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Completely splittable representations of affine Hecke-Clifford algebras

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Abstract We classify and construct irreducible completely splittable representations of affine and finite Hecke-Clifford algebras over an algebraically closed field of characteristic not equal to 2.

Keywords Completely splittable · Affine Hecke-Clifford algebra

1 Introduction

Let $\mathbb{F}$ be an algebraically closed field of characteristic p and denote by S_n the symmetric group on n letters. In [12], Mathieu gave the dimension of the irreducible $\mathbb{F}S_n$-modules associated to the partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n with length l and $\lambda_1 - \lambda_l \leq (l - p)$ by using the well-known Schur-Weyl duality. Subsequently, Kleshchev [7] showed that these representations are exactly these whose restrictions to the subgroup S_k are semi-simple for any $k \leq n$ or equivalently on which the Jucys-Murphy elements in $\mathbb{F}S_n$ act semisimply. These $\mathbb{F}S_n$-modules are called *completely splittable* in [7]. By using the modular branching rules for $\mathbb{F}S_n$ (cf. [8]), a formula for the dimensions of completely splittable modules was obtained in terms of the paths in Young modular graphs, which recovers Mathieu's result [12]. Generalizing the work in [7, 12], Ruff [17] classified the irreducible completely splittable representations of degenerate affine Hecke algebras $\mathcal{H}_n$ (introduced by Drinfeld [4] and Lusztig [11]). Over the complex field $\mathbb{C}$, these $\mathcal{H}_n$-modules were constructed and classified originally by Cherednik [2]. Generalizations were established to affine Hecke algebras of type A in [3, 16] and to Khovanov-Lauda-Rouquier algebras in [9].

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From now on let us assume $p \neq 2$. This paper aims to classify and construct completely splittable representations of affine Hecke-Clifford algebras $\mathfrak{H}_n^{\mathfrak{c}}$ over $\mathbb{F}$. The algebra $\mathfrak{H}_n^{\mathfrak{c}}$ was introduced by Nazarov [14] (called affine Sergeev algebra) to study the spin (or projective) representations of the symmetric group S_n or equivalently to study the representations of the spin symmetric group algebra $\mathbb{F}S_n^-$. Our construction is a generalization of Young's seminormal construction of the irreducible representations of symmetric groups and affine Hecke algebras of type A (cf. [3, 16]). The approach is similar in spirit to the technique introduced by Okounkov and Vershik [15] on symmetric groups over $\mathbb{C}$.

Let us denote by $x_1, \ldots, x_n$ the polynomial generators of the algebra $\mathfrak{H}_n^{\mathfrak{c}}$ (cf. subsection 2.2 for the definition). According to Brundan and Kleshchev [1] (cf. [8, Part II]), one can reduce the study of the finite dimensional $\mathfrak{H}_n^{\mathfrak{c}}$-modules to these so-called integral modules on which the eigenvalues of $x_1^2, \ldots, x_n^2$ are of the form $q(i)$ for $i \in \mathbb{I}$ (cf. (2.1) and (2.9) for notations). Then each finite dimensional $\mathfrak{H}_n^{\mathfrak{c}}$-module M admits a decomposition as $M = \oplus_{\underline{i} \in \mathbb{I}^n} M_{\underline{i}}$, where $M_{\underline{i}}$ is the simultaneous generalized eigenspace for the commuting operators $x_1^2, \ldots, x_n^2$ corresponding to the eigenvalues $q(i_1), \ldots, q(i_n)$. We call $\underline{i}$ a weight of M if $M_{\underline{i}} \neq 0$. By definition, a finite dimensional $\mathfrak{H}_n^{\mathfrak{c}}$-module is completely splittable if the polynomial generators $x_1, \ldots, x_n$ act semisimply.

Our work is based on several equivalent characterizations (cf. Proposition 3.6 for precise statements) of irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules. In particular, an irreducible $\mathfrak{H}_n^{\mathfrak{c}}$-module is completely splittable if and only if its restriction to the subalgebra $\mathfrak{H}_{(r,1^{n-r})}^{\mathfrak{c}}$ (cf. subsection 2.2 for notations) is semisimple for any $1 \leq r \leq n$. It follows that any irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-module is semisimple on restriction to the subalgebra of $\mathfrak{H}_n^{\mathfrak{c}}$ generated by $s_k, c_k, c_{k+1}, x_k, x_{k+1}$ (cf. subsection 2.2 for notations) which is isomorphic to $\mathfrak{H}_2^{\mathfrak{c}}$ for fixed $1 \leq k \leq n-1$. By exploring irreducible $\mathfrak{H}_2^{\mathfrak{c}}$-modules, we obtain an explicit description of the action of the simple transpositions s_k on irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules and identify all possible weights of irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules. This leads to the construction of a family of irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules. It turns out that these modules exhaust the non-isomorphic irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules. We further show that these representations are parameterized by skew shifted Young diagrams with precise constraints depending on p and give a dimension formula in terms of the associated standard Young tableaux. We remark that in the special case when $p = 0$, our result confirms a conjecture of Wang and it has been independently obtained by Hill, Kujawa, and Sussan [5].

Denote by $\mathcal{Y}_n$ the finite Hecke-Clifford algebra $\mathcal{Y}_n = \mathcal{C}_n \rtimes \mathbb{F}S_n$, where $\mathcal{C}_n$ is the Clifford algebra over $\mathbb{F}$ generated by $c_1, \ldots, c_n$ subject to the relations $c_k^2 = 1, c_k c_l = -c_l c_k$ for $1 \leq k \neq l \leq n$. A $\mathcal{Y}_n$-module is called completely splittable if the Jucys-Murphy elements $L_1, \ldots, L_n$ (cf. (6.1) for notations) act semisimply. There exists a surjective homomorphism (cf. [14]) from $\mathfrak{H}_n^{\mathfrak{c}}$ to $\mathcal{Y}_n$ which maps x_k to the Jucys-Murphy elements L_k for $1 \leq k \leq n$. By applying the results established for $\mathfrak{H}_n^{\mathfrak{c}}$ to $\mathcal{Y}_n$, we classify irreducible completely splittable $\mathcal{Y}_n$-modules and obtain a dimension formula for these modules. We understand that an unpublished work of Kleshchev and Ruff independently gave the classification of irreducible completely splittable $\mathcal{Y}_n$-modules. In [1], irreducible representations of $\mathcal{Y}_n$ over $\mathbb{F}$ are shown to be parameterized by p-restricted p-strict partitions of n. In this paper, we identify the subset

Γ of p-restricted p-strict partitions of n which parameterizes irreducible completely splittable $\mathcal{Y}_n$-modules. This together with a well-known Morita super-equivalence between the spin symmetric group algebra $\mathbb{F}S_n^-$ and $\mathcal{Y}_n$ leads to an interesting family of irreducible $\mathbb{F}S_n^-$-modules parameterized by Γ for which dimensions and characters can be explicitly described. In the special case when $p=0$, we recover the main result of [13] on the seminormal construction of all simple representations of $\mathbb{F}S_n^-$.

We observe that the L_k^2, $1 \le k \le n$, act semisimply on the basic spin $\mathcal{Y}_n$-module $I(n)$ (cf. [1, (9.11)]) which is not completely splittable. On the other hand, Wang [18] introduced the degenerate spin affine Hecke-Clifford algebras $\mathfrak{H}^-$ and established an isomorphism between $\mathfrak{H}_n^{\mathfrak{c}}$ and $\mathcal{C}_n \otimes \mathfrak{H}^-$ which sends x_k^2 to $2b_k^2$ (cf. Section 7 for notations). As the generators $b_1, \dots, b_n$ are anti-commutative, it is reasonable to study the $\mathfrak{H}^-$-modules on which the commuting operators $b_1^2, \dots, b_n^2$ act semisimply. This is equivalent to studying $\mathfrak{H}_n^{\mathfrak{c}}$-modules on which x_k^2, $1 \le k \le n$, act semisimply by using the isomorphism between $\mathfrak{H}_n^{\mathfrak{c}}$ and $\mathcal{C}_n \otimes \mathfrak{H}^-$. Motivated by these observations, we study and obtain a necessary condition in terms of weights for the classification of irreducible $\mathfrak{H}_n^{\mathfrak{c}}$-modules on which x_k^2, $1 \le k \le n$, act semisimply; moreover, this condition is conjectured to be sufficient, and the conjecture is verified when $n = 2, 3$.

The paper is organized as follows. In Section 2, we recall some basics about superalgebra and also the affine Hecke-Clifford algebras $\mathfrak{H}_n^{\mathfrak{c}}$. In Section 3, we analyze the structure of completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules by studying their weights and a classification of irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules is obtained in Section 4. In Section 5, we give a reinterpretation for weights of irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules in terms of shifted Young diagrams. In Section 6, we classify the irreducible completely splittable representations of finite Hecke-Clifford algebras. Finally, in Section 7 we introduce a larger category consisting of $\mathfrak{H}_n^{\mathfrak{c}}$-modules on which x_k^2 act semisimply and state a conjecture for classification of modules in this larger category.

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2 Affine Hecke-Clifford algebras $\mathfrak{H}_n^{\mathfrak{c}}$

Recall that $\mathbb{F}$ is an algebraically closed field of characteristic p with $p \neq 2$. Denote by $\mathbb{Z}_+$ the set of nonnegative integers and let

$$\mathbb{I} = \begin{cases} \mathbb{Z}_+, & \text{if } p = 0, \\ \{0, 1, \dots, \frac{p-1}{2}\}, & \text{if } p \ge 3. \end{cases} \tag{2.1}$$

2.1 Some basics about superalgebras

We shall recall some basic notions of superalgebras, referring the reader to [1, §2-b]. Let us denote by $\bar{v} \in \mathbb{Z}_2$ the parity of a homogeneous vector v of a vector superspace. By a superalgebra, we mean a $\mathbb{Z}_2$-graded associative algebra. Let $\mathcal{A}$ be a superalgebra. A $\mathcal{A}$-module means a $\mathbb{Z}_2$-graded left $\mathcal{A}$-module. A homomorphism $f: V \to W$

of $\mathcal{A}$-modules V and W means a linear map such that $f(av) = (-1)^{\bar{f}\bar{a}} af(v)$. Note that this and other such expressions only make sense for homogeneous a, f and the meaning for arbitrary elements is to be obtained by extending linearly from the homogeneous case. Let V be a finite dimensional $\mathcal{A}$-module. Let ΠV be the same underlying vector space but with the opposite $\mathbb{Z}_2$-grading. The new action of $a \in \mathcal{A}$ on $v \in \Pi V$ is defined in terms of the old action by $a \cdot v := (-1)^{\bar{a}} av$. Note that the identity map on V defines an isomorphism from V to ΠV.

A superalgebra analog of Schur's Lemma states that the endomorphism algebra of a finite dimensional irreducible module over a superalgebra is either one dimensional or two dimensional. In the former case, we call the module of *type* $\mathtt{M}$ while in the latter case the module is called of *type* $\mathtt{Q}$.

Given two superalgebras $\mathcal{A}$ and $\mathcal{B}$, we view the tensor product of superspaces $\mathcal{A} \otimes \mathcal{B}$ as a superalgebra with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\bar{b}\bar{a}'} (aa') \otimes (bb') \qquad (a, a' \in \mathcal{A}, b, b' \in \mathcal{B}).$$

Suppose V is an $\mathcal{A}$-module and W is a $\mathcal{B}$-module. Then $V \otimes W$ affords $A \otimes B$-module denoted by $V \boxtimes W$ via

$$(a \otimes b)(v \otimes w) = (-1)^{\bar{b}\bar{v}} av \otimes bw, \; a \in A, b \in B, v \in V, w \in W.$$

If V is an irreducible $\mathcal{A}$-module and W is an irreducible $\mathcal{B}$-module, $V \boxtimes W$ may not be irreducible. Indeed, we have the following standard lemma (cf. [7, Lemma 12.2.13]).

Lemma 2.1 *Let V be an irreducible $\mathcal{A}$-module and W be an irreducible $\mathcal{B}$-module.*

(1) *If both V and W are of type $\mathtt{M}$, then $V \boxtimes W$ is an irreducible $\mathcal{A} \otimes \mathcal{B}$-module of type $\mathtt{M}$.*
(2) *If one of V or W is of type $\mathtt{M}$ and the other is of type $\mathtt{Q}$, then $V \boxtimes W$ is an irreducible $\mathcal{A} \otimes \mathcal{B}$-module of type $\mathtt{Q}$.*
(3) *If both V and W are of type $\mathtt{Q}$, then $V \boxtimes W \cong X \oplus \Pi X$ for a type $\mathtt{M}$ irreducible $\mathcal{A} \otimes \mathcal{B}$-module X.*

Moreover, all irreducible $\mathcal{A} \otimes \mathcal{B}$-modules arise as constituents of $V \boxtimes W$ for some choice of irreducibles V, W.

If V is an irreducible $\mathcal{A}$-module and W is an irreducible $\mathcal{B}$-module, denote by $V \circledast W$ an irreducible component of $V \boxtimes W$. Thus,

$$V \boxtimes W = \begin{cases} V \circledast W \oplus \Pi(V \circledast W), & \text{if both } V \text{ and } W \text{ are of type } \mathtt{Q}, \\ V \circledast W, & \text{otherwise.} \end{cases}$$

2.2 Affine Hecke-Clifford algebras $\mathfrak{H}_n^{\mathfrak{c}}$

Now we proceed to define the superalgebra we will be interested in. For $n \in \mathbb{Z}_+$, the affine Hecke-Clifford algebra $\mathfrak{H}_n^{\mathfrak{c}}$ is the superalgebra generated by even generators $s_1, \dots, s_{n-1}, x_1, \dots, x_n$ and odd generators $c_1, \dots, c_n$ subject to the following relations

$$s_i^2=1,\quad s_is_j=s_js_i,\quad s_is_{i+1}s_i=s_{i+1}s_is_{i+1},\quad |i-j|>1,\tag{2.2}$$

$$x_ix_j=x_jx_i,\quad 1\le i,j\le n,\tag{2.3}$$

$$c_i^2=1, c_ic_j=-c_jc_i,\quad 1\le i\neq j\le n,\tag{2.4}$$

$$s_ix_i=x_{i+1}s_i-(1+c_ic_{i+1}),\tag{2.5}$$

$$s_ix_j=x_js_i,\quad j\neq i,i+1,\tag{2.6}$$

$$s_ic_i=c_{i+1}s_i, s_ic_{i+1}=c_is_i, s_ic_j=c_js_i,\quad j\neq i,i+1,\tag{2.7}$$

$$x_ic_i=-c_ix_i, x_ic_j=c_jx_i,\quad 1\le i\neq j\le n.\tag{2.8}$$

Remark 2.2 The affine Hecke-Clifford algebra $\mathfrak{H}_n^{\mathfrak{c}}$ was introduced by Nazarov [14] (called affine Sergeev algebra) to study the representations of $\mathbb{C}S_n^-$. The quantized version of the $\mathfrak{H}_n^{\mathfrak{c}}$ introduced later by Jones-Nazarov [6] to study the q-analogues of Young symmetrizers for projective representations of the symmetric group S_n is often also called affine Hecke-Clifford algebras.

For $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n$ and $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{Z}_2^n$, set $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha}$ and $c^\beta=c_1^{\beta_1}\cdots c_n^{\beta_n}$. Then we have the following.

Lemma 2.3 [1, *Theorem* 2.2] *The set* $\{x^\alpha c^\beta w\mid \alpha\in\mathbb{Z}_+^n,\beta\in\mathbb{Z}_2^n,w\in S_n\}$ *forms a basis of* $\mathfrak{H}_n^{\mathfrak{c}}$.

Denote by $\mathcal{P}_n^{\mathfrak{c}}$ the superalgebra generated by even generators $x_1,\dots,x_n$ and odd generators $c_1,\dots,c_n$ subject to the relations (2.3), (2.4) and (2.8). By Lemma 2.3, $\mathcal{P}_n^{\mathfrak{c}}$ can be identified with the subalgebra of $\mathfrak{H}_n^{\mathfrak{c}}$ generated by $x_1,\dots,x_n$ and $c_1,\dots,c_n$. For a composition $\mu=(\mu_1,\mu_2,\dots,\mu_r)$ of n, we define $\mathfrak{H}_\mu^{\mathfrak{c}}$ to be the subalgebra of $\mathfrak{H}_n^{\mathfrak{c}}$ generated by $\mathcal{P}_n^{\mathfrak{c}}$ and $s_j\in S_\mu=S_{\mu_1}\times\cdots\times S_{\mu_r}$. Note that $\mathcal{P}_n^{\mathfrak{c}}=\mathfrak{H}_{(1^n)}^{\mathfrak{c}}$. For each $i\in\mathbb{I}$, set

$$q(i)=i(i+1).\tag{2.9}$$

Let us denote by $\mathrm{Rep}_{\mathbb{I}}\,\mathfrak{H}_\mu^{\mathfrak{c}}$ the category of so-called *integral* finite dimensional $\mathfrak{H}_\mu^{\mathfrak{c}}$-modules on which the $x_1^2,\dots,x_n^2$ have eigenvalues of the form $q(i)$ for $i\in\mathbb{I}$. For each $i\in\mathbb{I}$, denote by $L(i)$ the 2-dimensional $\mathcal{P}_1^{\mathfrak{c}}$-module with $L(i)_{\bar 0}=\mathbb{F}v_0$ and $L(i)_{\bar 1}=\mathbb{F}v_1$ and

$$x_1v_0=\sqrt{q(i)}v_0,\quad x_1v_1=-\sqrt{q(i)}v_1,\quad c_1v_0=v_1,\quad c_1v_1=v_0.$$

Note that $L(i)$ is irreducible of type $\mathtt{M}$ if $i\neq 0$, and irreducible of type $\mathtt{Q}$ if $i=0$. Moreover $L(i), i\in\mathbb{I}$ form a complete set of pairwise non-isomorphic irreducible $\mathcal{P}_1^{\mathfrak{c}}$-module in the category $\mathrm{Rep}_{\mathbb{I}}\,\mathcal{P}_1^{\mathfrak{c}}$. Observe that $\mathcal{P}_n^{\mathfrak{c}}\cong\mathcal{P}_1^{\mathfrak{c}}\otimes\cdots\otimes\mathcal{P}_1^{\mathfrak{c}}$, and hence we have the following result by Lemma 2.1.

Lemma 2.4 [1, *Lemma* 4.8] *The $\mathcal{P}_n^{\mathfrak{c}}$-modules*

$$\{L(\underline{i}) = L(i_1) \circledast L(i_2) \circledast \cdots \circledast L(i_n) | \ \underline{i} = (i_1, \ldots, i_n) \in \mathbb{I}^n\}$$

form a complete set of pairwise non-isomorphic irreducible $\mathcal{P}_n^{\mathfrak{c}}$-module in the category $\mathrm{Rep}_{\mathbb{I}} \mathcal{P}_n^{\mathfrak{c}}$. *Moreover, denote by γ_0 the number of $1 \le j \le n$ with $i_j = 0$. Then $L(\underline{i})$ is of type* $\mathtt{M}$ *if γ_0 is even and type* $\mathtt{Q}$ *if γ_0 is odd. Furthermore, dim $L(\underline{i}) = 2^{n - \lfloor \frac{\gamma_0}{2} \rfloor}$, where $\lfloor \frac{\gamma_0}{2} \rfloor$ denotes the greatest integer less than or equal to $\frac{\gamma_0}{2}$.*

Remark 2.5 Note that each permutation $\tau \in S_n$ defines a superalgebra isomorphism $\tau : \mathcal{P}_n^{\mathfrak{c}} \to \mathcal{P}_n^{\mathfrak{c}}$ by mapping x_k to $x_{\tau(k)}$ and c_k to $c_{\tau(k)}$, for $1 \le k \le n$. For $\underline{i} \in \mathbb{I}^n$, the twist of the action of $\mathcal{P}_n^{\mathfrak{c}}$ on $L(\underline{i})$ with τ^{-1} leads to a new $\mathcal{P}_n^{\mathfrak{c}}$-module denoted by $L(\underline{i})^\tau$ with

$$L(\underline{i})^\tau = \{z^\tau \mid z \in L(\underline{i})\}, \quad f z^\tau = (\tau^{-1}(f) z)^\tau, \quad \text{for any } f \in \mathcal{P}_n^{\mathfrak{c}}, z \in L(\underline{i}).$$

So in particular we have $(x_k z)^\tau = x_{\tau(k)} z^\tau$ and $(c_k z)^\tau = c_{\tau(k)} z^\tau$. It is easy to see that $L(\underline{i})^\tau \cong L(\tau \cdot \underline{i})$, where $\tau \cdot \underline{i} = (i_{\tau^{-1}(1)}, \ldots, i_{\tau^{-1}(n)})$ for $\underline{i} = (i_1, \ldots, i_n) \in \mathbb{I}^n$ and $\tau \in S_n$.

2.3 Intertwining elements for $\mathfrak{H}_n^{\mathfrak{c}}$

Following [14], we define the intertwining elements as

$$\Phi_k := s_k(x_k^2 - x_{k+1}^2) + (x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1}), \quad 1 \le k \le n. \tag{2.10}$$

It is known that

$$\Phi_k^2 = 2(x_k^2 + x_{k+1}^2) - (x_k^2 - x_{k+1}^2)^2, \tag{2.11}$$

$$\Phi_k x_k = x_{k+1} \Phi_k, \Phi_k x_{k+1} = x_k \Phi_k, \Phi_k x_l = x_l \Phi_k, \tag{2.12}$$

$$\Phi_k c_k = c_{k+1} \Phi_k, \Phi_k c_{k+1} = c_k \Phi_k, \Phi_k c_l = c_l \Phi_k, \tag{2.13}$$

$$\Phi_j \Phi_k = \Phi_k \Phi_j, \Phi_k \Phi_{k+1} \Phi_k = \Phi_{k+1} \Phi_k \Phi_{k+1} \tag{2.14}$$

for all admissible j, k, l with $l \ne k, k+1$ and $|j - k| > 1$.

3 Weights of completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules

In this section, we shall describe the weights of completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules.

3.1 Structure of completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules

For $M \in \mathrm{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}}$ and $\underline{i} = (i_1, \ldots, i_n) \in \mathbb{I}^n$, set

$$M_{\underline{i}} = \{z \in M \mid (x_k^2 - q(i_k))^N z = 0 \text{ for } N \gg 0, 1 \le k \le n\}.$$

If $M_{\underline{i}} \neq 0$, then $\underline{i}$ is called a *weight* of M and $M_{\underline{i}}$ is called a weight space. Since the polynomial generators $x_1, \dots, x_n$ commute, we have

$$M = \bigoplus_{\underline{i} \in \mathbb{I}^n} M_{\underline{i}}. \tag{3.1}$$

For $i \in \mathbb{I}$ and $1 \leq m \leq n$, set

$$\Theta_{i^m} M = \{z \in M \mid (x_j^2 - q(i))^N z = 0, \text{ for } N \gg 0, n-m+1 \leq j \leq n\}.$$

One can show using (2.5) that

$$x_k^2 s_k = s_k x_{k+1}^2 - \big(x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1}) x_{k+1}\big) \tag{3.2}$$

$$x_{k+1}^2 s_k = s_k x_k^2 + \big(x_{k+1}(1 + c_k c_{k+1}) + (1 + c_k c_{k+1}) x_k\big). \tag{3.3}$$

Hence Θ_{i^m} defines an exact functor

$$\Theta_{i^m} : \operatorname{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}} \longrightarrow \operatorname{Rep}_{\mathbb{I}} \mathfrak{H}_{n-m,m}^{\mathfrak{c}}.$$

Moreover as $\mathfrak{H}_{n-1,1}^{\mathfrak{c}}$-modules, we have

$$\operatorname{res}^{\mathfrak{H}_n^{\mathfrak{c}}}_{\mathfrak{H}_{n-1,1}^{\mathfrak{c}}} M = \oplus_{i \in \mathbb{I}} \Theta_i M. \tag{3.4}$$

For $i \in \mathbb{I}$ and $M \in \operatorname{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}}$, define

$$\varepsilon_i(M) = \max\{m \geq 0 \mid \Theta_{i^m} M \neq 0\}.$$

Lemma 3.1 [1, *Lemma* 5.4] *Suppose that $M \in \operatorname{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}}$ is irreducible. Let $i \in \mathbb{I}$ and $m = \varepsilon_i(M)$. Then $\Theta_{i^m} M$ is isomorphic to $L \circledast \operatorname{ind}^{\mathfrak{H}_m^{\mathfrak{c}}}_{\mathcal{P}_m^{\mathfrak{c}}} L(i^m)$ for some irreducible $L \in \operatorname{Rep}_{\mathbb{I}} \mathfrak{H}_{n-m}^{\mathfrak{c}}$ with $\varepsilon_i(L) = 0$.*

Definition 3.2 A representation of $\mathfrak{H}_n^{\mathfrak{c}}$ is called *completely splittable* if $x_1, \dots, x_n$ act semisimply.

