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Terwilliger algebras of wreath products of one-class association schemes

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Abstract In this paper, we study the wreath product of one-class association schemes $K_n = H(1, n)$ for $n \geq 2$. We show that the d-class association scheme $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ formed by taking the wreath product of K_{n_i} (for $n_i \geq 2$) has the triple-regularity property. Then based on this fact, we determine the structure of the Terwilliger algebra of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ by studying its irreducible modules. In particular, we show that every non-primary module of this algebra is 1-dimensional.

Keywords Commutative association schemes · Terwilliger algebra · Wreath product

1 Introduction

The Terwilliger algebra, which is also known as the subconstituent algebra, of an association scheme was introduced by Terwilliger in 1992 as a new algebraic tool for the study of association schemes [16]. The Terwilliger algebra of a commutative association scheme is a finite dimensional, semi-simple $\mathbb{C}$-algebra, and is noncommutative in general. This algebra helps understanding the structure of the association schemes. It has been studied extensively for many classes of association schemes. For example, the Terwilliger algebra for P- and Q-polynomial association schemes has been studied in [6, 16–18]. The structure of the Terwilliger algebra of group association schemes has been studied in [1] and [3]. In [11] the structure of the Terwilliger

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algebra of a Hamming scheme $H(d,n)$ is given as symmetric d-tensors of the Terwilliger algebra of $H(1,n)$ which are all isomorphic for $n>2$. It is also shown that the Terwilliger algebra of $H(d,n)$ is decomposed as a direct sum of Terwilliger algebra of hypercubes $H(d,2)$ in [11]. There is a detailed study of the irreducible modules of the algebra for $H(d,2)$ in [9], and for Doob schemes (the schemes coming as direct products of copies of $H(2,4)$ and/or Shrikhande graphs), in [15].

In this paper, we study the Terwilliger algebras of association schemes which are obtained as wreath products of $H(1,n)$, also denoted K_n, for $n \geq 2$. We remark that the wreath power of K_n is also known as the kernel scheme and related to the study of ordered orthogonal arrays [12]. We find that the d-class association scheme $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ formed by taking the wreath product of one-class association schemes K_{n_i} has the triple-regularity property in the sense of [13] and [10]. Based on this fact, we show that the dimension of the Terwilliger algebra of $\mathcal{X}$ is $(d+1)^2 + \frac{1}{2}d(d+1) - b$ where b denotes the number of K_2 factors in the product, and that the Terwilliger algebra of $\mathcal{X}$ is isomorphic to $M_{d+1}(\mathbb{C}) \oplus M_1(\mathbb{C})^{\oplus \frac{1}{2}d(d+1)-b}$ in the notion of Wedderburn-Artin's decomposition theorem of semisimple algebra.

The remainder of the paper is organized as follows. In Section 2, we provide the notation and terminology as well as a few basic facts on the Terwilliger algebra and wreath product of association schemes that will be used throughout. In Section 3, we show that the wreath product of one-class association schemes has the triple-regularity property and calculate the dimension of its Terwilliger algebra. In Section 4, we study Terwilliger algebras of wreath product of one-class association schemes and their irreducible modules. In Section 5, we make a few remarks on further study in this direction.

2 Preliminaries

In this section we first briefly recall the notation and some basic facts about association schemes and the Terwilliger algebra of a scheme that are needed to deduce our results. Then we recall the definition of the wreath product of association schemes. For more information on the topics covered in this section, we refer the reader to [2, 4, 5, 14, 16].

2.1 Association schemes and their Terwilliger algebras

Let X denote an n-element set, and let $M_X(\mathbb{C})$ (or $M_n(\mathbb{C})$) denote the $\mathbb{C}$-algebra of matrices whose rows and columns are indexed by X.

Let $R_0, R_1, \ldots, R_d$ be nonempty subsets of $X \times X$ and $A_0, A_1, \ldots, A_d$ be matrices in $M_X(\mathbb{C})$ defined by $(A_i)_{xy} = 1$ if $(x,y) \in R_i$; 0 otherwise.

$\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is called a *d-class association scheme of order n* if the following hold:

(1) $A_0 = I$,
(2) $A_0 + A_1 + \cdots + A_d = J$,

(3) $A_i^t = A_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$,
(4) $A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h$ for some nonnegative integers p_{ij}^h, for all $i, j \in \{0, 1, \ldots, d\}$.

where $I = I_n$ and $J = J_n$ are the $n \times n$ identity matrix and all-ones matrix, respectively, and A^t denotes the transpose of A. The scheme $\mathcal{X}$ is *symmetric* if $A_{i'} = A_i$ for all i, and is *commutative* if $A_i A_j = A_j A_i$ for all i, j, that is, $p_{ij}^h = p_{ji}^h$ for all i, j, h. The numbers p_{ij}^h are called the *intersection numbers* and satisfy

$$p_{ij}^h = |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|,$$

where $(x, y) \in R_h$.

