J Algebr Comb (2009) 30: 491–514
DOI 10.1007/s10801-009-0174-3

A generalization of adjoint crystals for the quantized affine algebras of type $A_n^{(1)}$, $C_n^{(1)}$ and $D_{n+1}^{(2)}$

Ryosuke Kodera

Received: 17 March 2008 / Accepted: 17 February 2009 / Published online: 10 March 2009

Abstract We generalize Benkart-Frenkel-Kang-Lee's adjoint crystals and describe their crystal structure for type $A_n^{(1)}$, $C_n^{(1)}$ and $D_{n+1}^{(2)}$.

Keywords Crystal base · Quantized affine algebra · Kirillov-Reshetikhin module

1 Introduction

Let $U_q'(\mathfrak{g})$ be a quantized affine algebra without the degree operator. Let $\mathfrak{g}_0$ be the underlying simple Lie algebra of finite type contained in the affine Lie algebra $\mathfrak{g}$ and $U_q(\mathfrak{g}_0)$ the corresponding subalgebra of $U_q'(\mathfrak{g})$. Let i be a node of the Dynkin diagram of $\mathfrak{g}_0$ and l a nonnegative integer. We denote by $W^{i,l}$ the Kirillov-Reshetikhin module for i and l, which is a finite-dimensional irreducible module over $U_q'(\mathfrak{g})$. They are important objects from the viewpoint of the crystal base theory as it is conjectured that all Kirillov-Reshetikhin modules have crystal bases.

We assume that $\mathfrak{g}$ is not of type $A_{2n}^{(2)}$ for simplicity. We denote by $V(\lambda)$ the finite-dimensional irreducible module with highest weight λ over $U_q(\mathfrak{g}_0)$ and $B(\lambda)$ its crystal base. In [1], Benkart, Frenkel, Kang and Lee study a $U_q'(\mathfrak{g})$-module V with the following properties.

- V decomposes as $V(\theta) \oplus V(0)$ as a $U_q(\mathfrak{g}_0)$-module.
- V has a crystal base.

R. Kodera (✉)
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan
e-mail: kryosuke@ms.u-tokyo.ac.jp

Here θ is given by

$$\theta = \begin{cases} \varpi_1 + \varpi_n & \text{if } \mathfrak{g} \text{ is of type } A_n^{(1)}, \\ 2\varpi_1 & \text{if } \mathfrak{g} \text{ is of type } C_n^{(1)}, \\ \varpi_{i_0} & \text{otherwise,} \end{cases}$$

where ϖ_i denotes the i-th fundamental weight of $\mathfrak{g}_0$ and i_0 denotes the node of the Dynkin diagram of $\mathfrak{g}$ which is joined to the special node 0. If $\mathfrak{g}$ is an untwisted affine algebra, then θ is the highest root of $\mathfrak{g}_0$. Otherwise, namely if $\mathfrak{g}$ is a twisted affine algebra, then θ is the highest short root of $\mathfrak{g}_0$. The structure of the crystal base of V is determined by defining the action of the affine Kashiwara operator $\tilde{f}_0$ on $B(\theta) \oplus B(0)$ correctly for all quantized affine algebras in a uniform manner. We call them adjoint crystals here.

As a natural generalization of [1], we consider a family of $U_q'(\mathfrak{g})$-modules $\{V_l\}_{l \in \mathbb{Z}_{\geq 0}}$ which has the following properties.

- V_l decomposes as $\bigoplus_{k=0}^{l} V(k\theta)$ as a $U_q(\mathfrak{g}_0)$-module.
- V_l has a crystal base.

In fact, V_l is given as follows:

$$V_l = \begin{cases} W^{1,l} \otimes W^{n,l} & \text{if } \mathfrak{g} \text{ is of type } A_n^{(1)}, \\ W^{1,2l} & \text{if } \mathfrak{g} \text{ is of type } C_n^{(1)}, \\ W^{i_0,l} & \text{otherwise.} \end{cases}$$

The decomposition of V_l as a $U_q(\mathfrak{g}_0)$-module follows from results of [2] for untwisted cases and [3] for twisted cases. Moreover we can easily see that V_l satisfies the sufficient condition for the existence of a crystal base stated in [8, Proposition 3.4.5]. We denote by B_l the crystal base of V_l. Remark that the crystal B_l has been studied for some cases. For example, the case of type $C_n^{(1)}$ appears in [7], $D_n^{(1)}$ in [14], $D_{n+1}^{(2)}$ in [8], $G_2^{(1)}$ in [16] and $D_4^{(3)}$ in [10]. Inspired by [1], we expect that (i) the crystal graph of B_{l-1} is a full subgraph of that of B_l and (ii) there exists a simple rule for extending the action of $\tilde{f}_0$ on B_{l-1} to that on B_l. We consider in the present paper the case of $A_n^{(1)}$, $C_n^{(1)}$ and $D_{n+1}^{(2)}$ to show that the above expectations (i) and (ii) are true in these cases.

Let us explain results of this paper for type $A_n^{(1)}$. Similar assertions also hold for type $C_n^{(1)}$ and $D_{n+1}^{(2)}$. By the decomposition of B_l as a $U_q(\mathfrak{g}_0)$-crystal, we regard $B(k\theta)$ as a subset of B_l. We define $n+1$ maps $\Theta_1, \dots, \Theta_{n+1}$ from B_{l-1} to B_l in Subsection 3.3. Our results are summarized as follows.

Theorem 1.1

(i) *The crystal graph of B_{l-1} is regarded as a full subgraph of that of B_l via Θ_1.*
(ii) *For $j = 2, \dots, n+1$, the map Θ_j and the Kashiwara operator $\tilde{f}_0$ commute with each other.*

(iii) *We have*

$$\bigcup_{j=2}^{n+1} \operatorname{Im}\Theta_j = \bigsqcup_{k=0}^{l} \{b \in B(k\theta) \subset B_l \mid \operatorname{wt} b \in \operatorname{wt} B((k-1)\theta)\}$$

and the space $B_l \setminus \bigcup_{j=2}^{n+1} \operatorname{Im}\Theta_j$ *is weight multiplicity free.*

(iv) *Let* $b \in B_l \setminus \bigcup_{j=2}^{n+1} \operatorname{Im}\Theta_j$. *Then* $\tilde{f}_0 b = 0$ *if* $\operatorname{wt} b + \theta \notin \operatorname{wt} B(l\theta)$, *and* $\tilde{f}_0 b = b'$ *if* $\operatorname{wt} b + \theta \in \operatorname{wt} B(l\theta)$ *where* b' *is the unique element of* $B_l \setminus \bigcup_{j=2}^{n+1} \operatorname{Im}\Theta_j$ *satisfying* $\operatorname{wt} b' = \operatorname{wt} b + \theta$.

2 Preliminaries

2.1 Quantized universal enveloping algebras and crystal bases

We shall review on quantized universal enveloping algebras and the crystal base theory based on [9]. We also refer [4].

Suppose that the following data are given:

P: a free $\mathbb{Z}$-module,
I: an index set,
$\Pi = \{\alpha_i \mid i \in I\} \subset P$,
$\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee = \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$,
$(\cdot\,,\cdot)$: a $\mathbb{Q}$-valued symmetric bilinear form on P.

These data are supposed to satisfy the following conditions:

$$(\alpha_i, \alpha_i) > 0 \quad \text{for any } i \in I,$$
$$(\alpha_i, \alpha_j) \le 0 \quad \text{for any } i, j \in I \text{ with } i \ne j,$$
$$\langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for any } i \in I \text{ and } \lambda \in P.$$

We do not assume Π and $\Pi^\vee$ to be linearly independent sets in general. Note that $(\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is a symmetrizable generalized Cartan matrix. Let $\mathfrak{g}$ be the associated Kac-Moody Lie algebra.

Let γ be the minimal positive integer such that $(\alpha_i, \alpha_i)/2 \in \gamma^{-1}\mathbb{Z}$ for any $i \in I$. Let q be an indeterminate over $\mathbb{Q}$ and put $q_s = q^{1/\gamma}$. We use the notation:

$$q_i = q^{(\alpha_i,\alpha_i)/2},\ [k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}},\ [k]_i\,! = \prod_{r=1}^{k} [r]_i.$$

Definition 2.1 The quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with $(P, P^\vee, \Pi, \Pi^\vee)$ is the unital associative algebra over the rational fraction field $\mathbb{Q}(q_s)$ generated by e_i and f_i for $i \in I$ and q^h for $h \in \gamma^{-1}P^\vee$ with the following defining relations:

(i) $q^0 = 1,\ q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in \gamma^{-1}P^\vee$,

(ii) $q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i$ for $i \in I$ and $h \in \gamma^{-1}P^\vee$,
(iii) $q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i$ for $i \in I$ and $h \in \gamma^{-1}P^\vee$,
(iv) $[e_i, f_j] = \delta_{ij}\dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$,
(v) $\sum_{k=0}^{1-\langle h_i,\alpha_j\rangle} (-1)^k e_i^{(k)} e_j e_i^{(1-\langle h_i,\alpha_j\rangle-k)} = 0$ for $i, j \in I$ with $i \neq j$,
(vi) $\sum_{k=0}^{1-\langle h_i,\alpha_j\rangle} (-1)^k f_i^{(k)} f_j f_i^{(1-\langle h_i,\alpha_j\rangle-k)} = 0$ for $i, j \in I$ with $i \neq j$.