Remark 3.3 Observe that if $M \in \operatorname{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}}$ is completely splittable, then for $\underline{i} \in \mathbb{I}^n$,

$$M_{\underline{i}} = \{z \in M \mid x_k^2 z = q(i_k) z, 1 \leq k \leq n\}.$$

Lemma 3.4 *Suppose that $M \in \operatorname{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}}$ is completely splittable and that $M_{\underline{i}} \neq 0$ for some $\underline{i} \in \mathbb{I}^n$. Then $i_k \neq i_{k+1}$ for all $1 \leq k \leq n-1$.*

Proof Suppose $i_k = i_{k+1}$ for some $1 \leq k \leq n-1$. Let $0 \neq z \in M_{\underline{i}}$. Since M is completely splittable, $(x_k^2 - q(i_k))z = 0 = (x_{k+1}^2 - q(i_{k+1}))z$. This together with (3.2) shows that

$$(x_k^2 - q(i_k)) s_k z = (x_k^2 - q(i_{k+1})) s_k z = -\big(x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1}) x_{k+1}\big) z.$$

and hence

$$(x_k^2 - q(i_k))^2 s_k z = -\big(x_k(1-c_kc_{k+1}) + (1-c_kc_{k+1})x_{k+1}\big)(x_k^2 - q(i_k))z = 0.$$

Similarly, we see that

$$(x_{k+1}^2 - q(i_{k+1}))^2 s_k z = 0.$$

Hence $s_k z \in M_{\underline{i}}$. By Remark 3.3, we deduce that $(x_k^2 - q(i_k))s_k z = 0$ and therefore

$$\big(x_k(1-c_kc_{k+1}) + (1-c_kc_{k+1})x_{k+1}\big)z = 0.$$

This implies

$$2(x_k^2 + x_{k+1}^2)z = \big(x_k(1-c_kc_{k+1}) + (1-c_kc_{k+1})x_{k+1}\big)^2 z = 0.$$

This means $q(i_{k+1}) = -q(i_k)$ and hence $q(i_k) = q(i_{k+1}) = 0$ since $i_k = i_{k+1}$. Therefore $x_k^2 = 0 = x_{k+1}^2$ on $M_{\underline{i}}$. Since x_k, x_{k+1} act semisimply on $M_{\underline{i}}$, $x_k = 0 = x_{k+1}$ on $M_{\underline{i}}$. This implies $x_{k+1}s_k z = 0$ since $s_k z \in M_{\underline{i}}$ as shown above. Then

$$(1 + c_kc_{k+1})z = x_{k+1}s_k z - s_k x_k z = 0.$$

This means $2z = (1 - c_kc_{k+1})(1 + c_kc_{k+1})z = 0$. Hence $z = 0$ since $p \neq 2$. This contradicts the assumption that $z \neq 0$. □

Corollary 3.5 *Suppose that $M \in \mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}_n^{\mathfrak{c}}$ is completely splittable. Then $\varepsilon_i(M) \leq 1$ for any $i \in \mathbb{I}$.*

Proposition 3.6 *Let $M \in \mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}_n^{\mathfrak{c}}$ be irreducible. The following are equivalent.*

(1) *M is completely splittable.*
(2) *For any $\underline{i} \in \mathbb{I}^n$ with $M_{\underline{i}} \neq 0$, we have $i_k \neq i_{k+1}$ for all $1 \leq k \leq n-1$.*
(3) *The restriction $\mathrm{res}^{\mathfrak{H}_n^{\mathfrak{c}}}_{\mathfrak{H}^{\mathfrak{c}}_{(r,1^{n-r})}} M$ is semisimple for any $1 \leq r \leq n$.*
(4) *For any $\underline{i} \in \mathbb{I}^n$ with $M_{\underline{i}} \neq 0$, we have $M_{\underline{i}} \cong L(\underline{i})$ as $\mathcal{P}_n^{\mathfrak{c}}$-modules.*

Proof By Lemma 3.4, (1) implies (2). Suppose (2) holds, then by Lemma 3.1 and Corollary 3.5 we have $\Theta_i M$ is either zero or irreducible for each $i \in \mathbb{I}$ and hence by (3.4) $\mathrm{res}^{\mathfrak{H}_n^{\mathfrak{c}}}_{\mathfrak{H}^{\mathfrak{c}}_{(n-1,1)}} M$ is semisimple. Observe that if $\Theta_i M \cong N \circledast L(i)$ for some irreducible $N \in \mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}_{n-1}^{\mathfrak{c}}$, then (2) also holds for N. This implies $\mathrm{res}^{\mathfrak{H}_{n-1}^{\mathfrak{c}}}_{\mathfrak{H}^{\mathfrak{c}}_{(n-2,1)}} N$ is semisimple. Therefore $\mathrm{res}^{\mathfrak{H}_n^{\mathfrak{c}}}_{\mathfrak{H}^{\mathfrak{c}}_{(n-2,1,1)}} M$ is semisimple by (3.4). Continuing in this way we see that the restriction $\mathrm{res}^{\mathfrak{H}_n^{\mathfrak{c}}}_{\mathfrak{H}^{\mathfrak{c}}_{(r,1^{n-r})}} M$ is semisimple for any $1 \leq r \leq n$, whence (3).

Now assume (3) holds. In particular, $\mathrm{res}^{\mathfrak{H}_n^{\mathfrak{c}}}_{\mathfrak{H}^{\mathfrak{c}}_{(1^n)}} M$ is semisimple, that is, M is isomorphic to a direct sum of $L(\underline{i})$ as $\mathcal{P}_n^{\mathfrak{c}}$-modules. It is clear that $x_1, \ldots, x_n$ act semisimply on $L(\underline{i})$ for each $\underline{i} \in \mathbb{I}^n$, whence (1).

Clearly (1) holds if (4) is true. Now suppose (1) holds and we shall prove (4) by induction on n. Suppose $M_{\underline{i}} \neq 0$ for some $\underline{i} \in \mathbb{I}^n$. Observe that $M_{\underline{i}} \subseteq \Theta_{i_n} M \neq 0$. By Lemma 3.1 and Corollary 3.5, $\Theta_{i_n} M \cong N \circledast L(i_n)$ for some irreducible $N \in \mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}^{\mathfrak{c}}_{n-1}$. This means $M_{\underline{i}} \cong N_{\underline{i}'} \circledast L(i)$, where $\underline{i}' = (i_1, \ldots, i_{n-1})$. Note that N is completely splittable and hence by induction $N_{\underline{i}'} \cong L(i_1) \circledast \cdots \circledast L(i_{n-1})$. Therefore $M_{\underline{i}} \cong L(i_1) \circledast \cdots \circledast L(i_n)$. □

Remark 3.7 Note that $\mathfrak{H}^{\mathfrak{c}}_n$ possesses an automorphism σ_n which sends s_k to $-s_{n-k}$, x_l to x_{n+1-k} and c_l to c_{n+1-l} for $1 \le k \le n-1$ and $1 \le l \le n$. Moreover σ_n induces an algebra isomorphism for each composition $\mu = (\mu_1, \ldots, \mu_m)$ of n

$$\sigma_\mu : \mathfrak{H}^{\mathfrak{c}}_\mu \longrightarrow \mathfrak{H}^{\mathfrak{c}}_{\mu^t},$$

where $\mu^t = (\mu_m, \ldots, \mu_1)$. Given $M \in \mathfrak{H}^{\mathfrak{c}}_{\mu^t}$, we can twist with σ_μ to get a $\mathfrak{H}^{\mathfrak{c}}_\mu$-module M^{σ_μ}. Observe that for $\mathfrak{H}^{\mathfrak{c}}_n$-module M, we have

$$\left(\mathrm{res}^{\mathfrak{H}^{\mathfrak{c}}_n}_{\mathfrak{H}^{\mathfrak{c}}_{(r,1^{n-r})}} M^{\sigma_n}\right)^{\sigma_{(1^{n-r},r)}} \cong \mathrm{res}^{\mathfrak{H}^{\mathfrak{c}}_n}_{\mathfrak{H}^{\mathfrak{c}}_{(1^{n-r},r)}} M.$$

Hence $M \in \mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}^{\mathfrak{c}}_n$ is irreducible completely splittable if and only if $\mathrm{res}^{\mathfrak{H}^{\mathfrak{c}}_n}_{\mathfrak{H}^{\mathfrak{c}}_{(1^{n-r},r)}} M$ is semisimple for any $1 \le r \le n$ by Proposition 3.6.

Corollary 3.8 *Let $M \in \mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}^{\mathfrak{c}}_n$ be irreducible completely splittable. Then the restriction $\mathrm{res}^{\mathfrak{H}^{\mathfrak{c}}_n}_{\mathfrak{H}^{\mathfrak{c}}_{(1^{k-1},2,1^{n-k-1})}} M$ is semisimple for any $1 \le k \le n-1$. Hence M is semisimple on restriction to the subalgebra generated by $s_k, x_k, x_{k+1}, c_k, c_{k+1}$ which is isomorphic to $\mathfrak{H}^{\mathfrak{c}}_2$ for fixed $1 \le k \le n-1$.*

Proof By Proposition 3.6, $\mathrm{res}^{\mathfrak{H}^{\mathfrak{c}}_n}_{\mathfrak{H}^{\mathfrak{c}}_{(k+1,1^{n-k-1})}} M$ is semisimple. Hence

$$\mathrm{res}^{\mathfrak{H}^{\mathfrak{c}}_n}_{\mathfrak{H}^{\mathfrak{c}}_{(1^{k-1},2,1^{n-k-1})}} M = \mathrm{res}^{\mathfrak{H}^{\mathfrak{c}}_{(k+1,1^{n-k-1})}}_{\mathfrak{H}^{\mathfrak{c}}_{(1^{k-1},2,1^{n-k-1})}} \left(\mathrm{res}^{\mathfrak{H}^{\mathfrak{c}}_n}_{\mathfrak{H}^{\mathfrak{c}}_{(k+1,1^{n-k-1})}} M\right)$$

is semisimple by Remark 3.7. □

3.2 The weight constraints

Suppose that $M \in \mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}^{\mathfrak{c}}_n$ is completely splittable and that $M_{\underline{i}} \neq 0$ for some $\underline{i} \in \mathbb{I}^n$. By Lemma 3.4, $i_k \neq i_{k+1}$ for $1 \le k \le n-1$. It follows from Remark 3.3 that $x_k^2 - x_{k+1}^2$ acts as the nonzero scalar $q(i_k) - q(i_{k+1})$ on $M_{\underline{i}}$ for each $1 \le k \le n-1$. So we define linear operators Ξ_k and Ω_k on $M_{\underline{i}}$ such that for any $z \in M_{\underline{i}}$,

$$\Xi_k z := -\Big(\frac{x_k + x_{k+1}}{x_k^2 - x_{k+1}^2} + c_k c_{k+1} \frac{x_k - x_{k+1}}{x_k^2 - x_{k+1}^2}\Big) z, \tag{3.5}$$

$$\Omega_k z := \left(\sqrt{1 - \frac{2(x_k^2 + x_{k+1}^2)}{(x_k^2 - x_{k+1}^2)^2}}\right) z = \left(\sqrt{1 - \frac{2(q(i_k) + q(i_{k+1}))}{(q(i_k) - q(i_{k+1}))^2}}\right) z. \tag{3.6}$$

Both Ξ_k and Ω_k make sense as linear operators on $L(\underline{i})$ for $\underline{i} \in \mathbb{I}^n$ whenever $i_k \neq i_{k+1}$ for $1 \le k \le n$.

Proposition 3.9 *The following holds for $i, j \in \mathbb{I}$.*

(1) *If $i = j \pm 1$, then the irreducible $\mathcal{P}_2^{\mathfrak{c}}$-module $L(i) \circledast L(j)$ affords an irreducible $\mathfrak{H}_2^{\mathfrak{c}}$-module denoted by $V(i, j)$ with the action $s_1 z = \Xi_1 z$ for any $z \in L(i) \circledast L(j)$. The $\mathfrak{H}_2^{\mathfrak{c}}$-module $V(i, j)$ has the same type as the $\mathcal{P}_2^{\mathfrak{c}}$-module $L(i) \circledast L(j)$. Moreover, it is always completely splittable.*

(2) *If $i \neq j \pm 1$, the $\mathfrak{H}_2^{\mathfrak{c}}$-module $V(i, j) := \mathrm{ind}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} L(i) \circledast L(j)$ is irreducible and has the same type as the $\mathcal{P}_2^{\mathfrak{c}}$-module $L(i) \circledast L(j)$. It is completely splittable if and only if $i \neq j$ (and recall $i \neq j \pm 1$).*

(3) *Every irreducible module in the category $\mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}_2^{\mathfrak{c}}$ is isomorphic to some $V(i, j)$.*

Proof (1). It is routine to check $s_1 x_1 = x_2 s_1 - (1 + c_1 c_2)$ and $s_1 c_1 = c_2 s_1$, hence it remains to prove $s_1^2 = 1$ on $V(i, j)$. Indeed, for $z \in L(i) \circledast L(j)$, we have

$$s_1^2 z = \Xi_1^2 z = \frac{2(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} v = \frac{2(q(i) + q(j))}{(q(i) - q(j))^2} z = z,$$

where the last identity follows from the definition of $q(i)$ and the assumption $i = j \pm 1$. It is clear that $\mathrm{End}_{\mathcal{P}_2^{\mathfrak{c}}}(L(i) \circledast L(j)) \cong \mathrm{End}_{\mathfrak{H}_2^{\mathfrak{c}}}(V(i, j))$. Hence $V(i, j)$ has the same type as the $\mathcal{P}_2^{\mathfrak{c}}$-module $L(i) \circledast L(j)$. Since x_1, x_2 act semisimply on $L(i) \circledast L(j)$, $V(i, j)$ is completely splittable.

(2). Assume that $i \neq j \pm 1$ and that M is a nonzero proper submodule of $V(i, j) = \mathrm{ind}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} L(i) \circledast L(j)$. Observe that $V(i, j) = 1 \otimes (L(i) \circledast L(j)) \oplus s_1 \otimes (L(i) \circledast L(j))$ as vector spaces. Without loss of generality, we can assume M contains a nonzero vector v of the form $v = 1 \otimes u + s_1 \otimes u$ or $v = 1 \otimes u - s_1 \otimes u$ for some $0 \neq u \in L(i) \circledast L(j)$. Otherwise, we can replace v by $v + s_1 v$ or $v - s_1 v$ since either of them is nonzero. By (3.2),

$$\begin{aligned} x_1^2 v &= 1 \otimes x_1^2 u \pm s_1 \otimes x_2^2 u \mp 1 \otimes \big(x_1(1 - c_1 c_2) + (1 - c_1 c_2) x_2\big) u \\ &= 1 \otimes q(i) u \pm q(j) s_1 \otimes u \mp 1 \otimes \big(x_1(1 - c_1 c_2) + (1 - c_1 c_2) x_2\big) u. \end{aligned}$$

This together with $(x_1^2 - q(j)) v \in M$ shows that

$$1 \otimes \Big((q(i) - q(j)) u \pm \big(x_1(1 - c_1 c_2) + (1 - c_1 c_2) x_2\big) u\Big) \in M.$$

Since $1 \otimes ((q(i) - q(j)) u \pm (x_1(1 - c_1 c_2) + (1 - c_1 c_2) x_2) u) \in L(i) \circledast L(j)$ and M is a proper $\mathfrak{H}_2^{\mathfrak{c}}$-submodule of $V(i, j)$, we have

$$(q(i) - q(j)) u \pm [x_1(1 - c_1 c_2) + (1 - c_1 c_2) x_2] u = 0$$

and therefore

$$(q(i) - q(j))^2 u = (x_1(1 - c_1 c_2) + (1 - c_1 c_2) x_2)^2 u.$$

This together with $(x_1(1-c_1c_2)+(1-c_1c_2)x_2)^2u = 2(x_1^2+x_2^2)u$ shows that

$$2(q(i)+q(j)) = (q(i)-q(j))^2.$$

This contradicts the assumption $i \neq j \pm 1$ and hence $V(i,j)$ is irreducible.

Note that if $i \neq j$, then $V(i,j)$ has two weights, that is, (i,j) and (j,i). By Proposition 3.6, we see that $\mathrm{res}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} V(i,j)$ is semisimple and is isomorphic to the direct sum of $L(i) \circledast L(j)$ and $L(j) \circledast L(i)$. This means

$$\mathrm{Hom}_{\mathcal{P}_2^{\mathfrak{c}}}\big(L(i) \circledast L(j), \mathrm{res}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} V(i,j)\big) \cong \mathrm{End}_{\mathcal{P}_2^{\mathfrak{c}}}\big(L(i) \circledast L(j)\big).$$

By Frobenius reciprocity we obtain

$$\mathrm{End}_{\mathfrak{H}_2^{\mathfrak{c}}}\big(V(i,j)\big) \cong \mathrm{Hom}_{\mathcal{P}_2^{\mathfrak{c}}}\big(L(i) \circledast L(j), \mathrm{res}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} V(i,j)\big) \cong \mathrm{End}_{\mathcal{P}_2^{\mathfrak{c}}}\big(L(i) \circledast L(j)\big).$$

Hence $V(i,j)$ has the same type as the $\mathcal{P}_2^{\mathfrak{c}}$-module $L(i) \circledast L(j)$.

Now suppose $i = j$. This implies that (i,i) is a weight of $V(i,i)$ and hence $V(i,i)$ is not completely splittable by Lemma 3.4. By Proposition 3.6, $\mathrm{res}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} V(i,i)$ is not semisimple. Note that $\mathrm{res}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} V(i,i)$ has two composition factors and both of them are isomorphic to $L(i) \circledast L(i)$. Therefore the socle of $\mathrm{res}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} V(i,i)$ is simple and isomorphic to $L(i) \circledast L(i)$. Hence $\mathrm{Hom}_{\mathcal{P}_2^{\mathfrak{c}}}(L(i) \circledast L(i), \mathrm{res}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} V(i,i)) \cong \mathrm{End}_{\mathcal{P}_2^{\mathfrak{c}}}(L(i) \circledast L(i))$. By Frobenius reciprocity we obtain

$$\mathrm{End}_{\mathfrak{H}_2^{\mathfrak{c}}}\big(V(i,i)\big) \cong \mathrm{Hom}_{\mathcal{P}_2^{\mathfrak{c}}}\big(L(i) \circledast L(i), \mathrm{res}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} V(i,i)\big) \cong \mathrm{End}_{\mathcal{P}_2^{\mathfrak{c}}}\big(L(i) \circledast L(i)\big).$$

Hence $V(i,i)$ has the same type as the $\mathcal{P}_2^{\mathfrak{c}}$-module $L(i) \circledast L(i)$.

(3). Suppose $M \in \mathrm{Rep}_{\mathbb{I}} \mathfrak{H}_2^{\mathfrak{c}}$ is irreducible, then there exist $i, j \in \mathbb{I}$ such that $L(i) \circledast L(j) \subseteq \mathrm{res}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} M$. By Frobenius reciprocity M is an irreducible quotient of the induced module $\mathrm{ind}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} L(i) \circledast L(j)$. If $i \neq j \pm 1$, then $M \cong \mathrm{ind}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} L(i) \circledast L(j)$ since $\mathrm{ind}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} L(i) \circledast L(j)$ is irreducible by (2); otherwise using the fact that $\Xi_1^2 = 1$ on $L(i) \circledast L(j)$ one can show that the vector space

$$L := \mathrm{span}\big\{s_1 \otimes u - 1 \otimes \Xi_1 u \mid u \in L(i) \circledast L(j)\big\}$$

is a $\mathfrak{H}_2^{\mathfrak{c}}$-submodule of $\mathrm{ind}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} L(i) \circledast L(j)$ and it is isomorphic to $V(j,i)$. It is easy to check the quotient $\mathrm{ind}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} L(i) \circledast L(j)/L$ is isomorphic to $V(i,j)$. Hence $M \cong V(i,j)$. □

Observe from the proof above that if $i \neq j, j \pm 1$ then the completely splittable $\mathfrak{H}_2^{\mathfrak{c}}$-module $V(i,j)$ has two weights (i,j) and (j,i) and moreover $s_1 - \Xi_1$ gives a bijection between the associated weight spaces. This together with Corollary 3.8 and Proposition 3.9 leads to the following.

Corollary 3.10 *Let $M \in \mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}_n^{\mathfrak{c}}$ be irreducible completely splittable. Suppose $0 \neq v \in M_{\underline{i}}$ for some $\underline{i} = (i_1, \dots, i_n) \in \mathbb{I}^n$. The following holds for $1 \leq k \leq n-1$.*

(1) *If $i_k = i_{k+1} \pm 1$, then $s_k v = \Xi_k v$.*
(2) *If $i_k \neq i_{k+1} \pm 1$, then $0 \neq (s_k - \Xi_k)v \in M_{s_k \cdot \underline{i}}$ and hence $s_k \cdot \underline{i}$ is a weight of M.*

Definition 3.11 Let $\underline{i} \in \mathbb{I}^n$. For $1 \leq k \leq n-1$, the simple transposition s_k is called *admissible* with respect to $\underline{i}$ if $i_k \neq i_{k+1} \pm 1$.

Let $W(\mathfrak{H}_n^{\mathfrak{c}})$ be the set of weights $\underline{i} \in \mathbb{I}^n$ of irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules. By Corollary 3.10, if $\underline{i} \in W(\mathfrak{H}_n^{\mathfrak{c}})$ and s_k is admissible with respect to $\underline{i}$, then $s_k \cdot \underline{i} \in W(\mathfrak{H}_n^{\mathfrak{c}})$; moreover $\underline{i}$ and $s_k \cdot \underline{i}$ must occur as weights in an irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-module simultaneously.