If $\mathcal{X}$ is commutative, the adjacency matrices generate a $(d+1)$-dimensional commutative subalgebra $\mathcal{M} = \langle A_0, A_1, \ldots, A_d \rangle$ of $M_X(\mathbb{C})$. The algebra $\mathcal{M}$ is known as the Bose-Mesner algebra of the scheme, being semi-simple, admits a second basis $E_0, E_1, \ldots, E_d$ of primitive idempotents.

Given X and $M_X(\mathbb{C})$, by the standard module of X, we mean the n-dimensional vector space $V = \mathbb{C}^X = \bigoplus_{x \in X} \mathbb{C}\hat{x}$ of column vectors whose coordinates are indexed by X. Here for each $x \in X$, we denote by $\hat{x}$ the column vector with 1 in the xth position, and 0 elsewhere. Observe that $M_X(\mathbb{C})$ acts on V by left multiplication. We endow V with the Hermitian inner product defined by $\langle u, v \rangle = u^t \overline{v}$ $(u, v \in V)$. For a given association scheme $\mathcal{X}$, V can be written as the direct sum of $V_i = E_i V$ where V_i are the maximal common eigenspaces of $A_0, A_1, \ldots, A_d$. Given an element $x \in X$, let $R_i(x) = \{y \in X : (x, y) \in R_i\}$. Let $V_i^* = V_i^*(x) = \bigoplus_{y \in R_i(x)} \mathbb{C}\hat{y}$. Both $R_i(x)$ and V_i^* are referred to as the ith *subconstituent* of $\mathcal{X}$ with respect to x. Let $E_i^* = E_i^*(x)$ be the orthogonal projection map from $V = \bigoplus_{i=0}^{d} V_i^*$ to the ith subconstituent V_i^*. So, E_i^* can be represented by the diagonal matrix given by

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if}(x, y) \in R_i, \\ 0 & \text{if}(x, y) \notin R_i. \end{cases}$$

The matrices $E_0^*, E_1^*, \ldots, E_d^*$ are linearly independent, and they form a basis for a subalgebra $\mathcal{M}^* = \mathcal{M}^*(x) = \langle E_0^*, E_1^*, \ldots, E_d^* \rangle$ of $M_X(\mathbb{C})$. The algebra $\mathcal{M}^*$ is shown to be commutative, semi-simple subalgebra of $M_X(\mathbb{C})$. This algebra is called the dual Bose-Mesner algebra of $\mathcal{X}$ with respect to x. Let $\mathcal{T} = \mathcal{T}(x)$ denote the subalgebra of $M_X(\mathbb{C})$ generated by the Bose-Mesner algebra $\mathcal{M}$ and the dual Bose-Mesner algebra $\mathcal{M}^*$. We call $\mathcal{T}$ the *Terwilliger algebra of $\mathcal{X}$ with respect to x*.

The set of triple products $E_i^* A_j E_h^*$ in $\mathcal{T}(x)$ plays a special role in our study. Terwilliger proved the following key fact:

Proposition 2.1 [16, Lemma 3.2] *For $0 \le h, i, j \le d$, $E_i^* A_j E_h^* = 0$ if and only if $p_{ij}^h = 0$.*

Let $\mathcal{T}_0 = \mathcal{T}_0(x)$ be the subspace of $\mathcal{T} = \mathcal{T}(x)$ spanned by $\{E_i^* A_j E_h^* : 0 \le i, j, h \le d\}$. It is easy to see that $\mathcal{T}$ is generated by $\mathcal{T}_0$ as an algebra since $\mathcal{T}_0$ contains A_i and E_i^* for all i, but in general, $\mathcal{T}_0$ may be a proper linear subspace of $\mathcal{T}$.

We recall the concept of triple-regularity, which was first studied by Jaeger ([10, p. 120]); an association scheme $\mathcal{X}$ is called *triply-regular* if the size $p_{ijh}^{lmn}(x,y,z)$ of the set $R_i(x) \cap R_j(y) \cap R_h(z)$ depends only on i, j, h, l, m, n where $(x,y) \in R_l$, $(x,z) \in R_m$ and $(y,z) \in R_n$. Munemasa gave the following interpretation of the triple-regularity in connection with its Terwilliger algebra.

Proposition 2.2 [13] *Let $\mathcal{X}$ be a commutative association scheme. Then $\mathcal{X}$ is triply-regular if and only if $\mathcal{T}(x) = \mathcal{T}_0(x)$ for every $x \in X$.*

2.2 The wreath product of association schemes

We briefly recall the notion of the wreath product. Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ and $\mathcal{Y} = (Y, \{S_j\}_{0\le j\le e})$ be association schemes of order $|X| = m$ and $|Y| = n$. The wreath product $\mathcal{X} \wr \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ is defined on the set $X \times Y$; but we take $Y = \{y_1, y_2, \ldots, y_n\}$, and regard $X \times Y$ as the disjoint union of n copies $X_1, X_2, \ldots, X_n$ of X, where $X_j = X \times \{y_j\}$. The relations on $X_1 \cup X_2 \cup \cdots \cup X_n$ is defined by the following rule:

> For any j, the relations between the elements of X_j are determined by the association relations between the first coordinates in $\mathcal{X}$. For any i and j, the relations between X_i and X_j are determined by the association relation of the second coordinates y_i and y_j in $\mathcal{Y}$ and the relation is independent from the first coordinates.