Here $t_i = q^{((\alpha_i,\alpha_i)/2)h_i}$, $e_i^{(k)} = \dfrac{e_i^k}{[k]_i!}$ and $f_i^{(k)} = \dfrac{f_i^k}{[k]_i!}$.

We call P, $P^\vee$, α_i and h_i the weight lattice, the coweight lattice, a simple root and a simple coroot, respectively. We denote by P^+ the set of dominant weights, that is, $P^+ = \{\lambda \in P \mid \langle h_i, \lambda\rangle \geq 0$ for any $i \in I\}$. We define the root lattice Q and its positive part Q^+ by $Q = \sum_{i\in I}\mathbb{Z}\alpha_i$ and $Q^+ = \sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$. For $\lambda, \mu \in P$, we say $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

Let M be an integrable $U_q(\mathfrak{g})$-module and $M = \bigoplus_{\lambda\in P} M_\lambda$ its weight space decomposition. According to the representation theory of $U_q(\mathfrak{sl}_2)$, M_λ decomposes as

$$M_\lambda = \bigoplus_{n\geq 0} f_i^{(n)}(\operatorname{Ker} e_i \cap M_{\lambda+n\alpha_i})$$

for each i. We define endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ of M by

$$\tilde{e}_i(f_i^{(n)}u) = f_i^{(n-1)}u$$

and

$$\tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}u$$

for $u \in \operatorname{Ker} e_i \cap M_{\lambda+n\alpha_i}$. They are called Kashiwara operators or modified root operators.

Let A be the subring of $\mathbb{Q}(q_s)$ that consists of rational fractions without a pole at $q_s = 0$.

Definition 2.2 A pair (L, B) is called a crystal base of an integrable $U_q(\mathfrak{g})$-module M if it satisfies the following conditions:

(i) L is a free A-submodule of M such that $L \otimes_A \mathbb{Q}(q_s) \simeq M$,
(ii) B is a $\mathbb{Q}$-basis of L/q_sL,
(iii) $\tilde{e}_iL \subset L$ and $\tilde{f}_iL \subset L$ for any $i \in I$, hence $\tilde{e}_i$ and $\tilde{f}_i$ act on L/q_sL,
(iv) $\tilde{e}_iB \subset B \sqcup \{0\}$ and $\tilde{f}_iB \subset B \sqcup \{0\}$ for any $i \in I$,
(v) $L = \bigoplus_\lambda L_\lambda$ where $L_\lambda = L \cap M_\lambda$,
(vi) $B = \bigsqcup_\lambda B_\lambda$ where $B_\lambda = B \cap (L_\lambda/q_sL_\lambda)$,
(vii) for $b, b' \in B$, $b' = \tilde{f}_ib$ if and only if $b = \tilde{e}_ib'$.

We often regard B as a crystal base rather than (L, B). For a crystal base B, we define a colored oriented graph called the crystal graph as follows. The vertices of the graph are elements of B. For $b, b' \in B$, draw an arrow labeled by i from b to b' if $b' = \tilde{f}_i b$.

Let $\mathrm{wt}\colon B \to P$ be the map such that $\mathrm{wt}\, b = \lambda$ for $b \in B_\lambda$. For $b \in B$ and $i \in I$, we set

$$\varepsilon_i(b) = \max\{n \mid \tilde{e}_i^n b \neq 0\} \quad \text{and} \quad \varphi_i(b) = \max\{n \mid \tilde{f}_i^n b \neq 0\}.$$

Let M_1 and M_2 be integrable $U_q(\mathfrak{g})$-modules with crystal bases (L_1, B_1) and (L_2, B_2). We denote by $B_1 \oplus B_2$ a direct sum $B_1 \sqcup B_2$. Then $(L_1 \oplus_A L_2, B_1 \oplus B_2)$ gives a crystal base of $M_1 \oplus M_2$. We denote by $B_1 \otimes B_2$ a direct product $B_1 \times B_2$, which is a $\mathbb{Q}$-basis of $L_1/q_s L_1 \otimes_{\mathbb{Q}} L_2/q_s L_2 \simeq (L_1 \otimes_A L_2)/q_s(L_1 \otimes_A L_2)$. Then $(L_1 \otimes_A L_2, B_1 \otimes B_2)$ is a crystal base of $M_1 \otimes M_2$. Kashiwara operators act by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} (\tilde{e}_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes (\tilde{e}_i b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} (\tilde{f}_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes (\tilde{f}_i b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

The notion of crystals is a combinatorial generalization of crystal bases. We shall not review the theory of abstract crystals here since all crystals appearing in this article come from crystal bases. It can be found in [4, Section 4.5].

A bijection between two crystal bases is called an isomorphism of crystals if it commutes with Kashiwara operators and preserves weights.

2.2 Quantized affine algebras

Now we shall turn to affine situations. See [6, Chapter 6–8] for a reference on affine Lie algebras. Let $\mathfrak{g}$ be an affine Lie algebra with a Cartan matrix indexed by I. We assume that $\mathfrak{g}$ is not of type $A_{2n}^{(2)}$ for simplicity. Let δ be the generator of imaginary roots, c the canonical central element and d the degree operator. Choose $0 \in I$ such that $\delta - \alpha_0 \in \sum_{i \in I, i \neq 0} \mathbb{Z}\alpha_i$ and set $I_0 = I \setminus \{0\}$. Let $\mathfrak{g}_0$ be the underlying simple Lie algebra of finite type. Let Λ_i be the fundamental weight for $i \in I$ and define the weight lattice P by

$$P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta.$$

We normalize the symmetric invariant bilinear form on P so that $(\delta, \lambda) = \langle c, \lambda \rangle$ for any $\lambda \in P$. Set $P_{\mathrm{cl}} = P/\mathbb{Z}\delta$ and denote by the same letter the image of Λ_i for each i. Then we have

$$P_{\mathrm{cl}} = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i.$$

We define the level-zero fundamental weight ϖ_i for $i \in I_0$, which is an element of P_{cl}, by $\varpi_i = \Lambda_i - \langle c, \Lambda_i \rangle \Lambda_0$. We set $P_0 = P_{\mathrm{cl}}/\mathbb{Z}\Lambda_0$ and denote by the same letter the image of ϖ_i for each i. Then we have

$$P_0 = \bigoplus_{i \in I_0} \mathbb{Z}\varpi_i.$$

We identify P_0 with the weight lattice of $\mathfrak{g}_0$ and ϖ_i its i-th fundamental weight. Then quantized algebras $U_q(\mathfrak{g})$, $U_q'(\mathfrak{g})$ and $U_q(\mathfrak{g}_0)$ are defined associated with weight lattices P, P_{cl} and P_0. Note that simple roots in P_{cl} are linearly dependent while they are linearly independent in P and P_0. However we use the same letters for simple roots in P_{cl} and P_0. For the coweight lattices, we have

$$P_0^\vee = \bigoplus_{i \in I_0} \mathbb{Z}h_i \subset P_{\mathrm{cl}}^\vee = \bigoplus_{i \in I} \mathbb{Z}h_i \subset P^\vee = \bigoplus_{i \in I} \mathbb{Z}h_i \oplus \mathbb{Z}d.$$

Hence $U_q'(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by e_i and f_i for $i \in I$ and q^h for $h \in \gamma^{-1}P_{\mathrm{cl}}^\vee$ and $U_q(\mathfrak{g}_0)$ is the subalgebra of $U_q'(\mathfrak{g})$ generated by e_i and f_i for $i \in I_0$ and q^h for $h \in \gamma^{-1}P_0^\vee$. When we need to clarify in which weight lattice we work, write a $U_q(\mathfrak{g}_0)$-crystal, a $U_q(\mathfrak{g}_0)$-weight, etc.

Since the definition of Kirillov-Reshetikhin modules is not used in this paper, we omit it and refer [13, Section 3]. They are finite-dimensional irreducible $U_q'(\mathfrak{g})$-modules. We denote by $W^{i,l}$ the Kirillov-Reshetikhin module for $i \in I_0$ and a nonnegative integer l. For the quantized affine algebras of nonexceptional types, it is proved in [8] and [13] that any Kirillov-Reshetikhin module has a crystal base. They are called Kirillov-Reshetikhin crystals and denoted by $B^{i,l}$.

2.3 Some lemmas on weights

Assume that $\mathfrak{g}_0$ is either of type A_n, C_n or B_n in this subsection. For $\lambda \in P_0^+$, we denote by $B(\lambda)$ the crystal base of the finite-dimensional irreducible $U_q(\mathfrak{g}_0)$-module with highest weight λ. If $\mathfrak{g}_0$ is of type A_n or C_n, we denote by θ the highest root of $\mathfrak{g}_0$. If $\mathfrak{g}_0$ is of type B_n, denote by θ the highest short root of $\mathfrak{g}_0$. We shall prove some lemmas on weights of $B(k\theta)$.

It is well known that

$$\mathrm{wt}\, B(\lambda) = W_0 \cdot \{\nu \in P_0^+ \mid \nu \le \lambda\}$$

for any $\lambda \in P_0^+$. Here W_0 is the Weyl group of $\mathfrak{g}_0$. (See e.g. [5, 21.3 Proposition].) Hence the following lemma is immediate.

