

q, t -Fuß–Catalan numbers for finite reflection groups

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Abstract In type A , the q, t -Fuß–Catalan numbers can be defined as the bigraded Hilbert series of a module associated to the symmetric group. We generalize this construction to (finite) complex reflection groups and, based on computer experiments, we exhibit several conjectured algebraic and combinatorial properties of these polynomials with nonnegative integer coefficients. We prove the conjectures for the dihedral groups and for the cyclic groups. Finally, we present several ideas on how the q, t -Fuß–Catalan numbers could be related to some graded Hilbert series of modules arising in the context of rational Cherednik algebras and thereby generalize known connections.

Keywords Catalan number · Fuß–Catalan number · q, t -Catalan number · Nonnesting partition · Dyck path · Shi arrangement · Cherednik algebra

1 Introduction

The q, t -Catalan numbers and later the q, t -Fuß–Catalan numbers arose within the last 15 years in more and more contexts in different areas of mathematics, namely in *symmetric functions theory*, *algebraic* and *enumerative combinatorics*, *representation theory*, and *algebraic geometry*. They first appeared in a paper by Haiman [25] as the Hilbert series of the alternating component of the *space of diagonal coinvariants*. Garsia and Haiman [18] defined them as a rational function in the context of *modified Macdonald polynomials*. Later, in his work on the $n!$ - and the $(n + 1)^{n-1}$ -conjectures, Haiman [29] showed that both definitions coincide. Haglund [22] found a very interesting combinatorial interpretation of the q, t -Catalan numbers, which he

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proved together with Garsia in [17]. Loehr [32] conjectured a generalization of this combinatorial interpretation for the q, t -Fuß–Catalan numbers. This conjecture is still open.

The q, t -Fuß–Catalan numbers have many interesting algebraic and combinatorial properties. To mention some: they are symmetric functions in q and t with nonnegative integer coefficients and specialize for $q = t = 1$ to the well-known Fuß–Catalan numbers

$$\text{Cat}_n^{(m)} := \frac{1}{mn + 1} \binom{(m+1)n}{n}.$$

Furthermore, specializing $t = 1$ reduces them to the combinatorial q -Fuß–Catalan numbers introduced by Förlinger and Hofbauer [16]; specializing $t = q^{-1}$ reduces them, up to a power of q , to the q -Fuß–Catalan numbers introduced for $m = 1$ by MacMahon [34, p. 1345].

The Fuß–Catalan numbers $\text{Cat}_n^{(m)}$ have a generalization to all well-generated complex reflection groups. A standard reference for background on real reflection groups is [31]; for further information on complex reflection groups, see [10, 11, 37, 38, 41]. Let W be such a well-generated complex reflection group, having rank ℓ , degrees $d_1 \leq \dots \leq d_\ell$, and Coxeter number $h := d_\ell$. The Fuß–Catalan numbers associated to W are then defined by

$$\text{Cat}^{(m)}(W) := \prod_{i=1}^{\ell} \frac{d_i + mh}{d_i}.$$

In the case of $W = A_{n-1}$, we have $\ell = n - 1$, $d_i = i + 1$, and $h = n$. This gives

$$\text{Cat}^{(m)}(A_{n-1}) = \text{Cat}_n^{(m)}.$$

For $m = 1$, $\text{Cat}^{(m)}(W)$ first appeared in the paper by Reiner [36], who proved, for the classical reflection groups, case-by-case that the number of *noncrossing partitions* equals the number of *nonnesting partitions* and that both are counted by this product. In full generality of well-generated complex reflection groups, $\text{Cat}^{(m)}(W)$ was considered by Bessis [9], who studied chains in the noncrossing partition lattice.

It turns out that the interpretation of the q, t -Fuß–Catalan numbers in terms of the space of diagonal coinvariants is attached to the reflection group of type A , whereas the other interpretations can—so far—not be generalized to other reflection groups. We define the space of diagonal coinvariants for any (finite) complex reflection group and define q, t -Fuß–Catalan numbers in terms of this module. Moreover, we explore several conjectured properties of those polynomials in this generalized context. In particular, we conjecture that the q, t -Fuß–Catalan numbers reduce for well-generated complex reflection groups and the specialization $q = t = 1$ to the Fuß–Catalan numbers $\text{Cat}^{(m)}(W)$.