We may arrange the relations $W_0, W_1, \ldots, W_{d+e}$ of $\mathcal{X} \wr \mathcal{Y}$ as follows:

$W_0 = \{((x,y),(x,y)) : (x,y) \in X \times Y\}$;
$W_k = \{((x_1,y),(x_2,y)) : (x_1,x_2) \in R_k, y \in Y\}$, for $1 \le k \le d$; and
$W_k = \{((x_1,y_1),(x_2,y_2)) : x_1, x_2 \in X, (y_1,y_2) \in S_{k-d}\}$ for $d+1 \le k \le d+e$.

Then the wreath product $\mathcal{X} \wr \mathcal{Y} = (X \times Y, \{W_k\}_{0\le k\le d+e})$ is a $(d+e)$-class association scheme. It is clear that $\mathcal{X} \wr \mathcal{Y}$ is commutative (resp. symmetric) if and only if $\mathcal{X}$ and $\mathcal{Y}$ are. With the above ordering of the association relations of $\mathcal{X} \wr \mathcal{Y}$, the relation table of the wreath product is described by

$$R(\mathcal{X} \wr \mathcal{Y}) = \sum_{k=0}^{d+e} k \cdot A_k = I_n \otimes R(\mathcal{X}) + \{R(\mathcal{Y}) + d(J_n - I_n)\} \otimes J_m.$$

Let $A_0, A_1, \ldots, A_d$ and $C_0, C_1, \ldots, C_e$ be the adjacency matrices of $\mathcal{X}$ and those of $\mathcal{Y}$, respectively. Then the adjacency matrices W_k of $\mathcal{X} \wr \mathcal{Y}$ are given by

$$W_0 = C_0 \otimes A_0, \quad W_1 = C_0 \otimes A_1, \ldots, \quad W_d = C_0 \otimes A_d,$$
$$W_{d+1} = C_1 \otimes J_m, \ldots, \quad W_{d+e} = C_e \otimes J_m,$$

where "$\otimes$" denotes the Kronecker product: $A \otimes B = (a_{ij}B)$ of two matrices $A = (a_{ij})$ and B.

In what follows, we adopt the common practice of referring to the tensor product $A \otimes B$ of $A \in M_X(\mathbb{C})$ and $B \in M_Y(\mathbb{C})$ as the Kronecker product of A and B in

$M_{X\times Y}(\mathbb{C})$. Henceforth, in keeping with this practice, we will write $(A_1 \otimes B_1) \times (A_2 \otimes B_2) = A_1A_2 \otimes B_1B_2$ with ordinary matrix multiplication, and $(A \otimes B) \times (u \otimes v) = Au \otimes Bv$ for the action of linear operator $A \otimes B$ on elementary tensors $u \otimes v \in \mathbb{C}^X \otimes \mathbb{C}^Y$, and so on.

3 The dimension of the Terwilliger algebra of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$

In this section, we show that the wreath product $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ is triply-regular. We then determine the dimension of its Terwilliger algebra.

Throughout the remainder of this paper, we denote the set $\{1, 2, \ldots, n\}$ of integers by $[n]$, and both the one-class association scheme $([n], \{R_0, R_1\})$, and the complete graph on n vertices by K_n. Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ denote the wreath product $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ of $K_{n_1}, K_{n_2}, \ldots, K_{n_d}$ ($d \ge 2, n_i \ge 2$). Then the scheme $\mathcal{X}$ has vertex set

$$X = [n_1] \times [n_2] \times \cdots \times [n_d].$$

For $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d) \in X$ with $x \ne y$, we have $(x, y) \in R_i$ whenever $i = \max\{j : 1 \le j \le d, x_j \ne y_j\}$. Also, for $1 \le i \le d$, we have

$$R_i(x) = \{(y_1, \ldots, y_i, x_{i+1}, \ldots, x_d) : y_i \in [n_i] \setminus \{x_i\}, y_j \in [n_j] \text{ for } 1 \le j \le i-1\}.$$

Note that $\mathcal{X}$ is symmetric and commutative. We obtain the following lemma immediately.