Lemma 2.1 *Let k and k' be nonnegative integers with $k' \le k$. Then* $\mathrm{wt}\, B(k\theta)$ *contains* $\mathrm{wt}\, B(k'\theta)$.

We use the following standard realization for the finite root systems of type A_n, C_n and B_n. In the case of type A_n, simple roots and fundamental weights are defined by

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad \text{for } i \in I_0,$$

$$\varpi_i = \sum_{j=1}^{i} \epsilon_j \quad \text{for } i \in I_0.$$

Here we define the element $\epsilon_j \in P_0$ for $j = 1, \ldots, n+1$ by

$$\langle h_i, \epsilon_j \rangle = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\sum_{j=1}^{n+1} \epsilon_j = 0$. The highest root θ is given by

$$\theta = \alpha_1 + \cdots + \alpha_n = \epsilon_1 - \epsilon_{n+1} = \varpi_1 + \varpi_n.$$

In the case of type C_n,

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \quad \text{for } i = 1, \ldots, n-1, \\ \alpha_n &= 2\epsilon_n, \\ \varpi_i &= \sum_{j=1}^{i} \epsilon_j \quad \text{for } i \in I_0, \end{aligned}$$

where $\epsilon_j \in P_0$ for $j = 1, \ldots, n$ is defined by

$$\langle h_i, \epsilon_j \rangle = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

The highest root θ is given by

$$\theta = 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n = 2\epsilon_1 = 2\varpi_1.$$

In the case of type B_n,

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \quad \text{for } i = 1, \ldots, n-1, \\ \alpha_n &= \epsilon_n, \\ \varpi_i &= \sum_{j=1}^{i} \epsilon_j \quad \text{for } i = 1, \ldots, n-1, \\ \varpi_n &= \frac{1}{2} \sum_{j=1}^{n} \epsilon_j, \end{aligned}$$

where $\epsilon_j \in P_0$ for $j = 1, \ldots, n$ is defined by

$$\langle h_i, \epsilon_j \rangle = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i+1, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n-1$ and

$$\langle h_n, \epsilon_j \rangle = \begin{cases} 2 & \text{if } j = n, \\ 0 & \text{otherwise.} \end{cases}$$

The highest short root θ is given by

$$\theta = \alpha_1 + \cdots + \alpha_n = \epsilon_1 = \varpi_1.$$

Let $\mu \in P_0$. We denote by $m_j(\mu)$ the coefficient of ϵ_j in μ. For type A_n, we normalize them so that $\sum_{j=1}^{n+1} m_j(\mu) = 0$. Set

$$|\mu| = \sum_{1 \le j \le n+1,\, m_j(\mu) > 0} m_j(\mu)$$

if $\mathfrak{g}_0$ is of type A_n, and

$$|\mu| = \sum_{j=1}^{n} |m_j(\mu)|$$

if $\mathfrak{g}_0$ is of type C_n or B_n.

Lemma 2.2 *Let k be a nonnegative integer. Then we have*

$$\mathrm{wt}\, B(k\theta) = \{\mu \in P_0 \mid |\mu| \le k\}$$

if $\mathfrak{g}_0$ is of type A_n or B_n, and

$$\mathrm{wt}\, B(k\theta) = \{\mu \in P_0 \mid |\mu| = 2s, 0 \le s \le k\}$$

if $\mathfrak{g}_0$ is of type C_n.

Proof We prove the case of type A_n. Recall the Weyl group W_0 is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$. For $\mu \in P_0$, the coefficient of α_i in μ is given by $\sum_{j=1}^{i} m_j(\mu)$. Therefore $\mu \in \mathrm{wt}\, B(k\theta)$ if and only if there exists an element τ of W_0 satisfying

$$m_{\tau(1)}(\mu) \ge m_{\tau(2)}(\mu) \ge \cdots \ge m_{\tau(n+1)}(\mu)$$

and

$$k \ge \sum_{j=1}^{i} m_{\tau(j)}(\mu)$$

for $i = 1, \dots, n$. Since

$$|\mu| = \max\{\sum_{j=1}^{i} m_{\tau(j)}(\mu) \mid 1 \le i \le n\},$$

$\mu \in \mathrm{wt}\, B(k\theta)$ if and only if $|\mu| \le k$.

We prove the case of type C_n. The Weyl group W_0 is isomorphic to the semidirect product $\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. The coefficient of α_i in μ is given by $\sum_{j=1}^{i} m_j(\mu)$ for $i \neq n$ and $(\sum_{j=1}^{n} m_j(\mu))/2$ for $i = n$. Assume that μ is dominant, that is,

$$m_1(\mu) \ge m_2(\mu) \ge \cdots \ge m_n(\mu) \ge 0.$$

Then $\mu \in \mathrm{wt}\, B(k\theta)$ if and only if

$$2k \ge \sum_{j=1}^{i} m_j(\mu)$$

for $i = 1, \dots, n-1$ and

$$k - \frac{1}{2}\sum_{j=1}^{n} m_j(\mu) \in \mathbb{Z}_{\ge 0}.$$

This condition is equivalent to

$$2k - \sum_{j=1}^{n} m_j(\mu) \in 2\mathbb{Z}_{\ge 0}.$$

For general μ, μ is W_0-conjugate to a dominant weight with the above condition if and only if μ satisfies

$$2k - \sum_{j=1}^{n} |m_j(\mu)| \in 2\mathbb{Z}_{\ge 0}.$$

We prove the case of type B_n. The Weyl group W_0 is same as the case of type C_n. The coefficient of α_i in μ is given by $\sum_{j=1}^{i} m_j(\mu)$. Assume that μ is dominant, that is,

$$m_1(\mu) \ge m_2(\mu) \ge \cdots \ge m_n(\mu) \ge 0.$$

Then $\mu \in \mathrm{wt}\, B(k\theta)$ if and only if

$$k \ge \sum_{j=1}^{i} m_j(\mu)$$

for $i = 1, \dots, n$. This condition is equivalent to

$$k \ge \sum_{j=1}^{n} m_j(\mu).$$

For general μ, μ is W_0-conjugate to a dominant weight with the above condition if and only if μ satisfies

$$k \geq \sum_{j=1}^{n} |m_j(\mu)|.$$

$\square$

Let $\mu \in \operatorname{wt} B(k\theta) \setminus \operatorname{wt} B((k-1)\theta)$. Now we investigate the behavior of $\mu+\theta$. Let $\mathfrak{g}_0$ be of type A_n. There are the following four cases:

(a) $m_1(\mu) \geq 0$ and $m_{n+1}(\mu) \leq 0$,
(b) $m_1(\mu) \geq 0$ and $m_{n+1}(\mu) > 0$,
(c) $m_1(\mu) < 0$ and $m_{n+1}(\mu) \leq 0$,
(d) $m_1(\mu) < 0$ and $m_{n+1}(\mu) > 0$.

Lemma 2.3 *Let $\mathfrak{g}_0$ be of type A_n and assume $\mu \in \operatorname{wt} B(k\theta) \setminus \operatorname{wt} B((k-1)\theta)$. Then we have*

(i) $\mu+\theta \in \operatorname{wt} B((k+1)\theta) \setminus \operatorname{wt} B(k\theta)$ *if and only if μ satisfies* (a).
(ii) $\mu+\theta \in \operatorname{wt} B(k\theta) \setminus \operatorname{wt} B((k-1)\theta)$ *if and only if μ satisfies* (b) *or* (c).
(iii) $\mu+\theta \in \operatorname{wt} B((k-1)\theta) \setminus \operatorname{wt} B((k-2)\theta)$ *if and only if μ satisfies* (d).

Proof Since $\theta = \epsilon_1 - \epsilon_{n+1}$, we have $m_1(\mu+\theta) = m_1(\mu)+1$, $m_{n+1}(\mu+\theta) = m_{n+1}(\mu)-1$ and $m_j(\mu+\theta) = m_j(\mu)$ for $j \neq 1, n+1$.

If (a) is satisfied, then

$$\begin{aligned} |\mu+\theta| &= |\mu|+1 \\ &= k+1. \end{aligned}$$

This implies $\mu+\theta \in \operatorname{wt} B((k+1)\theta) \setminus \operatorname{wt} B(k\theta)$ by Lemma 2.2.

If (b) is satisfied, then

$$\begin{aligned} |\mu+\theta| &= |\mu|+1-1 \\ &= |\mu| \\ &= k. \end{aligned}$$

This implies $\mu+\theta \in \operatorname{wt} B(k\theta) \setminus \operatorname{wt} B((k-1)\theta)$.

If (c) is satisfied, then

$$\begin{aligned} |\mu+\theta| &= |\mu| \\ &= k. \end{aligned}$$

This implies $\mu+\theta \in \operatorname{wt} B(k\theta) \setminus \operatorname{wt} B((k-1)\theta)$.

If (d) is satisfied, then

$$|\mu+\theta| = |\mu|-1$$

$$= k - 1.$$

This implies $\mu + \theta \in \mathrm{wt}\, B((k-1)\theta) \setminus \mathrm{wt}\, B((k-2)\theta)$. □

Similar results for type C_n and B_n are verified as follows.

