tag{IIC}$$

(14) $H(D, 2)'$ (D even) [76].
If D is even, $H(D, 2)$ has another P-polynomial structure R_0, R_{D-1}, R_2, R_{D-3}, … and another Q-polynomial structure E_0, E_{D-1}, E_2, E_{D-3}, … in terms of the original ones. With respect to the original P-polynomial structure and the new Q-polynomial structure, or with respect to the new P-polynomial structure and the original Q-polynomial structure:

$$p^i_{1\ i-1} = i, \qquad p^i_{1\ i+1} = D - i \qquad (0 \le i \le D),$$

$$r_1 = r_2 = -(D+1)/2, \qquad s = s^* = D + 1. \tag{III}$$

(With respect to the new P-polynomial structure and the new Q-polynomial structure, we get the original scheme).

(15) $\frac{1}{2}H(2D, 2)$ [52], [71], [73].

$$p^i_{1\ i-1} = i(2i-1), \qquad p^i_{1\ i+1} = (D-i)(2D-1-2i) \qquad (0 \le i \le D),$$
$$r = -D - \frac{1}{2}, \qquad s = -2D-1, \qquad s^* = -4. \qquad \text{(IIA)}$$

(16) $\frac{1}{2}H(2D+1, 2)$ [52], [71], [73].

$$p^i_{1\ i-1} = i(2i-1), \qquad p^i_{1\ i+1} = (D-i)(2D+1-2i) \qquad (0 \le i \le D),$$
$$r = -D - \frac{3}{2}, \qquad s = -2D-2, \qquad s^* = -4. \qquad \text{(IIA)}$$

$\frac{1}{2}H(2D+1, 2)$ has another P-polynomial structure $R_0, R_D, R_1, R_{D-1}, \ldots$ and another Q-polynomial structure $E_0, E_2, E_4, \ldots, E_3, E_1$ in terms of the original ones.

(17) $\frac{1}{2}H(2D+1, 2)'$ [76].
With respect to the new P-polynomial structure and the original Q-polynomial structure:

$$p^i_{1\ i-1} = i \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = 2D+1-i \qquad (0 \le i \le D-1),$$
$$r_1 = -D-1, \qquad r_2 = -2D-2, \qquad s = s^* = 2D+2. \qquad \text{(III)}$$

(18) $\frac{1}{2}H(2D+1, 2)''$ [67].
With respect to the original P-polynomial structure and the new Q-polynomial structure:

$$p^i_{1\ i-1} = i(2i-1) \qquad p^i_{1\ i+1} = (D-i)(2D+1-2i) \qquad (0 \le i \le D)$$
$$r_1 = -\frac{1}{2}D - \frac{3}{4}, \qquad r_2 = -\frac{1}{2}D - \frac{5}{4}, \qquad s = s^* = -D - \frac{3}{2}. \qquad \text{(II)}$$

(19) $\frac{1}{2}H(2D+1, 2)'''$ [72].
With respect to the new P-polynomial structure and the new Q-polynomial structure:

$$p^i_{1\ i-1} = i \qquad (1 \le i \le D),$$
$$p^i_{1\ i+1} = 2D+1-i \qquad (0 \le i \le D-1),$$
$$r = -D - \frac{3}{2}, \qquad s = -4, \qquad s^* = -2D-2. \qquad \text{(IIB)}$$

(20) $\widetilde{H}(2D, 2)$ [72].

$$p^i_{1\ i-1} = i \qquad (1 \le i \le D-1),$$
$$p^D_{1\ D-1} = 2D,$$
$$p^i_{1\ i+1} = 2D-i \qquad (0 \le i \le D-1),$$
$$r = -D - \frac{1}{2}, \qquad s = -4, \qquad s^* = -2D-1. \qquad \text{(IIB)}$$

We note $\widetilde{H}(2D+1,2)$ is isomorphic to $\frac{1}{2}H(2D+1,2)$.

(21) $\frac{1}{2}\widetilde{H}(4D,2)$ [67].

$$\begin{aligned} p^i_{1\ i-1} &= i(2i-1) \qquad (1 \le i \le D-1), \\ p^D_{1\ D-1} &= 2D(2D-1), \\ p^i_{1\ i+1} &= (2D-i)(4D-2i-1) \qquad (0 \le i \le D-1), \\ r_1 &= -D-\frac{1}{2}, \qquad r_2 = -2D-\frac{1}{2}, \qquad s = s^* = -2D-1. \end{aligned} \tag{II}$$

(22) $\frac{1}{2}\widetilde{H}(4D+2,2)$ [67].