Lemma 3.1 *For $1 \le i \le d$ and $x \in X$, $R_i(x)$ induces an association scheme which is isomorphic to $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_{i-1}} \wr K_{n_i-1}$ if $n_i > 2$ and is isomorphic to $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_{i-1}}$ if $n_i = 2$.*

Lemma 3.2 *Suppose $i > j$, $y \in R_i(x)$ and $z \in R_j(x)$ for $x \in X$. Then $(y, z) \in R_i$.*

Proof Since $i > j$, we have $y_i \ne x_i = z_i$ and $y_l = x_l = z_l$ for $i < l \le d$. Hence $(y, z) \in R_i$. □

Lemma 3.3 *For $0 \le i, j, h \le d$, $p^h_{ij} = 0$ if and only if one of the following hold:*

(1) *$i = j = h \ne 0$ and $n_i = 2$.*
(2) *$i = j < h$, $j = h < i$, or $h = i < j$.*
(3) *i, j, h are all distinct.*

Proof Note that $p^h_{ij} = 0$ if and only if there is no configuration (x, y, z) of vertices $x, y, z \in X$ such that $(x, y) \in R_h, (x, z) \in R_i, (z, y) \in R_j$. *Case 1a*: Suppose $i = j = h = 0$. Then $x = y = z$ and $p^h_{ij} = 1$. *Case 1b*: Suppose $i = j = h \ne 0$. Then y, z belong to $R_i(x)$. By Lemma 3.1, $R_i(x)$ is the association scheme of class i if $n_i \ge 3$, and of class $i - 1$ if $n_i = 2$. So $p^h_{ij} = 0$ if and only if $n_i = 2$. *Case 2*: Suppose exactly two of i, j, h are equal. Without loss of generality we may assume $y, z \in R_i(x)$. We have $p^h_{ij} = 0$ if and only if $i < j$ by the same reason as Case 1b. *Case 3*: Suppose i, j, h are all distinct. Then by Lemma 3.2, $p^h_{ij} = 0$. □

Theorem 3.4 *$\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ is triply-regular.*

Proof For $0 \leq i, j, h, l, m, n \leq d$ and $x, y, z \in X$, let $p_{ijh}^{lmn}(x, y, z)$ be the size of the set $R_i(x) \cap R_j(y) \cap R_h(z)$, where $(x, y) \in R_l$, $(x, z) \in R_m$ and $(y, z) \in R_n$. Without loss of generality, we assume that $l \geq m \geq n$. This means that we are indeed assuming that $l = m \geq n$ as l cannot be strictly larger than m due to Lemma 3.2. So it suffices to prove that for given l, m, n with $l = m \geq n$ the number $p_{ijh}^{lmn}(x, y, z) = |R_i(x) \cap R_j(y) \cap R_h(z)|$ is independent of the choice of x, y, z as long as x, y, z satisfy the constraints $(x, y) \in R_l$, $(x, z) \in R_m$ and $(y, z) \in R_n$. We prove this by considering the cases when $i = l$ and when $i \neq l$ separately.

(i) Suppose $i = l$. Then $p_{ijh}^{lmn}(x, y, z) = p_{ijh}^{iin}(x, y, z)$. If either $j > i$ or $h > i$, then $p_{ij}^{i}(x, y) = |R_i(x) \cap R_j(y)| = 0$ or $p_{ih}^{i}(x, z) = |R_i(x) \cap R_h(z)| = 0$ by Lemma 3.3(2), so $p_{ijh}^{iin}(x, y, z) = 0$ for any x, y, z with $(x, y) \in R_l$, $(x, z) \in R_m$ and $(y, z) \in R_n$. On the other hand, if both $j \leq i$ and $h \leq i$, then $R_j(y) \cap R_h(z)$ is contained in $R_i(x)$, and $p_{ijh}^{iin}(x, y, z)$ is equal to the intersection number of all jth associates of y and all hth associates of z in the association scheme induced on $R_i(x)$ in terms of Lemma 3.1. Clearly the number does not depend on the choice of x, y, z.
(ii) Suppose $i \neq l$. Then by Lemma 3.3, unless $j = h = \max\{i, l\}$, $p_{ijh}^{lmn}(x, y, z) = 0$ as either $|R_i(x) \cap R_j(y)| = 0$ or $|R_i(x) \cap R_h(z)| = 0$. For the case when $j = h = \max\{i, l\}$, by denoting $\max\{i, l\}$ by s, we have $p_{ijh}^{lmn}(x, y, z) = p_{iss}^{lln}(x, y, z) = |R_i(x)| = p_{ii'}^{0}$ as $R_i(x) \subset R_s(y)$ and $R_i(x) \subset R_s(z)$. After all, $p_{ijh}^{lmn}(x, y, z)$ is independent of the choice of x, y, z with the given constraints. □

From Lemma 2.1 and Lemma 3.3, we have the following list of non-zero triple products in $\mathcal{T}$.

Theorem 3.5 *The complete list of nonzero triple products $E_i^* A_j E_h^*$ among all $h, i, j \in \{0, 1, 2, \ldots, d\}$ in $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ is given as follows.*

(1) $E_i^* A_i E_i^*$ *for* $i = 0$ *and for* $1 \leq i \leq d$ *with* $n_i \geq 3$.
(2) $E_i^* A_i E_h^*$ *for* $0 \leq h < i \leq d$.
(3) $E_i^* A_h E_h^*$ *for* $0 \leq i < h \leq d$.
(4) $E_i^* A_j E_i^*$ *for* $0 \leq j < i \leq d$.