Lemma 2.4 *Let* $\mathfrak{g}_0$ *be of type* C_n *and assume* $\mu \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$. *Then we have*

(i) $\mu + \theta \in \mathrm{wt}\, B((k+1)\theta) \setminus \mathrm{wt}\, B(k\theta)$ *if and only if* $m_1(\mu) \geq 0$.
(ii) $\mu + \theta \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$ *if and only if* $m_1(\mu) = -1$.
(iii) $\mu + \theta \in \mathrm{wt}\, B((k-1)\theta) \setminus \mathrm{wt}\, B((k-2)\theta)$ *if and only if* $m_1(\mu) \leq -2$.

Proof The assumption $\mu \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$ implies $|\mu| = 2k$ by Lemma 2.2. Since $\theta = 2\epsilon_1$, we have

$$\begin{aligned}|\mu + \theta| &= |m_1(\mu) + 2| + \sum_{j=2}^{n} |m_j(\mu)| \\ &= |m_1(\mu) + 2| - |m_1(\mu)| + |\mu| \\ &= |m_1(\mu) + 2| - |m_1(\mu)| + 2k.\end{aligned}$$

Then

$|\mu + \theta| = 2(k+1)$ if and only if $m_1(\mu) \geq 0$,
$|\mu + \theta| = 2k$ if and only if $m_1(\mu) = -1$,
$|\mu + \theta| = 2(k-1)$ if and only if $m_1(\mu) \leq -2$.

This completes the proof. □

Lemma 2.5 *Let* $\mathfrak{g}_0$ *be of type* B_n *and assume* $\mu \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$. *Then we have*

(i) $\mu + \theta \in \mathrm{wt}\, B((k+1)\theta) \setminus \mathrm{wt}\, B(k\theta)$ *if and only if* $m_1(\mu) \geq 0$.
(ii) $\mu + \theta \in \mathrm{wt}\, B((k-1)\theta) \setminus \mathrm{wt}\, B((k-2)\theta)$ *if and only if* $m_1(\mu) < 0$.

Proof By Lemma 2.2, $|\mu| = k$. Since $\theta = \epsilon_1$, we have

$$\begin{aligned}|\mu + \theta| &= |m_1(\mu) + 1| + \sum_{j=2}^{n} |m_j(\mu)| \\ &= |m_1(\mu) + 1| - |m_1(\mu)| + |\mu| \\ &= |m_1(\mu) + 1| - |m_1(\mu)| + k.\end{aligned}$$

Then

$|\mu + \theta| = k + 1$ if and only if $m_1(\mu) \geq 0$,
$|\mu + \theta| = k - 1$ if and only if $m_1(\mu) < 0$.

This completes the proof. □

3 The case of type $A_n^{(1)}$

We assume that $\mathfrak{g}$ is the affine Lie algebra of type $A_n^{(1)}$ in this section.

3.1 The crystal structure of $B(\lambda)$

We recall the structure of the $U_q(\mathfrak{g}_0)$-crystal $B(\lambda)$ following [11]. We identify the set of dominant weights P_0^+ with the set of Young diagrams with depth at most n as follows. For the Young diagram corresponding to a partition $(\lambda_1, \dots, \lambda_n)$, associate $\lambda = \sum_{i=1}^n (\lambda_i - \lambda_{i+1})\varpi_i \in P_0^+$ where $\lambda_{n+1} = 0$. We regard the crystal $B(\varpi_1)$ as the set consisting of letters $1, \dots, n+1$ with the following crystal structure:

$$\mathrm{wt}(j) = \epsilon_j,$$

$$\tilde{e}_i(i+1) = i,\ \tilde{e}_i(j) = 0 \text{ for } j \neq i+1,$$

$$\tilde{f}_i(i) = i+1,\ \tilde{f}_i(j) = 0 \text{ for } j \neq i.$$

Let $\lambda \in P_0^+$. As a set, $B(\lambda)$ is identified with the set of semistandard tableaux of shape λ with entries $1, \dots, n+1$. For $b \in B(\lambda)$, $\mathrm{Jw}(b)$ denotes the Japanese reading word of b, that is, the word obtained by reading entries of b from the top to the bottom in each column, from the right-most column to the left. If $\mathrm{Jw}(b) = b_1 \cdots b_N$ then we regard b as $b_1 \otimes \cdots \otimes b_N$, an element of $B(\varpi_1)^{\otimes N}$. Then Kashiwara operators act on b by the tensor product rule. More explicitly, one can calculate $\tilde{e}_i b$ and $\tilde{f}_i b$ as follows. Fix $i \in I$. Eliminate every letter which is neither i nor $i+1$ from $\mathrm{Jw}(b)$. If the resulting word has adjacent pairs $i \cdot (i+1)$, cancel out them, and repeat the procedure until we obtain the word $(i+1)^r \cdot i^s$ for some r, s. Conclude that $\tilde{e}_i b = 0$ if $r = 0$ and $\tilde{f}_i b = 0$ if $s = 0$. Otherwise, we determine b_j and b_k such that

$$\tilde{e}_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes (\tilde{e}_i b_j) \otimes \cdots \otimes b_N$$

and

$$\tilde{f}_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes (\tilde{f}_i b_k) \otimes \cdots \otimes b_N$$

as b_j corresponds to the right-most $i+1$ and b_k the left-most i in the word. When we deal with a tensor product $B(\lambda) \otimes B(\mu)$ for $\lambda, \mu \in P_0^+$, define $\mathrm{Jw}(b_1 \otimes b_2) = \mathrm{Jw}(b_1)\mathrm{Jw}(b_2)$ for $b_1 \in B(\lambda)$ and $b_2 \in B(\mu)$. Then one can calculate the actions of Kashiwara operators in the same way.

3.2 Kirillov-Reshetikhin crystals for type A

We recall the crystal structure of $B^{i,l}$ following [15]. Since $W^{i,l}$ is isomorphic to $V(l\varpi_i)$ as a $U_q(\mathfrak{g}_0)$-module, $B^{i,l}$ is isomorphic to $B(l\varpi_i)$ as a $U_q(\mathfrak{g}_0)$-crystal. For the description of the actions of $\tilde{e}_0$ and $\tilde{f}_0$, we use the promotion operator σ. The operator σ is a bijection from $B^{i,l}$ to $B^{i,l}$ which satisfy

$$\sigma \tilde{e}_j = \tilde{e}_{j+1} \sigma$$

and

$$\sigma \tilde{f}_j = \tilde{f}_{j+1}\sigma$$

for each j modulo $n+1$. This map corresponds to the automorphism of the Dynkin diagram of type $A_n^{(1)}$ which takes j to $j+1$ modulo $n+1$. Using σ, we obtain

$$\tilde{e}_0 = \sigma^{-1}\tilde{e}_1\sigma$$

and

$$\tilde{f}_0 = \sigma^{-1}\tilde{f}_1\sigma.$$

We shall describe the action of σ explicitly only for $B^{1,l}$ and $B^{n,l}$, and omit the general definition. See [15, 3.3] for details.

For $b \in B^{1,l}$, we denote the number of entries j appearing in the tableau b by $x_j(b)$. The map which takes $b \in B^{1,l}$ to $(x_j(b))_{j=1,\dots,n+1} \in (\mathbb{Z}_{\geq 0})^{n+1}$ is injective. The action of the promotion operator is given by

$$x_1(\sigma(b)) = x_{n+1}(b)$$

and

$$x_j(\sigma(b)) = x_{j-1}(b)$$

for $j = 2, \dots, n+1$. Hence the actions of Kashiwara operators are described as follows:

$$\begin{aligned}
x_j(\tilde{e}_0 b) &= x_j(b) - \delta_{j1} + \delta_{j\,n+1},\\
x_j(\tilde{f}_0 b) &= x_j(b) + \delta_{j1} - \delta_{j\,n+1},\\
x_j(\tilde{e}_i b) &= x_j(b) - \delta_{j\,i+1} + \delta_{ji} \quad \text{for } i \in I_0,\\
x_j(\tilde{f}_i b) &= x_j(b) + \delta_{j\,i+1} - \delta_{ji} \quad \text{for } i \in I_0.
\end{aligned}$$

Here we define $b = 0$ if $x_j(b) < 0$ for some j. Immediately we have

$$\begin{aligned}
\varepsilon_0(b) &= x_1(b),\\
\varphi_0(b) &= x_{n+1}(b),\\
\varepsilon_i(b) &= x_{i+1}(b) \quad \text{for } i \in I_0,\\
\varphi_i(b) &= x_i(b) \quad \text{for } i \in I_0.
\end{aligned}$$

For $j = 1, \dots, n+1$, we denote by C_j the semistandard tableau consisting of one column with depth n which has no entry j. Each column in a semistandard tableau of shape (l^n) is C_j for some j. These columns are arranged as, from left to right, some (possibly 0) C_{n+1}'s, some C_n's, ..., and some C_1's. For $b \in B^{n,l}$, we denote the number of columns C_j in b by $y_j(b)$. The map which takes $b \in B^{n,l}$ to $(y_j(b))_{j=1,\dots,n+1} \in (\mathbb{Z}_{\geq 0})^{n+1}$ is injective. The action of the promotion operator is given by

$$y_1(\sigma(b)) = y_{n+1}(b)$$

and

$$y_j(\sigma(b)) = y_{j-1}(b)$$

for $j = 2, \dots, n+1$. Hence the actions of Kashiwara operators are described as follows:

$$\begin{aligned}
y_j(\tilde{e}_0 b) &= y_j(b) - \delta_{j\,n+1} + \delta_{j1}, \\
y_j(\tilde{f}_0 b) &= y_j(b) + \delta_{j\,n+1} - \delta_{j1}, \\
y_j(\tilde{e}_i b) &= y_j(b) - \delta_{ji} + \delta_{j\,i+1} \quad \text{for } i \in I_0, \\
y_j(\tilde{f}_i b) &= y_j(b) + \delta_{ji} - \delta_{j\,i+1} \quad \text{for } i \in I_0.
\end{aligned}$$

