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Unmixed bipartite graphs and sublattices of the Boolean lattices

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Abstract The correspondence between unmixed bipartite graphs and sublattices of the Boolean lattice is discussed. By using this correspondence, we show existence of squarefree quadratic initial ideals of toric ideals arising from minimal vertex covers of unmixed bipartite graphs.

Keywords Toric ideal · Gröbner basis · Bipartite graph · Vertex cover · Cohen–Macaulay ring

Introduction

Let G be a finite graph on the vertex set $[N] = \{1, \ldots, N\}$ with no loops, no multiple edges and no isolated vertices. Let $E(G)$ denote the edge set of G. A *vertex cover* of G is a subset $C \subset [N]$ such that, for each edge $\{i, j\}$ of G, one has either $i \in C$ or $j \in C$. Such a vertex cover C is called *minimal* if no subset $C' \subsetneq C$ is a vertex cover of G. We say that a finite graph G is *unmixed* if all minimal vertex covers of G have the same cardinality. Let $A = K[z_1, \ldots, z_N]$ be the polynomial ring in N variables over a field K. The *edge ideal* of G is the monomial ideal $I(G)$ of A generated by

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those quadratic monomials $z_i z_j$ such that $\{i, j\}$ is an edge of G. It is well-known that the primary decomposition of the edge ideal of G is

$$I(G) = \bigcap_{C \in \mathcal{M}(G)} \langle z_i \mid i \in C \rangle$$

where $\mathcal{M}(G)$ is the set of all minimal vertex covers of G. We say that G is *Cohen–Macaulay* (over K) if the quotient ring $A/I(G)$ is Cohen–Macaulay. Every Cohen–Macaulay graph is unmixed. A graph-theoretical characterization of Cohen–Macaulay bipartite graphs was given in [1] and that of unmixed bipartite graphs was given in [5].

By relabeling the vertices, every unmixed bipartite graph G has the vertex set $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ and satisfies that $\{x_i, y_i\} \in E(G)$ for all i (See Section 1). In Sections 1 and 2, we study the correspondence between unmixed bipartite graphs and sublattices of the Boolean lattice $\mathcal{L}_n$ on $\{x_1, \dots, x_n\}$:

- There exists a one-to-one correspondence between unmixed bipartite graphs on $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ where $\{x_i, y_i\} \in E(G)$ for all i and sublattices $\mathcal{L}$ of $\mathcal{L}_n$ with $\emptyset \in \mathcal{L}$ and $\{x_1, \dots, x_n\} \in \mathcal{L}$. (Theorem 1.2.)
- There exists a one-to-one correspondence between Cohen–Macaulay bipartite graphs on $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ where $\{x_i, y_i\} \in E(G)$ for all i and full sublattices of $\mathcal{L}_n$. (Theorem 2.2.)

In Section 3, we study toric ideals arising from the set of minimal vertex covers of unmixed bipartite graphs. Let G be an unmixed bipartite graph on the vertex set $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ and let $K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring in $2n$ variables over a field K with each $\deg x_i = \deg y_i = 1$. We associate each minimal vertex cover C of G with the squarefree monomial $u_C = \prod_{v \in C} v \in K[x_1, \dots, x_n, y_1, \dots, y_n]$ of degree n. Let $\mathcal{R}_G$ denote the semigroup ring generated by all monomials u_C with $C \in \mathcal{M}(G)$ over K. Let $S_G = K[\{z_C\}_{C \in \mathcal{M}(G)}]$ denote the polynomial ring in $|\mathcal{M}(G)|$ variables over K. The *toric ideal* I_G of $\mathcal{R}_G$ is the kernel of the surjective homomorphism $\pi : S_G \to \mathcal{R}_G$ defined by $\pi(z_C) = u_C$. In Section 3, by using the correspondence given in Section 1, we show that:

- The toric ideal arising from an unmixed bipartite graph possesses a squarefree quadratic initial ideal. (Theorem 3.1.)