As a consequence of Theorem 3.4, Proposition 2.2, and Theorem 3.5, we have the following.

Theorem 3.6 *The dimension of the Terwilliger algebra $\mathcal{T}$ of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ is*

$$\dim(\mathcal{T}) = (d+1)^2 + \frac{1}{2} d(d+1) - b$$

where $b = |\{i : n_i = 2, 1 \leq i \leq d\}|$.

Proof By Theorem 3.4 and Proposition 2.2, we have $\mathcal{T} = \mathcal{T}_0$. The dimension of $\mathcal{T}_0$, which is the number of non-zero triple products, can be, by Theorem 3.5, calculated as $1 + d - b$ from (1), $\frac{1}{2}(d + 1)d \times 3$ from (2), (3) and (4). As they are independent of each other we have the $\dim(\mathcal{T}_0(x)) = 1 + d - b + \frac{3}{2}d(d + 1) = (d + 1)^2 + \frac{1}{2}d(d + 1) - b$ as desired. The case when all $n_i = 2$ gives the lower bound as in this case $b = d$. The upper bound is given by the case where $n_i \geq 3$ for all i. In such a situation $b = 0$. This completes the proof. □

We will use the following lemma in Section 4.

Lemma 3.7 *If $i \neq h$ and $j = max\{i, h\}$, then $E_i^* A_j E_h^* = E_i^* J E_h^*$.*

Proof By Lemma 2.1 and Lemma 3.3, $E_i^* A_j E_h^*$ is nonzero if and only if $j = \max\{i, h\}$. Hence $E_i^* J E_h^* = E_i^*(A_0 + \cdots + A_d)E_h^* = E_i^* A_j E_h^*$. □

4 The Terwilliger algebra of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$

In this section we describe the structure of the Terwilliger algebra of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ with $n_i \geq 2$ for all $i \in \{1, 2, \ldots, d\}$. In particular, we show that all non-primary irreducible $\mathcal{T}$-modules of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ are of dimension 1.

4.1 The Terwilliger algebra of K_n

We begin with the description of the Terwilliger algebra of the one-class association scheme $K_n = ([n], \{R_0, R_1\})$ with $A_1 = J - I$.

Remark 4.1 By Theorem 3.6, we know that the dimension of the Terwilliger algebra of K_n is 5 if $n > 2$ and 4 if $n = 2$. Also by Theorem 3.5, all the matrices in the Terwilliger algebra of K_n is a linear combination of the matrices $E_0^* A_0 E_0^*$, $E_0^* A_1 E_1^*$, $E_1^* A_1 E_0^*$, $E_1^* A_0 E_1^*$, and $E_1^* A_1 E_1^*$. (If $n = 2$, then $E_1^* A_1 E_1^* = 0$.)

If we set

$$E_{11} = E_0^* A_0 E_0^*, \quad E_{12} = E_0^* A_1 E_1^*, \quad E_{21} = \frac{1}{n-1} E_1^* A_1 E_0^*,$$
$$E_{22} = \frac{1}{n-1}(E_1^* A_0 E_1^* + E_1^* A_1 E_1^*),$$

then these matrices form a subalgebra $\mathcal{U}$ of $\mathcal{T}(x)$ as its multiplication table is given by

	E_{11}	E_{12}	E_{21}	E_{22}
E_{11}	E_{11}	E_{12}	0	0
E_{12}	0	0	E_{11}	E_{12}
E_{21}	E_{21}	E_{22}	0	0
E_{22}	0	0	E_{21}	E_{22}

Considering the isomorphism between $\mathcal{U}$ and $M_2(\mathbb{C})$ that takes

$$E_{11} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad E_{12} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad E_{21} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad E_{22} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

we see that $\mathcal{T}(x) = M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$ by Wedderburn-Artin's Theorem (cf. [7, Sec. 2.4]). Specifically, if we set $F = E_1^* A_0 E_1^* - E_{22}$, then it turns out that $FX = 0$ for all $X \in \mathcal{U}$. This gives us $\mathcal{T}(x) = \mathbb{C}F \oplus \mathcal{U}$. While $F = 0$ for $n = 2$, $F \neq 0$ for all $n > 2$. Therefore, we reasserted the following.