$$\begin{aligned} p^i_{1\ i-1} &= i(2i-1) \qquad (1 \le i \le D), \\ p^i_{1\ i+1} &= (2D+1-i)(4D+1-2i) \qquad (0 \le i \le D-1), \\ r_1 &= -D-\frac{3}{2}, \qquad r_2 = -2D-\frac{3}{2}, \qquad s = s^* = -2D-2. \end{aligned} \tag{II}$$

(23) Ordinary $2D$-cycle.

$$\begin{aligned} r_1 &= q^{-\frac{1}{2}}, \qquad r_2 = -q^{-\frac{1}{2}}, \qquad s = s^* = q^{-1}, \\ q &= \text{primitive } 2D^{\text{th}} \text{ root of unity.} \end{aligned} \tag{I}$$

(24) Ordinary $(2D+1)$-cycle.

$$\begin{aligned} r_1 &= q^{-1}, \qquad r_2 = q^{-\frac{1}{2}}, \qquad s = s^* = q^{-1}, \\ q &= \text{primitive } (2D+1)^{\text{th}} \text{ root of unity.} \end{aligned} \tag{I}$$

Proof. It is well known that the above schemes are P- and Q-polynomial. A recent reference is [11, p253]. Also, the given schemes are thin. Indeed, the examples other than 10, 11, and 12 can be shown to satisfy condition G of Theorem 5.1. Example 10 is thin by Corollary 5.7, and Examples 11, 12 can be shown to satisfy condition VWS of Theorem 5.1.

Example 6.1, continued. Let the scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ be as in Example 6.1. Fix any $x \in X$, and let W denote an irreducible $T(x)$-module with diameter $d \ge 1$. Then W is strong. Let μ, ν, e denote the endpoint, dual endpoint, and auxiliary parameter of W, respectively, and recall

$$0 \le (D-d)/2 \le \mu, \nu \le D-d < D, \tag{270}$$

$$e+d+D \text{ is even}, \qquad |e| \le 2\mu - D + d, \qquad |e| \le 2\nu - D + d \tag{271}$$

by (93), (98), (204). Then additional restrictions are given below. The parameters $a_i(W)$ $(0 \le i \le d)$, $b_i(W)$ $(0 \le i \le d-1)$, $c_i(W)$ $(1 \le i \le d)$ of $LS(W)$ are also given.

(1) $J(D, N)$.

$$\nu \le \mu, \qquad e = D - d - 2\nu,$$
$$d \in \{D - 2\nu,\ \min\{D - \mu, N - D - 2\nu\}\}.$$

$$\begin{aligned} a_i(W) &= D(N-D) + \mu(\mu + d - N - 1) + d(d - N + 2\nu) \\ &\quad + i(N - 4\nu - 2i) \qquad (0 \le i \le d), \\ b_i(W) &= (d-i)(N - d - 2\nu - \mu - i) \qquad (0 \le i \le d-1), \\ c_i(W) &= i(i + 2\nu - \mu) \qquad (1 \le i \le d). \end{aligned}$$

(2) O_{D+1}.

$$d = D - \nu, \qquad e = D - d - 2\mu.$$

$$a_i(W) = 0 \qquad (0 \le i \le d-1),$$

$$a_d(W) = \begin{cases} (1 - \mu + \frac{1}{2}(D+\nu))(-1)^{\mu} & \text{if } d \text{ is even}, \\ \frac{1}{2}(D - \nu + 1)(-1)^{\mu} & \text{if } d \text{ is odd}, \end{cases}$$

$$b_i(W) = \begin{cases} (D - \mu + 1 - \frac{1}{2}i)(-1)^{\mu} & \text{if } i \text{ is even}, \\ (D - \nu + \frac{1-i}{2})(-1)^{\mu} & \text{if } i \text{ is odd}, \end{cases} \quad (0 \le i \le d-1),$$

$$c_i(W) = \begin{cases} \frac{1}{2}i(-1)^{\mu} & \text{if } i \text{ is even}, \\ (\nu - \mu + \frac{i+1}{2})(-1)^{\mu} & \text{if } i \text{ is odd}. \end{cases} \quad (1 \le i \le d).$$

(3) $\tilde{J}(2D, 4D)$.

$$e = D - d - 2\mu, \qquad \nu \in \{D - d,\ D - d - 1\}.$$

If $\nu = D - d$:

$$\begin{aligned} a_i(W) &= (2D - i - 2\mu + 1)(2D - 2d - 2\mu + i) \\ &\quad + (i+1)(2d - i) - 2D \qquad (0 \le i \le d), \\ b_i(W) &= (2d - i)(2D - 2\mu - i) \qquad (0 \le i \le d-1), \\ c_i(W) &= i(2D - 2d - 2\mu + i)) \qquad (1 \le i \le d-1), \\ c_d(W) &= 2d(2D - d - 2\mu). \end{aligned}$$