1 Minimal vertex covers of unmixed bipartite graphs

First we recall a fact stated in [1, p.300]. Let G be a bipartite graph without isolated vertices and let $V(G) = \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$ denote the set of vertices of G. Suppose that G is unmixed. Since $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ are minimal vertex covers of G, we have $m = n$. Moreover, thanks to the "marriage theorem," $\{x_i, y_i\} \in E(G)$ for all i by relabeling the vertices.

Thanks to this fact, it follows that each minimal vertex cover of G is of the form $\{x_{i_1}, \dots, x_{i_s}, y_{i_{s+1}}, \dots, y_{i_n}\}$ where $\{i_1, \dots, i_n\} = [n]$. For a minimal vertex cover

$C = \{x_{i_1}, \ldots, x_{i_s}, y_{i_{s+1}}, \ldots, y_{i_n}\}$ of G, we set $\overline{C} = \{x_{i_1}, \ldots, x_{i_s}\}$. Let $\mathcal{L}_n$ denote the Boolean lattice on the set $\{x_1, \ldots, x_n\}$ and let

$$\mathcal{L}_G = \{\overline{C} \mid C \text{ is a minimal vertex cover of } G\} \ (\subset \mathcal{L}_n).$$

Remark 1.1 Let G and G' be unmixed bipartite graphs on $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$.

(i) Both $\emptyset$ and $\{x_1, \ldots, x_n\}$ belong to $\mathcal{L}_G$.
(ii) If $G \neq G'$, then we have $I(G) \neq I(G')$. Hence $\mathcal{L}_G \neq \mathcal{L}_{G'}$ follows from the primary decomposition of the edge ideals.

Theorem 1.2 *Let $\mathcal{L}$ be a subset of $\mathcal{L}_n$. Then there exists a* (*unique*) *unmixed bipartite graph G on $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ such that $\mathcal{L} = \mathcal{L}_G$ if and only if $\emptyset$ and $\{x_1, \ldots, x_n\}$ belong to $\mathcal{L}$ and $\mathcal{L}$ is a sublattice of $\mathcal{L}_n$.*

Proof (**"Only if"**) Suppose that both $C = \{x_{i_1}, \ldots, x_{i_s}, y_{i_{s+1}}, \ldots, y_{i_n}\}$ and $C' = \{x_{j_1}, \ldots, x_{j_t}, y_{j_{t+1}}, \ldots, y_{j_n}\}$ are minimal vertex covers of G. Then

$$\{y_k \mid x_k \notin \overline{C} \cap \overline{C'}\} = \{y_{i_{s+1}}, \ldots, y_{i_n}\} \cup \{y_{j_{t+1}}, \ldots, y_{j_n}\},$$

$$\{y_k \mid x_k \notin \overline{C} \cup \overline{C'}\} = \{y_{i_{s+1}}, \ldots, y_{i_n}\} \cap \{y_{j_{t+1}}, \ldots, y_{j_n}\}.$$

First we show that $\overline{C} \cap \overline{C'} \in \mathcal{L}_G$, that is, $C_1 = (\overline{C} \cap \overline{C'}) \cup \{y_k \mid x_k \notin \overline{C} \cap \overline{C'}\}$ is a minimal vertex cover of G. Suppose that an edge $\{x_i, y_j\}$ of G satisfies $y_j \notin \{y_k \mid x_k \notin \overline{C} \cap \overline{C'}\} = \{y_{i_{s+1}}, \ldots, y_{i_n}\} \cup \{y_{j_{t+1}}, \ldots, y_{j_n}\}$. Since C (resp. C') is a vertex cover of G, we have $x_i \in \overline{C}$ (resp. $x_i \in \overline{C'}$). Hence $x_i \in \overline{C} \cap \overline{C'}$. Thus C_1 is a minimal vertex cover of G.