Theorem 4.1 [11] *The Terwilliger algebra of K_n can be described as follows*:

$$\mathcal{T}(x) \cong \begin{cases} M_2(\mathbb{C}) & \text{if } n = 2, \\ M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) & \text{if } n > 2. \end{cases}$$

4.2 Irreducible modules of the Terwilliger algebra

We now consider the d-class association scheme $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d}) = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ of order $n = |X| = \prod_{i=1}^{d} n_i$ $(d \ge 2)$, and describe the irreducible $\mathcal{T}$-modules for $\mathcal{X}$. We define an n-dimensional vector space V and n_i-dimensional vector spaces V_i for $1 \le i \le d$ over $\mathbb{C}$ as follows:

$$V := \mathrm{Span}\{\hat{z} : z \in X\}, \quad V_i := \mathrm{Span}\{\hat{z} : z \in [n_i]\}.$$

Let $\mathbf{1}$ and $\mathbf{1}_i$ denote the all-ones column vectors in V and V_i, respectively. Fix an element $x = (x_1, x_2, \ldots, x_d) \in X$. We can rearrange the ordering of X so that

$$V = V_d \otimes V_{d-1} \otimes \cdots \otimes V_1, \quad \hat{x} = \hat{x}_d \otimes \hat{x}_{d-1} \otimes \cdots \otimes \hat{x}_1.$$

For $1 \le i \le d$, we can easily verify that

$$A_i = I_{n_d} \otimes \cdots \otimes I_{n_{i+1}} \otimes (J_{n_i} - I_{n_i}) \otimes J_{n_{i-1}} \otimes \cdots \otimes J_{n_1}. \tag{1}$$

For $1 \le i \le d$, by (1), we have

$$\begin{aligned} A_i \hat{x} &= (I_{n_d} \otimes \cdots \otimes I_{n_{i+1}} \otimes (J_{n_i} - I_{n_i}) \otimes J_{n_{i-1}} \otimes \cdots \otimes J_{n_1}) \\ &\quad \times (\hat{x}_d \otimes \hat{x}_{d-1} \otimes \cdots \otimes \hat{x}_1) \\ &= \hat{x}_d \otimes \cdots \otimes \hat{x}_{i+1} \otimes (\mathbf{1}_i - \hat{x}_i) \otimes \mathbf{1}_{i-1} \otimes \cdots \otimes \mathbf{1}_1; \end{aligned}$$

and thus,

$$\begin{aligned} E_i^* = \mathrm{diag}(A_i \hat{x}) &= \mathrm{diag}(\hat{x}_d \otimes \cdots \otimes \hat{x}_{i+1} \otimes (\mathbf{1}_i - \hat{x}_i) \otimes \mathbf{1}_{i-1} \otimes \cdots \otimes \mathbf{1}_1) \\ &= \mathrm{diag}(\hat{x}_d) \otimes \cdots \otimes \mathrm{diag}(\hat{x}_{i+1}) \\ &\quad \otimes \mathrm{diag}(\mathbf{1}_i - \hat{x}_i) \otimes I_{n_{i-1}} \otimes \cdots \otimes I_{n_1}. \end{aligned} \tag{2}$$

where $\mathrm{diag}(u)$ denotes the diagonal matrix whose diagonal entries are the entries of a vector u.

For $1 \le i \le d$, if we define a vector space $\widetilde{V}_i$ by

$$\widetilde{V}_i := \mathrm{diag}(\mathbf{1}_i - \hat{x}_i)V_i \subset V_i,$$

then, by (2), we have

$$V_i^* = E_i^* V = \mathbb{C}\hat{x}_d \otimes \cdots \otimes \mathbb{C}\hat{x}_{i+1} \otimes \widetilde{V}_i \otimes V_{i-1} \otimes \cdots \otimes V_1.$$

We observe that for each $1 \le i \le d$, the subset $B_i(x)$ of V_i defined by

$$B_i(x) := \{\mathbf{1}_i, \hat{x}_i - \hat{z}_i : z_i \in [n_i], z_i \neq x_i\}$$

is a basis of V_i consisting of eigenvectors of K_{n_i}.

Next, fix $y = (y_1, y_2, \ldots, y_d) \in X$ such that $x_i \neq y_i$ for all i, and define, for each $1 \le i \le d$, the subset $\widetilde{B}_i(x)$ of $\widetilde{V}_i$ by

$$\widetilde{B}_i(x) := \begin{cases} \{\mathbf{1}_i - \hat{x}_i, \hat{y}_i - \hat{z}_i : z_i \in [n_i] \setminus \{x_i, y_i\}\} & \text{if } n_i > 2, \\ \{\mathbf{1}_i - \hat{x}_i\} & \text{if } n_i = 2. \end{cases}$$

Then clearly $\widetilde{B}_i(x)$ is a basis of $\widetilde{V}_i$ consisting of eigenvectors of K_{n_i-1}.

Moreover, for $1 \le i \le d$, the subset $B(x; i)$ of V_i^* defined by

$$B(x;i) := \{\hat{x}_d \otimes \cdots \otimes \hat{x}_{i+1} \otimes \tilde{u}_i \otimes u_{i-1} \otimes \cdots \otimes u_1 : \tilde{u}_i \in \widetilde{B}_i(x), u_j \in B_j(x) \text{ for } 1 \le j \le i-1\}$$

forms a basis of V_i^*. We also observe that

$$E_i^*\mathbf{1} = \hat{x}_d \otimes \cdots \otimes \hat{x}_{i+1} \otimes (\mathbf{1}_i - \hat{x}_i) \otimes \mathbf{1}_{i-1} \otimes \cdots \otimes \mathbf{1}_1 \ \in \ B(x;i).$$

It is shown that $W_0 := \mathrm{Span}\{E_0^*\mathbf{1}, E_1^*\mathbf{1}, \ldots, E_d^*\mathbf{1}\}$ is a $(d+1)$-dimensional irreducible $\mathcal{T}(x)$-module, known as the *primary module* (cf. [16]). With the above notations, we now have the following theorem.