Lemma 3.12 *Let $\underline{i} \in W(\mathfrak{H}_n^{\mathfrak{c}})$. Suppose that $i_k = i_{k+2}$ for some $1 \leq k \leq n-2$.*

(1) *If $p = 0$, then $i_k = i_{k+2} = 0, i_{k+1} = 1$.*
(2) *If $p \geq 3$, then either $i_k = i_{k+2} = 0, i_{k+1} = 1$ or $i_k = i_{k+2} = \frac{p-3}{2}, i_{k+1} = \frac{p-1}{2}$.*

Proof Suppose $\underline{i}$ occurs in the irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-module M and $i_k = i_{k+2}$ for some $1 \leq k \leq n-2$. If $i_k \neq i_{k+1} \pm 1$, then $s_k \cdot \underline{i}$ is a weight of M with the form $(\cdots, u, u, \cdots)$ by Corollary 3.10. This contradicts Lemma 3.4. Hence $i_k = i_{k+1} \pm 1$. This together with Corollary 3.10 shows that $s_k = \Xi_k$ and $s_{k+1} = \Xi_{k+1}$ on $M_{\underline{i}}$ and by (2.8) we have

$$
\begin{aligned}
&s_k s_{k+1} s_k - s_{k+1} s_k s_{k+1} \\
&\quad = \frac{1}{(a-b)(b-a)(a-b)}(x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) \\
&\qquad + \frac{1}{(a-b)(b-a)(a-b)} c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2})
\end{aligned} \tag{3.7}
$$

on $M_{\underline{i}}$, where $a = q(i_k) = q(i_{k+2})$ and $b = q(i_{k+1})$. This implies that for $z \in M_{\underline{i}}$,

$$
(x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2})z + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2})z = 0. \tag{3.8}
$$

On $M_{\underline{i}}$, x_k, x_{k+2} act semisimply and x_k^2, x_{k+2}^2 act as scalars $q(i_k), q(i_{k+2})$. Hence $M_{\underline{i}}$ admits a decomposition $M_{\underline{i}} = N_1 \oplus N_2$, where $N_1 = \{z \in M_{\underline{i}} \mid x_k z = x_{k+2} z = \pm\sqrt{q(i_k)} z\}$ and $N_2 = \{z \in M_{\underline{i}} \mid x_k z = -x_{k+2} z = \pm\sqrt{q(i_k)} z\}$. Applying the identity (3.8) to N_1 and N_2, we obtain

$$
2\sqrt{q(i_k)}\big(6q(i_{k+1}) + 2q(i_k)\big) = 0. \tag{3.9}
$$

By the fact that $i_{k+1} = i_k \pm 1$, and the definition of $q(i_k)$ and $q(i_{k+1})$, one can check that (3.9) is equivalent to the following

$$
i_{k+1} = i_k - 1, \quad \sqrt{i_k(i_k+1)}(4i_k - 2)i_k = 0 \tag{3.10}
$$

or

$$i_{k+1} = i_k + 1, \quad \sqrt{i_k(i_k+1)}(4i_k+6)(i_k+1) = 0. \tag{3.11}$$

(1). If $p = 0$, since i_k, i_{k+1} are nonnegative there is no solution for the equation (3.10) and the solution of (3.11) is $i_k = 0, i_{k+1} = 1$.
(2). If $p \geq 3$, since $1 \leq i_k, i_{k+1} \leq \frac{p-3}{2}$ there is no solution for the equation (3.10) and the solutions of (3.11) are $i_k = 0, i_{k+1} = 1$ or $i_k = \frac{p-3}{2}, i_{k+1} = \frac{p-1}{2}$. □

Lemma 3.13 *Let $\underline{i} \in W(\mathfrak{H}_n^{\mathfrak{c}})$. Suppose $i_k = i_l$ for some $1 \leq k < l \leq n$. Then $i_k + 1 \in \{i_{k+1}, \ldots, i_{l-1}\}$.*

Proof Suppose $i_k = i_l = u$ for some $1 \leq k < l \leq n$. Without loss of generality, we can assume $u \notin \{i_{k+1}, \ldots, i_{l-1}\}$. If $u = 0$, then $1 \in \{i_{k+1}, \ldots, i_{l-1}\}$; otherwise we can apply admissible transpositions to $\underline{i}$ to obtain an element in $W(\mathfrak{H}_n^{\mathfrak{c}})$ of the form $(\cdots, 0, 0, \cdots)$, which contradicts Lemma 3.4.

Now assume $u \geq 1$ and $u + 1 \notin \{i_{k+1}, \ldots, i_{l-1}\}$. If $u - 1$ does not appear between i_{k+1} and i_{l-1} in $\underline{i}$, then we can apply admissible transpositions to $\underline{i}$ to obtain an element in $W(\mathfrak{H}_n^{\mathfrak{c}})$ of the form $(\cdots, u, u, \cdots)$, which contradicts Lemma 3.4. If $u - 1$ appears only once between i_{k+1} and i_{l-1} in $\underline{i}$, then we can apply admissible transpositions to $\underline{i}$ to obtain an element in $W(\mathfrak{H}_n^{\mathfrak{c}})$ of the form $(\cdots, u, u-1, u, \cdots)$, which contradicts Lemma 3.12. Hence $u - 1$ appears at least twice between i_{k+1} and i_{l-1} in $\underline{i}$. This implies that there exist $k < k_1 < l_1 < l$ such that

$$i_{k_1} = i_{l_1} = u - 1, \{u, u-1\} \cap \{i_{k_1+1}, \ldots, i_{l_1-1}\} = \emptyset.$$

An identical argument shows that there exist $k_1 < k_2 < l_2 < l_1$ such that

$$i_{k_2} = i_{l_2} = u - 2, \{u, u-1, u-2\} \cap \{i_{k_2+1}, \ldots, i_{l_2-1}\} = \emptyset.$$

Continuing in this way, we obtain $k < s < t < l$ such that

$$i_s = i_t = 0, \{u, u-1, \ldots, 1, 0\} \cap \{i_{s+1}, \ldots, i_{t-1}\} = \emptyset,$$

which is impossible as shown at the beginning. □

Proposition 3.14 *Let $\underline{i} \in W(\mathfrak{H}_n^{\mathfrak{c}})$. Then*

(1) *$i_k \neq i_{k+1}$ for all $1 \leq k \leq n - 1$.*
(2) *If $p \geq 3$, then $\frac{p-1}{2}$ appears at most once in $\underline{i}$.*
(3) *If $i_k = i_l = 0$ for some $1 \leq k < l \leq n$, then $1 \in \{i_{k+1}, \ldots, i_{l-1}\}$.*
(4) *If $p = 0$ and $i_k = i_l \geq 1$ for some $1 \leq k < l \leq n$, then $\{i_k - 1, i_k + 1\} \subseteq \{i_{k+1}, \ldots, i_{l-1}\}$.*
(5) *If $p \geq 3$ and $i_k = i_l \geq 1$ for some $1 \leq k < l \leq n$, then either of the following holds*:
    (a) $\{i_k - 1, i_k + 1\} \subseteq \{i_{k+1}, \ldots, i_{l-1}\}$,

(b) *there exists a sequence of integers* $k \le r_0 < r_1 < \cdots < r_{\frac{p-3}{2}-i_k} < q < t_{\frac{p-3}{2}-i_k} < \cdots < t_1 < t_0 \le l$ *such that* $i_q = \frac{p-1}{2}, i_{r_j} = i_{t_j} = i_k + j$ *and* $i_k + j$ *does not appear between* i_{r_j} *and* i_{t_j} *in* $\underline{i}$ *for each* $0 \le j \le \frac{p-3}{2} - i_k$.

Proof (1). It follows from Lemma 3.4.

(2). If $\frac{p-1}{2}$ appears more than once in $\underline{i}$, then it follows from Lemma 3.13 that $\frac{p+1}{2}$ appears in $\underline{i}$ which is impossible since $\frac{p+1}{2} \notin \mathbb{I}$.

(3). It follows from Lemma 3.13.

(4). Now suppose $p = 0$ and $i_k = i_l = u \ge 1$ for some $1 \le k < l \le n$. Without loss of generality, we can assume $u \notin \{i_{k+1}, \ldots, i_{l-1}\}$. By Lemma 3.13 we have $u + 1 \in \{i_{k+1}, \ldots, i_{l-1}\}$ and hence it suffices to show $u - 1 \in \{i_{k+1}, \ldots, i_{l-1}\}$. Now assume $u - 1 \notin \{i_{k+1}, \ldots, i_{l-1}\}$. Then $u + 1$ must appear in the subsequence $(i_{k+1}, \ldots, i_{l-1})$ at least twice, otherwise we can apply admissible transpositions to $\underline{i}$ to obtain an element in $W(\mathfrak{H}_n^{\mathfrak{c}})$ of the form $(\cdots, u, u+1, u \cdots)$ which contradicts Lemma 3.12. Hence there exist $k < k_1 < l_1 < l$ such that

$$i_{k_1} = i_{l_1} = u + 1, \quad u + 1 \text{ does not appear between } i_{k_1} \text{ and } i_{l_1} \text{ in } \underline{i}.$$

Since $u \notin \{i_{k+1}, \ldots, i_{l-1}\} \supseteq \{i_{k_1+1}, \ldots, i_{l_1-1}\}$, a similar argument gives k_2, l_2 with $k_1 < k_2 < l_2 < l_1$ such that

$$i_{k_2} = i_{l_2} = u + 2, \quad u + 2 \text{ does not appear between } i_{k_2} \text{ and } i_{l_2} \text{ in } \underline{i}.$$

Continuing in this way we see that any integer greater than u will appear in the subsequence $(i_{k+1}, \ldots, i_{l-1})$ which is impossible. Hence $u - 1 \in \{i_{k+1}, \ldots, i_{l-1}\}$.

(5). Suppose $p \ge 3$ and $1 \le i_k = i_l = u \le \frac{p-3}{2}$ for some $1 \le k < l \le n$ and $u - 1 \notin \{i_{k+1}, \ldots, i_{l-1}\}$. Clearly there exist $k \le r_0 < t_0 \le l$ such that

$$i_{r_0} = i_{t_0} = u, u \notin \{i_{r_0+1}, \ldots, i_{t_0-1}\}.$$

An identical argument used for proving (2) shows that there exists a sequence of integers

$$k \le r_0 < r_1 < \cdots < r_{\frac{p-3}{2}-u} < t_{\frac{p-3}{2}-u} < \cdots < t_1 < t_0 \le l$$

such that

$$r_j = t_j = u + j, \quad \{u, u+1, \ldots, u+j\} \cap \{i_{r_j+1}, \ldots, i_{t_j-1}\} = \emptyset$$

for each $0 \le j \le \frac{p-3}{2} - u$. Since $i_{r_{\frac{p-3}{2}-u}} = i_{t_{\frac{p-3}{2}-u}} = \frac{p-3}{2}$, by Lemma 3.13 there exists $r_{\frac{p-3}{2}-u} < q < t_{\frac{p-3}{2}-u}$ such that $i_q = \frac{p-1}{2}$. □

4 Classification of irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules

In this section, we shall give an explicit construction and a classification of irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules.

Recall that for $\underline{i} \in \mathbb{I}^n$ and $1 \leq k \leq n-1$, the simple transposition s_k is said to be admissible with respect to $\underline{i}$ if $i_k \neq i_{k+1} \pm 1$. Define an equivalence relation $\sim$ on $\mathbb{I}^n$ by declaring that $\underline{i} \sim \underline{j}$ if there exist $s_{k_1}, \ldots, s_{k_t}$ for some $t \in \mathbb{Z}_+$ such that $\underline{j} = (s_{k_t} \cdots s_{k_1}) \cdot \underline{i}$ and s_{k_l} is admissible with respect to $(s_{k_{l-1}} \cdots s_{k_1}) \cdot \underline{i}$ for $1 \leq l \leq t$.

Denote by $W'(\mathfrak{H}_n^{\mathfrak{c}})$ the set of $\underline{i} \in \mathbb{I}^n$ satisfying the properties (3), (4) and (5) in Proposition 3.14. Observe that if $\underline{i} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$ and s_k is admissible with respect to $\underline{i}$, then the properties in Proposition 3.14 hold for $s_k \cdot \underline{i}$ and hence $s_k \cdot \underline{i} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$. This means there is an equivalence relation denoted by $\sim$ on $W'(\mathfrak{H}_n^{\mathfrak{c}})$ inherited from the equivalence relation $\sim$ on $\mathbb{I}^n$. For each $\underline{i} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$, set

$$P_{\underline{i}} = \{\tau = s_{k_t} \cdots s_{k_1} \mid s_{k_l} \text{ is admissible with respect to } s_{k_{l-1}} \cdots s_{k_1} \cdot \underline{i},\ 1 \leq l \leq t,\ t \in \mathbb{Z}_+\}. \tag{4.1}$$

Lemma 4.1 *Let $\Lambda \in W'(\mathfrak{H}_n^{\mathfrak{c}})/\sim$ and $\underline{i} \in \Lambda$. Then the map*

$$\varphi : P_{\underline{i}} \to \Lambda, \tau \mapsto \tau \cdot \underline{i}$$

is bijective.

Proof By the definitions of $P_{\underline{i}}$ and the equivalence relation $\sim$ on $W'(\mathfrak{H}_n^{\mathfrak{c}})$, one can check that φ is surjective. Note that if $\tau, \sigma \in P_{\underline{i}}$ then $\sigma^{-1}\tau \in P_{\underline{i}}$. Therefore, to check the injectivity of φ, it suffices to show that for $\tau \in P_{\underline{i}}$ if $\tau \cdot \underline{i} = \underline{i}$ then $\tau = 1$. Associated to each $\underline{j} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$, there exists a unique table $\Gamma(\underline{j})$ whose ath column consists of all numbers k with $j_k = a$ and is increasing for each $a \in \mathbb{I}$. Since $\underline{j} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$, $j_k \neq j_{k+1}$ and hence k and $k+1$ are in different columns in $\Gamma(\underline{j})$ for each $1 \leq k \leq n-1$. This means each simple transposition s_k can naturally act on the table $\Gamma(\underline{j})$ by switching k and $k+1$ to obtain a new table denoted by $s_k \cdot \Gamma(\underline{j})$. It is clear that

$$s_k \cdot \Gamma(\underline{j}) = \Gamma(s_k \cdot \underline{j}), \quad 1 \leq k \leq n-1. \tag{4.2}$$

Since $\tau \in P_{\underline{i}}$, we can write $\tau = s_{k_t} s_{k_{t-1}} \cdots s_{k_1}$ so that s_{k_l} is admissible with respect to $s_{k_{l-1}} \cdots s_{k_1} \cdot \underline{i}$ for each $1 \leq l \leq t$. Observe that $s_{k_{l-1}} \cdots s_{k_1} \cdot \underline{i} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$ and hence there exists a table $\Gamma(s_{k_{l-1}} \cdots s_{k_1} \cdot \underline{i})$ as defined above for $1 \leq l \leq t$. By (4.2) we have

$$s_{k_l} \cdot \Gamma(s_{k_{l-1}} \cdots s_{k_1} \cdot \underline{i}) = \Gamma(s_{k_l} s_{k_{l-1}} \cdots s_{k_1} \cdot \underline{i})$$

for $1 \leq l \leq t$. This implies

$$\tau \cdot \Gamma(\underline{i}) = s_{k_t} \cdots s_{k_1} \cdot \Gamma(\underline{i}) = \Gamma(s_{k_t} \cdots s_{k_1} \cdot \underline{i}) = \Gamma(\underline{i}).$$

Therefore $\tau = 1$. □

Before stating the main theorem of this section, we need the following two lemmas. Let $M \in \mathrm{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}}$ be irreducible completely splittable and suppose $M_{\underline{i}} \neq 0$ for some $\underline{i} = (i_1, \ldots, i_n) \in \mathbb{I}^n$. Recall the linear operators Ξ_k and Ω_k on $M_{\underline{i}}$ from (3.5) and (3.6). If s_k is admissible with respect to $\underline{i}$, then $i_k \neq i_{k+1} \pm 1$ and hence $2(q(i_k) + q(i_{k+1})) \neq (q(i_k) - q(i_{k+1}))^2$. This implies that on $M_{\underline{i}}$ the linear operator Ω_k acts as a nonzero scalar and hence is invertible. Therefore we can define the linear map $\widehat{\Phi}_k$ as follows:

$$\widehat{\Phi}_k : M_{\underline{i}} \longrightarrow M,$$
$$z \mapsto (s_k - \Xi_k)\frac{1}{\Omega_k}z.$$

Lemma 4.2 *Let $M \in \mathrm{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}}$ be irreducible completely splittable. Assume that $M_{\underline{i}} \neq 0$ and that s_k is admissible with respect to $\underline{i}$ for some $\underline{i} = (i_1, \dots, i_n) \in \mathbb{I}^n$ and $1 \le k \le n-1$. Then,*

(1) *$\widehat{\Phi}_k$ satisfies*

$$\widehat{\Phi}_k x_k = x_{k+1}\widehat{\Phi}_k, \ \widehat{\Phi}_k x_{k+1} = x_k\widehat{\Phi}_k, \ \widehat{\Phi}_k x_l = x_l\widehat{\Phi}_k, \tag{4.3}$$

$$\widehat{\Phi}_k c_k = c_{k+1}\widehat{\Phi}_k, \ \widehat{\Phi}_k c_{k+1} = c_k\widehat{\Phi}_k, \ \widehat{\Phi}_k c_l = c_l\widehat{\Phi}_k, \tag{4.4}$$

for $1 \le l \le n$ with $|k-l| > 1$. Hence for each $z \in M_{\underline{i}}$, $\widehat{\Phi}_k(z) \in M_{s_k \cdot \underline{i}}$.

(2) *$\widehat{\Phi}_k^2 = 1$, and hence $\widehat{\Phi}_k : M_{\underline{i}} \to M_{s_k \cdot \underline{i}}$ is a bijection.*

(3)

$$\widehat{\Phi}_j\widehat{\Phi}_l = \widehat{\Phi}_l\widehat{\Phi}_j \quad \text{if } |j-l| > 1, \tag{4.5}$$

$$\widehat{\Phi}_j\widehat{\Phi}_{j+1}\widehat{\Phi}_j = \widehat{\Phi}_{j+1}\widehat{\Phi}_j\widehat{\Phi}_{j+1}. \tag{4.6}$$

whenever both sides are well-defined.

Proof (1) Recalling the intertwining element Φ_k from (2.10), we see that

$$\widehat{\Phi}_k = \Phi_k \frac{1}{x_k^2 - x_{k+1}^2}\frac{1}{\Omega_k}. \tag{4.7}$$

This together with (2.12) and (2.13) implies (4.3) and (4.4). By (4.3), we have for any $z \in M_{\underline{i}}$,

$$(x_k^2 - q(i_{k+1}))\widehat{\Phi}_k z = 0, \ (x_{k+1}^2 - q(i_k))\widehat{\Phi}_k z = 0, \ (x_l^2 - q(i_l))\widehat{\Phi}_k z = 0,$$
$$\text{for all } l \neq k, k+1.$$

This means $\widehat{\Phi}_k z \in M_{s_k \cdot \underline{i}}$.

(2) By (2.11) and (4.7), one can check that for $z \in M_{\underline{i}}$,

$$\widehat{\Phi}_k^2 z = \Phi_k^2 \frac{1}{(x_k^2 - x_{k+1}^2)(x_{k+1}^2 - x_k^2)}\frac{1}{\Omega_k^2}z = \left(1 - \frac{2(x_k^2 + x_{k+1}^2)}{(x_k^2 - x_{k+1}^2)^2}\right)\frac{1}{\Omega_k^2}z = z.$$

Hence $\widehat{\Phi}_k^2 = 1$ and so $\widehat{\Phi}_k$ is bijective.

(3). If $|j-l| > 1$ and both $\widehat{\Phi}_j\widehat{\Phi}_l$ and $\widehat{\Phi}_l\widehat{\Phi}_j$ are well-defined on $M_{\underline{i}}$, then by (2.12) and (4.7) we see that

$$\widehat{\Phi}_j\widehat{\Phi}_l = \Phi_j\Phi_l \frac{1}{\Omega_j\Omega_l(x_j^2 - x_{j+1}^2)(x_l^2 - x_{l+1}^2)},$$

$$\widehat{\Phi}_l\widehat{\Phi}_j = \Phi_l\Phi_j \frac{1}{\Omega_l\Omega_j(x_l^2 - x_{l+1}^2)(x_j^2 - x_{j+1}^2)}.$$

This together with (2.14) implies (4.5). By (4.7), one can check that if both $\widehat{\Phi}_k\widehat{\Phi}_{k+1}\widehat{\Phi}_k$ and $\widehat{\Phi}_{k+1}\widehat{\Phi}_k\widehat{\Phi}_{k+1}$ are well-defined on $M_{\underline{i}}$ then

$$\widehat{\Phi}_k\widehat{\Phi}_{k+1}\widehat{\Phi}_k = C\Phi_k\Phi_{k+1}\Phi_k,$$
$$\widehat{\Phi}_{k+1}\widehat{\Phi}_k\widehat{\Phi}_{k+1} = C\Phi_{k+1}\Phi_k\Phi_{k+1},$$

where C is the scalar

$$C = \frac{1}{(a-b)(a-c)(b-c)}\sqrt{1 - \frac{2(a+b)}{(a-b)^2}}\sqrt{1 - \frac{2(a+c)}{(a-c)^2}}\sqrt{1 - \frac{2(b+c)}{(b-c)^2}}$$

with $a = q(i_k), b = q(i_{k+1}), c = q(i_{k+2})$. Hence (4.6) follows from (2.14). □

Remark 4.3 Suppose that $M \in \mathrm{Rep}_{\mathbb{I}}\mathfrak{H}_n^{\mathfrak{c}}$ is completely splittable. By Lemma 4.2, if $M_{\underline{i}} \neq 0$ and $\underline{j} \sim \underline{i}$, then $M_{\underline{j}} \neq 0$.

Lemma 4.4 *Let $M \in \mathrm{Rep}_{\mathbb{I}}\mathfrak{H}_n^{\mathfrak{c}}$ be irreducible completely splittable. Suppose that $M_{\underline{i}} \neq 0$ for some $\underline{i} \in \mathbb{I}^n$ and $\tau \in P_{\underline{i}}$. Write $\tau = s_{k_t}\cdots s_{k_1}$ so that s_{k_l} is admissible with respect to $s_{k_{l-1}}\cdots s_{k_1}\cdot \underline{i}$ for $1 \leq l \leq t$. Then*

$$\widehat{\Phi}_\tau := \widehat{\Phi}_{k_t}\cdots\widehat{\Phi}_{k_1} : M_{\underline{i}} \longrightarrow M_{\tau\cdot\underline{i}}$$

is a bijection satisfying $x_k\widehat{\Phi}_\tau = \widehat{\Phi}_\tau x_{\tau(k)}$ and $c_k\widehat{\Phi}_\tau = \widehat{\Phi}_\tau c_{\tau(k)}$ for $1 \leq k \leq n$. Moreover $\widehat{\Phi}_\tau$ does not depend on the choice of the expression $s_{k_t}\cdots s_{k_1}$ for τ.

Proof Since s_{k_l} is admissible with respect to $s_{k_{l-1}}\cdots s_{k_1}\cdot\underline{i}$ for $1 \leq l \leq t$, each $\widehat{\Phi}_{k_l}$ is a well-defined bijection from $M_{s_{k_{l-1}}\cdot s_{k_1}\cdot\underline{i}}$ to $M_{s_{k_l}\cdot s_{k_1}\cdot\underline{i}}$ by Lemma 4.2 and hence $\widehat{\Phi}_\tau$ is bijective. By (4.5) and (4.6), $\widehat{\Phi}_\tau$ does not depend on the choice of the expression $s_{k_t}\cdots s_{k_1}$ for τ. Using (4.3) and (4.4), we obtain $x_k\widehat{\Phi}_\tau = \widehat{\Phi}_\tau x_{\tau(k)}$ and $c_k\widehat{\Phi}_\tau = \widehat{\Phi}_\tau c_{\tau(k)}$ for $1 \leq k \leq n$. □

Suppose $\underline{i} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$. Recall the definition of $L(\underline{i})^\tau$ from Remark 3.7 for $\tau \in P_{\underline{i}}$. Denote by $D^{\underline{i}}$ the $\mathcal{P}_n^{\mathfrak{c}}$-module defined by

$$D^{\underline{i}} = \oplus_{\tau\in P_{\underline{i}}} L(\underline{i})^\tau. \tag{4.8}$$

The following theorem is the main result of this paper.