If $\nu = D - d - 1$:

$$\begin{aligned} a_i(W) &= (2D - i - 2\mu + 2)(2D - 2d - 2\mu - 1 + i) \\ &\quad +(i+1)(2d+2-i) - 2D \qquad (0 \le i \le d), \end{aligned}$$

$$b_i(W) = \frac{(2d+2-i)(2D-2\mu+1-i)(d-i)}{d+1-i} \qquad (0 \le i \le d-1),$$

$$c_i(W) = \frac{i(2D-2d-2\mu-1+i)(d+2-i)}{d+1-i} \qquad (1 \le i \le d).$$

(4) $\widetilde{J}(2D+1, 4D+2)$.

$$\nu = D - d, \qquad e \in \{D-d-2\mu,\ D-d-2\mu+2\}.$$

If $e = D - d - 2\mu$:

$$\begin{aligned} a_i(W) &= (2D - 2\mu + 2 - i)(2D - 2d - 2\mu + i) \\ &\quad +(i+1)(2d+1-i) - 2D - 1 \qquad (0 \le i \le d-1), \\ a_d(W) &= d^2 - 2dD + 1 + 4D(D+1) - 2\mu(4D - d - 2\mu + 3), \\ b_i(W) &= (2d+1-i)(2D+1-2\mu-i) \qquad (0 \le i \le d-1), \\ c_i(W) &= i(2D - 2d - 2\mu + i) \qquad (1 \le i \le d), \end{aligned}$$

If $e = D - d - 2\mu + 2$:

$$\begin{aligned} a_i(W) &= (2D - 2\mu + 3 - i)(2D - 2d - 2\mu + 1 + i) \\ &\quad +(i+1)(2d+1-i) - 2D - 1 \qquad (0 \le i \le d-1), \end{aligned}$$

$$a_d(W) = 3d^2 - 3d + 1 + 2D(2D - 3d + 2) + 2\mu(2\mu - 4D + 3d - 3),$$

$$b_i(W) = \frac{(2d+1-i)(2D-2\mu+2-i)(2d-2i-1)}{2d-2i+1} \qquad (0 \le i \le d-1),$$

$$c_i(W) = \frac{i(2D-2d-2\mu+1+i)(2d-2i+3)}{2d-2i+1} \qquad (1 \le i \le d).$$

(5) $J_q(D, N)$.

$$\mu \ge \nu, \qquad d \in \{e + D - 2\nu,\ \min\{D - \mu,\ e + D - 2\nu + 2(N - 2D)\}\}.$$

$$\begin{aligned} a_i(W) &= \frac{q(q^D-1)(q^{N-D}-1)}{(q-1)^2} - \frac{(q^\mu-1)(q^{N+1-\mu}-1)}{(q-1)^2} \\ &\quad -b_i(W) - c_i(W), \qquad (0 \le i \le d), \qquad (b_d(W) = c_0(W) = 0), \end{aligned}$$

where

$$b_i(W) = \frac{q^{2i+1+\nu+\frac{D-d-e}{2}}(q^{d-i}-1)(q^{N-i-\mu-\nu+\frac{e-D-d}{2}}-1)}{(q-1)^2} \qquad (0 \le i \le d-1),$$

$$c_i(W) = \frac{q^{\mu}(q^i-1)(q^{i+\nu-\mu+\frac{D-d-e}{2}}-1)}{(q-1)^2} \qquad (1 \le i \le d).$$

(6) Dual polar spaces.

$$\mu \le \nu, \qquad \mu \le \frac{1}{2}(D-d)+1+\varepsilon,$$

$$e = \begin{cases} 0 & \text{if } D+d \text{ is even,} \\ -1 & \text{if } D+d \text{ is odd.} \end{cases}$$

$$a_i(W) = \frac{q^{\varepsilon+1}(q^{D-d-\mu+i}-1)}{q-1} - \frac{q^{\mu+i}-1}{q-1} \qquad (0 \le i \le d),$$

$$b_i(W) = \frac{q^{D-d+1+\varepsilon-\mu+i}(q^{d-i}-1)}{q-1} \qquad (0 \le i \le d-1),$$

$$c_i(W) = \frac{q^{\mu}(q^i-1)}{q-1} \qquad (1 \le i \le d).$$

(7) ${}^2A_{2D-1}(p^n)'$.

$$\mu = D-d, \qquad e = 2\nu - D + d.$$

$$a_i(W) = -\frac{q^{D-d+2i}-1}{q-1} \qquad (0 \le i \le d),$$

$$b_i(W) = -\frac{q^{D-d+2i+1}(q^{2d-2i}-1)}{q^2-1} \qquad (0 \le i \le d-1),$$

$$c_i(W) = \frac{q^{D-d}(q^{2i}-1)}{q^2-1} \qquad (1 \le i \le d).$$

(8) $\frac{1}{2}D_{2D}(p^n)$.