Theorem 4.2 *For $1 \le i \le d$, the following hold.*

(1) *For every $u \in B(x;i) \setminus \{E_i^*\mathbf{1}\}$, Span$(u)$ is $E_j^* A_h E_i^*$-invariant for $0 \le j, h \le d$. In particular, Span(u) is a one-dimensional irreducible $\mathcal{T}(x)$-module.*

(2) *For $u \in B(x;i) \setminus \{E_i^*\mathbf{1}\}$, define*

$$cl(u) := \begin{cases} i & \text{if } u_j = \mathbf{1}_j \text{ for all } 1 \le j \le i-1; \\ \min\{j : 1 \le j \le i-1, u_j \neq \mathbf{1}_j\} & \text{otherwise.} \end{cases}$$

Then for $u, v \in B(x;i) \setminus \{E_i^*\mathbf{1}\}$, Span$(u)$ and Span(v) are isomorphic as $\mathcal{T}(x)$-modules if and only if $cl(u) = cl(v)$. We refer to $(i, cl(u))$ as the type of Span(u).

(3) *The number of non-isomorphic non-primary irreducible $\mathcal{T}(x)$-modules in E_i^*V is i if $n_i > 2$, and $i-1$ if $n_i = 2$.*

(4) *The number of non-isomorphic non-primary irreducible $\mathcal{T}(x)$-modules in V is $\frac{1}{2}d(d+1) - b$ where b denotes the number of K_2 factors in the wreath product.*

Proof Using Equations (1) and (2), we can express $E_j^* A_h E_i^*$ as the product of

$$\mathrm{diag}(\hat{x}_d) \otimes \cdots \otimes \mathrm{diag}(\hat{x}_{j+1}) \otimes \mathrm{diag}(\mathbf{1}_j - \hat{x}_j) \otimes I_{n_{j-1}} \otimes \cdots \otimes I_{n_1},$$

$$I_{n_d} \otimes \cdots \otimes I_{n_{h+1}} \otimes (J_{n_h} - I_{n_h}) \otimes J_{n_{h-1}} \otimes \cdots \otimes J_{n_1},$$

and

$$\mathrm{diag}(\hat{x}_d) \otimes \cdots \otimes \mathrm{diag}(\hat{x}_{i+1}) \otimes \mathrm{diag}(\mathbf{1}_i - \hat{x}_i) \otimes I_{n_{i-1}} \otimes \cdots \otimes I_{n_1}.$$

To prove Part (1), we will show that $\mathrm{Span}(u)$ is closed under the left multiplication by $E_j^* A_h E_i^*$. By Lemma 2.1 and Lemma 3.3, $E_j^* A_h E_i^*$ is nonzero only if $i = j$, $i = h$ or $j = h$.

First, we consider the case when $i = j$. By Lemma 2.1 and Lemma 3.3, we have $h \le i$. If $h < i$, as $E_i^* A_h E_i^*$ becomes

$$\begin{aligned} &\mathrm{diag}(\hat{x}_d) \otimes \cdots \otimes \mathrm{diag}(\hat{x}_{i+1}) \otimes \mathrm{diag}(\mathbf{1}_i - \hat{x}_i) \otimes I_{n_{i-1}} \otimes \cdots \otimes I_{n_{h+1}} \otimes (J_{n_h} - I_{n_h}) \\ &\quad \otimes J_{n_{h-1}} \otimes \cdots \otimes J_{n_1}, \end{aligned}$$

we have $E_i^* A_h E_i^* u$ equals

$$\hat{x}_d \otimes \cdots \otimes \hat{x}_{i+1} \otimes \tilde{u}_i \otimes u_{i-1} \otimes \cdots \otimes u_{h+1} \otimes (J_{n_h} - I_{n_h}) u_h \otimes J_{n_{h-1}} u_{h-1} \otimes \cdots \otimes J_{n_1} u_1,$$

which implies that

$$E_i^* A_h E_i^* u = \begin{cases} 0 & \text{if } u_g \neq \mathbf{1}_g \text{ for some } 1 \le g \le h-1; \\ (-1) n_{h-1} \cdots n_1 u & \text{if } u_h \neq \mathbf{1}_h, u_g = \mathbf{1}_g \text{ for all } 1 \le g \le h-1; \\ (n_h - 1) n_{h-1} \cdots n_1 u & \text{if } u_g = \mathbf{1}_g \text{ for all } 1 \le g \le h. \end{cases} \tag{3}$$