Second we show that $\overline{C} \cup \overline{C'} \in \mathcal{L}_G$, that is, $C_2 = (\overline{C} \cup \overline{C'}) \cup \{y_k \mid x_k \notin \overline{C} \cup \overline{C'}\}$ is a minimal vertex cover of G. Suppose that an edge $\{x_i, y_j\}$ of G satisfies $x_i \notin \overline{C} \cup \overline{C'}$. Since C (resp. C') is a vertex cover of G, we have $y_j \in \{y_{i_{s+1}}, \ldots, y_{i_n}\}$ (resp. $y_j \in \{y_{j_{t+1}}, \ldots, y_{j_n}\}$). Thus $y_j \in \{y_{i_{s+1}}, \ldots, y_{i_n}\} \cap \{y_{j_{t+1}}, \ldots, y_{j_n}\} = \{y_k \mid x_k \notin \overline{C} \cup \overline{C'}\}$ and hence C_2 is a minimal vertex cover of G.

(**"If"**) For each element $S \in \mathcal{L}$, let S^* denote the set $\{y_j \mid x_j \notin S\}$. Let I be an ideal $\bigcap_{S \in \mathcal{L}} \langle S \cup S^* \rangle$. We will show that there exists an unmixed bipartite graph G such that $I = \langle x_i y_j \mid \{x_i, y_j\} \in E(G) \rangle$.

Since $\emptyset \in \mathcal{L}$ and $\{x_1, \ldots, x_n\} \in \mathcal{L}$, $I \subset \langle x_i y_j \mid 1 \leq i, j \leq n \rangle$. Suppose that a monomial M of degree ≥ 3 belongs to the minimal set of generators of I.

Suppose that $M = x_i x_j u$ where $i \neq j$ and u is a (squarefree) monomial. Since $x_j u \notin I$, there exists $S \in L$ such that $x_j u \notin \langle S \cup S^* \rangle$. Moreover since $x_i u \notin I$, there exists $S' \in L$ such that $x_i u \notin \langle S' \cup S'^* \rangle$. Then we have $x_i, x_j \notin S \cap S'$ and $u \notin \langle S \cup S' \cup S^* \cup S'^* \rangle$. Since $\mathcal{L}$ is a sublattice of $\mathcal{L}_n$, $S \cap S' \in \mathcal{L}$. Note that $(S \cap S')^* = S^* \cup S'^*$. Hence we have $I \subset \langle (S \cap S') \cup (S^* \cup S'^*) \rangle$. However, none of the variables in $x_i x_j u$ appear in the set $(S \cap S') \cup (S^* \cup S'^*)$ contradicting the fact that $x_i x_j u \in I$.

Suppose that $M = y_i y_j u$ where $i \neq j$ and u is a (squarefree) monomial. Since $y_j u \notin I$, there exists $S \in L$ such that $y_j u \notin \langle S \cup S^* \rangle$. Moreover since $y_i u \notin I$, there exists $S' \in L$ such that $y_i u \notin \langle S' \cup S'^* \rangle$. Then we have $y_i, y_j \notin S^* \cap S'^*$ and $u \notin \langle S \cup S' \cup S^* \cup S'^* \rangle$. Since $\mathcal{L}$ is a sublattice of $\mathcal{L}_n$, $S \cup S' \in \mathcal{L}$. Note that $(S \cup S')^* =$

$S^* \cap S'^*$. Hence we have $I \subset \langle (S \cup S') \cup (S^* \cap S'^*) \rangle$. However, none of the variables in $y_i y_j u$ appear in the set $(S \cup S') \cup (S^* \cap S'^*)$ contradicting the fact that $y_i y_j u \in I$.

Thus the minimal set of generators of I is a subset of $\{x_i y_j \mid 1 \leq i, j \leq n\}$. Moreover, since either x_i or y_i belongs to $S \cup S^*$ for all $S \in \mathcal{L}$, $x_i y_i \in I$ for all i. Hence there exists a bipartite graph G such that $I = I(G)$ and $\{x_i, y_i\} \in E(G)$. Since the primary decomposition of the edge ideal $I(G)$ of G is $I = \bigcap_{C \in \mathcal{M}(G)} \langle C \rangle$, it follows that $\mathcal{M}(G) = \{S \cup S^* \mid S \in \mathcal{L}\}$. Thus we have $\mathcal{L} = \mathcal{L}_G$. Since the cardinality of each $S \cup S^*$ with $S \in \mathcal{L}$ is n, G is unmixed as desired. □

2 Minimal vertex covers of Cohen–Macaulay bipartite graphs

As before, let G be a finite graph on $[N]$ and $A = K[z_1, \ldots, z_N]$ the polynomial ring in N variables over a field K. The *edge ideal* of G is the monomial ideal $I(G)$ of A generated by those quadratic monomials $z_i z_j$ such that $\{i, j\}$ is an edge of G. We say that G is *Cohen–Macaulay* (over K) if the quotient ring $A/I(G)$ is Cohen–Macaulay. Every Cohen–Macaulay graph is unmixed.