$$e = 2\nu - D + d, \qquad \mu \in \{D-d, D-d-1\}.$$

If $\mu = D-d$:

$$a_i(W) = \frac{q^{D-d+i-\frac{1}{2}}(q^i-1)(q^{d-i+\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)}$$

$$+\frac{q^{D-d+i}(q^{i+\frac{1}{2}}-1)(q^{d-i}-1)}{(q-1)(q^{\frac{1}{2}}-1)}-\frac{q^D-1}{q-1} \qquad (0\le i\le d),$$

$$b_i(W)=\frac{q^{D-d+2i+\frac{1}{2}}(q^{d-i}-1)(q^{d-i-\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (0\le i\le d-1),$$

$$c_i(W)=\frac{q^{D-d}(q^i-1)(q^{i-\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (1\le i\le d),$$

If $\mu=D-d-1$:

$$a_i(W)=\frac{q^{D-d+i-\frac{1}{2}}(q^{i+1}-1)(q^{d-i+\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)}$$
$$+\frac{q^{D-d+i-1}(q^{i+\frac{1}{2}}-1)(q^{d+1-i}-1)}{(q-1)(q^{\frac{1}{2}}-1)}-\frac{q^D-1}{q-1}, \qquad (0\le i\le d),$$

$$b_i(W)=\frac{q^{D-d+2i+\frac{1}{2}}(q^{d-i+\frac{1}{2}}-1)(q^{d-i}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (0\le i\le d-1),$$

$$c_i(W)=\frac{q^{D-d-1}(q^{i+\frac{1}{2}}-1)(q^i-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (1\le i\le d).$$

(9) $\frac{1}{2}D_{2D+1}(p^n)$.

$$\mu=D-d, \qquad e\in\{2\nu-D+d,\ 2\nu-D+d-2\}.$$

If $e=2\nu-D+d$:

$$a_i(W)=\frac{q^{D-d+i}(q^{i+\frac{1}{2}}-1)(q^{d-i+\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)}$$
$$+\frac{q^{D-d+i-\frac{1}{2}}(q^i-1)(q^{d-i+1}-1)}{(q-1)(q^{\frac{1}{2}}-1)}-\frac{q^{D+\frac{1}{2}}-1}{q-1} \qquad (0\le i\le d),$$

$$b_i(W)=\frac{q^{D-d+2i+\frac{1}{2}}(q^{d-i+\frac{1}{2}}-1)(q^{d-i}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (0\le i\le d-1),$$

$$c_i(W)=\frac{q^{D-d}(q^i-1)(q^{i-\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (1\le i\le d).$$

If $e=2\nu-D+d-2$:

$$a_i(W) = \frac{q^{D-d+i+\frac{1}{2}}(q^{i+1}-1)(q^{d-i}-1)}{(q-1)(q^{\frac{1}{2}}-1)} + \frac{q^{D-d+i}(q^{i+\frac{1}{2}}-1)(q^{d-i+\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)} - \frac{q^{D+\frac{1}{2}}-1}{q-1} \qquad (0 \le i \le d),$$

$$b_i(W) = \frac{q^{D-d+2i+\frac{3}{2}}(q^{d-i}-1)(q^{d-i-\frac{1}{2}}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (0 \le i \le d-1),$$

$$c_i(W) = \frac{q^{D-d}(q^{i+\frac{1}{2}}-1)(q^{i}-1)}{(q-1)(q^{\frac{1}{2}}-1)} \qquad (1 \le i \le d).$$

(10) $Hem_D(p^n)$.

$$\mu = (D-d)/2, \qquad e = 0.$$
$$a_i(W) = 0 \qquad (0 \le i \le d),$$

$$b_i(W) = \frac{q^{i+\frac{D-d}{2}}(q^{d-i}-1)}{q-1} \qquad (0 \le i \le d-1),$$

$$c_i(W) = \frac{q^{\frac{D-d}{2}}(q^{i}-1)}{q-1} \qquad (1 \le i \le d),$$

(11) $\frac{1}{2}Hem_{2D}(p^n)$. Same as (8).
(12) $\frac{1}{2}Hem_{2D+1}(p^n)$. Same as (9).
(13) $H(D, q)$.