Here we define $b = 0$ if $y_j(b) < 0$ for some j. Immediately we have

$$\begin{aligned}
\varepsilon_0(b) &= y_{n+1}(b), \\
\varphi_0(b) &= y_1(b), \\
\varepsilon_i(b) &= y_i(b) \quad \text{for } i \in I_0, \\
\varphi_i(b) &= y_{i+1}(b) \quad \text{for } i \in I_0.
\end{aligned}$$

3.3 The crystal structure of B_l

For a fixed nonnegative integer l, we set $B_l = B^{1,l} \otimes B^{n,l}$. We shall describe the decomposition of the tensor product $B(l\varpi_1) \otimes B(l\varpi_n)$. Let b_1 be a semistandard tableau of shape (k) and b_2 that of shape (k^n), and assume $x_1(b_1) = 0$ or $y_1(b_2) = 0$. We define $b_1 \cdot b_2$ as the semistandard tableau of shape $(2k, k^{n-1})$ such that $\mathrm{Jw}(b_1 \cdot b_2) = \mathrm{Jw}(b_1)\,\mathrm{Jw}(b_2)$. Define a map $\alpha\colon B(l\varpi_1) \otimes B(l\varpi_n) \to \bigoplus_{k=0}^{l} B(k\theta)$ by $\alpha(b_1 \otimes b_2) = \tilde{b}_1 \cdot \tilde{b}_2 \in B(k\theta)$ where $k = l - \min\{x_1(b_1), y_1(b_2)\}$, $\tilde{b}_1$ is the semistandard tableau of shape (k) which is obtained by removing $\min\{x_1(b_1), y_1(b_2)\}$ 1's from b_1, and $\tilde{b}_2$ is the semistandard tableau of shape (k^n) obtained by removing $\min\{x_1(b_1), y_1(b_2)\}$ C_1's from b_2. Then we have the following lemma.

Lemma 3.1 *The map α is an isomorphism of $U_q(\mathfrak{g}_0)$-crystals between $B(l\varpi_1) \otimes B(l\varpi_n)$ and $\bigoplus_{k=0}^{l} B(k\theta)$.*

Proof It is obvious that α is bijective and preserves weights. Suppose $\alpha(b_1 \otimes b_2) = \tilde{b}_1 \cdot \tilde{b}_2 \in B(k\theta)$. Then we have $\mathrm{Jw}(b_1 \otimes b_2) = \mathrm{Jw}(\tilde{b}_1) \cdot 1^{l-k} \cdot (2 \cdot 3 \cdots (n+1))^{l-k} \cdot \mathrm{Jw}(\tilde{b}_2)$. Since the element $1 \otimes 2 \otimes \cdots \otimes (n+1)$ is annihilated by all $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I_0$, α commutes with Kashiwara operators. □

We write $b \in B(k\theta) \subset B_l$ for $b \in B_l$, if $\alpha(b) \in B(k\theta)$. Here B_l is identified with $B(l\varpi_1) \otimes B(l\varpi_n)$ as a set.

For $j = 1, \dots, n+1$, we define an injective map $\Theta_j\colon B_{l-1} \to B_l$ as follows. For $b_1 \otimes b_2 \in B_l$, set $\Theta_j(b_1 \otimes b_2) = b_1' \otimes b_2'$, where b_1' is the semistandard tableau of shape (l) which is obtained by adding an entry j to b_1, and b_2' is the semistandard tableau of shape (l^n) obtained by adding a column C_j to b_2. These maps preserve $U_q(\mathfrak{g}_0)$-weights by definition.

Proposition 3.1

(i) *For* $k=0,\dots,l-1$, $\Theta_1(B(k\theta))\subset B(k\theta)$.
(ii) *For* $i\in I_0$, *the map* Θ_1 *and the Kashiwara operator* $\tilde{f}_i$ *commute with each other.*
(iii) *Let* $b\in B_{l-1}$ *and assume* $\tilde{f}_0 b\neq 0$. *Then* $\Theta_1(\tilde{f}_0 b)=\tilde{f}_0\Theta_1(b)$.
(iv) *Let* $b\in B_{l-1}$ *and assume* $\tilde{f}_0 b=0$. *Then* $\varphi_0(\Theta_1(b))=1$ *and* $\tilde{f}_0\Theta_1(b)\in B(l\theta)\subset B_l$.

Proof Let $b_1\otimes b_2\in B_{l-1}$ and suppose $\Theta_1(b_1\otimes b_2)=b_1'\otimes b_2'$. We have $x_1(b_1')=x_1(b_1)+1$ and $y_1(b_2')=y_1(b_2)+1$ by the definition of Θ_1. This implies (i).

Since $\mathrm{Jw}(b_1'\otimes b_2')=\mathrm{Jw}(b_1)\cdot 1\cdot 2\cdots(n+1)\cdot\mathrm{Jw}(b_2)$, (ii) is immediate.

We prove (iii). By the definition of Θ_1, we have

$$\varphi_0(b_1')=x_{n+1}(b_1')=x_{n+1}(b_1)=\varphi_0(b_1)$$

and

$$\varepsilon_0(b_2')=y_{n+1}(b_2')=y_{n+1}(b_2)=\varepsilon_0(b_2).$$

There are the following two cases:

$$\tilde{f}_0(b_1\otimes b_2)=\begin{cases}(\tilde{f}_0 b_1)\otimes b_2 & \text{if } \varphi_0(b_1)>\varepsilon_0(b_2),\\ b_1\otimes(\tilde{f}_0 b_2) & \text{if } \varphi_0(b_1)\le\varepsilon_0(b_2).\end{cases}$$

First we assume $\tilde{f}_0(b_1\otimes b_2)=(\tilde{f}_0 b_1)\otimes b_2$. Then $\tilde{f}_0(b_1'\otimes b_2')=(\tilde{f}_0 b_1')\otimes b_2'$ since $\varphi_0(b_1')=\varphi_0(b_1)$ and $\varepsilon_0(b_2')=\varepsilon_0(b_2)$. Recalling the action of $\tilde{f}_0$, the tableau $\tilde{f}_0 b_1$ (resp. $\tilde{f}_0 b_1'$) is obtained from b_1 (resp. b_1') by removing an entry $n+1$ and adding an entry 1. We have

$$\begin{aligned}\Theta_1(\tilde{f}_0(b_1\otimes b_2))&=\Theta_1((\tilde{f}_0 b_1)\otimes b_2)\\&=(\tilde{f}_0 b_1)'\otimes b_2'\end{aligned}$$

where $(\tilde{f}_0 b_1)'$ is the tableau obtained from $\tilde{f}_0 b_1$ by adding an entry 1. Hence the assertion is true since $(\tilde{f}_0 b_1)'$ coincides with $\tilde{f}_0 b_1'$: the semistandard tableau of shape (l) which is obtained from b_1 by removing one entry $n+1$ and adding two 1's. Next we assume $\tilde{f}_0(b_1\otimes b_2)=b_1\otimes(\tilde{f}_0 b_2)$. Similarly, we have $\tilde{f}_0(b_1'\otimes b_2')=b_1'\otimes(\tilde{f}_0 b_2')$. In this case, we have

$$\begin{aligned}\Theta_1(\tilde{f}_0(b_1\otimes b_2))&=\Theta_1(b_1\otimes(\tilde{f}_0 b_2))\\&=b_1'\otimes(\tilde{f}_0 b_2)'\end{aligned}$$

where $(\tilde{f}_0 b_2)'$, unless $\tilde{f}_0 b_2=0$, is obtained from b_2 by removing one column C_1, adding one C_{n+1}, and adding one C_1 in this order. It coincides with $\tilde{f}_0 b_2'$ whenever b_2 has at least one column C_1. Thus (iii) is proved.

To prove (iv), suppose $b = b_1 \otimes b_2 \in B_{l-1}$ and $\tilde{f}_0 b = 0$. We have $\varphi_0(b_2) = 0$ and $\varphi_0(b_1) \le \varepsilon_0(b_2)$ by the tensor product rule. Then $\tilde{f}_0(b'_1 \otimes b'_2) = b'_1 \otimes (\tilde{f}_0 b'_2)$ and $\tilde{f}_0 b'_2 \neq 0$ since b'_2 has exactly one column C_1. As $\tilde{f}_0 b'_2$ has no more C_1, $b'_1 \otimes (\tilde{f}_0 b'_2) \in B(l\theta) \subset B_l$ and $\tilde{f}_0^2(b'_1 \otimes b'_2) = 0$. □

By Proposition 3.1, the crystal graph of B_{l-1} can be naturally regarded as a full subgraph of that of B_l.