For real reflection groups, we finally explore connections between the q, t -Fuß–Catalan numbers and a module which naturally arises in the context of *rational Cherednik algebras*. We construct a surjection from the space of diagonal coinvariants to the module in question. This construction was, for $m = 1$, exhibited by Gordon [19].

For background on representation theory, we refer to [15]; for background on the classical q, t -Fuß–Catalan numbers, we refer to a series of papers and survey articles by Garsia and Haiman [18, 25, 26, 30] and to a recent book by Haglund [23].

This paper is organized as follows:

In Sect. 2, we recall some background on classical q, t -Fuß–Catalan numbers.

In Sect. 3, we define q, t -Fuß–Catalan numbers for all complex reflection groups (Definition 3) and present several conjectures concerning them (Conjectures 2, 3, and 4). Moreover, we explicitly compute the q, t -Fuß–Catalan numbers for the dihedral groups (Theorem 6) and thereby prove the conjectures in this case (Corollary 6 and Theorem 7). Finally, we compute the q, t -Fuß–Catalan numbers for the cyclic groups as a first example of a nonreal reflection group (Corollary 7).

In Sect. 4, we present some background on rational Cherednik algebras, prove a generalization of a theorem of Gordon which connects the q, t -Fuß–Catalan numbers to those (Theorem 10), and finally, we present a conjecture (Conjecture 5) in this context which would imply Conjectures 2 and 3.

2 Background on classical q, t -Fuß–Catalan numbers

The symmetric group \mathcal{S}_n acts *diagonally* on the polynomial ring

$$\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$$

by

$$\sigma(x_i) := x_{\sigma(i)}, \quad \sigma(y_i) := y_{\sigma(i)} \quad \text{for } \sigma \in \mathcal{S}_n. \tag{1}$$

Note that $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ is bigraded by degree in \mathbf{x} and degree in \mathbf{y} and that this diagonal action preserves the bigrading.

The diagonal coinvariant ring DR_n is defined to be $\mathbb{C}[\mathbf{x}, \mathbf{y}]/\mathcal{I}$, where \mathcal{I} is the ideal in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ generated by all invariant polynomials without constant term, i.e., all polynomials $p \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ such that $\sigma(p) = p$ for all $\sigma \in \mathcal{S}_n$ and $p(0) = 0$. This ring has a closely related extension for any integer m : let \mathcal{A} be the ideal generated by all alternating polynomials, i.e., all polynomials $p \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ such that $\sigma(p) = \text{sgn}(\sigma)p$ for all $\sigma \in \mathcal{S}_n$, where $\text{sgn}(\sigma)$ denotes the sign of the permutation σ . Then the space $DR_n^{(m)}$ was defined by Garsia and Haiman [18] as

$$DR_n^{(m)} := (\mathcal{A}^{m-1} / \mathcal{A}^{m-1}\mathcal{I}) \otimes \epsilon^{\otimes(m-1)},$$

where ϵ is the one-dimensional sign representation defined by $\sigma(z) := \text{sgn}(\sigma)z$ for $z \in \mathbb{C}$, and where $\epsilon^{\otimes k}$ is its k th tensor power. As $DR_n^{(m)}$ is a module that reduces for $m = 1$ to the diagonal coinvariant ring, we call it the space of generalized diagonal coinvariants. Haiman proved that the dimension of $DR_n^{(m)}$ is equal to $(mn + 1)^{n-1}$, see, e.g., [28, Theorem 1.4]. For $m = 1$, Haglund and Loehr [24] found a conjectured combinatorial interpretation of its Hilbert series in terms of certain statistics on parking functions.

Observe that the natural S_n -action on $DR_n^{(m)}$ is twisted by the $(m - 1)$ -st power of the sign representation such that the generators of this module, which are the minimal generators of \mathcal{A}^{m-1} , become invariant. One can show that the alternating component of $DR_n^{(m)}$ is, except for the sign-twist, naturally isomorphic to $\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m$, where $\langle \mathbf{x}, \mathbf{y} \rangle = \langle x_1, y_1, \dots, x_n, y_n \rangle$ is the ideal of all polynomials without constant term. Let $M^{(m)}$ denote this alternating component of $DR_n^{(m)}$,

$$M^{(m)} := \mathbf{e}_\epsilon (DR_n^{(m)}) \cong (\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m) \otimes \epsilon^{\otimes(m-1)}, \tag{2}$$

where \mathbf{e}_ϵ is the sign idempotent defined by

$$\mathbf{e}_\epsilon(p) := \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(p). \tag{3}$$

This alternating component was first considered in [18, Sect. 3], but a proof of (2) was left to the reader. It can be deduced from the following well-known lemma. We will prove the identity in a more general context in Sect. 3.2.