$$\mu = \nu, \qquad e = \begin{cases} 0, & \text{if } D-d \text{ is even}, \\ -1, & \text{if } D-d \text{ is odd}. \end{cases}$$
$$\mu = (D-d)/2 \text{ if } q = 2.$$

$$a_i(W) = (q-1)(D-d+i) - q\mu - i \qquad (0 \le i \le d),$$
$$b_i(W) = (q-1)(d-i) \qquad (0 \le i \le d-1),$$
$$c_i(W) = i \qquad (1 \le i \le d).$$

(14) $H(D, 2)'$

$$\mu = \nu = (D-d)/2, \qquad e = 0,$$
$$a_i(W) = 0 \qquad (0 \le i \le d),$$
$$b_i(W) = (d-i)(-1)^{\mu} \qquad (0 \le i \le d-1),$$
$$c_i(W) = i(-1)^{\mu} \qquad (1 \le i \le d).$$

(15) $\frac{1}{2}H(2D, 2)$.

$$\nu = (D-d)/2, \qquad e = 0, \qquad \mu \in \{D-d,\ D-d-1\}.$$

If $\mu = D - d$:

$$\begin{aligned} a_i(W) &= d - D + 4id - 4i^2 \qquad (0 \le i \le d), \\ b_i(W) &= (d-i)(2d-2i-1) \qquad (0 \le i \le d-1), \\ c_i(W) &= i(2i-1) \qquad (1 \le i \le d). \end{aligned}$$

If $\mu = D - d - 1$:

$$\begin{aligned} a_i(W) &= 3d - D + 2 + 4id - 4i^2 \qquad (0 \le i \le d), \\ b_i(W) &= (d-i)(2d-2i+1) \qquad (0 \le i \le d-1), \\ c_i(W) &= i(2i+1) \qquad (1 \le i \le d). \end{aligned}$$

(16) $\frac{1}{2}H(2D+1, 2)$.

$$\mu = D - d, \qquad \nu = (D-d-e)/2,$$
$$e = \begin{cases} 0 & \text{if } D-d \text{ is even,} \\ -1 & \text{if } D-d \text{ is odd.} \end{cases}$$

If $D - d$ is even:

$$\begin{aligned} a_i(W) &= d - D + i(4d+2) - 4i^2 \qquad (0 \le i \le d), \\ b_i(W) &= (d-i)(2d-2i+1) \qquad (0 \le i \le d-1), \\ c_i(W) &= i(2i-1) \qquad (1 \le i \le d). \end{aligned}$$

If $D - d$ is odd:

$$\begin{aligned} a_i(W) &= 3d - D + i(4d-2) - 4i^2 \qquad (0 \le i \le d), \\ b_i(W) &= (d-i)(2d-2i-1) \qquad (0 \le i \le d-1), \\ c_i(W) &= i(2i+1) \qquad (1 \le i \le d). \end{aligned}$$

(17) $\frac{1}{2}H(2D+1, 2)'$.

$$\mu = \nu = -e = D - d.$$

$$\begin{aligned} a_i(W) &= 0 \qquad (0 \le i \le d-1), \\ a_d(W) &= (d+1)(-1)^\mu, \\ b_i(W) &= (2d+1-i)(-1)^\mu \qquad (0 \le i \le d-1), \\ c_i(W) &= i(-1)^\mu \qquad (1 \le i \le d). \end{aligned}$$

(18) $\frac{1}{2}H(2D+1,2)''$.

$$\mu = \nu, \qquad \mu = (D-d-e)/2$$

$$e = \begin{cases} 0 & \text{if } D-d \text{ is even,} \\ -1 & \text{if } D-d \text{ is odd.} \end{cases}$$

If $D-d$ is even:

$$a_i(W) = d - D + i(4d+2) - 4i^2 \qquad (0 \le i \le d),$$
$$b_i(W) = (d-i)(2d+1-2i) \qquad (0 \le i \le d-1),$$
$$c_i(W) = i(2i-1) \qquad (1 \le i \le d).$$

If $D-d$ is odd:

$$a_i(W) = 3d - D + 2i(2d-1) - 4i^2 \qquad (0 \le i \le d),$$
$$b_i(W) = \frac{(d-i)(2d-2i-1)(2d-4i-5)}{2d-4i-1} \qquad (0 \le i \le d-1),$$
$$c_i(W) = \frac{i(2i+1)(2d-4i+3)}{2d-4i-1} \qquad (1 \le i \le d).$$

(19) $\frac{1}{2}H(2D+1,2)'''$.

$$\nu = D-d, \qquad \mu = (D-d-e)/2$$

$$e = \begin{cases} 0 & \text{if } D-d \text{ is even,} \\ -1 & \text{if } D-d \text{ is odd.} \end{cases}$$

If $D-d$ is even:

$$a_i(W) = 0 \qquad (0 \le i \le d),$$
$$a_d(W) = d+1,$$
$$b_i(W) = 2d+1-i \qquad (0 \le i \le d-1),$$
$$c_i(W) = i \qquad (1 \le i \le d).$$