Proposition 3.2 *Let $j = 2, \dots, n+1$.*

(i) *For $k = 0, \dots, l-1$, $\Theta_j(B(k\theta)) \subset B((k+1)\theta)$.*
(ii) *Let $i \in I_0$ and $b \in B_{l-1}$. If $\tilde{f}_i b \neq 0$, then $\Theta_j(\tilde{f}_i b) = \tilde{f}_i \Theta_j(b)$.*
(iii) *The map Θ_j and the Kashiwara operator $\tilde{f}_0$ commute with each other.*

Proof For $b \in B_{l-1}$, we suppose $b = b_1 \otimes b_2$ and $\Theta_j(b) = b'_1 \otimes b'_2$. Then (i) follows from $x_1(b'_1) = x_1(b_1)$ and $y_1(b'_2) = y_1(b_2)$.

Recalling $\varphi_i(b_1) = x_i(b_1)$, $\varepsilon_i(b_2) = y_i(b_2)$ etc., we have

$$\varphi_i(b'_1) = \varphi_i(b_1) + \delta_{ij} \text{ for } i \in I_0,$$

$$\varepsilon_i(b'_2) = \varepsilon_i(b_2) + \delta_{ij} \text{ for } i \in I_0,$$

$$\varphi_0(b'_1) = \varphi_0(b_1) + \delta_{j\,n+1},$$

$$\varepsilon_0(b'_2) = \varepsilon_0(b_2) + \delta_{j\,n+1}.$$

Then (ii) and the case $\tilde{f}_0 b \neq 0$ of (iii) are obtained from the above formulas by arguments similar to that in the proof of Proposition 3.1 (iii). In the case $\tilde{f}_0 b = 0$, we have $\tilde{f}_0(b'_1 \otimes b'_2) = b'_1 \otimes (\tilde{f}_0 b'_2) = 0$ since $\varphi_0(b'_2) = y_1(b'_2) = y_1(b_2) = \varphi_0(b_2) = 0$. □

Hence the action of $\tilde{f}_0$ on the set $\bigcup_{j=2}^{n+1} \mathrm{Im}\,\Theta_j \subset B_l$ is completely determined if we know that on B_{l-1}.

Remark 3.1 For Proposition 3.1 and 3.2, the similar statements hold when we replace $\tilde{f}_i$ by $\tilde{e}_i$ for $i \in I$ and φ_0 by ε_0. We can prove them in a similar way.

From now on, we study the action of $\tilde{f}_0$ on $B_l \setminus \bigcup_{j=2}^{n+1} \mathrm{Im}\,\Theta_j$. Let $\mu \in P_0$. In Subsection 2.3, we defined $m_j(\mu)$ as the coefficient of ϵ_j in μ with $\sum_{j=1}^{n+1} m_j(\mu) = 0$. The number of entries j in a semistandard tableau of shape $k\theta$ and weight μ is equal to $m_j(\mu) + k$. Put

$$m_j(\mu, k) = m_j(\mu) + k$$

and

$$J(\mu) = \{j \mid m_j(\mu) > 0\}.$$

Then we can rewrite Lemma 2.2 as follows.

Lemma 3.2 *Let k be a nonnegative integer. For $\mu \in P_0$, $\mu \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$ if and only if $\sum_{j\in J(\mu)} m_j(\mu,k) = k|J(\mu)| + k$.*

Proposition 3.3 *We have*

$$\bigcup_{j=2}^{n+1} \Theta_j(B((k-1)\theta)) = \{b \in B(k\theta) \mid \mathrm{wt}\, b \in \mathrm{wt}\, B((k-1)\theta)\}.$$

Proof Proposition 3.2 (i) implies that the right-hand side contains the left. Suppose b does not belong to the left-hand side and $\alpha(b) = \tilde{b}_1 \cdot \tilde{b}_2 \in B(k\theta)$. Put $\mu = \mathrm{wt}\, b$. Let J be the set of the letters appearing in $\tilde{b}_1$. Then $\tilde{b}_2$ has no column C_j if $j \in J$. This means the number of entries j in $\tilde{b}_2$ is equal to k if $j \in J$. Therefore $J(\mu) = J$ and $\sum_{j\in J} m_j(\mu,k) = k|J| + k$. We obtain $\mu \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$ by Lemma 3.2. □

Although the following fact seems to be known, we give a proof.

Lemma 3.3 *Let k be a nonnegative integer. The multiplicity of every element of $\mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$ in $B(k\theta)$ is one.*

Proof Suppose $b = b_1 \cdot b_2 \in B(k\theta)$ where $b_1 \in B(k\varpi_1)$ and $b_2 \in B(k\varpi_n)$, and $\mu = \mathrm{wt}\, b \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$. By Lemma 3.2, the number of boxes of b filled with elements of $J(\mu)$ is equal to $k|J(\mu)| + k$. Therefore b_1 should have only elements of $J(\mu)$ with its entries and the number of entries j in b_2 should be equal to k for $j \in J(\mu)$. When the weight of b is given, the numbering of b is uniquely determined by that of b_1. Hence b is uniquely determined by its weight. □

The following proposition immediately follows from Proposition 3.3 and Lemma 3.3.

Proposition 3.4 *The restriction $\mathrm{wt}\,|_{B_l \setminus \bigcup_{j=2}^{n+1} \mathrm{Im}\,\Theta_j}$ is injective.*

Let $b \in B(k\theta) \subset B_l$ and $b \notin \bigcup_{j=2}^{n+1} \mathrm{Im}\,\Theta_j$. By Lemma 2.3, exactly one of the following occurs:

- $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B((k+1)\theta) \setminus \mathrm{wt}\, B(k\theta)$.
- $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$.
- $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B((k-1)\theta) \setminus \mathrm{wt}\, B((k-2)\theta)$.

Theorem 3.1 *Let $b \in B(k\theta) \subset B_l$ and assume $b \notin \bigcup_{j=2}^{n+1} \mathrm{Im}\,\Theta_j$.*

(i) *We have $\tilde{f}_0 b = 0$ if and only if $k = l$ and $\mathrm{wt}\, b + \theta \notin \mathrm{wt}\, B(l\theta)$.*
(ii) *If $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B((k+1)\theta) \setminus \mathrm{wt}\, B(k\theta)$, then $\tilde{f}_0 b \in B((k+1)\theta) \subset B_l$.*
(iii) *If $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$, then $\tilde{f}_0 b \in B(k\theta) \subset B_l$.*
(iv) *If $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B((k-1)\theta) \setminus \mathrm{wt}\, B((k-2)\theta)$, then $\tilde{f}_0 b \in B((k-1)\theta) \subset B_l$.*

Moreover, the element $\tilde{f}_0 b$ is uniquely determined by its weight in each case.

Proof We prove (i). Since $\alpha_0 + \theta = 0$ in the weight lattice P_{cl}, $\tilde{f}_0$ raises $U_q(\mathfrak{g}_0)$-weights by θ. Hence the sufficiency is obvious. Suppose $b = b_1 \otimes b_2$. If $\tilde{f}_0 b = 0$, we have $\varphi_0(b_2) = 0$ and $\varphi_0(b_1) \le \varepsilon_0(b_2)$ by the tensor product rule. Since $y_1(b_2) = \varphi_0(b_2) = 0$, we have $k = l$. The number of entries 1 in b is

$$\begin{aligned} x_1(b_1) + (l - y_1(b_2)) &= l + x_1(b_1) \\ &\ge l \end{aligned}$$

and the number of entries $n+1$ is

$$\begin{aligned} x_{n+1}(b_1) + (l - y_{n+1}(b_2)) &= l + (\varphi_0(b_1) - \varepsilon_0(b_2)) \\ &\le l. \end{aligned}$$

Hence we have $m_1(\mathrm{wt}\, b) \ge 0$ and $m_{n+1}(\mathrm{wt}\, b) \le 0$. By Lemma 2.3 (i), we obtain $\mathrm{wt}\, b + \theta \notin \mathrm{wt}\, B(l\theta)$.

By (i), $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B(l\theta)$ implies $\tilde{f}_0 b \ne 0$. Then we see $\tilde{f}_0 b \in B_l \setminus \bigcup_{j=2}^{n+1} \mathrm{Im}\, \Theta_j$ by Proposition 3.2 (iii) and Remark 3.1. By Proposition 3.4, $\tilde{f}_0 b$ is uniquely determined by its weight $\mathrm{wt}\, b + \theta$. This proves the remaining assertions. □

4 The case of $C_n^{(1)}$

Assume that $\mathfrak{g}$ is of type $C_n^{(1)}$ in this section. The crystal structure of $B_l = B^{1,2l}$ is given in [7]. We recall explicit formulas on the actions of Kashiwara operators in [12]. As sets,

$$B(k\theta) = \{(x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in (\mathbb{Z}_{\ge 0})^{2n} \mid \sum_{j=1}^{n} (x_j + \bar{x}_j) = 2k\}$$

and

$$B_l = \bigsqcup_{k=0}^{l} B(k\theta).$$

For $b = (x_1, \dots, \bar{x}_1) \in B_l$, the $U_q(\mathfrak{g}_0)$-weight of b and the actions of Kashiwara operators are given as follows:

$$\mathrm{wt}\, b = \sum_{j=1}^{n} (x_j - \bar{x}_j)\epsilon_j,$$

$$\tilde{e}_0 b = \begin{cases} (x_1 - 2, x_2, \dots, \bar{x}_2, \bar{x}_1) & \text{if } x_1 \ge \bar{x}_1 + 2, \\ (x_1 - 1, x_2, \dots, \bar{x}_2, \bar{x}_1 + 1) & \text{if } x_1 = \bar{x}_1 + 1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 + 2) & \text{if } x_1 \le \bar{x}_1, \end{cases}$$

$$\tilde{e}_i b = \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} > \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \le \bar{x}_{i+1}, \end{cases}$$

$$\tilde{e}_n b = (x_1, \dots, x_n + 1, \bar{x}_n - 1, \dots, \bar{x}_1),$$

$$\tilde{f}_0 b = \begin{cases} (x_1 + 2, x_2, \dots, \bar{x}_2, \bar{x}_1) & \text{if } x_1 \ge \bar{x}_1, \\ (x_1 + 1, x_2, \dots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_1 = \bar{x}_1 - 1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 - 2) & \text{if } x_1 \le \bar{x}_1 - 2, \end{cases}$$