Note 1 The notions of ϵ for the sign representation and \mathbf{e}_ϵ for the sign idempotent will become clear in Sect. 3.1, where we generalize the notions to all complex reflection groups.

Lemma 1 (Graded version of Nakayama’s Lemma) *Let $R = \bigoplus_{i \geq 0} R_i$ be an \mathbb{N} -graded k -algebra for some field k , and let M be a graded R -module, bounded below in degree. Then $\{m_1, \dots, m_t\}$ generate M as an R -module if and only if their images $\{\overline{m}_1, \dots, \overline{m}_t\}$ k -linearly span the k -vector space M/R_+M , where $R_+ := \bigoplus_{n \geq 1} R_i$. In particular, $\{m_1, \dots, m_t\}$ generate M minimally as an R -module if and only if $\{\overline{m}_1, \dots, \overline{m}_t\}$ is a basis of M/R_+M as a k -vector space.*

Remark 1 Nakayama’s Lemma implies that $\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m$ has a vector space basis given by (the images of) any minimal generating set of \mathcal{A}^m as a $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ -module. Therefore, it is often called the *minimal generating space* of \mathcal{A}^m .

For $X = \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\} \subseteq \mathbb{N} \times \mathbb{N}$, define the bivariate Vandermonde determinant by

$$\Delta_X(\mathbf{x}, \mathbf{y}) := \det \begin{pmatrix} x_1^{\alpha_1} y_1^{\beta_1} & \cdots & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & \cdots & x_n^{\alpha_n} y_n^{\beta_n} \end{pmatrix}.$$

As a vector space, the space $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ of all alternating polynomials has a well-known basis given by

$$\mathcal{B} = \{ \Delta_X : X \subseteq \mathbb{N} \times \mathbb{N}, |X| = n \}. \tag{4}$$

In particular, the ideal generated by these elements equals \mathcal{A} ; compare [30]. Again by Nakayama’s Lemma, $M^{(1)}$ has a vector space basis given by (the images of) any maximal linearly independent subset of \mathcal{B} with coefficients in $\mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{S_n}$. Unfortunately, no general construction of such an independent subset is known so far.

The q, t -Fuß–Catalan numbers were first defined by Haiman [25] as the bigraded Hilbert series of the alternating component of the space of generalized diagonal coinvariants,

$$\text{Cat}_n^{(m)}(q, t) := \mathcal{H}(M^{(m)}; q, t), \tag{5}$$

where $\mathcal{H}(M; q, t) = \sum_{i,j \geq 0} \dim(M_{i,j})q^i t^j$ is the bigraded Hilbert series of the bigraded module M , and where $M_{i,j}$ denotes the bihomogeneous component of M in bidegree (i, j) . He moreover conjectured that $\text{Cat}_n^{(m)}(q, t)$ is in fact a q, t -extension of the Fuß–Catalan numbers $\text{Cat}_n^{(m)}$. Using subtle results from algebraic geometry, he was finally able to prove this conjecture in the context of the $n!$ - and the $(n + 1)^{n-1}$ -conjectures [29]. From this work it follows that $\text{Cat}_n^{(m)}(q, t)$ is equal to a complicated rational function in the context of *modified Macdonald polynomials*. This rational function was studied by Garsia and Haiman [18]. They were able to prove the specializations $t = 1$ and $t = q^{-1}$ in $\text{Cat}_n^{(m)}(q, t)$. Those specializations were already conjectured by Haiman [25] and turn out to be equal to well-known q -extensions of the Fuß–Catalan numbers, namely the generating function for the area statistic on m -Dyck paths considered by Furlinger and Hofbauer [16],

$$\text{Cat}_n^{(m)}(q, 1) = \sum_{D \in \mathcal{D}_n^{(m)}} q^{\text{area}(D)}, \tag{6}$$

and, up to a power of q , MacMahon’s q -Catalan numbers,

$$q^{m \binom{n}{2}} \text{Cat}_n^{(m)}(q, q^{-1}) = \frac{1}{[mn + 1]_q} \left[\begin{matrix} (m + 1)n \\ n \end{matrix} \right]_q. \tag{7}$$