Given a finite poset (partially ordered set) $P = \{p_1, \ldots, p_n\}$, we introduce the bipartite graph G_P on the vertex set $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ whose edges are those 2-element subsets $\{x_i, y_j\}$ with $p_i \leq p_j$. In particular for each $i \in [n]$ the edge $\{x_i, y_i\}$ belongs to G_P. It is known [1] that G_P is Cohen–Macaulay. Conversely, given a Cohen–Macaulay bipartite graph G, there is a finite poset P with $G = G_P$.

A subset $\alpha \subset P$ is called a *poset ideal* of P if α enjoys the property that if $p_i \in \alpha$ and $p_j \leq p_i$, then $p_j \in \alpha$. In particular the empty set and P itself are poset ideals of P. For each poset ideal α of P, we set $\alpha_x = \{x_i \mid p_i \in \alpha\}$ and $\alpha_y = \{y_j \mid p_j \notin \alpha\}$.

Lemma 2.1 *The set $\alpha_x \cup \alpha_y$ is a minimal vertex cover of G_P. Conversely, every minimal vertex cover of G_P is of the form $\beta_x \cup \beta_y$ for some poset ideal β of P.*

Proof Let α be a poset ideal of P. We show that $C = \alpha_x \cup \alpha_y$ is a minimal vertex cover of G_P. Let $\{x_i, y_j\}$ be an edge of G. Then $p_i \leq p_j$. Suppose $x_i \notin \alpha_x$. Then $p_i \notin \alpha$. Since α is a poset ideal of P, it follows that $p_j \notin \alpha$. Thus $y_j \in \alpha_y$. Hence C is a vertex cover of G. Since G_P is unmixed and $|C| = n$, it follows that C is a minimal vertex cover.

Conversely, given a minimal vertex cover $C = \{x_{i_1}, \ldots, x_{i_s}\} \cup \{y_{i_{s+1}}, \ldots, y_{i_n}\}$ of G_P, where $\{i_1, \ldots, i_n\} = [n]$, we prove that $\alpha = \{p_{i_1}, \ldots, p_{i_s}\}$ is a poset ideal of P. Let $p_{i_j} \in \alpha$ and $p_a < p_{i_j}$ in P. Then $\{x_a, y_{i_j}\}$ is an edge of G_P. Suppose $p_a \notin \alpha$. Then $x_a \notin C$. Since $x_{i_j} \in C$, one has $y_{i_j} \notin C$. Thus neither x_a nor y_{i_j} belongs to C. However, $\{x_a, y_{i_j}\}$ is an edge of G_P. Thus C cannot be a vertex cover of G_P. Hence $p_a \in \alpha$. Consequently, α is a poset ideal of G_P, as desired. □

As before, let $\mathcal{L}_n$ denote the Boolean lattice on $\{x_1, \ldots, x_n\}$. A sublattice $\mathcal{L}$ of $\mathcal{L}_n$ is called *full* if the rank of $\mathcal{L}$ is equal to n. Here the rank of $\mathcal{L}$ is defined to be the nonnegative integer $\ell - 1$, where ℓ is the maximal cardinality of chains (totally ordered subsets) of $\mathcal{L}$.

Let P be a finite poset with $|P| = n$ and $\mathcal{J}(P)$ the set of all poset ideals of P. It turns out that the subset $\mathcal{J}(P)$ of $\mathcal{L}_n$ is a full sublattice of $\mathcal{L}_n$. Conversely, the classical fundamental structure theorem for finite distributive lattices [3, pp. 118–119] guarantees that every full sublattice of $\mathcal{L}_n$ is of the form $\mathcal{J}(P)$ for a unique poset P with $|P| = n$.