Theorem 4.5 *Suppose $\underline{i}, \underline{j} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$. Then,*

(1) *$D^{\underline{i}}$ affords an irreducible $\mathfrak{H}_n^{\mathfrak{c}}$-module via*

$$s_k z^\tau = \begin{cases} \Xi_k z^\tau + \Omega_k z^{s_k\tau}, & \text{if } s_k \text{ is admissible with respect to } \tau\cdot\underline{i}, \\ \Xi_k z^\tau, & \text{otherwise,} \end{cases} \tag{4.9}$$

for $1 \le k \le n-1$, $z \in L(\underline{i})$ and $\tau \in P_{\underline{i}}$. It has the same type as the irreducible $\mathcal{P}_n^{\mathfrak{c}}$-module $L(\underline{i})$.

(2) *$D^{\underline{i}} \cong D^{\underline{j}}$ if and only if $\underline{i} \sim \underline{j}$.*

(3) *Every irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-module in $\mathrm{Rep}_{\mathbb{I}}\,\mathfrak{H}_n^{\mathfrak{c}}$ is isomorphic to $D^{\underline{i}}$ for some $\underline{i} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$. Hence the equivalence classes $W'(\mathfrak{H}_n^{\mathfrak{c}})/\sim$ parametrize irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules in the category $\mathrm{Rep}_{\mathbb{I}}\,\mathfrak{H}_n^{\mathfrak{c}}$.*

Proof (1). To show the formula (4.9) defines a $\mathfrak{H}_n^{\mathfrak{c}}$-module structure on $D^{\underline{i}}$, we need to check the defining relations (2.2), (2.5), (2.6) and (2.7) on $L(\underline{i})^\tau$ for each $\tau \in P_{\underline{i}}$. One can show using (2.8) that

$$\Xi_k x_k - x_{k+1}\Xi_k = -(1+c_kc_{k+1}). \tag{4.10}$$

For $1 \le k \le n-1$, $(x_{\tau^{-1}(k)}z)^{s_k\tau} = x_{k+1}z^{s_k\tau}$ by Remark 2.5 and hence if s_k is admissible with respect to $\tau\cdot\underline{i}$, then

$$\begin{aligned}
s_kx_kz^\tau &= s_k(x_{\tau^{-1}(k)}z)^\tau\\
&= \Xi_k(x_{\tau^{-1}(k)}z)^\tau + \Omega_k(x_{\tau^{-1}(k)}z)^{s_k\tau}\\
&= \Xi_kx_kz^\tau + x_{k+1}\Omega_kz^{s_k\tau}\\
&= (\Xi_kx_k - x_{k+1}\Xi_k)z^\tau + x_{k+1}(\Xi_kz^\tau + \Omega_kz^{s_k\tau})\\
&= -(1+c_kc_{k+1})z^\tau + x_{k+1}s_kz^\tau \quad \text{by (4.10)}.
\end{aligned}$$

Otherwise we have

$$\begin{aligned}
s_kx_kz^\tau = s_k(x_{\tau^{-1}(k)}z)^\tau &= \Xi_k(x_kz^\tau)\\
&= (\Xi_kx_k - x_{k+1}\Xi_k)z^\tau + x_{k+1}\Xi_kz^\tau\\
&= -(1+c_kc_{k+1})z^\tau + x_{k+1}s_kz^\tau \quad \text{by (4.10)}.
\end{aligned}$$

Therefore (2.5) holds. It is routine to check (2.6) and (2.7).

It remains to prove (2.2). It is clear by (2.6) that $s_ks_l = s_ls_k$ if $|l-k|>1$, so it suffices to prove $s_k^2=1$ and $s_ks_{k+1}s_k = s_{k+1}s_ks_{k+1}$. For the remaining of the proof, let us fix $\tau \in P_{\underline{i}}$ and set $\underline{j} = \tau\cdot\underline{i}$. One can check using (2.8) and (4.9) that

$$s_k^2z^\tau = \begin{cases} (\Xi_k^2+\Omega_k^2)z^\tau, & \text{if } s_k \text{ is admissible with respect to } \underline{j}=\tau\cdot\underline{i}\\ \Xi_k^2z^\tau, & \text{otherwise.}\end{cases}$$

Hence if s_k is admissible with respect to $\underline{j}=\tau\cdot\underline{i}$, then

$$s_k^2z^\tau = \Xi_k^2z^\tau + \Omega_k^2z^\tau = \Big(\frac{2(x_k^2+x_{k+1}^2)}{(x_k^2-x_{k+1}^2)^2}\Big)z^\tau + \Big(1-\frac{2(x_k^2+x_{k+1}^2)}{(x_k^2-x_{k+1}^2)^2}\Big)z^\tau = z^\tau.$$

Otherwise we have $j_k = j_{k+1}\pm 1$. This implies $2(q(j_k)+q(j_{k+1})) = (q(j_k)-q(j_{k+1}))^2$ and hence

$$s_k^2z^\tau = \Xi_k^2z^\tau = \frac{2(q(j_k)+q(j_{k+1}))}{(q(j_k)-q(j_{k+1}))^2}z^\tau = z^\tau.$$

Therefore $s_k^2 = 1$ on $D^{\underline{i}}$ for $1 \le k \le n-1$. Next we shall prove $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$ for $1 \le k \le n-2$. Set $\widehat{s}_k = s_k - \Xi_k$ for $1 \le k \le n-1$. It is clear by (4.9) that

$$\widehat{s}_k z^\tau = \begin{cases} \Omega_k z^{s_k \tau}, & \text{if } s_k \text{ is admissible with respect to } \underline{j} = \tau \cdot \underline{i}, \\ 0, & \text{otherwise.} \end{cases}$$

If $j_k - j_{k+1} = \pm 1$, $j_{k+1} - j_{k+2} = \pm 1$ or $j_k - j_{k+2} = \pm 1$, then $\widehat{s}_k \widehat{s}_{k+1} \widehat{s}_k = 0 = \widehat{s}_{k+1} \widehat{s}_k \widehat{s}_{k+1}$ on $L(\underline{i})^\tau$; otherwise, one can show using (3.6) that

$$\widehat{s}_k \widehat{s}_{k+1} \widehat{s}_k z^\tau = \left(\sqrt{1 - \frac{2(a+b)}{(a-b)^2}} \sqrt{1 - \frac{2(b+c)}{(b-c)^2}} \sqrt{1 - \frac{2(a+c)}{(a-c)^2}}\right) z^\tau = \widehat{s}_{k+1} \widehat{s}_k \widehat{s}_{k+1} z^\tau,$$

for any $z \in L(\underline{i})$, where $a = q(j_k)$, $b = q(j_{k+1})$, $c = q(j_{k+2})$. Hence

$$\widehat{s}_k \widehat{s}_{k+1} \widehat{s}_k z^\tau = \widehat{s}_{k+1} \widehat{s}_k \widehat{s}_{k+1} z^\tau, \text{ for any } z \in L(\underline{i}), 1 \le k \le n-2. \tag{4.11}$$

Fix $1 \le k \le n-2$. If $j_k \ne j_{k+2}$, then $\frac{1}{(x_k^2 - x_{k+1}^2)(x_k^2 - x_{k+2}^2)(x_{k+1}^2 - x_{k+2}^2)}$ acts as the nonzero scalar $\frac{1}{(a-b)(a-c)(b-c)}$ on $L(\underline{i})^\tau$. Recalling the intertwining elements Φ_k from (2.10), we see that

$$\widehat{s}_k = \Phi_k \frac{1}{x_k^2 - x_{k+1}^2}.$$

This together with (2.14) shows that for any $z \in L(\underline{i})$,

$$\widehat{s}_k \widehat{s}_{k+1} \widehat{s}_k z^\tau = \Phi_k \Phi_{k+1} \Phi_k \frac{1}{(x_k^2 - x_{k+1}^2)(x_k^2 - x_{k+2}^2)(x_{k+1}^2 - x_{k+2}^2)} z^\tau,$$

and

$$\widehat{s}_{k+1} \widehat{s}_k \widehat{s}_{k+1} z^\tau = \Phi_{k+1} \Phi_k \Phi_{k+1} \frac{1}{(x_k^2 - x_{k+1}^2)(x_k^2 - x_{k+2}^2)(x_{k+1}^2 - x_{k+2}^2)} z^\tau.$$

Hence by (4.11) we see that for any $z \in L(\underline{i})$,

$$(\Phi_k \Phi_{k+1} \Phi_k - \Phi_{k+1} \Phi_k \Phi_{k+1}) \frac{1}{(x_k^2 - x_{k+1}^2)(x_k^2 - x_{k+2}^2)(x_{k+1}^2 - x_{k+2}^2)} z^\tau = 0,$$

A tedious calculation shows that

$$\begin{aligned} &\Phi_k \Phi_{k+1} \Phi_k - \Phi_{k+1} \Phi_k \Phi_{k+1} \\ &\quad = (s_k s_{k+1} s_k - s_{k+1} s_k s_{k+1})(x_k^2 - x_{k+1}^2)(x_k^2 - x_{k+2}^2)(x_{k+1}^2 - x_{k+2}^2). \end{aligned}$$

Therefore we obtain that if $j_k \ne j_{k+2}$ then

$$s_k s_{k+1} s_k z^\tau = s_{k+1} s_k s_{k+1} z^\tau, \quad \text{for any } z \in L(\underline{i}).$$

Now assume $j_k = j_{k+2}$, then by Lemma 3.12 we have either $j_k = j_{k+2} = 0$, $j_{k+1} = 1$ or $j_k = j_{k+2} = \frac{p-3}{2}$, $j_{k+1} = \frac{p-1}{2}$. Hence $s_k = \Xi_k$ and $s_{k+1} = \Xi_{k+1}$ on $L(\underline{i})^\tau$. We see from the proof of Lemma 3.12 that $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$. Therefore $D^{\underline{i}}$ affords a $\mathfrak{H}_n^{\mathfrak{c}}$-module by the formula (4.9).

Suppose N is a nonzero irreducible submodule of $D^{\underline{i}}$, then $N_{\underline{j}} \neq 0$ for some $\underline{j} \in \mathbb{I}^n$. This implies $(D^{\underline{i}})_{\underline{j}} \neq 0$ and hence $\underline{j} \sim \underline{i}$. Since $\tau \cdot \underline{i} \sim \underline{i} \sim \underline{j}$, by Remark 4.3 we see that $N_{\tau\cdot\underline{i}} \neq 0$ for all $\tau \in P_{\underline{i}}$. Observe that $(D^{\underline{i}})_{\tau\cdot\underline{i}} \cong L(\tau \cdot \underline{i})$ is irreducible as a $\mathcal{P}_n^{\mathfrak{c}}$-module for $\tau \in P_{\underline{i}}$. Therefore $N_{\tau\cdot\underline{i}} = (D^{\underline{i}})_{\tau\cdot\underline{i}}$ for $\tau \in P_{\underline{i}}$ and hence $N = D^{\underline{i}}$. This means $D^{\underline{i}}$ is irreducible.

We shall show that $D^{\underline{i}}$ has the same type as $L(\underline{i})$. Suppose $\Psi \in \mathrm{End}_{\mathfrak{H}_n^{\mathfrak{c}}}(D^{\underline{i}})$. Note that for each $\tau \in P_{\underline{i}}$ and $1 \leq k \leq n-1$, if s_k is admissible with respect to $\tau \cdot \underline{i}$, then for any $z \in L(\underline{i})$,

$$\Omega_k \Psi(z^{s_k\tau}) = \Psi(\Omega_k z^{s_k\tau}) = \Psi(s_k z^\tau - \Xi_k z^\tau) = s_k \Psi(z^\tau) - \Xi_k \Psi(z^\tau). \tag{4.12}$$

Since s_k is admissible with respect to $\underline{j} := \tau \cdot \underline{i}$, $j_k \neq j_{k+1} \pm 1$ and hence Ω_k acts as a nonzero scalar on $L(\underline{i})^{s_k\tau}$. By (4.12) we see that $\Psi(z^{s_k\tau})$ is uniquely determined by $\Psi(z^\tau)$ for any $\tau \in P_{\underline{i}}$. Since each τ can be written as $\tau = s_{k_t} \cdots s_{k_1}$ so that s_{k_l} is admissible with respect to $s_{k_{l-1}} \cdots s_{k_1} \cdot \underline{i}$, we deduce $\Psi(z^\tau)$ is uniquely determined by $\Psi(z)$ for any $z \in L(\underline{i})$. Therefore Ψ is uniquely determined by its restriction to the $\mathcal{P}_n^{\mathfrak{c}}$-submodule $L(\underline{i})$. Clearly the image of restriction of Ψ to $L(\underline{i})$ is contained in $L(\underline{i})$ by Lemma 4.1. This implies

$$\dim_{\mathbb{F}} \mathrm{End}_{\mathfrak{H}_n^{\mathfrak{c}}}(D^{\underline{i}}) \leq \dim_{\mathbb{F}} \mathrm{End}_{\mathcal{P}_n^{\mathfrak{c}}}(L(\underline{i})). \tag{4.13}$$

One the other hand, it is routine to check that each $\mathcal{P}_n^{\mathfrak{c}}$-endomorphism $\rho : L(\underline{i}) \to L(\underline{i})$ induces a $\mathfrak{H}_n^{\mathfrak{c}}$-endomorphism $\oplus_{\tau\in P_{\underline{i}}} \rho^\tau : D^{\underline{i}} \to D^{\underline{i}}$, where $\rho^\tau(z^\tau) = (\rho(z))^\tau$. Therefore

$$\dim_{\mathbb{F}} \mathrm{End}_{\mathfrak{H}_n^{\mathfrak{c}}}(D^{\underline{i}}) \geq \dim_{\mathbb{F}} \mathrm{End}_{\mathcal{P}_n^{\mathfrak{c}}}(L(\underline{i})).$$

This together with (4.13) shows $\dim_{\mathbb{F}} \mathrm{End}_{\mathfrak{H}_n^{\mathfrak{c}}}(D^{\underline{i}}) = \dim_{\mathbb{F}} \mathrm{End}_{\mathcal{P}_n^{\mathfrak{c}}}(L(\underline{i}))$ and hence $D^{\underline{i}}$ has the same type as $\mathcal{P}_n^{\mathfrak{c}}$-module $L(\underline{i})$.

(2). If $D^{\underline{i}} \cong D^{\underline{j}}$, then $(D^{\underline{i}})_{\underline{j}} \neq 0$ and hence $\underline{i} \sim \underline{j}$. Conversely, by Lemma 4.1, there exists $\sigma \in P_{\underline{i}}$ such that $\underline{j} = \sigma \cdot \underline{i}$. By Remark 2.5, we have $L(\underline{j}) \cong L(\underline{i})^\sigma$ and hence there exists a linear map $\phi : L(\underline{j}) \to L(\underline{i})$ such that the map $L(\underline{j}) \to L(\underline{i})^\sigma$, $u \mapsto (\phi(u))^\sigma$ is a $\mathcal{P}_n^{\mathfrak{c}}$-isomorphism. For each $\pi \in P_{\underline{j}}$, set

$$\begin{aligned} \phi^\pi : L(\underline{j})^\pi &\longrightarrow L(\underline{i})^{\pi\sigma} \\ u^\pi &\mapsto (\phi(u))^{\pi\sigma}. \end{aligned}$$

It is routine to check that

$$\oplus_{\pi\in P_{\underline{j}}} \phi^\pi : D^{\underline{j}} \longrightarrow D^{\underline{i}}$$

is a nonzero $\mathfrak{H}_n^{\mathfrak{c}}$-homomorphism. This means $D^{\underline{i}} \cong D^{\underline{j}}$ since both of them are irreducible.

(3). Suppose $M \in \mathrm{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}}$ is irreducible completely splittable with $M_{\underline{i}} \neq 0$ for some $\underline{i} \in \mathbb{I}^n$. By Proposition 3.6, there exists a $\mathcal{P}_n^{\mathfrak{c}}$-isomorphism $\psi : M_{\underline{i}} \to L(\underline{i})$. By

Lemma 4.4, for each $\tau \in P_{\underline{i}}$, there exists a bijection $\widehat{\Phi}_\tau : M_{\underline{i}} \to M_{\tau \cdot \underline{i}}$. Now for $\tau \in P_{\underline{i}}$, define

$$\psi^\tau : L(\underline{i})^\tau \longrightarrow M_{\tau \cdot \underline{i}}, \quad z^\tau \mapsto \widehat{\Phi}_\tau(\psi(z)).$$

By Lemma 4.4, the bijection $\widehat{\Phi}_\tau$ satisfies $\widehat{\Phi}_\tau x_k = x_{\tau(k)} \widehat{\Phi}_\tau$, $\widehat{\Phi}_\tau c_k = c_{\tau(k)} \widehat{\Phi}_\tau$ for $1 \le k \le n$. Hence for $z \in L(\underline{i})$, $\tau \in P_{\underline{i}}$ and $1 \le k \le n$,

$$\begin{aligned}\psi^\tau(x_k z^\tau) &= \psi^\tau((x_{\tau^{-1}(k)} z)^\tau) = \widehat{\Phi}_\tau(\psi(x_{\tau^{-1}(k)} z)) \\ &= \widehat{\Phi}_\tau(x_{\tau^{-1}(k)})\psi(z) = x_k \widehat{\Phi}_\tau(\psi(z)) = x_k \psi^\tau(z^\tau).\end{aligned}$$

Similarly one can show that $\psi^\tau(c_k z^\tau) = c_k \psi^\tau(z^\tau)$. Therefore ψ^τ is a $\mathcal{P}_n^{\mathfrak{c}}$-homomorphism. By Proposition 3.14 we have $W(\mathfrak{H}_n^{\mathfrak{c}}) \subseteq W'(\mathfrak{H}_n^{\mathfrak{c}})$ and hence $\underline{i} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$. By the fact that ψ^τ is a $\mathcal{P}_n^{\mathfrak{c}}$-module homomorphism for each $\tau \in P_{\underline{i}}$, one can easily check that

$$\oplus_{\tau \in P_{\underline{i}}} \psi^\tau : D^{\underline{i}} \longrightarrow M$$

is a $\mathfrak{H}_n^{\mathfrak{c}}$-module isomorphism.

Remark 4.6 Observe that Theorem 4.5 confirms a slightly modified version of [10, Conjecture 52]. Leclerc defined a completely splittable representation to be one on which the x_k^2, $1 \le k \le n$ act semisimply.

□

By Proposition 3.14 we have $W(\mathfrak{H}_n^{\mathfrak{c}}) \subseteq W'(\mathfrak{H}_n^{\mathfrak{c}})$. By Theorem 4.5 we obtain the following.

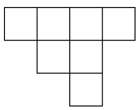
Corollary 4.7 *We have* $W(\mathfrak{H}_n^{\mathfrak{c}}) = W'(\mathfrak{H}_n^{\mathfrak{c}})$.

5 A diagrammatic classification

In this section, we shall give a reinterpretation of irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules in terms of Young diagrams.

Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of the integer $|\lambda| = \lambda_1 + \cdots + \lambda_l$, where $\lambda_1 \ge \cdots \ge \lambda_l \ge 1$. Denote by $l(\lambda)$ the number of nonzero parts in λ. It is known that the partition λ can be drawn as Young diagrams.

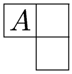
A strict partition λ (i.e. with distinct parts) can be identified with the shifted Young diagram which is obtained from the ordinary Young diagram by shifting the kth row to the right by $k-1$ squares, for all $k > 1$. For example, let $\lambda = (4, 2, 1)$, the corresponding shifted Young diagram is



From now on, we shall always identify strict partitions with their shifted Young diagrams. If λ and μ are strict partitions such that $\mu_k \le \lambda_k$ for all k, we write $\mu \subseteq \lambda$.

A *skew shifted Young diagram* λ/μ is defined to be the diagram obtained by removing the shifted Young diagram μ from λ for some strict partitions $\mu \subseteq \lambda$ (see examples below). Note that any skew shifted Young diagram is a union of connected components. Moreover, different pairs of strict partitions may give an identical skew shifted Young diagram.

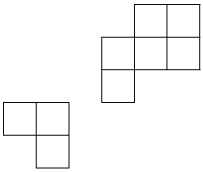
A *placed skew shifted Young diagram* $(c, \lambda/\mu)$ consists of a skew shifted Young diagram λ/μ and a *content function* $c : \{\text{boxes of } \lambda/\mu\} \longrightarrow \mathbb{Z}_+$ which is increasing from southwest to northeast in each connected component of λ/μ and satisfies the following:

(1) $c(A) = c(B)$, if and only if A and B are on the same diagonal,
(2) $c(A) = c(B) + 1$, if and only if A and B are on the adjacent diagonals,
(3) $c(A) = 0$, if the box A is located in λ/μ as 

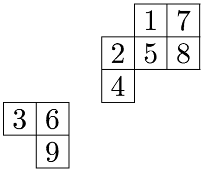
and there is no box below A.

A standard tableau of the shape λ/μ is a labeling of the skew shifted Young diagram λ/μ with the numbers $1, 2, \ldots, |\lambda| - |\mu|$ such that the numbers strictly increase from left to right along each row and down each column. If T is a tableau of the shape λ/μ, denote by $T(k)$ the box of λ/μ labeled by k in T for $1 \le k \le |\lambda| - |\mu|$.

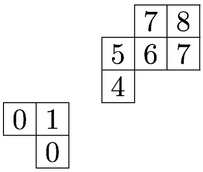
Example 5.1 Let $\lambda = (9, 8, 5, 2, 1)$ and $\mu = (7, 5, 4)$. The skew shifted Young diagram λ/μ is as follows:



A standard tableau T of shape λ/μ:



A placed skew shifted Young diagram $(c, \lambda/\mu)$:



satisfying $(c(T(1)), \ldots, c(T(9))) = (7, 5, 0, 4, 6, 1, 8, 7, 0)$.

Remark 5.2 For each shifted Young diagram λ, there exists one and only one content function c_λ defined by setting the contents of boxes on the first diagonal to be 0. Moreover, each placed skew shifted Young diagram can be obtained by removing a shifted Young diagram μ associated with c_μ from the shifted Young diagram λ associated with c_λ for some strict partitions $\mu \subseteq \lambda$.

If we modify the definition of placed skew shifted Young diagram by allowing non-integer contents and by adding that the difference between contents of two boxes is an integer if and only if they belong to the same connected component, then placed skew shifted Young diagrams may be used for the study of "non-integral" $\mathfrak{H}_n^{\mathfrak{c}}$-modules.

For each $n \in \mathbb{Z}_+$, denote by $\mathcal{PS}(n)$ the set of placed skew shifted Young diagrams with n boxes and set

$$\Delta(n) := \{((c, \lambda/\mu), T) \mid (c, \lambda/\mu) \in \mathcal{PS}(n), T \text{ is a standard tableau of shape } \lambda/\mu\}.$$

For each $((c, \lambda/\mu), T) \in \Delta(n)$, define

$$\mathcal{F}((c, \lambda/\mu), T) := (c(T(1)), \dots, c(T(n))). \tag{5.1}$$

A vector $\underline{i} \in \mathbb{Z}_+^n$ is said to be *splittable* if it satisfies that if $i_k = i_l = u$ for some $1 \le k < l \le n$ then $u = 0$ implies $1 \in \{i_{k+1}, \dots, i_{l-1}\}$ and $u \ge 1$ implies $\{i_k - 1, i_k + 1\} \subseteq \{i_{k+1}, \dots, i_{l-1}\}$. Denote by $\nabla(n)$ the subset of $\mathbb{I}^n$ consisting of splittable vectors.