$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \ge \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

$$\tilde{f}_n b = (x_1, \dots, x_n - 1, \bar{x}_n + 1, \dots, \bar{x}_1).$$

In each case, if the right-hand side does not belong to the set B_l, regard it as zero. It is obvious that the crystal graph of B_{l-1} is a full subgraph of that of B_l from the above formulas. For $b \in B(k\theta) \subset B_l$, we have

$$\varepsilon_0(b) = (l - k) + \max\{0, x_1 - \bar{x}_1\},$$

$$\varphi_0(b) = (l - k) + \max\{0, \bar{x}_1 - x_1\},$$

$$\varepsilon_i(b) = \bar{x}_i + \max\{0, x_{i+1} - \bar{x}_{i+1}\},$$

$$\varphi_i(b) = x_i + \max\{0, \bar{x}_{i+1} - x_{i+1}\},$$

$$\varepsilon_n(b) = \bar{x}_n,$$

$$\varphi_n(b) = x_n.$$

For $j = 1, \dots, n$, we define a map $\Phi_j : B_{l-1} \to B_l$ by

$$\Phi_j(x_1, \dots, \bar{x}_1) = (\dots, x_j + 1, \dots, \bar{x}_j + 1, \dots).$$

Proposition 4.1 *Let $j = 1, \dots, n$.*

(i) *For $k = 0, \dots, l - 1$, $\Phi_j(B(k\theta)) \subset B((k + 1)\theta)$.*
(ii) *Let $i \in I_0$ and $b \in B_{l-1}$. If $\tilde{f}_i b \neq 0$, then $\Phi_j(\tilde{f}_i b) = \tilde{f}_i \Phi_j(b)$.*
(iii) *The map Φ_j and the Kashiwara operator $\tilde{f}_0$ commute with each other.*

Proof The assertion of (i) is obvious from the definition of Φ_j.

Suppose $b = (x_1, \dots, \bar{x}_1)$ and $\Phi_j(b) = (y_1, \dots, \bar{y}_1)$. Then $y_i - \bar{y}_i = x_i - \bar{x}_i$ for $i = 1, \dots, n$. Hence (ii) and the case $\tilde{f}_0 b \neq 0$ of (iii) are immediate from the formulas on the actions of Kashiwara operators. If $\tilde{f}_0 b = 0$, then $b \in B((l - 1)\theta) \subset B_{l-1}$ and $x_1 \ge \bar{x}_1$ by the formula on φ_0. Hence $\Phi_j(b) \in B(l\theta)$ and $\tilde{f}_0 \Phi_j(b) = 0$. □

Proposition 4.2 *We have*

$$\bigcup_{j=1}^{n} \Phi_j(B((k - 1)\theta)) = \{b \in B(k\theta) \mid \operatorname{wt} b \in \operatorname{wt} B((k - 1)\theta)\}.$$

Proof Let $b = (x_1, \dots, \bar{x}_1) \in B(k\theta)$. By Lemma 2.2, $\mathrm{wt}\, b \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$ if and only if $x_j = 0$ or $\bar{x}_j = 0$ for each $j = 1, \dots, n$. This condition is equivalent to that b does not belong to the left-hand side. $\square$

Lemma 4.1 *Let k be a nonnegative integer. The multiplicity of every element of* $\mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$ *in* $B(k\theta)$ *is one.*

Proof Let $b = (x_1, \dots, \bar{x}_1) \in B(k\theta)$ and $\mu = \mathrm{wt}\, b \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$. Set

$$J_+ = \{j \mid m_j(\mu) > 0\},$$

$$J_- = \{j \mid m_j(\mu) < 0\}.$$

Then we have

$$x_j = \begin{cases} m_j(\mu) & \text{if } j \in J_+, \\ 0 & \text{if } j \notin J_+, \end{cases}$$

$$\bar{x}_j = \begin{cases} -m_j(\mu) & \text{if } j \in J_-, \\ 0 & \text{if } j \notin J_- \end{cases}$$

by Proposition 4.2. This means that b is uniquely determined by its weight. $\square$

Theorem 4.1 *Let $b \in B(k\theta) \subset B_l$ and assume $b \notin \bigcup_{j=1}^{n} \mathrm{Im}\, \Phi_j$.*

(i) *We have $\tilde{f}_0 b = 0$ if and only if $k = l$ and $\mathrm{wt}\, b + \theta \notin \mathrm{wt}\, B(l\theta)$.*
(ii) *If $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B((k+1)\theta) \setminus \mathrm{wt}\, B(k\theta)$, then $\tilde{f}_0 b \in B((k+1)\theta) \subset B_l$.*
(iii) *If $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B(k\theta) \setminus \mathrm{wt}\, B((k-1)\theta)$, then $\tilde{f}_0 b \in B(k\theta) \subset B_l$.*
(iv) *If $\mathrm{wt}\, b + \theta \in \mathrm{wt}\, B((k-1)\theta) \setminus \mathrm{wt}\, B((k-2)\theta)$, then $\tilde{f}_0 b \in B((k-1)\theta) \subset B_l$.*

Moreover, the element $\tilde{f}_0 b$ is uniquely determined by its weight in each case.

Proof Suppose $b = (x_1, \dots, \bar{x}_1)$. Then $m_1(\mathrm{wt}\, b) = x_1 - \bar{x}_1$. Hence the assertions are immediate from the formula on the action of $\tilde{f}_0$, Lemma 2.4 and Lemma 4.1. $\square$

5 The case of $D_{n+1}^{(2)}$

Assume that $\mathfrak{g}$ is of type $D_{n+1}^{(2)}$ in this section. Thus $\mathfrak{g}_0$ is of type B_n. The crystal structure of $B_l = B^{1,l}$ is given in [8]. Explicit formulas are available in [12] and we follow it. As sets,

$$B(k\theta) = \{(x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \in (\mathbb{Z}_{\geq 0})^{2n+1} \mid \\ x_0 = 0 \text{ or } 1, \sum_{j=1}^{n} (x_j + \bar{x}_j) + x_0 = k\}$$

and

$$B_l = \bigsqcup_{k=0}^{l} B(k\theta).$$

For $b = (x_1, \dots, \bar{x}_1) \in B_l$, the $U_q(\mathfrak{g}_0)$-weight of b and the actions of Kashiwara operators are given as follows:

$$\mathrm{wt}\, b = \sum_{j=1}^{n} (x_j - \bar{x}_j)\epsilon_j,$$

$$\tilde{e}_0 b = \begin{cases} (x_1 - 1, x_2, \dots, \bar{x}_2, \bar{x}_1) & \text{if } x_1 > \bar{x}_1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 + 1) & \text{if } x_1 \le \bar{x}_1, \end{cases}$$

$$\tilde{e}_i b = \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} > \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \le \bar{x}_{i+1}, \end{cases}$$

$$\tilde{e}_n b = \begin{cases} (x_1, \dots, x_n, x_0 + 1, \bar{x}_n - 1, \dots, \bar{x}_1) & \text{if } x_0 = 0, \\ (x_1, \dots, x_n + 1, x_0 - 1, \bar{x}_n, \dots, \bar{x}_1) & \text{if } x_0 = 1, \end{cases}$$

$$\tilde{f}_0 b = \begin{cases} (x_1 + 1, x_2, \dots, \bar{x}_2, \bar{x}_1) & \text{if } x_1 \ge \bar{x}_1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_1 < \bar{x}_1, \end{cases}$$

$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \ge \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

$$\tilde{f}_n b = \begin{cases} (x_1, \dots, x_n - 1, x_0 + 1, \bar{x}_n, \dots, \bar{x}_1) & \text{if } x_0 = 0, \\ (x_1, \dots, x_n, x_0 - 1, \bar{x}_n + 1, \dots, \bar{x}_1) & \text{if } x_0 = 1. \end{cases}$$

In each case, if the right-hand side does not belong to B_l, regard it as zero. The crystal graph of B_{l-1} is a full subgraph of that of B_l. For $b \in B(k\theta) \subset B_l$, we have

$$\begin{aligned}
\varepsilon_0(b) &= (l-k) + 2\max\{0, x_1 - \bar{x}_1\}, \\
\varphi_0(b) &= (l-k) + 2\max\{0, \bar{x}_1 - x_1\}, \\
\varepsilon_i(b) &= \bar{x}_i + \max\{0, x_{i+1} - \bar{x}_{i+1}\}, \\
\varphi_i(b) &= x_i + \max\{0, \bar{x}_{i+1} - x_{i+1}\}, \\
\varepsilon_n(b) &= 2\bar{x}_n + x_0, \\
\varphi_n(b) &= 2x_n + x_0.
\end{aligned}$$