Here, $\mathcal{D}_n^{(m)}$ denotes the set of all m -Dyck paths of semilength n which are north-east lattice paths from $(0, 0)$ to (mn, n) that stay above the diagonal $x = my$. Moreover, the area is defined to be the number of full lattice squares which lie between a path and the diagonal. See Fig. 1 for an example.

Haglund [22] defined the *bounce statistic* on 1-Dyck paths, and Loehr [32] generalized the definition to m -Dyck paths. They conjectured that the q, t -Fuß–Catalan numbers can be described combinatorially in the manner of (6) using the bounce statistic as the t -exponent. Garsia and Haglund [17] were able to prove this conjecture for $m = 1$; it remains open for $m \geq 2$.

3 q, t -Fuß–Catalan numbers for complex reflection groups

In this section, we generalize the definition of q, t -Fuß–Catalan numbers to arbitrary (finite) complex reflection groups. As we only deal with finite reflection groups, we

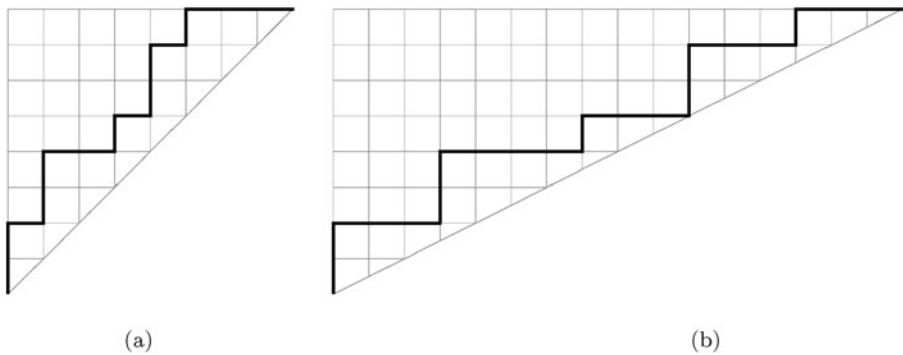


Fig. 1 A 1-Dyck path and a 2-Dyck path; both have semilength 8 and area 10

usually suppress the term “finite.” Moreover, we present several conjectures concerning this generalization. They are based on computer experiments which are listed in Tables 1–4 in the Appendix. Moreover, we prove the conjectures for the dihedral groups $I_2(k) = G(k, k, 2)$ and for the cyclic groups $C_k = G(k, 1, 1)$. Here and below, $G(k, p, \ell)$ refers to the infinite family in the Shephard–Todd classification of complex reflection groups [38].

3.1 The space of generalized diagonal coinvariants

The definition of the space of generalized diagonal coinvariants makes sense for any complex reflection group. For real reflection groups and for $m = 1$, it can be found in [25, Sect. 7].

Any complex reflection group W of rank ℓ acts naturally as a matrix group on V (i.e., V is a reflection representation) and moreover on $V \oplus V^*$ by

$$\omega(v \oplus v^*) := \omega v \oplus {}^t(\omega^{-1})v^*.$$

W is a subgroup of the unitary group $U(V)$; thus ${}^t(\omega^{-1})$ is the *complex conjugate* of $\omega \in W$. This action induces a diagonal contragredient action of W on $V^* \oplus V$ and thereby on its symmetric algebra $S(V^* \oplus V)$, which is the ring of polynomial functions on $V \oplus V^*$. After fixing a basis for V , this ring of polynomial functions can be identified with

$$\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, y_1, \dots, x_\ell, y_\ell] = \mathbb{C}[V \oplus V^*].$$

Observe that, as for the symmetric group, the W -action on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ preserves the bi-grading on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ and moreover, that

$$\omega(pq) = \omega(p)\omega(q) \quad \text{for all } p, q \in \mathbb{C}[\mathbf{x}, \mathbf{y}] \text{ and } \omega \in W. \quad (8)$$

If W is a real reflection group, ${}^t\omega = \omega^{-1}$, and W therefore acts identically on \mathbf{x} and on \mathbf{y} . In particular, this action generalizes the action described in (1) for the symmetric group.