Lemma 5.3 *The map $\mathcal{F}$ in (5.1) sends $\Delta(n)$ to $\nabla(n)$.*

Proof Suppose $((c, \lambda/\mu), T) \in \Delta(n)$, we need to show that $(c(T(1)), \dots, c(T(n)))$ is splittable. Suppose $c(T(k)) = c(T(l)) = u$ for some $1 \le k < l \le n$. Without loss of generality, we can assume that $u \notin \{c(T(k+1)), \dots, c(T(l-1))\}$. This means that there is a configuration in T of the form

$$\begin{array}{|c|c}\hline k & \multicolumn{1}{|c|}{} \\ \hline \multicolumn{1}{c|}{} & l \\ \cline{2-2}\end{array}.$$

Since $(c, \lambda/\mu)$ is a placed skew shifted Young diagram and T is standard, there exists a box labeled by j located in T as in the configuration

$$\begin{array}{|c|c|}\hline k & j \\ \hline \multicolumn{1}{c|}{} & l \\ \cline{2-2}\end{array}$$

for some $k < j < l$ and moreover $c(T(j)) = u + 1$. If $u = 0$, then there is no box below the box labeled by k and $c(T(j)) = 1$. This implies $1 \in \{c(T(k+1)), \dots, c(T(l-1))\}$. If $u \ge 1$, then there is a box labeled by t below the box labeled by k and $c(T(t)) = u - 1$, that is, T contains the following configuration

$$\begin{array}{|c|c|}\hline k & s \\ \hline t & l \\ \hline\end{array}$$

for some $k < s \neq t < l$. This implies that $\{u-1, u+1\} \subseteq \{c(T(k+1)), \ldots, c(T(l-1))\}$. Hence $(c(T(1)), \ldots, c(T(n))) \in \nabla(n)$. $\square$

Given $\underline{i} \in \nabla(n)$, by induction on n we can produce a pair $\mathcal{G}(\underline{i}) = ((c, \lambda/\mu), T) \in \Delta(n)$ satisfying $c(T(k)) = i_k$ for $1 \leq k \leq n$. If $n = 1$, let $\mathcal{G}(\underline{i})$ be a box labeled by 1 with content i_1. Assume inductively that $\mathcal{G}(\underline{i}') = ((c', \lambda'/\mu'), T') \in \Delta(n-1)$ is already defined, where $\underline{i}' = (i_1, \ldots, i_{n-1}) \in \nabla(n-1)$. Set $u = i_n$.
Case 1: $(c', \lambda'/\mu')$ contains neither a box with content $u-1$ nor a box with content $u+1$. Adding a new component consisting of one box labeled by n with content u to T', we obtain a new placed skew shifted Young diagram $(c, \lambda/\mu)$ and a standard tableau T of shape λ/μ. Set $\mathcal{G}(\underline{i}) = ((c, \lambda/\mu), T)$.
Case 2: $(c', \lambda'/\mu')$ contains boxes with content $u-1$ but no box with content $u+1$. This implies $u+1 \notin \{i_1, \ldots, i_n\}$. Since $(i_1, \ldots, i_n)$ is splittable, u does not appear in $\underline{i}'$ and hence $u-1$ appears only once in $\underline{i}'$ by Lemma 3.13. Therefore there is no box of content u and only one box denoted by A with content $u-1$ in $((c', \lambda'/\mu'), T')$. So we can add a new box labeled by n with content u to the right of A to obtain a new tableau T of shape $(c, \lambda/\mu)$. Set $\mathcal{G}(\underline{i}) = ((c, \lambda/\mu), T)$. Observe that there is no box above A in the column containing A since there is no box of content u in $((c', \lambda'/\mu'), T')$. Hence $\mathcal{G}(\underline{i}) \in \Delta(n)$.
Case 3: $(c', \lambda'/\mu')$ contains boxes with content $u+1$ but no box with content $u-1$. This implies $u-1 \notin \{i_1, \ldots, i_n\}$. Since $(i_1, \ldots, i_n)$ is splittable, u does not appear in $\underline{i}'$ and hence $u+1$ appears only once in $\underline{i}'$ by Lemma 3.13. Therefore $((c', \lambda'/\mu'), T')$ contains only one box denoted by B with content $u+1$ and no box with content u. This means there is no box below B in $((c', \lambda'/\mu'), T')$. Adding a new box labeled by n with content u below B, we obtain a new tableau T of shape $(c, \lambda/\mu)$. Set $\mathcal{G}(\underline{i}) = ((c, \lambda/\mu), T)$. Clearly $\mathcal{G}(\underline{i}) \in \Delta(n)$.
Case 4: $(c', \lambda'/\mu')$ contains boxes with contents $u-1$ and $u+1$. Let C and D be the last boxes on the diagonals with content $u-1$ and $u+1$, respectively. Suppose C is labeled by s and D is labeled by t. Then $i_s = u-1, i_t = u+1$ and moreover $u-1 \notin \{i_{t+1}, \ldots, i_{n-1}\}, u+1 \notin \{i_{s+1}, \ldots, i_{n-1}\}$. Since $i_n = u$, by Lemma 3.13 we see that $u \notin \{i_{t+1}, \ldots, i_{n-1}\}$ and $u \notin \{i_{s+1}, \ldots, i_{n-1}\}$. This implies that there is no box below C and no box to the right of D in $((c', \lambda'/\mu'), T')$. Moreover C and D must be of the following shape

$$\begin{array}{cc} & \boxed{C} \\ \boxed{D} & \end{array} .$$

Add a new box labeled by n to the right of D and below C to obtain a new tableau T of shape $(c, \lambda/\mu)$. Set $\mathcal{G}(\underline{i}) = ((c, \lambda/\mu), T)$. It is clear that $\mathcal{G}(\underline{i}) \in \Delta(n)$.

Therefore we obtain a map

$$\mathcal{G} : \nabla(n) \longrightarrow \Delta(n) \tag{5.2}$$

satisfying $\underline{i} = (c(T(1)), \ldots, c(T(n)))$ if $\mathcal{G}(\underline{i}) = ((c, \lambda/\mu), T)$. In this case, we will say that $\mathcal{G}(\underline{i})$ affords the placed skew shifted Young diagram $(c, \lambda/\mu)$.

Example 5.4 Suppose $n = 5$. The map $\mathcal{G}$ maps the splittable vector $\underline{i} = (1, 2, 0, 1, 0) \in \nabla(5)$ to the pair $((c, \lambda/\mu), T) \in \Delta(5)$ with

$$(c, \lambda/\mu) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 0 \\ \cline{2-2}\end{array}, \qquad T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 \\ \cline{2-2}\end{array}.$$

Proposition 5.5 *The map $\mathcal{G}$ in (5.2) is a bijection from $\nabla(n)$ to $\Delta(n)$ with inverse $\mathcal{F}$.*

Proof It is clear that $\mathcal{F} \circ \mathcal{G}(\underline{i}) = \underline{i}$ for any $\underline{i} \in \nabla(n)$ by (5.2). It remains to prove that $\mathcal{G} \circ \mathcal{F}((c, \lambda/\mu), T) = ((c, \lambda/\mu), T)$ for any $((c, \lambda), T) \in \Delta(n)$. We shall proceed by induction on n. Denote by A the box labeled by n in T. Removing A from $(c, \lambda/\mu)$ and T, we obtain a new pair $((c', \lambda'/\mu'), T') \in \Delta(n-1)$. By induction we see that

$$\mathcal{G} \circ \mathcal{F}(((c', \lambda'/\mu'), T')) = ((c', \lambda'/\mu'), T').$$

This means $\mathcal{G}((c(T(1)), \dots, c(T(n-1)))) = ((c', \lambda'/\mu'), T')$. By adding a box denoted by B labeled by n with content $c(T(n))$ to $((c', \lambda'/\mu'), T')$ by the procedure for defining $\mathcal{G}$, we obtain $\mathcal{G}((c(T(1)), \dots, c(T(n))))$. One can check case by case that B coincides with A and hence $\mathcal{G}((c(T(1)), \dots, c(T(n)))) = ((c, \lambda/\mu), T)$. This means $\mathcal{G} \circ \mathcal{F}((c, \lambda/\mu), T) = \mathcal{G}((c(T(1)), \dots, c(T(n)))) = ((c, \lambda/\mu), T)$. $\square$

Lemma 5.6 *Suppose $\underline{i}, \underline{j} \in \nabla(n)$. Then $\underline{i} \sim \underline{j}$ if and only if $\mathcal{G}(\underline{i})$ and $\mathcal{G}(\underline{j})$ afford the same placed skew shifted Young diagram.*

Proof Suppose that $\mathcal{G}(\underline{i})$ and $\mathcal{G}(\underline{j})$ afford the same placed skew shifted Young diagram $(c, \lambda/\mu)$. This means that there exist standard tableaux T and S of shape λ/μ such that $(i_1, \dots, i_n) = (c(T(1)), \dots, c(T(n)))$ and $(j_1, \dots, j_n) = (c(S(1)), \dots, c(S(n)))$. We shall prove $\underline{i} \sim \underline{j}$ by induction on n. Let T_0 be the tableau of shape λ/μ obtained by filling in the numbers $1, \dots, n$ from left to right along the rows, starting from the first row and going down. Clearly T_0 is standard and hence we have $(c(T_0(1)), \dots, c(T_0(n))) \in \nabla(n)$ by Lemma 5.3. Let A be the last box of the last row of λ/μ. Then in T_0, A is occupied by n. Suppose in T, A is occupied by the number k. Clearly $k+1$ and k do not lie on adjacent diagonals in T, hence the transposition s_k is admissible with respect to $\underline{i}$. So we can apply s_k to swap k and $k+1$, then to swap $k+1$ and $k+2$, and finally we obtain a new standard tableau T_1 in which A is occupied by n and moreover $\underline{i} \sim (c(T_1(1)), \dots, c(T_1(n)))$. Observe that A is occupied by n in both T_1 and T_0. Hence both $\mathcal{G}((c(T_1(1)), \dots, c(T_1(n-1))))$ and $\mathcal{G}((c(T_0(1)), \dots, c(T_0(n-1))))$ contains the placed skew shifted Young diagram obtained by removing A from $(c, \lambda/\mu)$. By induction we have $(c(T_1(1)), \dots, c(T_1(n-1)))$ is equivalent to $(c(T_0(1)), \dots, c(T_0(n-1)))$ and then $(c(T_1(1)), \dots, c(T_1(n))) \sim (c(T_0(1)), \dots, c(T_0(n)))$. Therefore we obtain $\underline{i} \sim (c(T_0(1)), \dots, c(T_0(n)))$. Similarly, we can apply the above argument to P to obtain $\underline{j} \sim (c(T_0(1)), \dots, c(T_0(n)))$. Hence $\underline{i} \sim \underline{j}$.

Conversely, it suffices to check the case when $\underline{j} = s_k \cdot \underline{i}$, where s_k is admissible with respect to $\underline{i}$ for some $1 \le k \le n-1$. This is reduced to show that $\mathcal{G}((i_1, \dots, i_{k-1}, i_k, i_{k+1}))$ and $\mathcal{G}((i_1, \dots, i_{k-1}, i_{k+1}, i_k))$ afford the same placed skew shifted Young diagram. Suppose $\mathcal{G}((i_1,\dots,i_{k-1}))$ affords the placed skew shifted Young diagram $(c,\lambda/\mu)$. Since s_k is admissible with respect to $\underline{i}$, we have $i_k \neq i_{k+1}\pm 1$ and hence the resulting placed skew shifted Young diagram obtained by adding two boxes with contents i_k, i_{k+1} in two different orders to $(c,\lambda/\mu)$ via the procedure for defining $\mathcal{G}$ are identical. □

5.1 A diagrammatic classification for $p=0$

In this subsection, we assume that $p=0$. By Proposition 3.14, $W'(\mathfrak{H}_n^{\mathfrak{c}})$ consists of all splittable vectors in $\mathbb{Z}_+^n$ and hence $W'(\mathfrak{H}_n^{\mathfrak{c}})=\nabla(n)$. Recall the definition of $\mathfrak{H}_n^{\mathfrak{c}}$-module $D^{\underline{i}}$ from Theorem 4.5 for $\underline{i}\in W'(\mathfrak{H}_n^{\mathfrak{c}})$. Suppose $(c,\lambda/\mu)\in\mathcal{PS}(n)$, by Proposition 5.5 there exists $\underline{i}\in W'(\mathfrak{H}_n^{\mathfrak{c}})$ such that $\mathcal{G}(\underline{i})$ affords $(c,\lambda/\mu)$. Let

$$D(c,\lambda/\mu)=D^{\underline{i}}. \tag{5.3}$$

Note that if $\underline{j}\in W'(\mathfrak{H}_n^{\mathfrak{c}})$ satisfies that $\mathcal{G}(\underline{j})$ also affords $(c,\lambda/\mu)$, then $\underline{i}\sim\underline{j}$ by Lemma 5.6 and hence the $\mathfrak{H}_n^{\mathfrak{c}}$-module $D(c,\lambda/\mu)$ is unique (up to isomorphism) by Theorem 4.5(2).

For $(c,\lambda/\mu)\in\mathcal{PS}(n)$, denote by $\gamma_0(c,\lambda/\mu)$ the number of boxes with content zero in $(c,\lambda/\mu)$ and let $f^{\lambda/\mu}$ be the number of standard tableaux of shape λ/μ.

The following is a Young diagrammatic reformulation of Theorem 4.5 for $p=0$.

Theorem 5.7 *Suppose that $(c,\lambda/\mu)\in\mathcal{PS}(n)$ and write $\gamma_0=\gamma_0(c,\lambda/\mu)$.*

(1) *$D(c,\lambda/\mu)$ is type* $\mathtt{M}$ *if γ_0 is even and is type* $\mathtt{Q}$ *if γ_0 is odd. Moreover,* $\dim D(c,\lambda/\mu)=2^{n-\lfloor\frac{\gamma_0}{2}\rfloor}f^{\lambda/\mu}$.

(2) *The $\mathfrak{H}_n^{\mathfrak{c}}$-modules $D(c,\lambda/\mu)$ for $(c,\lambda/\mu)\in\mathcal{PS}(n)$ form a complete set of pairwise non-isomorphic irreducible completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules in* $\mathrm{Rep}_{\mathbb{I}}\,\mathfrak{H}_n^{\mathfrak{c}}$.

Proof (1) Suppose $(c,\lambda/\mu)\in\mathcal{PS}(n)$ and $\mathcal{G}(\underline{i})$ affords $(c,\lambda/\mu)$ for some $\underline{i}\in W(\mathfrak{H}_n^{\mathfrak{c}})$. By Proposition 5.5, we have $\underline{i}=(c(T(1)),\dots,c(T(n)))$ and hence the number of $1\le k\le n$ with $i_k=0$ is equal to γ_0. This together with Lemma 2.4 and Theorem 4.5(1) shows that $D^{\underline{i}}$ is type $\mathtt{M}$ if γ_0 is even and is type $\mathtt{Q}$ if γ_0 is odd. Denote by $|P_{\underline{i}}|$ the number of elements contained in $P_{\underline{i}}$. By Lemma 5.6, there exists a one-to-one correspondence between the set of weights in $W(\mathfrak{H}_n^{\mathfrak{c}})$ equivalent to $\underline{i}$ and the set of standard tableaux of shape λ/μ. Hence $f^{\lambda/\mu}=|P_{\underline{i}}|$ by Lemma 4.1. This together with Lemma 2.4 and Theorem 4.5 shows that

$$\dim D(c,\lambda/\mu)=\dim D^{\underline{i}}=2^{n-\lfloor\frac{\gamma_0}{2}\rfloor}|P_{\underline{i}}|=2^{n-\lfloor\frac{\gamma_0}{2}\rfloor}f^{\lambda/\mu}.$$

(2) It follows from Proposition 5.5, Lemma 5.6 and Theorem 4.5. □

5.2 A diagrammatic classification for $p\ge 3$

In this subsection, we assume $p\ge 3$. Set

$$W_1(\mathfrak{H}_n^{\mathfrak{c}})=\Big\{\underline{i}\in W(\mathfrak{H}_n^{\mathfrak{c}})\ |i_k-1\in\{i_{k+1},\dots,i_{l-1}\}\text{ whenever}$$

$$1 \le i_k = i_l \le \frac{p-3}{2} \text{ with } 1 \le k < l \le n\Big\},$$

$$W_2(\mathfrak{H}_n^{\mathfrak{c}}) = \Big\{\underline{i} \in W(\mathfrak{H}_n^{\mathfrak{c}}) \mid \text{there exist } 1 \le k < l \le n \text{ such that } 1 \le i_k = i_l \le \frac{p-3}{2},$$
$$i_k - 1 \notin \{i_{k+1}, \dots, i_{l-1}\}\Big\}.$$

Observe that $W(\mathfrak{H}_n^{\mathfrak{c}})$ is the disjoint union of $W_1(\mathfrak{H}_n^{\mathfrak{c}})$ and $W_2(\mathfrak{H}_n^{\mathfrak{c}})$. Moreover if $\underline{i} \in W_k(\mathfrak{H}_n^{\mathfrak{c}})$ and $\underline{j} \sim \underline{i}$, then $\underline{j} \in W_k(\mathfrak{H}_n^{\mathfrak{c}})$ for $k = 1, 2$. For each $u \in \mathbb{Z}_+$ and $m \ge 1$, let $\mathcal{PS}_u(m)$ be the set of placed skew shifted Young diagrams $(c, \lambda/\mu)$ with m boxes such that the contents of boxes of λ/μ are smaller than or equal to u. For $n \in \mathbb{Z}_+$, set

$$\Delta_1(n) = \{((c, \lambda/\mu), T) \mid (c, \lambda/\mu) \in \mathcal{PS}_{\frac{p-1}{2}}(n),$$
$$T \text{ is a standard tableau of shape } \lambda/\mu\}.$$

By Lemma 3.13, we see that $W_1(\mathfrak{H}_n^{\mathfrak{c}}) \subseteq \nabla(n)$.

Proposition 5.8 *The restriction of the map $\mathcal{G}$ in (5.2) to $W_1(\mathfrak{H}_n^{\mathfrak{c}})$ gives a bijection $\mathcal{G}_1 : W_1(\mathfrak{H}_n^{\mathfrak{c}}) \to \Delta_1(n)$. Moreover, $\underline{i} \sim \underline{j} \in W_1(\mathfrak{H}_n^{\mathfrak{c}})$ if and only if $\mathcal{G}_1(\underline{i})$ and $\mathcal{G}_1(\underline{j})$ afford the same placed skew shifted Young diagram.*

Proof Observe that $W_1(\mathfrak{H}_n^{\mathfrak{c}})$ can be identified with the subset of $\nabla(n)$ consisting of splittable vectors whose parts are less than or equal to $\frac{p-1}{2}$. Hence by Proposition 5.5, the restriction $\mathcal{G}_1$ of the map $\mathcal{G}$ establishes a bijection between $W_1(\mathfrak{H}_n^{\mathfrak{c}})$ and $\Delta_1(n)$. Now the rest of the Proposition follows from Lemma 5.6. □

For each $\underline{i} \in W_2(\mathfrak{H}_n^{\mathfrak{c}})$, denote by $1 \le u_{\underline{i}} \le \frac{p-3}{2}$ the minimal integer such that there exist $1 \le k < l \le n$ satisfying $i_k = i_l = u_{\underline{i}}$ and $u_{\underline{i}} - 1 \notin \{i_{k+1}, \dots, i_{l-1}\}$. By the definition of $W_2(\mathfrak{H}_n^{\mathfrak{c}})$, we see that $u_{\underline{i}}$ always exists.

Lemma 5.9 *Let $\underline{i} \in W_2(\mathfrak{H}_n^{\mathfrak{c}})$ and write $u = u_{\underline{i}}$.*

(1) *There exists a unique sequence of integers $1 \le r_0 < r_1 < \ldots < r_{\frac{p-3}{2}-u} < q < t_{\frac{p-3}{2}-u} < \ldots < t_1 < t_0 \le n$ such that*
   (a) *$i_q = \frac{p-1}{2}, i_{r_j} = i_{t_j} = u + j$ for $0 \le j \le \frac{p-3}{2}$,*
   (b) *$i_a \ne u - 1$ for all $r_0 \le a \le t_0$,*
   (c) *$i_b \le u - 1$ for all $b \ne r_0, r_1, \dots, r_{\frac{p-3}{2}-u}, q, t_{\frac{p-3}{2}-u}, \dots, t_1, t_0$.*

(2) *$\underline{i} \sim (\underline{i}', u, u+1, \dots, \frac{p-3}{2}, \frac{p-1}{2}, \frac{p-3}{2}, \dots, u+1, u, u-1, \dots, u-m)$ for some $\underline{i}' \in \nabla(n - p + 2u - m)$ whose parts are less than u and some $0 \le m \le u$.*

Proof (1) By Proposition 3.14, there exists a sequence of integers $r_0 < r_1 < \cdots < r_{\frac{p-3}{2}-u} < q < t_{\frac{p-3}{2}-u} < \cdots < t_1 < t_0$ such that $i_q = \frac{p-1}{2}, i_{r_j} = i_{t_j} = u + j$, and $u + j$ does not appear between i_{r_j} and i_{t_j} in $\underline{i}$ for each $0 \le j \le \frac{p-3}{2} - u$. Hence it suffices to prove (1)(c). Assume that $i_b = u + k$ for some $0 \le k \le \frac{p-3}{2} - u$ and some $b \notin$

$\{r_0,\ldots,r_{\frac{p-3}{2}-u},q,t_{\frac{p-3}{2}-u},\ldots,t_0\}$. Since $u+k$ does not appear between i_{r_k} and i_{t_k}, we see that either $b<r_k$ or $b>t_k$. Now assume $b<r_k$. Since $i_b=u+k=i_{r_k}$, by Lemma 3.13 there exists b_1 with $b<b_1<r_k$ and $i_{b_1}=u+k+1$. Again since $i_{b_1}=u+k+1=i_{r_{k+1}}$, using Lemma 3.13 there exists b_2 with $b_1<b_2<r_{k+1}$ such that $i_{b_2}=u+k+2$. Continuing in this way, we finally obtain an integer f satisfying $f<r_{\frac{p-3}{2}-u-1}$ and $i_f=\frac{p-3}{2}$. By Lemma 3.13, $\frac{p-1}{2}$ appears between i_f and $i_{r_{\frac{p-3}{2}-u}}$. So $\frac{p-1}{2}$ appears at least twice in $\underline{i}$ since $i_q=\frac{p-3}{2}$ and $q>r_{\frac{p-3}{2}-u}$. This contradicts Proposition 3.14. An identical argument holds for the case when $b>t_k$. Therefore $i_b\le u-1$ for all $b\notin\{r_0,\ldots,r_{\frac{p-3}{2}-u},q,t_{\frac{p-3}{2}-u},\ldots,t_0\}$.