If $D-d$ is odd:

$$a_i(W) = 0 \qquad (0 \le i \le d-1),$$
$$a_d(W) = -d-1,$$
$$b_i(W) = \frac{(2d+1-i)(2d-1-2i)}{2d+1-2i} \qquad (0 \le i \le d-1),$$
$$c_i(W) = \frac{i(2d-2i+3)}{2d-2i+1} \qquad (1 \le i \le d).$$

(20) $\widetilde{H}(2D,2)$.

$$\mu = (D-d)/2, \qquad e = 0, \qquad \nu \in \{D-d,\ D-d-1\}.$$

If $\nu = D - d$:

$$\begin{aligned} a_i(W) &= 0 \qquad (0 \le i \le d), \\ b_i(W) &= 2d - i \qquad (0 \le i \le d-1), \\ c_i(W) &= i \qquad (1 \le i \le d-1), \\ c_d(W) &= 2d. \end{aligned}$$

If $\nu = D - d - 1$:

$$\begin{aligned} a_i(W) &= 0 \qquad (0 \le i \le d), \\ b_i(W) &= \frac{(2d+2-i)(d-i)}{d+1-i} \qquad (0 \le i \le d-1), \\ c_i(W) &= \frac{i(d+2-i)}{d+1-i} \qquad (1 \le i \le d). \end{aligned}$$

(21) $\frac{1}{2}\widetilde{H}(4D, 2)$.

$$e = D - d - 2\nu, \qquad \nu = \mu \qquad \nu \in \{D-d, D-d-1\}.$$

If $\nu = D - d$:

$$\begin{aligned} a_i(W) &= 2(d-D) + 8id - 4i^2 \qquad (0 \le i \le d), \\ b_i(W) &= (2d-i)(4d-2i-1) \qquad (0 \le i \le d-1), \\ c_i(W) &= i(2i-1) \qquad (1 \le i \le d-1), \\ c_d(W) &= 2d(2d-1). \end{aligned}$$

If $\nu = D - d - 1$:

$$\begin{aligned} a_i(W) &= 2(d-D+4) + 8i(d+1) - 4i^2 \qquad (0 \le i \le d), \\ b_i(W) &= \frac{(d-i)(2d+2-i)(4d+5-2i)}{d+1-i} \qquad (0 \le i \le d-1), \\ c_i(W) &= \frac{i(2i+1)(d-i+2)}{d-i+1} \qquad (1 \le i \le d). \end{aligned}$$

(22) $\frac{1}{2}\widetilde{H}(4D+2, 2)$.

$$\mu = \nu = D - d, \qquad e \in \{d-D,\ d-D+2\}.$$

If $e = d - D$:

$$a_i(W) = 2(d - D) + 4i(2d + 1) - 4i^2 \qquad (0 \le i \le d - 1),$$
$$a_d(W) = 6d^2 + 9d + 1 - 2D,$$
$$b_i(W) = (2d + 1 - i)(4d + 1 - 2i) \qquad (0 \le i \le d - 1),$$
$$c_i(W) = i(2i - 1) \qquad (1 \le i \le d).$$

If $e = d - D + 2$:

$$a_i(W) = 2(3d - D + 2) + 4i(2d + 1) - 4i^2 \qquad (0 \le i \le d - 1),$$
$$a_d(W) = 2d^2 + 5d + 1 - 2D,$$
$$b_i(W) = \frac{(2d + 1 - i)(2d - 1 - 2i)(4d + 3 - 2i)}{2d + 1 - 2i} \qquad (0 \le i \le d - 1),$$
$$c_i(W) = \frac{i(2i + 1)(2d + 3 - 2i)}{2d + 1 - 2i} \qquad (1 \le i \le d).$$

(23) Ordinary $2D$-cycle.

$$(\mu, \nu, d, e) = (0, 0, D, 0) \text{ or } (1, 1, D - 2, 0).$$

If $(\mu, \nu, d, e) = (1, 1, D - 2, 0)$:

$$a_i(W) = 0 \qquad (0 \le i \le d),$$
$$b_i(W) = \frac{(q^{2i+4} - 1)}{q(q^{2i+2} - 1)} \qquad (0 \le i \le d - 1),$$
$$c_i(W) = \frac{q(q^{2i} - 1)}{q^{2i+2} - 1} \qquad (1 \le i \le d).$$

(24) Ordinary $(2D + 1)$-cycle.