We define maps $\Psi_j \colon B_{l-2} \to B_l$ for $j = 1, \dots, n-1$ and $\Psi_n \colon B_{l-1} \to B_l$ by

$$\Psi_j(x_1, \dots, \bar{x}_1) = (\dots, x_j + 1, \dots, \bar{x}_j + 1, \dots) \ \text{ for } j = 1, \dots, n-1,$$

$$\Psi_n(x_1,\dots,\bar{x}_1)=\begin{cases}(x_1,\dots,x_n,x_0+1,\bar{x}_n,\dots,\bar{x}_1) & \text{if } x_0=0,\\ (x_1,\dots,x_n+1,x_0-1,\bar{x}_n+1,\dots,\bar{x}_1) & \text{if } x_0=1.\end{cases}$$

Proposition 5.1

(i) *Let* $j=1,\dots,n-1$. *Then* $\Psi_j(B(k\theta))\subset B((k+2)\theta)$ *for* $k=0,\dots,l-2$.
(ii) *For* $k=0,\dots,l-1$, $\Psi_n(B(k\theta))\subset B((k+1)\theta)$.
(iii) *Let* $j=1,\dots,n-1$ (*resp.* $j=n$) *and* $i\in I_0$. *If we take* $b\in B_{l-2}$ (*resp.* $b\in B_{l-1}$) *with* $\tilde{f}_ib\neq 0$, *then* $\Psi_j(\tilde{f}_ib)=\tilde{f}_i\Psi_j(b)$.
(iv) *For* $j=1,\dots,n$, *the map* Ψ_j *and the Kashiwara operator* $\tilde{f}_0$ *commute with each other.*

Proof The assertions of (i) and (ii) are obvious.

The assertions of (iii) and (iv) are immediate except for the commutativity of Ψ_n and $\tilde{f}_n$. We show that $\Psi_n(\tilde{f}_nb)=\tilde{f}_n\Psi_n(b)$ when $\tilde{f}_nb\neq 0$. Let $b=(x_1,\dots,\bar{x}_1)$. First we assume $x_0=0$. Then

$$\begin{aligned}\Psi_n(\tilde{f}_nb)&=\Psi_n(x_1,\dots,x_n-1,x_0+1,\bar{x}_n,\dots,\bar{x}_1)\\&=(x_1,\dots,x_n,x_0,\bar{x}_n+1,\dots,\bar{x}_1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\tilde{f}_n\Psi_n(b)&=\tilde{f}_n(x_1,\dots,x_n,x_0+1,\bar{x}_n,\dots,\bar{x}_1)\\&=(x_1,\dots,x_n,x_0,\bar{x}_n+1,\dots,\bar{x}_1).\end{aligned}$$

Hence the assertion is true. Next assume $x_0=1$. Then we obtain

$$\begin{aligned}\Psi_n(\tilde{f}_nb)&=\Psi_n(x_1,\dots,x_n,x_0-1,\bar{x}_n+1,\dots,\bar{x}_1)\\&=(x_1,\dots,x_n,x_0,\bar{x}_n+1,\dots,\bar{x}_1)\end{aligned}$$

and

$$\begin{aligned}\tilde{f}_n\Psi_n(b)&=\tilde{f}_n(x_1,\dots,x_n+1,x_0-1,\bar{x}_n+1,\dots,\bar{x}_1)\\&=(x_1,\dots,x_n,x_0,\bar{x}_n+1,\dots,\bar{x}_1).\end{aligned}$$

□

Proposition 5.2 *We have*

$$\left(\bigcup_{j=1}^{n-1}\Psi_j(B((k-2)\theta))\right)\cup\Psi_n(B((k-1)\theta))$$
$$=\{b\in B(k\theta)\mid \operatorname{wt}b\in\operatorname{wt}B((k-1)\theta)\}.$$

Proof Let $b=(x_1,\dots,\bar{x}_1)\in B(k\theta)$. By Lemma 2.2, $\operatorname{wt}b\in\operatorname{wt}B(k\theta)\setminus\operatorname{wt}B((k-1)\theta)$ if and only if $x_0=0$ and ($x_j=0$ or $\bar{x}_j=0$) for each $j=1,\dots,n$. This condition is equivalent to that b does not belong to the left-hand side. □

Lemma 5.1 *Let k be a nonnegative integer. The multiplicity of every element of* $\mathrm{wt}\,B(k\theta) \setminus \mathrm{wt}\,B((k-1)\theta)$ *in* $B(k\theta)$ *is one.*

Proof Let $b = (x_1, \dots, \bar{x}_1) \in B(k\theta)$ and $\mu = \mathrm{wt}\,b \in \mathrm{wt}\,B(k\theta) \setminus \mathrm{wt}\,B((k-1)\theta)$. If we set

$$J_+ = \{j \mid m_j(\mu) > 0\},$$

$$J_- = \{j \mid m_j(\mu) < 0\},$$

then

$$x_j = \begin{cases} m_j(\mu) & \text{if } j \in J_+, \\ 0 & \text{if } j \notin J_+, \end{cases}$$

$$\bar{x}_j = \begin{cases} -m_j(\mu) & \text{if } j \in J_-, \\ 0 & \text{if } j \notin J_-, \end{cases}$$

$$x_0 = 0.$$

Hence b is uniquely determined by its weight. □

Theorem 5.1 *Let $b \in B(k\theta) \subset B_l$ and assume $b \notin \bigcup_{j=1}^n \mathrm{Im}\,\Psi_j$.*

(i) *We have $\tilde{f}_0 b = 0$ if and only if $k = l$ and $\mathrm{wt}\,b + \theta \notin \mathrm{wt}\,B(l\theta)$.*
(ii) *If $\mathrm{wt}\,b + \theta \in \mathrm{wt}\,B((k+1)\theta) \setminus \mathrm{wt}\,B(k\theta)$, then $\tilde{f}_0 b \in B((k+1)\theta) \subset B_l$.*
(iii) *If $\mathrm{wt}\,b + \theta \in \mathrm{wt}\,B((k-1)\theta) \setminus \mathrm{wt}\,B((k-2)\theta)$, then $\tilde{f}_0 b \in B((k-1)\theta) \subset B_l$.*

Moreover, the element $\tilde{f}_0 b$ is uniquely determined by its weight in each case.

Proof The assertions are immediate from the formula on the action of $\tilde{f}_0$, Lemma 2.5 and Lemma 5.1. □

Acknowledgements The author would like to thank his advisor Yoshihisa Saito for his guidance and valuable comments. He also would like to thank Masato Okado for answering the author's questions about Kirillov-Reshetikhin crystals, and Noriyuki Abe for providing a computer program for drawing crystal graphs.

References

1. Benkart, G., Frenkel, I., Kang, S.-J., Lee, H.: Level 1 perfect crystals and path realizations of basic representations at $q = 0$. Int. Math. Res. Not., 10312 (2006)
2. Chari, V.: On the fermionic formula and the Kirillov-Reshetikhin conjecture. Int. Math. Res. Not. **12**, 629–654 (2001)
3. Hernandez, D.: Kirillov-Reshetikhin conjecture: the general case. arXiv:0704.2838
4. Hong, J., Kang, S.-J.: Introduction to Quantum Groups and Crystal Bases. Graduate Studies in Mathematics, vol. 42. American Mathematical Society, Providence (2002)
5. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Graduate Texts in Mathematics, vol. 9. Springer, New York (1978); second printing, revised
6. Kac, V.G.: Infinite-Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)

7. Kang, S.-J., Kashiwara, M., Misra, K.C.: Crystal bases of Verma modules for quantum affine Lie algebras. Compos. Math. **92**(3), 299–325 (1994)
8. Kang, S.-J., Kashiwara, M., Misra, K.C., Miwa, T., Nakashima, T., Nakayashiki, A.: Perfect crystals of quantum affine Lie algebras. Duke Math. J. **68**(3), 499–607 (1992)
9. Kashiwara, M.: On crystal bases of the Q-analogue of universal enveloping algebras. Duke Math. J. **63**(2), 465–516 (1991)
10. Kashiwara, M., Misra, K.C., Okado, M., Yamada, D.: Perfect crystals for $U_q(D_4^{(3)})$. J. Algebra **317**(1), 392–423 (2007)
11. Kashiwara, M., Nakashima, T.: Crystal graphs for representations of the q-analogue of classical Lie algebras. J. Algebra **165**(2), 295–345 (1994)
12. Okado, M.: $X = M$ conjecture, Combinatorial aspect of integrable systems. MSJ Mem., Math. Soc. Jpn., Tokyo **17**, 43–73 (2007)
13. Okado, M., Schilling, A.: Existence of Kirillov-Reshetikhin crystals for nonexceptional types. Represent. Theory **12**, 186–207 (2008)
14. Schilling, A., Sternberg, P.: Finite-dimensional crystals $B^{2,s}$ for quantum affine algebras of type $D_n^{(1)}$. J. Algebr. Comb. **23**(4), 317–354 (2006)
15. Shimozono, M.: Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties. J. Algebr. Comb. **15**(2), 151–187 (2002)
16. Yamane, S.: Perfect crystals of $U_q(G_2^{(1)})$. J. Algebra **210**(2), 440–486 (1998)