(2) As shown in (1), there exists a sequence of integers $r_0<r_1<\ldots<r_{\frac{p-3}{2}-u}<q<t_{\frac{p-3}{2}-u}<\ldots<t_1<t_0$ such that $i_q=\frac{p-1}{2}$, $i_{r_j}=i_{t_j}=u+j$ for $0\le j\le\frac{p-3}{2}$. If $u-1$ does not appear after i_{t_0} in $\underline{i}$, then $i_a\le u-2$ for all $a\in\{r_0,r_0+1,\ldots,n\}\setminus\{r_0,r_1,\ldots,r_{\frac{p-3}{2}-u},q,t_{\frac{p-3}{2}-u},\ldots,t_0\}$ by $(1)(c)$. By applying admissible transpositions we can swap i_k with i_l in $\underline{i}$ for all $k\in\{r_0,r_0+1,\ldots,n\}\setminus\{r_0,r_1,\ldots,r_{\frac{p-3}{2}-u},q,t_{\frac{p-3}{2}-u},\ldots,t_0\}$ and $l\in\{r_0,r_1,\ldots,r_{\frac{p-3}{2}-u},q,t_{\frac{p-3}{2}-u},\ldots,t_0\}$. Finally we obtain an element of the form $(\ldots,u,u+1,\ldots,\frac{p-3}{2},\frac{p-1}{2},\frac{p-3}{2},\ldots,u+1,u)$.

Now assume $u-1$ appears after i_{t_0} in $\underline{i}$. Since $i_b\neq u$ for all $b>t_0$ by $(1)(c)$, we see that $u-1$ appears at most once after i_{t_0} in $\underline{i}$ by Lemma 3.13. Therefore there exists a unique $l_1>t_0$ such that $i_{l_1}=u-1$. If $u-2$ does not appear after i_{l_1}, then $i_a\le u-3$ for all $a\in\{r_0,r_0+1,\ldots,n\}\setminus\{r_0,r_1,\ldots,r_{\frac{p-3}{2}-u},q,t_{\frac{p-3}{2}-u},\ldots,t_0,l_1\}$ by $(1)(c)$. Hence we can apply admissible transpositions to $\underline{i}$ to obtain an element of the form $(\ldots,u,u+1,\ldots,\frac{p-3}{2},\frac{p-1}{2},\frac{p-3}{2},\ldots,u+1,u,u-1)$.

Now assume $u-2$ appears after i_{l_1}. Since $u-1$ appears only once with $i_{l_1}=u-1$ after i_{t_0} in $\underline{i}$, we see that $i_b\neq u-1$ for all $b>l_1$ and hence $u-2$ appears at most once after i_{l_1} in $\underline{i}$ by Lemma 3.13. This means there exists a unique $l_2>l_1$ such that $i_{l_2}=u-2$. By repeating the above process, we arrive at the claim in (2). □

By Lemma 5.9, for $\underline{i}\in W_2(\mathfrak{H}_n^{\mathfrak{c}})$, there exists a unique vector $\widehat{\underline{i}}$ as follows

$$\widehat{\underline{i}}=(i_1,\ldots,i_{r_0},\ldots,\widehat{i_{r_1}},\ldots,\widehat{i_q},\ldots,\widehat{i_{t_1}},\ldots,\widehat{i_{t_0}},\ldots,i_n), \tag{5.4}$$

which is obtained by removing $i_{r_1},\ldots,i_{r_{\frac{p-3}{2}-u_{\underline{i}}}},i_q,i_{t_{\frac{p-3}{2}-u_{\underline{i}}}},\ldots,i_{t_1},i_{t_0}$ from $\underline{i}$.

Lemma 5.10 *The following holds for $\underline{i},\underline{j}\in W_2(\mathfrak{H}_n^{\mathfrak{c}})$.*

(1) *$\widehat{\underline{i}}$ has a unique part equal to $u_{\underline{i}}$ and all other parts are less than $u_{\underline{i}}$.*

(2) *$\widehat{\underline{i}}$ is splittable.*

(3) *$\widehat{\underline{i}}\sim\widehat{\underline{j}}$ if $\underline{i}\sim\underline{j}$.*

Proof (1). It follows from the definition of $\widehat{\underline{i}}$.

(2). Suppose $\underline{i}\in W_2(\mathfrak{H}_n^{\mathfrak{c}})$. By Lemma 5.9, there exists a unique sequence of integers $r_0<r_1<\ldots<r_{\frac{p-3}{2}-u}<q<t_{\frac{p-3}{2}-u}<\ldots<t_1<t_0$ such that $i_q=\frac{p-1}{2}$, $i_{r_j}=$

$i_{t_j} = u + j$ for $0 \le j \le \frac{p-3}{2}$ and $i_a \neq u-1, i_b \le u-1$ for all $r_0 \le a \le t_0$ and $b \neq r_0, r_1, \ldots, r_{\frac{p-3}{2}-u}, q, t_{\frac{p-3}{2}-u}, \ldots, t_1, t_0$. Assume $i_k = i_l = v$ for some $k < l \notin \{r_1, \ldots, r_{\frac{p-3}{2}-u}, q, t_{\frac{p-3}{2}-u}, \ldots, t_0\}$. To show $\widehat{\underline{i}}$ is splittable, we need to show that if $v = 0$ then 1 appears between i_k and i_l in $\widehat{\underline{i}}$ and if $1 \le v \le u-1$ then $v-1$ and $v+1$ appear between i_k and i_l in $\widehat{\underline{i}}$. One can easily check the case when $v = 0$. Now assume $v \ge 1$. If $1 \le v < u-1$, by the choice of u there exist $k < s, t < l$ such that $i_s = v-1$, $i_t = v+1$. Observe that $v-1 < u-2$, $v+1 \le u-1$. Hence $s, t \neq r_1, \ldots, r_{\frac{p-3}{2}-u}, q, t_{\frac{p-3}{2}-u}, \ldots, t_0$. This means $v-1$ and $v+1$ appear between i_k and i_l in $\widehat{\underline{i}}$. Now assume $v = u-1$. Since there are no parts equal to u before i_{r_0} and after i_{t_0} in $\underline{i}$, it follows from Lemma 3.13 that $u-1$ appears at most once before i_{r_0} and after i_{t_0} in $\underline{i}$, respectively. This together with the fact that $i_a \neq u-1$ for $r_0 < a < t_0$ shows $a < r_0$ and $b > t_0$. By the choice of u, there exists $a < c < b$ such that $i_c = u-2$. This together with $i_{r_0} = u$ shows that $v-1 = u-2$ and $v+1 = u$ appear between i_k and i_l in $\widehat{\underline{i}}$.

(2). It suffices to check the case when $\underline{j} = s_k \cdot \underline{i} \in W_2(\mathfrak{H}_n^{\mathfrak{c}})$, where s_k is admissible with respect to $\underline{i}$. If $\{k, k+1\} \cap \{r_1, \ldots, r_{\frac{p-3}{2}-u}, q, t_{\frac{p-3}{2}-u}, \ldots, t_1, t_0\} = \emptyset$, then $\widehat{\underline{j}} = s_k \cdot \widehat{\underline{i}}$ and hence $\widehat{\underline{j}} \sim \widehat{\underline{i}}$. Otherwise we see that $\widehat{\underline{j}} = \widehat{\underline{i}}$. □

Suppose $1 \le u \le \frac{p-3}{2}$ and $(c, \lambda/\mu) \in \mathcal{PS}_u(m)$ for some $m \in \mathbb{Z}_+$. Observe that there exists at most one box with content u in $(c, \lambda/\mu)$. Let us denote by $\mathcal{PS}_u^*(m) \subseteq \mathcal{PS}_u(m)$ the subset consisting of placed skew shifted Young diagrams which contain a unique box of content u. Suppose $(c, \lambda/\mu) \in \mathcal{PS}_u^*(m)$ and let $A_{(c,\lambda/\mu)}$ be the unique box of content u. Add $p - 2u - 1$ boxes to the right of $A_{(c,\lambda/\mu)}$ in the row containing $A_{(c,\lambda/\mu)}$ to obtain a skew shifted Young diagram denoted by $\overline{\lambda/\mu}$. A standard tableau of shape $\overline{\lambda/\mu}$ is said to be *p-standard* if it satisfies that if there exists a box below $A_{(c,\lambda/\mu)}$ then it is labeled by a number greater than the one in the last box in the row containing $A_{(c,\lambda/\mu)}$. For $n \in \mathbb{Z}_+$, set

$$\Delta_2(n) = \Big\{((c, \lambda/\mu), S) \,|\, (c, \lambda/\mu) \in \mathcal{PS}_u^*(n - p + 2u + 1), S \text{ is a } p\text{-standard tableau of shape } \overline{\lambda/\mu}, 1 \le u \le \frac{p-3}{2}\Big\}.$$

Suppose $\underline{i} \in W_2(\mathfrak{H}_n^{\mathfrak{c}})$ and set $u = u_{\underline{i}}$. By Lemma 5.9, there exists a unique sequence of integers $r_0 < r_1 < \ldots < r_{\frac{p-3}{2}-u} < q < t_{\frac{p-3}{2}-u} < \ldots < t_1 < t_0$ such that $i_q = \frac{p-1}{2}, i_{r_j} = i_{t_j} = u + j$ for $0 \le j \le \frac{p-3}{2}$. Since the vector $\widehat{\underline{i}}$ in (5.4) is splittable, by Lemma 5.10(2) we apply the map $\mathcal{G}$ in (5.2) to $\widehat{\underline{i}}$ to get a placed skew shifted Young diagram $(c, \lambda/\mu)$ and a standard tableau T of shape λ/μ whose boxes are labeled by $\{1, \ldots, n\} \setminus \{r_1, \ldots, r_{\frac{p-3}{2}-u}, \ldots, t_1, t_0\}$. By Lemma 5.10(1), we have $(c, \lambda/\mu) \in \mathcal{PS}_u^*(n - p + 2u + 1)$. Label the boxes in $\overline{\lambda/\mu}$ on the right of $A_{(c,\lambda/\mu)}$ by $r_1, \ldots, r_{\frac{p-3}{2}-u}, q, t_{\frac{p-3}{2}-u}, \ldots, t_1, t_0$ consecutively and denote the resulting tableau by S. Observe that $A_{(c,\lambda/\mu)}$ is labeled by r_0 and hence the row containing $A_{(c,\lambda/\mu)}$ in S is increasing. If there exists a box denoted by B below $A_{(c,\lambda/\mu)}$, then B has content $u-1$. Suppose B is labeled by e, then $i_e = u-1$ and $e > r_0$. Hence $e > t_0$ since

$i_k \neq u-1$ for $r_0 \leq k \leq t_0$. Therefore S is p-standard. Set $\mathcal{G}_2(\underline{i}) := ((c, \lambda/\mu), S)$. Hence we obtain a map

$$\mathcal{G}_2 : W_2(\mathfrak{H}_n^{\mathfrak{c}}) \longrightarrow \Delta_2(n). \tag{5.5}$$

If $\mathcal{G}_2(\underline{i}) = ((c, \lambda/\mu), S)$, we say $\mathcal{G}_2(\underline{i})$ affords the placed skew shifted Young diagram $(c, \lambda/\mu)$.

Example 5.11 Suppose $p = 7$ and $n = 7$. Note that the vector $\underline{i} = (1, 2, 0, 3, 2, 1, 0)$ belongs to $W_2(\mathfrak{H}_7^{\mathfrak{c}})$ with $u_{\underline{i}} = 2$ and $\widehat{\underline{i}} = (1, 2, 0, 1, 0)$. By Example 5.4, we see that the map $\mathcal{G}_2$ sends $\underline{i}$ to the pair $((c, \lambda/\mu), S)$ with

$$(c, \lambda/\mu) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 0 \\ \cline{2-2}\end{array}, \qquad \overline{\lambda/\mu} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \multicolumn{2}{c}{} \\ \cline{1-2} \multicolumn{1}{c|}{} & & \multicolumn{2}{c}{} \\ \cline{2-2}\end{array}, \qquad S = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \multicolumn{1}{c|}{} & 7 & \multicolumn{2}{c}{} \\ \cline{2-2}\end{array}.$$

On the other hand, suppose $(c, \lambda/\mu) \in \mathcal{PS}_u^*(n - p + 2u + 1)$ for $1 \leq u \leq \frac{p-3}{2}$ and S is p-standard tableau of shape $\overline{\lambda/\mu}$. Assume that the boxes on the right of $A_{(c,\lambda/\mu)}$ in S are labeled by $r_1, \ldots, r_{\frac{p-3}{2}-u}, q, t_{\frac{p-3}{2}-u}, \ldots, t_1, t_0$. Define the contents of these additional boxes by setting $c(S(q)) = \frac{p-1}{2}$, $c(S(t_0)) = u$ and $c(S(r_j)) = u + j = c(S(t_j))$ for $1 \leq j \leq \frac{p-3}{2} - u$. Set

$$\mathcal{F}_2((c, \lambda/\mu), S) := (c(S(1)), \ldots, c(S(n))). \tag{5.6}$$

Lemma 5.12 *The map $\mathcal{F}_2$ in (5.6) sends $\Delta_2(n)$ to $W_2(\mathfrak{H}_n^{\mathfrak{c}})$.*

Proof Suppose $((c, \lambda/\mu), S) \in \Delta_2(n)$ so that $(c, \lambda/\mu) \in \mathcal{PS}_u^*(n - p + 2u + 1)$ for $1 \leq u \leq \frac{p-3}{2}$. Assume that $c(S(k)) = c(S(l)) = v$ for some $1 \leq k < l \leq n$.

If $0 \leq v \leq u - 1$, both boxes $S(k)$ and $S(l)$ belong to $(c, \lambda/\mu)$. Hence $1 \in \{c(S(k+1)), \ldots, c(S(l-1))\}$ if $v = 0$, and $\{v-1, v+1\} \subseteq \{c(S(k+1)), \ldots, c(S(l-1))\}$ if $1 \leq v \leq u-1$. Now $v = u + m$ for some $0 \leq m \leq \frac{p-3}{2} - u$. Suppose the box $A_{(c,\lambda/\mu)}$ is labeled by r_0 and the boxes on its right in S are labeled by $r_1, \ldots, r_{\frac{p-3}{2}-u}, q, t_{\frac{p-3}{2}-u}, \ldots, t_1, t_0$. By the definition of $\mathcal{F}_2$, the boxes $S(k)$ and $S(l)$ coincide with $S(r_m)$ and $S(t_m)$, respectively. Therefore $(c(S(k+1)), \ldots, c(S(l-1)))$ contains the subsequence $(v+1, \ldots, \frac{p-3}{2}, \frac{p-1}{2}, \frac{p-3}{2}, \ldots, v+1)$, and $\mathcal{F}_2((c, \lambda/\mu), S) \in W_2(\mathfrak{H}_n^{\mathfrak{c}})$. □

Proposition 5.13 *The map $\mathcal{G}_2 : W_2(\mathfrak{H}_n^{\mathfrak{c}}) \to \Delta_2(n)$ is a bijection with inverse $\mathcal{F}_2$.*

Proof It is clear that $\mathcal{F}_2 \circ \mathcal{G}_2(\underline{i}) = \underline{i}$ for $\underline{i} \in W_2(\mathfrak{H}_n^{\mathfrak{c}})$. Conversely, suppose $(c, \lambda/\mu) \in \mathcal{PS}_u^*(n - p + 2u + 1)$ and S is a p-standard tableau of shape $\overline{\lambda/\mu}$ for some $1 \leq u \leq \frac{p-3}{2}$. Set $\underline{i} = \mathcal{F}_2((c, \lambda/\mu), S) \in W_2(\mathfrak{H}_n^{\mathfrak{c}})$. Denote by T the standard tableau of shape λ/μ obtained by removing the $p - 2u - 1$ boxes on the right of $A_{(c,\lambda/\mu)}$ from S. Suppose the boxes of T are labeled by $l_1 < l_2 < \cdots < l_{n-p+2u+1}$. By the definition of $\mathcal{F}_2$ we have

$$\widehat{\underline{i}} = (c(T(l_1)), \ldots, c(T(l_{n-p+2u+1}))) = \mathcal{F}((c, \lambda/\mu), T).$$

Therefore $\mathcal{G}(\widehat{\underline{i}}) = \mathcal{G} \circ \mathcal{F}((c, \lambda/\mu), T) = ((c, \lambda/\mu), T)$ by Proposition 5.5. Note that $\mathcal{G}_2(\underline{i})$ is obtained by adding $p-2u-1$ boxes labeled by $\{1, \ldots\} \setminus \{l_1, \ldots, l_{n-p+2u+1}\}$ to the right of $A_{(c,\lambda/\mu)}$ in T. Since T is obtained by removing the $p-2u-1$ boxes on the right of $A_{(c,\lambda/\mu)}$ from S, $\mathcal{G}_2(\underline{i}) = ((c, \lambda/\mu), S)$ and hence $\mathcal{G}_2 \circ \mathcal{F}_2((c, \lambda/\mu), S) = ((c, \lambda/\mu), S)$. □

Lemma 5.14 *$\underline{i} \sim \underline{j} \in W_2(\mathfrak{H}_n^{\mathfrak{c}})$ if and only if $\mathcal{G}_2(\underline{i})$ and $\mathcal{G}_2(\underline{j})$ afford the same placed skew shifted Young diagram in $\mathcal{PS}_u^*(n-p+2u+1)$ for some $1 \le u \le \frac{p-3}{2}$.*

Proof By Lemma 5.10, if $\underline{i} \sim \underline{j}$, then $\widehat{\underline{i}} \sim \widehat{\underline{j}}$. By Lemma 5.6, $\mathcal{G}(\widehat{\underline{i}})$ and $\mathcal{G}(\widehat{\underline{j}})$ afford the same skew shifted Young diagram. Hence $\mathcal{G}_2(\underline{i})$ and $\mathcal{G}_2(\underline{j})$ afford the same placed skew shifted Young diagram $(c, \lambda/\mu) \in \mathcal{PS}_u^*(n-p+2u+1)$ for some $1 \le u \le \frac{p-3}{2}$.

Conversely, suppose $\mathcal{G}_2(\underline{i})$ and $\mathcal{G}_2(\underline{j})$ afford the same placed skew shifted Young diagram $(c, \lambda/\mu) \in \mathcal{PS}_u^*(n-p+2u+1)$ for some $1 \le u \le \frac{p-3}{2}$. Suppose there are m boxes below $A_{(c,\lambda/\mu)}$ in $(c, \lambda/\mu)$. By Lemma 5.9, we see that

$$\underline{i} \sim (\underline{i}', u, u+1, \cdots, \frac{p-3}{2}, \frac{p-1}{2}, \frac{p-3}{2}, \cdots, u+1, u, u-1, \cdots, u-m), \tag{5.7}$$

$$\underline{j} \sim (\underline{j}', u, u+1, \cdots, \frac{p-3}{2}, \frac{p-1}{2}, \frac{p-3}{2}, \cdots, u+1, u, u-1, \cdots, u-m). \tag{5.8}$$

for some $\underline{i}', \underline{j}' \in \nabla(n-p+2u-m)$. This together with Lemma 5.10 shows that

$$\widehat{\underline{i}} \sim (\underline{i}', u, u-1, \cdots, u-m)$$
$$\widehat{\underline{j}} \sim (\underline{j}', u, u-1, \cdots, u-m).$$

Observe that $\mathcal{G}(\widehat{\underline{i}})$ and $\mathcal{G}(\widehat{\underline{j}})$ afford the placed skew shifted Young diagram $(c, \lambda/\mu)$. Therefore $\mathcal{G}(\underline{i}')$ and $\mathcal{G}(\underline{j}')$ afford the same placed skew shifted Young diagram and hence $\underline{i}' \sim \underline{j}'$ by Lemma 5.6. This together with (5.7) and (5.8) shows that $\underline{i} \sim \underline{j}$. □

Suppose $(c, \lambda/\mu) \in \mathcal{PS}_{\frac{p-1}{2}}(n) \cup (\cup_{1 \le u \le \frac{p-3}{2}} \mathcal{PS}_u^*(n-p+2u+1))$. By Proposition 5.8 and Proposition 5.13, there exists $\underline{i} \in W_k(\mathfrak{H}_n^{\mathfrak{c}})$ such that $\mathcal{G}_k(\underline{i})$ affords $(c, \lambda/\mu)$ for $k = 1, 2$. Let

$$D_p(c, \lambda/\mu) := D^{\underline{i}}. \tag{5.9}$$

Note that if there exists $\underline{j} \in W'(\mathfrak{H}_n^{\mathfrak{c}})$ satisfying that $\mathcal{G}_k(\underline{j})$ also affords $(c, \lambda/\mu)$ for $k = 1, 2$, then $\underline{i} \sim \underline{j}$ by Proposition 5.8 and Lemma 5.14 and hence the $\mathfrak{H}_n^{\mathfrak{c}}$-module $D_p(c, \lambda/\mu)$ is unique (up to isomorphism) by Theorem 4.5(2).

For $(c, \lambda/\mu) \in \mathcal{PS}_{\frac{p-1}{2}}(n) \cup (\cup_{1 \le u \le \frac{p-3}{2}} \mathcal{PS}_u^*(n-p+2u+1))$, denote by $\gamma_0(c, \lambda/\mu)$ the number of boxes with content zero in $(c, \lambda/\mu)$. If $(c, \lambda/\mu) \in \mathcal{PS}_{\frac{p-1}{2}}(n)$, set $f^{\lambda/\mu}$ to be the number of standard tableaux of shape λ/μ. If

$(c,\lambda/\mu)\in\cup_{1\le u\le\frac{p-3}{2}}\mathcal{PS}^*_u(n-p+2u+1)$, let $f_p^{\lambda/\mu}$ be the number of p-standard tableaux of shape $\overline{\lambda/\mu}$.

The following is a Young diagrammatic reformulation of Theorem 4.5 for $p\ge 3$.

Theorem 5.15 *Suppose* $(c,\lambda/\mu)\in\mathcal{PS}_{\frac{p-1}{2}}(n)\cup(\cup_{1\le u\le\frac{p-3}{2}}\mathcal{PS}^*_u(n-p+2u+1))$ *and write* $\gamma_0=\gamma_0(c,\lambda/\mu)$. *Then,*

(1) $D_p(c,\lambda/\mu)$ *is type* $\mathtt{M}$ *if* γ_0 *is even and is type* $\mathtt{Q}$ *if* γ_0 *is odd. Moreover if* $(c,\lambda/\mu)\in\mathcal{PS}_{\frac{p-1}{2}}(n)$, *then* $\dim D_p(c,\lambda/\mu)=2^{n-\lfloor\frac{\gamma_0}{2}\rfloor}f^{\lambda/\mu}$; *if* $(c,\lambda/\mu)\in(\cup_{1\le u\le\frac{p-3}{2}}\mathcal{PS}^*_u(n-p+2u+1))$, *then* $\dim D_p(c,\lambda/\mu)=2^{n-\lfloor\frac{\gamma_0}{2}\rfloor}f_p^{\lambda/\mu}$.

(2) *The* $\mathfrak{H}^{\mathfrak{c}}_n$*-modules* $D_p(c,\lambda/\mu)$ *for* $(c,\lambda/\mu)\in\mathcal{PS}_{\frac{p-1}{2}}(n)\cup(\cup_{1\le u\le\frac{p-3}{2}}\mathcal{PS}^*_u(n-p+2u+1))$ *form a set of pairwise non-isomorphic irreducible completely splittable* $\mathfrak{H}^{\mathfrak{c}}_n$*-modules in* $\mathrm{Rep}_{\mathbb{I}}\,\mathfrak{H}^{\mathfrak{c}}_n$.

Proof (1) By Proposition 5.8 and Proposition 5.13, the number of $1\le k\le n$ with $i_k=0$ equals to γ_0. Hence by Lemma 2.4 and Theorem 4.5, $D_p(c,\lambda/\mu)$ is type $\mathtt{M}$ if γ_0 is even and is type $\mathtt{Q}$ if γ_0 is odd.