$$(\mu, \nu, d, e) = (0, 0, D, 0) \text{ or } (1, 1, D - 1, 1).$$

If $(\mu, \nu, d, e) = (1, 1, D - 1, 1)$:

$$a_i(W) = 0 \qquad (0 \le i \le d - 1),$$
$$a_d(W) = -1,$$
$$b_i(W) = \frac{q^{2i+4} - 1}{q(q^{2i+2} - 1)} \qquad (0 \le i \le d - 1),$$
$$c_i(W) = \frac{q(q^{2i} - 1)}{q^{2i+2} - 1} \qquad (1 \le i \le d).$$

Note 6.2. As indicated in [11, p195], there are 5 known infinite families of P- and Q-polynomial schemes with unbounded diameter D, that are not listed in Example 6.1. They are (i) the *Doob schemes* (IIC, $r = 12$, $s = s^* = -4$) [11, p27], [24], (ii) the schemes $H_q(D, N)(N \geq D)$ of bilinear forms (I, $s = s^* = r_1 = 0, r_2 = q^{-N-1}$) [3, p306], [23], [35], (iii) the schemes $Alt_q(N)$ $(D = \lfloor \frac{N}{2} \rfloor)$ of alternating forms (I, $s = s^* = r_1 = 0, r_2 = q^{-D-\frac{1}{2}}$ (if N is even), $r_2 = q^{-D-\frac{3}{2}}$ (if N is odd))[3, p307], [37], (iv) the schemes $Her_q(D)$ of Hermitean forms (I, $s = s^* = r_1 = 0, r_2 = -q^{-D-1}$) [3, p308], [42], and (v) the schemes $Quad_q(N)(D = \lfloor \frac{N+1}{2} \rfloor)$ of quadratic forms (I, $s = s^* = r_1 = 0, r_2 = q^{-D-\frac{3}{2}}$ (if N is even), $r_2 = q^{-D-\frac{1}{2}}$ (if N is odd) [3, p308], [26], [32], [33], [34]. The schemes (i)–(v) are not thin if $D \geq 3$, since they can be shown to violate condition VWS in Theorem 5.1.

Every known P- and Q-polynomial scheme with diameter at least 6 is listed in (i)–(v) above or in Example 6.1 [11, p253].

7. Directions for further research

In this section we give some conjectures and problems concerning a commutative association scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ with diameter $D \geq 3$. We refer the reader to Definitions 3.5, 3.7 and 3.10 for the meaning of *thin, P-polynomial,* and *Q-polynomial*, respectively.

Conjecture 1. Suppose Y is thin and imprimitive. Then the subschemes and quotient schemes of Y are thin. (See [3, p134, p140] for the definitions of *imprimitive, subscheme*, and *quotient scheme*).

For 2, 3 below, assume Y is P-polynomial, thin, but not Q-polynomial.

Conjecture 2. If D is sufficiently large, then either

(2a) Y is bipartite, and the bipartite half $\frac{1}{2}Y$ is thin and Q-polynomial, or

(2b) Y is antipodal, and the antipodal quotient $\widetilde{Y}$ is thin and Q-polynomial.

Problem 3. Find all examples that come under (2a), (2b) above.

For 4–6 below, assume Y is P- and Q-polynomial, but not thin.

Conjecture 4. Y has only one P-polynomial structure and only one Q-polynomial structure. Furthermore, $p^k_{ij} = q^k_{ij}$ for all integers i, j, k $(0 \leq i, j, k \leq D)$.

Problem 5. Fix any $x \in X$, and find the structure of those irreducible $T(x)$-modules that are not thin. If W is such a module, how big can the dimensions of the $E_i^*(x)W \quad (0 \le i \le D)$ be?

Conjecture 6. For each $x \in X$, there exists a nonthin irreducible $T(x)$-module with endpoint 1, and a nonthin irreducible $T(x)$-module with dual endpoint 1.

For 7–11 below, assume Y is thin, P- and Q-polynomial.

Conjecture 7. Either

(7a) The endpoint of W is at most the dual endpoint of W, for all $x \in X$ and all irreducible $T(x)$-modules W, or
(7b) the endpoint of W is at least the dual endpoint of W, for all $x \in X$ and all irreducible $T(x)$-modules W. (See (72), (79) for the definition of endpoint and dual endpoint).

Conjecture 8. Every irreducible $T(x)$-module with diameter at least 1 is strong, for every $x \in X$. See Definition 4.4 and Theorem 4.10.

Conjecture 9. For all integers μ, ν, d, e satisfying (270), (271), and all $x \in X$, let mult$[\mu, \nu, d, e](x)$ denote the multiplicity with which the irreducible $T(x)$-module with endpoint μ, dual endpoint ν, diameter d, and auxiliary parameter e appears in the standard module V. If there is no such module, set mult $[\mu, \nu, d, e](x) = 0$. If Conjectures 7, 8 hold, we further conjecture that mult $[\mu, \nu, d, e](x)$ is determined by μ, ν, d, e and the intersection numbers of Y (and hence is independent of x).