If $h = i$, since

$$\begin{aligned} E_i^* A_h E_i^* &= \mathrm{diag}(\hat{x}_d) \otimes \cdots \otimes \mathrm{diag}(\hat{x}_{i+1}) \otimes \mathrm{diag}(\mathbf{1}_i - \hat{x}_i)(J_{n_h} - I_{n_h}) \mathrm{diag}(\mathbf{1}_i - \hat{x}_i) \\ &\quad \otimes J_{n_{i-1}} \otimes \cdots \otimes J_{n_1}, \end{aligned}$$

we have

$$\begin{aligned} E_i^* A_i E_i^* u &= \hat{x}_d \otimes \cdots \otimes \hat{x}_{i+1} \otimes \mathrm{diag}(\mathbf{1}_i - \hat{x}_i)(J_{n_h} - I_{n_h}) \tilde{u}_i \\ &\quad \otimes J_{n_{i-1}} u_{i-1} \otimes \cdots \otimes J_{n_1} u_1 \\ &= \begin{cases} 0 & \text{if } u_g \neq \mathbf{1}_g \text{ for some } 1 \le g \le i-1; \\ (-1) n_{i-1} \cdots n_1 u & \text{if } u_g = \mathbf{1}_g \text{ for all } 1 \le g \le i-1. \end{cases} \end{aligned} \tag{4}$$

Next we consider the case when $i \neq j$. By Lemma 2.1 and Lemma 3.3, $E_j^* A_h E_i^*$ is nonzero only if $h = \max\{i, j\}$. By Lemma 3.7, then we have $E_j^* A_j E_i^* = E_j^* J E_i^*$. Since $u \neq E_i^* \mathbf{1}$, $J_{n_g} u_g = 0$ for some $1 \le g \le i-1$ or $J_{n_i} \mathrm{diag}(\mathbf{1}_i - \hat{x}_i) \tilde{u}_i = J_{n_i} \tilde{u}_i = 0$. Hence $E_j^* A_j E_i^* u = E_j^* A_i E_i^* u = 0$. This completes the proof of (1).

Part (2) can be verified by Equation (3) and Equation (4). Proofs of Part (3) and Part (4) are omitted as they are straightforward. □

Remark 4.2 Each vector $u \in B(x; i) \setminus \{E_i^* \mathbf{1}\}$ can be identified by a common eigenvector of the Bose-Mesner algebra of $K_{n_1} \wr \cdots \wr K_{n_{i-1}} \wr K_{n_i - 1}$.

With reference to Theorem 4.2, let $U_{i,j}$ be the sum of all non-primary irreducible modules of type (i, j) for $1 \le j \le i \le d$. Then we have the following decomposition of V:

$$V = W_0 \bigoplus \left(\bigoplus_{i:n_i=2} \bigoplus_{j=1}^{i-1} U_{i,j} \right) \bigoplus \left(\bigoplus_{i:n_i>2} \bigoplus_{j=1}^{i} U_{i,j} \right).$$

Now we can describe the structure of $\mathcal{T}$ as in the following.

Corollary 4.3 *Let $\mathcal{T}$ be the Terwilliger algebra of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ where $n_i \ge 2$ for all $1 \le i \le d$. Then*

$$\mathcal{T} \cong M_{d+1}(\mathbb{C}) \oplus M_1(\mathbb{C})^{\oplus \frac{1}{2} d(d+1) - b},$$

where $b = |\{i : n_i = 2, 1 \le i \le d\}|$.

5 Concluding remarks

There is further work that is needed on the theme related to our work. Here we state two problems that are of our interest.

1. As explained by Eric Egge [8] and introduction of [16] it is possible to define an "abstract version" of the Terwilliger algebra using generators and relations. In all cases the concrete Terwilliger algebra is a homomorphic image of the abstract Terwilliger algebra, and in some cases they are isomorphic. In the case of the wreath product of trivial schemes, the entire structure of the Terwilliger algebra is determined by the intersection numbers and Krein parameters, so it may be easy to see what is going on. Once all the vanishing intersection numbers and Krein parameters are worked out, we can obtain the defining relations for the algebra. Terwilliger believes that for the association schemes considered in the current paper, the abstract Terwilliger algebra and the concrete Terwilliger algebra are isomorphic. It is remained to study the Terwilliger algebras (basis, irreducible modules, dimension) from the generators/relations alone for the association schemes considered in this paper.
2. There are also other products besides the wreath product. It is also an interesting problem to look at the direct products of $H(1, q)$'s. We study the wreath product first because the direct product of two association schemes has a lot more classes than the wreath product. Namely, the direct product of a d-class association scheme and a e-class association scheme is of class $de + d + e$ while the wreath product becomes $(d + e)$-class association scheme. So a study of direct products may involve more work than that of wreath products. However, it may be worthy to look at it now as we know more about the schemes related to $H(1, q)$.

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