Set $|P_{\underline{i}}|$ to be the number of elements contained in $P_{\underline{i}}$. If $(c,\lambda/\mu)\in\mathcal{PS}_{\frac{p-1}{2}}(n)$, then by Proposition 5.8 there exists a one-to-one correspondence between the set of weights in $W(\mathfrak{H}^{\mathfrak{c}}_n)$ equivalent to $\underline{i}$ and the set of standard tableaux of shape λ/μ. This implies $|P_{\underline{i}}|=f^{\lambda/\mu}$ by Lemma 4.1. If $(c,\lambda/\mu)\in\mathcal{PS}^*_u(n-p+2u+1)$ for some $1\le u\le\frac{p-3}{2}$. By Lemma 5.14, there exists a one-to-one correspondence between the set of weights in $W(\mathfrak{H}^{\mathfrak{c}}_n)$ equivalent to $\underline{i}$ and the set of splittable standard tableaux of shape $\overline{\lambda/\mu}$. This implies $|P_{\underline{i}}|=f_p^{\lambda/\mu}$ by Lemma 4.1. Now the Proposition follows from Lemma 2.4 and Theorem 4.5.

(2) It follows from Proposition 5.8, Proposition 5.13, Lemma 5.14 and Theorem 4.5(3). □

Remark 5.16 Note that for fixed $p\ge 3$, $\mathcal{PS}_{\frac{p-1}{2}}(n)\ne\emptyset$ if and only if $n\le\frac{(p+1)(p+3)}{8}$. Moreover if $n>\frac{(p+1)(p+3)}{8}$, then $\mathcal{PS}^*_u(n-p+2u+1)=\emptyset$ for $1\le u\le\frac{p-3}{2}$. Hence there is no irreducible completely splittable supermodule in $\mathrm{Rep}_{\mathbb{I}}\,\mathfrak{H}^{\mathfrak{c}}_n$ if $n>\frac{(p+1)(p+3)}{8}$ for fixed $p\ge 3$.

6 Completely splittable representations of finite Hecke-Clifford algebras

Denote by $\mathcal{C}_n$ the subalgebra of $\mathfrak{H}^{\mathfrak{c}}_n$ generated by $c_1,\dots,c_n$, which is known as the Clifford algebra. The finite Hecke-Clifford algebra $\mathcal{Y}_n=\mathcal{C}_n\rtimes\mathbb{F}S_n$ is isomorphic to the subalgebra of $\mathfrak{H}^{\mathfrak{c}}_n$ generated by $c_1,\dots,c_n,s_1,\dots,s_{n-1}$. The Jucys-Murphy elements $L_k(1\le k\le n)$ in $\mathcal{Y}_n$ are defined as

$$L_k=\sum_{1\le j<k}(1+c_jc_k)(jk),\tag{6.1}$$

where (jk) is the transposition exchanging j and k and keeping all others fixed.

Definition 6.1 A $\mathcal{Y}_n$-module is called *completely splittable* if the Jucys-Murphy elements $L_k (1 \leq k \leq n)$ act semisimply.

It is well known that there exists a surjective homomorphism

$$\mathcal{F} : \mathfrak{H}_n^{\mathfrak{c}} \to \mathcal{Y}_n$$
$$c_k \mapsto c_k, s_l \mapsto s_l, x_k \mapsto L_k, \quad (1 \leq k \leq n, 1 \leq l \leq n-1)$$

whose kernel coincides with the ideal of $\mathfrak{H}_n^{\mathfrak{c}}$ generated by x_1. Hence the category of finite dimensional $\mathcal{Y}_n$-modules can be identified as the category of finite dimensional $\mathfrak{H}_n^{\mathfrak{c}}$-modules which are annihilated by x_1. By [1, Lemma 4.4] (cf. [8, Lemma 15.1.2]), a $\mathfrak{H}_n^{\mathfrak{c}}$-module M belongs to the category $\mathrm{Rep}_{\mathbb{I}}\mathfrak{H}_n^{\mathfrak{c}}$ if all of eigenvalues of x_j on M are of the form $q(i)$ for some $1 \leq j \leq n$. Hence the category of finite dimensional completely splittable $\mathcal{Y}_n$-module can be identified with the subcategory of $\mathrm{Rep}_{\mathbb{I}}\mathfrak{H}_n^{\mathfrak{c}}$ consisting of completely splittable $\mathfrak{H}_n^{\mathfrak{c}}$-modules on which $x_1 = 0$. By (3.1), we can decompose any finite dimensional $\mathcal{Y}_n$-module M as

$$M = \oplus_{\underline{i} \in \mathbb{I}^n} M_{\underline{i}},$$

where $M_{\underline{i}} = \{z \in M \mid (L_k^2 - q(i_k))^N z = 0, \text{ for } N \gg 0, 1 \leq k \leq n\}$. If $M_{\underline{i}} \neq 0$, then $\underline{i}$ is called a *weight* of M.

Definition 6.2 Define $W(\mathcal{Y}_n)$ to be the set of weights $\underline{i} = (i_1, \ldots, i_n) \in W(\mathfrak{H}_n^{\mathfrak{c}})$ satisfying the following additional conditions:

$$i_1 = 0, \quad \{i_k - 1, i_k + 1\} \cap \{i_1, \ldots, i_{k-1}\} \neq \emptyset \text{ for } 2 \leq k \leq n. \tag{6.2}$$

Proposition 6.3 *$W(\mathcal{Y}_n)$ is the set of weights occurring in irreducible completely splittable $\mathcal{Y}_n$-modules.*

Proof Suppose $\underline{i}$ occurs in some irreducible completely splittable representation M of $\mathcal{Y}_n$, then $i_1 = 0$ since $L_1 = 0$ on M. For $2 \leq k \leq n$, if $i_k = 0$, then by Lemma 3.13 we have $1 \in \{i_1, \ldots, i_{k-1}\}$ and hence $\{i_k - 1, i_k + 1\} \cap \{i_1, \ldots, i_{k-1}\} \neq \emptyset$. Now assume $i_k \geq 1$ and suppose $\{i_k - 1, i_k + 1\} \cap \{i_1, \ldots, i_{k-1}\} = \emptyset$. Then s_l is admissible with respect to $s_{l+1} \cdots s_{k-1} \cdot \underline{i}$ for $1 \leq l \leq k-1$ and hence $M_{s_1 \cdots s_{k-1} \cdot \underline{i}} \neq 0$. Set $\underline{j} = s_1 \cdots s_{k-1} \cdot \underline{i}$. Note $j_1 = i_k \neq 0$ and this contradicts the fact that $L_1 = 0$ on M.

Conversely, let $\underline{i} \in W(\mathcal{Y}_n)$. Recall $P_{\underline{i}}$ and $D^{\underline{i}}$ from (4.1) and (4.8), respectively. It can be easily checked that $\tau \cdot \underline{i} \in W(\mathcal{Y}_n)$ for each $\tau \in P_{\underline{i}}$ and hence $x_1 = 0$ on $D^{\underline{i}}$. This implies that $D^{\underline{i}}$ can be factored through the surjective map $\mathcal{F}$ and hence it gives an irreducible completely splittable $\mathcal{Y}_n$-module. The Proposition follows from the fact that $\underline{i}$ is a weight of $D^{\underline{i}}$. □

Denote by $\nabla^\circ(n)$ the subset of $\nabla(n)$ consisting of $\underline{i}$ satisfying (6.2).

Lemma 6.4 *The restriction $\mathcal{G}^\circ$ of the map $\mathcal{G}$ in (5.2) induces a bijection between $\nabla^\circ(n)$ and the set of pairs (λ, T) of strict partitions λ and standard tableaux T of shape λ.*

Proof Let us proceed by induction on n. Clearly the statement holds for $n = 1$. Let $\underline{i} \in \nabla^\circ(n)$. Then $\underline{i}' := (i_1, \dots, i_{n-1}) \in \nabla^\circ(n-1)$ and by induction we have $\mathcal{G}(\underline{i}') = (\widetilde{\lambda}, S)$ for some shifted Young diagram $\widetilde{\lambda}$ with $n-1$ boxes and a standard tableau S of shape $\widetilde{\lambda}$. Note that $\mathcal{G}(\underline{i})$ is obtained by adding a box labeled by n to the diagonal of content i_n in S. Since $\{i_n - 1, i_n + 1\} \cap \{i_1, \dots, i_{n-1}\} \neq \emptyset$, the resulting diagram is still a shifted Young diagram. □

Note that if $p = 0$, then $W(\mathcal{Y}_n)$ coincides with $\nabla^\circ(n)$. Hence by Theorem 5.7 we have the following which recovers Nazarov's result in [13].

Corollary 6.5 *Suppose that $p = 0$ and that λ is a strict partition of n. Then,*

(1) *There exists an irreducible $\mathcal{Y}_n$-module $D(\lambda)$ satisfying that* $\dim D(\lambda) = 2^{n - \lfloor \frac{l(\lambda)}{2} \rfloor} f^\lambda$, *where f^λ is the number of standard λ-tableaux. Moreover, $D(\lambda)$ is type* $\mathtt{M}$ *if $l(\lambda)$ is even and is type* $\mathtt{Q}$ *if $l(\lambda)$ is odd.*

(2) *The set of shifted Young diagrams with n boxes parameterizes the irreducible completely splittable $\mathcal{Y}_n$-modules.*

Proof Suppose λ is a strict partition of n. Recall the content function c_λ from Remark 5.2. Note that $(c_\lambda, \lambda) \in \mathcal{PS}(n)$. Recall the $\mathfrak{H}_n^{\mathfrak{c}}$-module $D(c_\lambda, \lambda)$ from (5.3) and let

$$D(\lambda) = D(c_\lambda, \lambda).$$

Now the Proposition follows from Theorem 5.7. □

In the remaining part of this section, let us assume that $p \geq 3$. Set $W_k(\mathcal{Y}_n) := W(\mathcal{Y}_n) \cap W_k(\mathfrak{H}_n^{\mathfrak{c}})$ for $k = 1, 2$.

Lemma 6.6 *The restriction $\mathcal{G}_1^\circ$ of $\mathcal{G}_1$ to $W_1(\mathcal{Y}_n)$ gives a bijection from $W_1(\mathcal{Y}_n)$ to the set of pairs (λ, T) of strict partitions λ of n boxes whose first part is less than or equal to $\frac{p+1}{2}$ and standard tableaux T of shape λ.*

Proof Observe that $W_1(\mathcal{Y}_n) \subseteq \nabla^\circ(n)$. By Lemma 6.4 and Proposition 5.8, there exists a one-to-one correspondence between $W(\mathcal{Y}_n) \cap W_1(\mathfrak{H}_n^{\mathfrak{c}})$ and the set consisting of pairs of shifted Young diagrams $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{PS}_{\frac{p-1}{2}}(n)$ and standard tableaux of shape λ with $c(T(k)) = i_k$ for each $1 \leq k \leq n$. Suppose the last box in the first row of T is labeled by l, then $c(T(l)) = \lambda_1 - 1$ and hence $\lambda_1 \leq \frac{p+1}{2}$ since $c(T(l)) = i_l \leq \frac{p-1}{2}$. □

Lemma 6.7 *The restriction $\mathcal{G}_2^\circ$ of the map $\mathcal{G}_2$ to $W_2(\mathcal{Y}_n)$ gives a bijection from $W_2(\mathfrak{H}_n^{\mathfrak{c}})$ to the set consisting of pairs (λ, T), where λ is a strict partition whose first part is equal to $p - u$ and second part is less than or equal to u for some $1 \leq u \leq \frac{p-3}{2}$, and T is a standard tableau of shape λ satisfying that if $\lambda_2 = u$ then the number in last box of the second row is greater than the number in the last box of the first row in T.*

Proof Suppose $\underline{i} \in W_2(\mathcal{Y}_n)$. It is clear that $\widehat{\underline{i}} \in \nabla^\circ(n-p+2u+1)$ for some $1 \le u \le \frac{p-3}{2}$. By Lemma 6.4, we have $\mathcal{G}^\circ(\widehat{\underline{i}}) = (\mu, S)$ for some shifted Young diagram $\mu \in \mathcal{PS}^*_u(n-p+2u+1)$ and splittable standard tableau S of shape μ. Suppose $\mu = (\mu_1, \dots, \mu_m)$ and set $\lambda = (\lambda_1, \dots, \lambda_m) := \overline{\mu}$. Observe that the last box in the first row of μ has content u. This implies $\mu_1 - 1 = u$ and hence $\mu_1 = u+1$, $\mu_2 \le u$. Therefore $\lambda_1 = \mu_1 + p - 2u - 1 = p - u$ and $\lambda_2 = \mu_2 \le u$. Note that if $\lambda_2 < u$, then the set of splittable standard tableaux of shape λ coincides with the set of standard λ-tableaux; otherwise the set of splittable standard tableaux of shape λ coincides with the set of standard λ-tableaux in which the number in last box in the second row is greater than the number in the last box in the first row. □

If $\lambda = (\lambda_1, \dots, \lambda_l)$ is strict partition of n satisfying $\lambda_1 \le \frac{p-3}{2}$, then $(c_\lambda, \lambda) \in \mathcal{PS}_{\frac{p-1}{2}}(n)$, where c_λ is the unique content function on λ by Remark 5.2. Recall the $\mathfrak{H}^{\mathfrak{c}}_n$-module $D_p(c_\lambda, \lambda)$ from (5.9) and let

$$D_p(\lambda) = D_p(c_\lambda, \lambda).$$

Let f^λ be the number of standard tableaux of shape λ. Recall $f^{\lambda/\emptyset}$ and $\gamma_0(c_\lambda, \lambda)$ from Theorem 5.15. Clearly $f^\lambda = f^{\lambda/\emptyset}$ and moreover $\gamma_0(c_\lambda, \lambda) = l(\lambda)$.

If $\lambda = (\lambda_1, \dots, \lambda_l)$ is a strict partition of n satisfying $\lambda_1 = p-u$ and $\lambda_2 \le u$ for some $1 \le u \le \frac{p-3}{2}$. Denote by $\widehat{\lambda}$ the strict partition obtained by removing the last $p-2u-1$ boxes in the first row of λ. Recall $c_{\widehat{\lambda}}$ from Remark 5.2. Note that $(c_{\widehat{\lambda}}, \widehat{\lambda}) \in \cup_{1 \le u \le \frac{p-3}{2}} \mathcal{PS}^*_u(n-p+2u+1)$. Recall the $\mathfrak{H}^{\mathfrak{c}}_n$-module $D_p(c_{\widehat{\lambda}}, \widehat{\lambda})$ from (5.9) and let

$$D_p(\lambda) = D_p(c_{\widehat{\lambda}}, \widehat{\lambda}).$$

Let f^λ_p be the number of standard λ-tableau T if $\lambda_1 = p-u$, $\lambda_2 < u$ for some $1 \le u \le \frac{p-3}{2}$; if $\lambda_1 = p-u$, $\lambda_2 = u$ for some $1 \le u \le \frac{p-3}{2}$ let f^λ_p be the number of standard λ-tableau T in which the number in last box of the second row is greater than the number in the last box of the first row. Recall $f^{\widehat{\lambda}/\emptyset}$ and $\gamma_0(c_{\widehat{\lambda}}, \widehat{\lambda})$ from Theorem 5.15. One can easily check that $f^\lambda_p = f^{\widehat{\lambda}/\emptyset}_p$ and moreover $\gamma_0(c_{\widehat{\lambda}}, \widehat{\lambda}) = l(\lambda)$.

Combining the above observations and Lemma 6.6, Lemma 6.7 and Theorem 5.15, we have the following.

Theorem 6.8 *Let $p \ge 3$. Suppose that $\lambda = (\lambda_1, \dots, \lambda_m)$ is strict partition with n boxes satisfying either ($\lambda_1 = p-u$ and $\lambda_2 \le u$ for some $1 \le u \le \frac{p-3}{2}$) or ($\lambda_1 \le \frac{p+1}{2}$).*

(1) $D_p(\lambda)$ is type $\mathtt{M}$ if $l(\lambda)$ is even and is type $\mathtt{Q}$ if $l(\lambda)$ is odd. If $\lambda_1 \le \frac{p+1}{2}$, then $\dim D_p(\lambda) = 2^{n - \lfloor \frac{l(\lambda)}{2} \rfloor} f^\lambda$; *if $\lambda_1 = p-u$ and $\lambda_2 \le u$, then* $\dim D_p(\lambda) = 2^{n-\lfloor \frac{l(\lambda)}{2} \rfloor} f^\lambda_p$.

(2) The $\mathcal{Y}_n$-modules $D_p(\lambda)$ for strict partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ with n boxes satisfying either ($\lambda_1 = p-u, \lambda_2 \le u$ for some $1 \le u \le \frac{p-3}{2}$) or ($\lambda_1 \le \frac{p+1}{2}$) form a complete set of non-isomorphic irreducible completely splittable $\mathcal{Y}_n$-modules.

Remark 6.9 (1) A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is called p-restricted p-strict if p divides λ_r whenever $\lambda_r = \lambda_{r+1}$ for $r \ge 1$ and in addition $\lambda_r - \lambda_{r+1} < p$ if $p \mid \lambda_r$ and $\lambda_r - \lambda_{r+1} \le p$ if $p \nmid \lambda_r$ (cf. [1, §9-a]). It is known from [1, §9-b] that there exists an irreducible $\mathcal{Y}_n$-module $M(\lambda)$ associated to each p-restricted p-strict partition λ of n and moreover $\{M(\lambda) \mid \lambda \text{ is a } p\text{-restricted } p\text{-strict partition of } n\}$ forms a complete set of pairwise non-isomorphic irreducible $\mathcal{Y}_n$-modules. If λ is a strict partition with either $\lambda_1 = p - u$ and $\lambda_2 \le u$ for some $1 \le u \le \frac{p-3}{2}$ or $\lambda_1 \le \frac{p+1}{2}$, then λ is p-restricted p-strict and moreover $D(\lambda) \cong M(\lambda)$ by claiming that they have the same set of weights.

(2) It is well known that the representation theory of the spin symmetric group algebra $\mathbb{F}S_n^-$ is essentially equivalent to that of $\mathcal{Y}_n$ due to the isomorphism $\mathcal{C}_n \otimes \mathbb{F}S_n^- \cong \mathcal{Y}_n$. Applying the representation theory of $\mathcal{Y}_n$ established so far, we can obtain a family of irreducible representations of the spin symmetric group algebra $\mathbb{F}S_n^-$ for which dimensions and characters can be explicitly described. Over the complex field $\mathbb{C}$, these modules were originally constructed by Nazarov in [13].

7 A larger category

Recall that $\mathcal{C}_n$ is the Clifford algebra generated by $c_1, \dots, c_n$ subject to the relation (2.4) and $\mathcal{Y}_n = \mathcal{C}_n \rtimes \mathbb{F}S_n$. The basic spin $\mathcal{Y}_n$-module $I(n)$ (cf. [1, (9.11)]) is defined by

$$I(n) := \operatorname{ind}_{\mathbb{F}S_n}^{\mathcal{Y}_n} \mathbf{1}, \tag{7.1}$$

where $\mathbf{1}$ is the trivial 1-dimensional $\mathbb{F}S_n$-module. Note that $\{c_1^{\alpha_1} \cdots c_n^{\alpha_n} \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_2^n\}$ forms a basis of $I(n)$. It can be easily checked that each element $c_1^{\alpha_1} \cdots c_n^{\alpha_n}$ is a simultaneous eigenvector for $L_1^2, \dots, L_n^2$. Hence all L_k^2, $1 \le k \le n$, act semisimply on $I(n)$. Define the p-restricted p-strict partition ω_n by

$$\omega_n = \begin{cases} (p^a, b), & \text{if } b \neq 0 \\ (p^{a-1}, p-1, 1), & \text{otherwise,} \end{cases}$$

where $n = ap + b$ with $0 \le b < p$. By [1, Lemma 9.7], we have $I(n) \cong M(\omega_n)$ if $p \nmid n$ and if $p \mid n$ then $I(n)$ is an indecomposable module with two composition factors both isomorphic to $M(\omega_n)$. By Remark 6.9, the Jucys-Murphy elements L_k do not act semisimply on $M(\omega_n)$. Hence L_k^2, $1 \le k \le n$, act semisimply on $M(\omega_n)$ which is not completely splittable. On the other hand, Wang [18] introduced the degenerate spin affine Hecke-Clifford algebra $\mathfrak{H}^-$, which is the superalgebra with odd generators $b_i (1 \le i \le n)$ and $t_i (1 \le i \le n-1)$ subject to the relations

$$t_i^2 = 1, t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, t_i t_i = -t_i t_i, \quad |i - j| > 1,$$

$$b_i b_j = -b_j b_i, \quad i \neq j,$$

$$t_i b_i = -b_{i+1} t_i + 1, t_i b_j = -b_j t_i, \quad j \neq i, i+1.$$

Moreover, an algebra isomorphism between $\mathfrak{H}_n^{\mathfrak{c}}$ and $\mathcal{C}_n \otimes \mathfrak{H}^-$ which maps x_k to $\sqrt{-2} c_k b_k$ is established. Since $b_1, \dots, b_n$ are anti-commutative, it is reasonable to study $\mathfrak{H}^-$-modules on which the commuting operators $b_1^2, \dots, b_n^2$ act semisimply. As x_k^2 is sent to $2b_k^2$, it is reduced to study the $\mathfrak{H}_n^{\mathfrak{c}}$-modules on which x_k^2 act semisimply.

Motivated by the above observations, in this section we shall study the category of $\mathfrak{H}_n^{\mathfrak{c}}$-modules on which all x_k^2, $1 \le k \le n$, act semisimply.

7.1 The case for $n = 2, 3$

Recall the irreducible $\mathfrak{H}_2^{\mathfrak{c}}$-module $V(i, j)$ for $i, j \in \mathbb{I}$ from Proposition 3.9.

Lemma 7.1 *Let $i, j \in \mathbb{I}$. Then x_1^2, x_2^2 act semisimply on the $\mathfrak{H}_2^{\mathfrak{c}}$-module $V(i, j)$ if and only if $i \neq j$ or $i = j = 0$.*

Proof By Proposition 3.9, if $i \neq j$ then $V(i, j)$ is completely splittable and hence x_1^2, x_2^2 act semisimply. It suffices to prove that if $i = j$, then x_1^2, x_2^2 act semisimply on $V(i, j)$ if and only if $i = j = 0$. Now assume $i = j$. By Proposition 3.9, $V(i, j) = \mathrm{ind}_{\mathcal{P}_2^{\mathfrak{c}}}^{\mathfrak{H}_2^{\mathfrak{c}}} L(i) \circledast L(j)$. Suppose x_1^2, x_2^2 act semisimply on $V(i, j)$ and let $0 \neq z \in V(i, j)$. Then $x_1^2 z = q(i)z = x_2^2 z$. This together with (3.2) shows that

$$\big(x_1(1 - c_1c_2) + (1 - c_1c_2)x_2\big)z = 0.$$

This implies

$$4q(i)z = 2(x_1^2 + x_2^2)z = \big(x_1(1 - c_1c_2) + (1 - c_1c_2)x_2\big)^2 z = 0.$$

This means $q(i) = 0$ and hence $i = 0$ since $p \neq 2$.

Conversely if $i = j = 0$, then $x_1 = 0 = x_2$ on $L(i) \circledast L(j)$ and hence $x_1^2 = 0 = x_2^2$ on $V(i, j)$ by the fact that $V(0, 0)$ has two composition factors isomorphic to $L(0) \circledast L(0)$ as $\mathcal{P}_2^{\mathfrak{c}}$-modules. □

Observe that the subalgebra generated by $x_k, x_{k+1}, c_k, c_{k+1}, s_k$ is isomorphic to $\mathfrak{H}_2^{\mathfrak{c}}$ for each fixed $1 \leq k \leq n - 1$. By Lemma 7.1, we have the following.