Conjecture 10. Fix any $x \in X$ and consider the ring

$$R := \{a | a \in T(x), \text{ the entries of } a \text{ are all integers}\}.$$

Then R contains (is generated by?) $E_i^*(x)$, A_i $(0 \le i \le D)$. Now let W denote a strong irreducible $T(x)$-module, with endpoint μ, dual endpoint ν, diameter d, and auxiliary parameter e. For each $a \in R$, the eigenvalues of the restriction of a to W can be computed in terms of μ, ν, d, e, and the intersection numbers of Y. Since a has integer entries, these eigenvalues must be algebraic integers. This "feasibility condition" restricts (μ, ν, d, e) beyond (270), (271) for many Y. We conjecture $a_i(W) \quad (0 \le i \le d)$ and $c_i(W)b_{i-1}(W) \quad (1 \le i \le d)$ (which are eigenvalues of $a = E^*_{\nu+i}(x)AE^*_{\nu+i}(x)$ and $a = E^*_{\nu+i}(x)AE^*_{\nu+i-1}(x)AE^*_{\nu+i}(x)$, respectively), are rational integers.

Problem 11. Find all the thin P- and Q-polynomial schemes Y with sufficiently large diameter (see [27], [41], [50], [52], [66], [71], [72], [75], [76]). If necessary, assume some combination of Conjectures 7, 8, 9, 10.

For 12–15 below, assume Y is P-polynomial.

Conjecture 12. Fix any vertex $x \in X$, write $E_i^* = E_i^*(x) \quad (0 \le i \le D)$, $T = T(x)$, and assume

$$[E_i^* A E_i^*,\ E_i^* A E_{i-1}^* A E_i^*] \in T_i \qquad (2 \le i \le D),$$
$$E_{i-1}^* A E_{i-2}^* A^2 E_i^* \in T_{i-1} \qquad (3 \le i \le D),$$
$$E_{i-2}^* A E_{i-2}^* A^2 E_i^* \in T_{i-\frac{3}{2}} \qquad (3 \le i \le D),$$

where T_η ($\eta \in \frac{1}{2}\mathbb{Z}$, $0 \le \eta \le D$) is the subalgebra of T from Definition 5.2. Then either Y is Q-polynomial, or else Y is antipodal, and the antipodal quotient $\widetilde{Y}$ is Q-polynomial (see Lemma 5.5).

Conjecture 13. Suppose the intersection numbers of Y satisfy $p^0_{DD} = 1$ (so that Y is antipodal). Then Y is thin.

Conjecture 14. Suppose the intersection numbers of Y satisfy $p^i_{1\ i+1} = p^0_{11} - (p^1_{11} + 1)p^i_{1\ i-1}$ $(0 \le i \le D-1)$. Suppose further that there does not exist vertices $x, y, z, w \in X$ with $(x, y), (y, z), (z, w), (w, x), (y, w) \in R_1$, $(x, z) \in R_2$ (i.e., Y is the point graph of a regular near polygon [11, p199]). Then Y is thin.

Conjecture 15. Suppose Y is primitive, but not a Hamming scheme (part 13 in Example 6.1) or an ordinary cycle. Suppose further that $G = Aut(Y)$ acts distance-transitively on X (see[11, p136] for a definition of distance-transitive). Then according to Praeger, Saxl, and Yokoyama [58], either

(i) G is *almost simple* (i.e., $S \subseteq G \subseteq Aut(S)$ for some nonabelian finite simple group S), or
(ii) G is *affine* (i.e., G has an elementary abelian normal subgroup which is regular on X).

We conjecture that (i) holds if and only if Y is thin.

Problems 16–17 refer to the algebra $\mathcal{T}$ defined in Section 1.

Problem 16. Find the commutative association schemes Y where $\mathcal{T}$ is a finite dimensional vector space over $\mathbb{C}$. Let us say these schemes are of *finite type.* We conjecture that if Y is P- and Q-polynomial with

$$p^i_{1i} = 0 \text{ for all integers } i \qquad (2 \le i \le D-1),$$

or

$$q^i_{1i} = 0 \text{ for all integers } i \qquad (2 \le i \le D-1),$$

then Y is finite type. (See Corollary 5.7).

Conjecture 17. Referring to the previous problem, suppose Y is of finite type. Then $\mathcal{T}$ is semisimple.

Note. A *distance biregular graph* is a certain generalization of a P-polynomial scheme [21], [31], [55], [57]. We expect most of the results of this paper can be extended to these objects.

It is requested that progress on the above conjectures and problems be reported to us.