Theorem 2.2 *A subset $\mathcal{L}$ of $\mathcal{L}_n$ is a full sublattice of $\mathcal{L}_n$ if and only if there exists a Cohen–Macaulay bipartite graph G on $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ with $\mathcal{L} = \mathcal{L}_G$.*

Proof **("If")** Let G be a Cohen–Macaulay bipartite graph on the set $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ and P a poset with $G = G_P$, where $|P| = n$. Lemma 2.1 says that $\mathcal{L}_G$ coincides with $\mathcal{J}(P)$. Thus $\mathcal{L}_G$ is a full sublattice of $\mathcal{L}_n$.

("Only if") Suppose that $\mathcal{L}$ is a full sublattice of $\mathcal{L}_n$. One has $\mathcal{L} = \mathcal{J}(P)$ for a unique poset P with $|P| = n$. Let $G = G_P$. Then G is a Cohen–Macaulay bipartite graph. Lemma 2.1 says that $\mathcal{L}_G$ coincides with $\mathcal{J}(P)$. Thus $\mathcal{L}_G = \mathcal{L}$, as required. □

3 Toric ideals arising from minimal vertex covers

Let G be an unmixed bipartite graph on the vertex set $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ and let $K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring in $2n$ variables over a field K with each $\deg x_i = \deg y_i = 1$. Let $\mathcal{M}(G)$ denote the set of all minimal vertex covers of G. We associate each minimal vertex cover C of G with the squarefree monomial $u_C = \prod_{v \in C} v \in K[x_1, \dots, x_n, y_1, \dots, y_n]$ of degree n. Let $\mathcal{R}_G$ denote the semigroup ring generated by all monomials u_C with $C \in \mathcal{M}(G)$ over K. Let $S_G = K[\{z_C\}_{C \in \mathcal{M}(G)}]$ denote the polynomial ring in $|\mathcal{M}(G)|$ variables over K. The *toric ideal* I_G of $\mathcal{R}_G$ is the kernel of the surjective homomorphism $\pi : S_G \to \mathcal{R}_G$ defined by $\pi(z_C) = u_C$.

Theorem 3.1 *Let G be an unmixed bipartite graph. Then the toric ideal I_G of $\mathcal{R}_G$ has a squarefree quadratic initial ideal with respect to a reverse lexicographic order.*

Proof Let G_0 denote the (unmixed) bipartite graph with the edge set $E(G) = \{\{x_1, y_1\}, \dots, \{x_n, y_n\}\}$. Then $\mathcal{L}_{G_0}$ is the Boolean lattice on $\{x_1, \dots, x_n\}$. It is known [2] that the reduced Gröbner basis of toric ideal of $\mathcal{R}_{G_0}$ with respect to a suitable reverse lexicographic order is

$$\mathcal{G}_0 = \{ \underline{z_C z_{C'}} - z_{C \cap C'} z_{C \cup C'} \mid C, C' \in \mathcal{M}(G),\ C \neq C' \}$$

where the initial monomial of each binomial of $\mathcal{G}_0$ is the first monomial.

Let G be an unmixed bipartite graph on the vertex set $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$. Then $\mathcal{L}_G$ is a sublattice of $\mathcal{L}_{G_0}$. Hence we have the following:

(i) $I_G = I_{G_0} \cap S_G$ (by [4, Proposition 4.13 (a)]);
(ii) If C and C' belong to $\mathcal{M}(G)$, then $C \cap C'$ and $C \cup C'$ belong to $\mathcal{M}(G)$. Thus, if $z_C z_{C'} \in S_G$, then we have $z_{C \cap C'} z_{C \cup C'} \in S_G$.

Thanks to the elimination property above, $\mathcal{G}_0 \cap S_G$ is a Gröbner basis of the toric ideal I_G of $\mathcal{R}_G$ as desired. □

Corollary 3.2 *Let G be an unmixed bipartite graph. Then the semigroup ring $\mathcal{R}_G$ is normal and Koszul.*

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