Corollary 7.2 *Suppose that $M \in \mathrm{Rep}_{\mathbb{I}} \mathfrak{H}_n^{\mathfrak{c}}$ and $x_k^2, 1 \leq k \leq n$ act semisimply. Let $\underline{i} \in \mathbb{I}^n$ be a weight of M. If $i_k = i_{k+1}$ for some $1 \leq k \leq n - 1$, then $i_k = i_{k+1} = 0$.*

Lemma 7.3 *For any $z \in V(0, 0)$, we have*

$$\big((1 + c_1c_2)x_1 + (1 - c_1c_2)x_2\big)z = 0, \quad x_1x_2z = 0.$$

Proof Let $z \in V(0, 0)$. By Lemma 7.1, $x_1^2 = 0 = x_2^2$ on $V(0, 0)$. This together with (3.2) shows that

$$\big((1 + c_1c_2)x_1 + (1 - c_1c_2)x_2\big)z = 0. \tag{7.2}$$

Multiplying both sides of (7.2) by $x_1(1 + c_1c_2)$, we obtain that

$$(2x_1c_1c_2x_1 + 2x_1x_2)z = 0.$$

This implies that $x_1x_2z = 0$ since $x_1^2 z = 0$. □

Recall that $\mathfrak{H}_{2,1}^{\mathfrak{c}}$ is the subalgebra of $\mathfrak{H}_3^{\mathfrak{c}}$ generated by $\mathcal{P}_3^{\mathfrak{c}}$ and S_2.

Lemma 7.4 *The irreducible $\mathfrak{H}^{\mathfrak{c}}_{2,1}$-module $V(0,0,1) := V(0,0) \circledast L(1)$ affords an irreducible $\mathfrak{H}^{\mathfrak{c}}_3$-module via $s_2 = \Xi_2$.*

Proof Since $L(1)$ is of type $\mathtt{M}$, by Lemma 2.1 $V(0,0,1) = V(0,0) \boxtimes L(1)$. It is routine to check that $s_2^2 = 1, s_2x_1 = x_1s_2, s_2x_2 = x_3s_2 - (1 + c_2c_3)$ and $s_2c_1 = c_1s_2, s_2c_2 = c_3s_2$ on $V(0,0,1)$. It remains to prove $s_1s_2s_1 = s_2s_1s_2$. Let $0 \neq z \in V(0,0,1)$. Note that

$$x_2^2z = 0, x_3^2z = 2z$$

and hence

$$s_2z = \frac{1}{2}\big((x_2 + x_3) + c_2c_3(x_2 - x_3)\big)z = \frac{1}{2}\big((1 + c_2c_3)x_2 + (1 - c_2c_3)x_3\big)z. \quad (7.3)$$

Using (7.3) with z replaced by s_1z and (2.5), we show by a straightforward calculation that

$$s_2s_1z = \frac{1}{2}s_1\big((1 + c_1c_3)x_1 + (1 - c_1c_3)x_3\big)z + \frac{1}{2}(1 + c_1c_2 + c_2c_3 - c_1c_3)z. \quad (7.4)$$

This implies that

$$s_1s_2s_1z = \frac{1}{2}\big((1 + c_1c_3)x_1 + (1 - c_1c_3)x_3\big)z + \frac{1}{2}s_1(1 + c_1c_2 + c_2c_3 - c_1c_3)z. \quad (7.5)$$

On the other hand, it follows from (7.3) with z replaced by s_2z and (7.4) that

$$\begin{aligned}
s_2s_1s_2z &= \frac{1}{4}s_1\big((1 + c_1c_3)x_1 + (1 - c_1c_3)x_3\big)\big((1 + c_2c_3)x_2 + (1 - c_2c_3)x_3\big)z \\
&\quad + \frac{1}{4}(1 + c_1c_2 + c_2c_3 - c_1c_3)\big((1 + c_2c_3)x_2 + (1 - c_2c_3)x_3\big)z \\
&= \frac{1}{4}s_1\big((1 + c_1c_3)x_1 + (1 - c_1c_3)x_3\big)\big((1 + c_2c_3)x_2 + (1 - c_2c_3)x_3\big)z \\
&\quad + \frac{1}{2}(c_1c_2 + c_2c_3)x_2z + \frac{1}{2}(1 - c_1c_3)x_3z. \quad (7.6)
\end{aligned}$$

The first term on the right hand side of (7.6) can be simplified as follows

$$\begin{aligned}
&\frac{1}{4}s_1\big((1 + c_1c_3)x_1 + (1 - c_1c_3)x_3\big)\big((1 + c_2c_3)x_2 + (1 - c_2c_3)x_3\big)z \\
&= \frac{1}{4}s_1\big((1 + c_1c_3)(1 + c_2c_3)x_1x_2z \\
&\quad + (1 - c_2c_3)x_3\big((1 + c_1c_2)x_1 + (1 - c_1c_2)x_2\big)z\big) \\
&\quad + \frac{1}{4}s_1(1 + c_1c_2 + c_2c_3 - c_1c_3)x_3^2z \\
&= \frac{1}{2}s_1(1 + c_1c_2 + c_2c_3 - c_1c_3)z \quad \text{by Lemma 7.3.}
\end{aligned}$$

This together with (7.5) and (7.6) shows that

$$\begin{aligned}(s_1s_2s_1 - s_2s_1s_2)z &= \frac{1}{2}((x_1+x_3)+c_1c_3(x_1-x_3))z\\ &\quad - \frac{1}{2}(c_1c_2+c_2c_3)x_2z - \frac{1}{2}(1-c_1c_3)x_3z\\ &= \frac{1}{2}(1+c_1c_3)x_1z - \frac{1}{2}(c_1c_2+c_2c_3)x_2z\\ &= \frac{1}{4}(1-c_1c_2-c_2c_3+c_1c_3)\Big((1+c_1c_2)x_1+(1-c_1c_2)x_2\Big)z\end{aligned}$$

which is zero by Lemma 7.3. □

An identical argument used for proving Lemma 7.4 shows that $\mathfrak{H}^{\mathfrak{c}}_{2,1}$-module $V(1,0,0) := L(1) \circledast V(0,0)$ affords an irreducible $\mathfrak{H}^{\mathfrak{c}}_3$-module via $s_1 = \Xi_1$.

Proposition 7.5 *Each irreducible $\mathfrak{H}^{\mathfrak{c}}_3$-module in* $\mathrm{Rep}_{\mathbb{I}}\,\mathfrak{H}^{\mathfrak{c}}_3$ *on which x_1^2, x_2^2, x_3^2 act semisimply is isomorphic to one of the following.*

(1) *A completely splittable $\mathfrak{H}^{\mathfrak{c}}_3$-module $D^{\underline{i}}$ for $\underline{i} \in W'(\mathfrak{H}^{\mathfrak{c}}_3)$ (see Theorem 4.5).*
(2) $V(0,0,1)$.
(3) $V(1,0,0)$.
(4) $\mathrm{ind}^{\mathfrak{H}^{\mathfrak{c}}_3}_{\mathfrak{H}^{\mathfrak{c}}_{2,1}} V(0,0) \circledast L(j)$ *with $j \neq 0, 1$.*

Proof We first show that listed pairwise non-isomorphic modules are irreducible and all x_k^2 act semisimply. The case (1), (2) and (3) are taken care of by Theorem 4.5 and Lemma 7.4. Using [1, Theorem 5.18], we have $\mathrm{ind}^{\mathfrak{H}^{\mathfrak{c}}_3}_{\mathfrak{H}^{\mathfrak{c}}_{2,1}} V(0,0) \circledast L(j)$ is irreducible if $j \neq 0, 1$. It is known that as vector spaces

$$\begin{aligned}&\mathrm{ind}^{\mathfrak{H}^{\mathfrak{c}}_3}_{\mathfrak{H}^{\mathfrak{c}}_{2,1}} V(0,0) \circledast L(j)\\ &\quad = V(0,0) \circledast L(j) \oplus s_2 \otimes (V(0,0) \circledast L(j)) \oplus s_1s_2 \otimes (V(0,0) \circledast L(j)).\end{aligned}$$

It is clear that for $z \in V(0,0) \circledast L(j)$,

$$x_1^2 z = 0 = x_2^2 z, \quad x_3^2 z = q(j)z. \tag{7.7}$$

This together with (2.6) implies $x_1^2 = 0$ on $s_2 \otimes (V(0,0) \circledast L(j))$. Using (3.2) and (3.3), we obtain that

$$\begin{aligned}(x_2^2 - q(j))\big(s_2 \otimes (V(0,0) \circledast L(j))\big) &\subseteq V(0,0) \circledast L(j)\\ x_3^2\big(s_2 \otimes (V(0,0) \circledast L(j))\big) &\subseteq V(0,0) \circledast L(j).\end{aligned}$$

This together with (7.7) shows that for any $v \in s_2 \otimes (V(0,0) \circledast L(j))$,

$$x_1^2 v = 0,\; x_2^2(x_2^2 - q(j))v = 0,\; x_3^2(x_3^2 - q(j))v = 0. \tag{7.8}$$

Similarly using (3.2), (3.3) and (7.7) we see that

$$(x_1^2 - q(j))s_1s_2 \otimes (V(0,0) \circledast L(j)) \subseteq V(0,0) \circledast L(j) \oplus s_2 \otimes (V(0,0) \circledast L(j))$$
$$(x_3^2)s_1s_2 \otimes (V(0,0) \circledast L(j)) \subseteq V(0,0) \circledast L(j).$$

Therefore it follows from (7.7) and (7.8) that for any $w \in s_1s_2 \otimes (V(0,0) \circledast L(j))$

$$x_1^2(x_1^2 - q(j))w = 0, \ x_3^2(x_3^2 - q(j))w = 0. \tag{7.9}$$

By (3.2) and (7.7), we obtain that for any $z \in V(0,0) \circledast L(j)$,

$$\begin{aligned} x_2^2 s_1s_2 \otimes z &= s_1s_2 \otimes x_1^2 z + \big((1+c_1c_2)x_1 + x_2(1+c_1c_2)\big)s_2 \otimes z \\ &= \big((1+c_1c_2)x_1 + x_2(1+c_1c_2)\big)s_2 \otimes z. \end{aligned} \tag{7.10}$$

This together with (3.2) and $x_3^2 = q(j)$ on $V(0,0) \circledast L(j)$ shows that for $z \in V(0,0) \circledast L(j)$

$$\begin{aligned} &(x_2^2 - q(j))x_2^2(s_1s_2 \otimes z) \\ &\quad = (x_2^2 - q(j))\big((1+c_1c_2)x_1 + x_2(1+c_1c_2)\big)s_2 \otimes z \\ &\quad = \big((1+c_1c_2)x_1 + x_2(1+c_1c_2)\big)(x_2^2 - q(j))s_2 \otimes z \\ &\quad = ((1+c_1c_2)x_1 + x_2(1+c_1c_2))(-x_2(1-c_2c_3) - (1-c_2c_3)x_3)z \\ &\quad = 0 \quad \text{by Lemma 7.3.} \end{aligned}$$

Therefore for any $w \in s_1s_2 \otimes (V(0,0) \circledast L(j))$,

$$(x_2^2 - q(j))x_2^2 w = 0. \tag{7.11}$$

Combining (7.7), (7.8), (7.9) and (7.11), we see that the actions of x_1, x_2, x_3 on the $\mathfrak{H}_3^{\mathfrak{c}}$-module $\mathrm{ind}_{\mathfrak{H}_{2,1}^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} V(0,0) \circledast L(j)$ satisfy

$$x_1^2(x_1^2 - q(j)) = 0, \ x_2^2(x_2^2 - q(j)) = 0, \ x_3^2(x_3^2 - q(j)) = 0.$$

It follows that x_1^2, x_2^2, x_3^2 act semisimply on $\mathrm{ind}_{\mathfrak{H}_{2,1}^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} L(0^2) \circledast L(j)$.

Now assume $M \in \mathrm{Rep}_{\mathbb{I}}\, \mathfrak{H}_3^{\mathfrak{c}}$ is irreducible, on which all x_k^2, $1 \le k \le n$ act semisimply. Let us assume M is not completely splittable, then by Proposition 3.6 M has a weight of the form (i,i,j) or (j,i,i) for some $i, j \in \mathbb{I}$. By Corollary 7.2 we obtain that $i = 0$. Hence by Frobenius reciprocity M is a quotient of $\mathrm{ind}_{\mathcal{P}_3^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} L(0) \circledast L(0) \circledast L(j)$ or $\mathrm{ind}_{\mathcal{P}_3^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} L(j) \circledast L(0) \circledast L(0)$.

If $j = 0$, then M is isomorphic to the Kato module $\mathrm{ind}_{\mathcal{P}_3^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} L(0) \circledast L(0) \otimes L(0)$. By [1, Lemma 4.15], all Jordan blocks of x_1 on M are of size 3. This means $x_1^4 = 0$ on M but not x_1^2. Hence x_1^2 does not act semisimply on M.

If $j=1$, then the weights of M belong to $S_3\cdot(0,0,1)$. By [1, §5-d], there are at most three non-isomorphic irreducible $\mathfrak{H}_3^{\mathfrak{c}}$-modules whose weights belong to the set $S_3\cdot(0,0,1)=\{(0,0,1),(0,1,0),(1,0,0)\}$. By Theorem 4.5, the $\mathcal{P}_3^{\mathfrak{c}}$-module $V(0,1,0)=L(0)\circledast L(1)\circledast L(0)$ affords an irreducible completely splittable $\mathfrak{H}_3^{\mathfrak{c}}$-module via $s_1=\Xi_1, s_2=\Xi_2$. Observe that the modules $V(0,0,1), V(1,0,0)$ and $V(0,1,0)$ are non-isomorphic and have weights belonging to $S_3\cdot(0,0,1)$. Since M is not completely splittable, $M\cong V(0,0,1)$ or $M\cong V(1,0,0)$.

If $j\neq 0,1$, by [1, Theorem 5.18] we have that

$$\begin{aligned}\operatorname{ind}_{\mathcal{P}_3^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} L(0)\circledast L(0)\otimes L(j) &\cong \operatorname{ind}_{\mathfrak{H}_{2,1}^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} V(0,0)\circledast L(j)\\ &\cong \operatorname{ind}_{\mathfrak{H}_{2,1}^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} L(j)\circledast V(0,0)\\ &\cong \operatorname{ind}_{\mathcal{P}_3^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} L(j)\circledast L(0)\otimes L(0)\end{aligned}$$

is irreducible. Hence $M\cong \operatorname{ind}_{\mathfrak{H}_{2,1}^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} V(0,0)\circledast L(j)$. □

Observe that the subalgebra generated by $x_k, x_{k+1}, x_{k+2}, c_k, c_{k+1}, c_{k+2}, s_k, s_{k+1}$ is isomorphic to $\mathfrak{H}_3^{\mathfrak{c}}$ for fixed $1\le k\le n-2$. By Proposition 7.5, we have the following.

Corollary 7.6 *Suppose that $M\in \operatorname{Rep}_{\mathbb{I}}\mathfrak{H}_n^{\mathfrak{c}}$, on which all x_k^2, $1\le k\le n$ act semisimply. Let $\underline{i}\in\mathbb{I}^n$ be a weight of M. Then there does not exist $1\le k\le n-2$ such that $i_k=i_{k+1}=i_{k+2}$.*

7.2 Conjecture for general n

Proposition 7.7 *Suppose that $M\in \operatorname{Rep}_{\mathbb{I}}\mathfrak{H}_n^{\mathfrak{c}}$ is irreducible and $M_{\underline{i}}\neq 0$ for some $\underline{i}\in\mathbb{I}^n$. If x_k^2, $1\le k\le n$, act semisimply on M, then $\underline{i}$ satisfies the following.*

(1) *If $i_k\neq i_{k+1}\pm 1$, then $s_k\cdot\underline{i}$ is a weight of M.*
(2) *If $i_k=i_{k+1}$ for some $1\le k\le n-1$, then $i_k=i_{k+1}=0$.*
(3) *There does not exist $1\le k\le n-2$ such that $i_k=i_{k+1}=i_{k+2}$.*
(4) *If $i_k=i_{k+2}$ for some $1\le k\le n-2$, then*
   (a) *If $p=0$, then $i_k=i_{k+2}=0$.*
   (b) *If $p\ge 3$, then either ($i_k=i_{k+2}=\frac{p-3}{2}$ and $i_{k+1}=\frac{p-1}{2}$) or ($i_k=i_{k+2}=0$).*

Proof (1) If $i_k\neq i_{k+1}\pm 1$, by Lemma 4.2 $\widehat{\Phi}_k$ is a well-defined bijection from $M_{\underline{i}}$ to $M_{s_k\cdot\underline{i}}$. Hence $M_{s_k\cdot\underline{i}}\neq 0$.

(2) It follows from Corollary 7.2.

(3) It follows from Corollary 7.6.

(4) Suppose $i_k=i_{k+2}=u$ and $i_{k+1}=v$ for some $1\le k\le n-2$. Observe that for each fixed $1\le k\le n-2$, $x_k^2, x_{k+1}^2, x_{k+2}^2$ act semisimply on the restriction of M to the subalgebra generated by $x_k, x_{k+1}, x_{k+2}, c_k, c_{k+1}, c_{k+2}, s_k, s_{k+1}$ which is isomorphic to $\mathfrak{H}_3^{\mathfrak{c}}$. This implies that (u,v,u) appears as a weight of a $\mathfrak{H}_3^{\mathfrak{c}}$-module on which

x_1^2, x_2^2, x_3^2 act semisimply. By Proposition 7.5, if $p=0$, then $u=0$; if $p\geq 3$, then either $u=0$, v is arbitrary or $u=\frac{p-3}{2}$, $v=\frac{p-1}{2}$. □

Corollary 7.8 *Suppose that $M\in \mathrm{Rep}_{\mathbb{I}}\mathfrak{H}_n^{\mathfrak{c}}$ is irreducible and $M_{\underline{i}}\neq 0$ for some $\underline{i}\in\mathbb{I}^n$. If all x_k^2, $1\leq k\leq n$ act semisimply on M, then $\underline{i}$ satisfies the following.*

(1) *If $p=0$ and $u=i_k=i_l\geq 1$ for some $1\leq k<l\leq n$, then*

$$\{u-1,u+1\}\subseteq\{i_{k+1},\ldots,i_{l-1}\},$$

or

$$(u,u-1,\ldots,1,0,0,1,\ldots,u-1,u) \text{ is a subsequence of } (i_{k+1},\ldots,i_{l-1}).$$

(2) *If $p\geq 3$ and $u=i_k=i_l\geq 1$ for some $1\leq k<l\leq n$, then*

$$\{u-1,u+1\}\subseteq\{i_{k+1},\ldots,i_{l-1}\},$$

or

$$(u,u-1,\ldots,1,0,0,1,\ldots,u-1,u)$$
$$\text{is a subsequence of } (i_{k+1},\ldots,i_{l-1}),$$

or

$$\left(u,u+1,\ldots,\tfrac{p-3}{2},\tfrac{p-1}{2},\tfrac{p-3}{2},\ldots,u+1,u\right)$$
$$\text{is a subsequence of } (i_{k+1},\ldots,i_{l-1}).$$

Proof (1) Without loss of generality, we can assume $u\notin\{i_{k+1},\ldots,i_{l-1}\}$. By the technique used in the proof of Proposition 3.14, one can show that $u-1\in\{i_{k+1},\ldots,i_{l-1}\}$. Now assume $u+1\notin\{i_{k+1},\ldots,i_{l-1}\}$. Then $u-1$ appears at least twice between i_{k+1} and i_{l-1} in $\underline{i}$; otherwise we can apply admissible transpositions to $\underline{i}$ to obtain a weight of M of the form $(\cdots,u,u-1,u,\cdots)$ which contradicts Proposition 7.7(4). Hence there exist $k<k_1<l_1<l$ such that

$$i_{k_1}=u-1=i_{l_1},\ \{u,u-1\}\cap\{i_{k_1+1},\ldots,i_{l_1-1}\}=\emptyset.$$

An identical argument shows that there exist $k_1<k_2<l_2<l_1$ such that

$$i_{k_2}=u-2=i_{l_2},\ \{u,u-1,u-2\}\cap\{i_{k_2+1},\ldots,i_{l_2-1}\}=\emptyset.$$

Continuing in this way, we achieve the claim.

(2) By the technique used in (1), one can easily show that if $u+1\notin\{i_{k+1},\ldots,i_{l-1}\}$ then $(i_{k+1},\ldots,i_{l-1})$ contains $(u,u-1,\ldots,1,0,0,1,\ldots,u-1,u)$ as a subsequence. If $u-1\notin\{i_{k+1},\ldots,i_{l-1}\}$, an identical argument used in the proof of Proposition 3.14(5) shows that $(i_{k+1},\ldots,i_{l-1})$ contains $(u,u+1,\ldots,\frac{p-3}{2},\frac{p-1}{2},\frac{p-3}{2},\ldots,u+1,u)$ as a subsequence. □

Conjecture 7.9 *Suppose that $M\in\mathrm{Rep}_{\mathbb{I}}\mathfrak{H}_n^{\mathfrak{c}}$ is irreducible. Then x_k^2, $1\leq k\leq n$, act semisimply on M if and only if each weight of M satisfies the list of properties stated in Proposition 7.7.*

Theorem 7.10 *The above conjecture holds for $n = 2, 3$.*

Proof Clearly the above conjecture holds for $n = 2$ by Lemma 7.1. Suppose M is an irreducible $\mathfrak{H}_3^{\mathfrak{c}}$-module whose weights satisfy the list of properties stated in Proposition 7.7. Let $(i_1, i_2, i_3) \in \mathbb{I}^3$ be a weight of M. Then by Frobenius reciprocity, M is isomorphic to a quotient of $\mathrm{ind}_{\mathcal{P}_3^{\mathfrak{c}}}^{\mathfrak{H}_3^{\mathfrak{c}}} L(i_1) \circledast L(i_2) \circledast L(i_3)$. Hence the weights of M are of the form $\sigma \cdot (i_1, i_2, i_3)$ for $\sigma \in S_3 \cdot$. If i_1, i_2, i_3 are distinct, then all weights $\underline{j}$ of M satisfy $j_k \neq j_{k+1}$ for $k = 1, 2$. By Proposition 3.6, M is completely splittable and hence all x_1^2, x_2^2, x_3^2 act semisimply on it.

Now assume i_1, i_2, i_3 are not distinct. If $p = 0$, by the properties in Proposition 7.7 we have that (i_1, i_2, i_3) is of the form $(0, 0, j)$, $(0, j, 0)$ or $(j, 0, 0)$ for some $j \geq 1$. By Proposition 7.5, all $x_k^2, 1 \leq k \leq 3$ act semisimply on M. If $p \geq 3$, by the properties in Proposition 7.7 we see that either $(i_1, i_2, i_3) = (\frac{p-3}{2}, \frac{p-1}{2}, \frac{p-3}{2})$ or (i_1, i_2, i_3) has the form $(0, 0, j)$, $(0, j, 0)$ or $(j, 0, 0)$ for some $j \geq 1$. In the latter case, by Proposition 7.5, all $x_k^2, 1 \leq k \leq 3$ act semisimply on M. Assume $(i_1, i_2, i_3) = (\frac{p-3}{2}, \frac{p-1}{2}, \frac{p-3}{2})$. Since M satisfies the properties in Proposition 7.7, $(\frac{p-3}{2}, \frac{p-3}{2}, \frac{p-1}{2})$ and $(\frac{p-3}{2}, \frac{p-3}{2}, \frac{p-1}{2})$ are not the weights of M. Hence M has only one weight, that is, $(\frac{p-3}{2}, \frac{p-1}{2}, \frac{p-3}{2})$. By Proposition 3.6, M is completely splittable and hence all x_1^2, x_2^2, x_3^2 act semisimply on it. □

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