Let S be any W -module. Define the trivial idempotent \mathbf{e} to be the linear operator on S defined by

$$\mathbf{e} := \frac{1}{|W|} \sum_{\omega \in W} \omega \in \text{End}(S),$$

and, generalizing (3), define the determinantal idempotent \mathbf{e}_ϵ to be the linear operator defined by

$$\mathbf{e}_\epsilon := \frac{1}{|W|} \sum_{\omega \in W} \det^{-1}(\omega)\omega \in \text{End}(S).$$

The trivial idempotent is a projection from S onto its trivial component

$$S^W := \{p \in S : \omega(p) = p \text{ for all } \omega \in W\},$$

which is the isotypic component of the trivial representation \mathbb{C} defined for $\omega \in W$ by $\omega(z) := z$. Analogously, the determinantal idempotent is a projection onto its determinantal component

$$S^\epsilon := \{p \in S : \omega(p) = \det(\omega)p \text{ for all } \omega \in W\},$$

which is the isotypic component of the determinantal representation ϵ defined for $\omega \in W$ by $\omega(z) := \det(\omega)z$. As above, $\epsilon^{\otimes k}$ denotes its k th tensor power which is given by $\omega(z) = \det^k(\omega)z$. Moreover, we can define $\epsilon^{\otimes k}$ for negative k to be the k th tensor of the inverse determinantal representation $\epsilon^{\otimes(-1)}$ defined by $\omega(z) := \det^{-1}(\omega)z$. Observe that the determinantal and the inverse determinantal representations coincide for real reflection groups and that in this case, $\epsilon^{\otimes 2} = \epsilon^{\otimes 0} = \mathbb{C}$ is the trivial representation.

$p \in S^W$ is called invariant in S , $p \in S^\epsilon$ is called determinantal in S , and we have

$$p \otimes 1^{\otimes k} \in S \otimes \epsilon^{\otimes k} \text{ invariant} \iff p \otimes 1^{\otimes(k+1)} \in S \otimes \epsilon^{\otimes(k+1)} \text{ determinantal}.$$

As $S \otimes \epsilon^{\otimes k}$ and $S \otimes \epsilon^{\otimes(k+1)}$ differ only by a determinantal factor in their W -actions, we often write p instead of $p \otimes 1^{\otimes \ell}$ and say, e.g., that p is invariant in $S \otimes \epsilon^{\otimes k}$ if and only if p is determinantal in $S \otimes \epsilon^{\otimes(k+1)}$.

Definition 1 Let W be a complex reflection group of rank ℓ acting diagonally on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. Let \mathcal{I} be the ideal in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ generated by all invariant polynomials without constant term, and let \mathcal{A} be the ideal generated by all determinantal polynomials. For any positive integer m , define the space of generalized diagonal coinvariants $DR^{(m)}(W)$ as

$$DR^{(m)}(W) := (\mathcal{A}^{m-1} / \mathcal{A}^{m-1}\mathcal{I}) \otimes \epsilon^{\otimes(1-m)}.$$

Observe that we have to twist the natural W -action on $DR^{(m)}$ by the $(m - 1)$ -st power of the *inverse determinantal representation*, rather than of the determinantal representation, so that the generators of this module, which are the minimal generators of \mathcal{A}^{m-1} , become invariant.

Fig. 2 The actual dimension of $DR^{(1)}(W)$ for the reflection groups B_4, B_5 and D_4

| | dim | $(h + 1)^\ell$ |
|-------|-------------|----------------|
| B_4 | $9^4 + 1$ | 9^4 |
| B_5 | $11^5 + 33$ | 11^5 |
| D_4 | $7^4 + 40$ | 7^4 |

As seen above, the dimension of $DR_n^{(m)}$ can be expressed in terms of the reflection group A_{n-1} , which is the symmetric group S_n , as

$$\dim DR^{(m)}(A_{n-1}) = (mh + 1)^\ell,$$

where $h = n$ is the Coxeter number of A_{n-1} , and $\ell = n - 1$ is its rank.

Haiman computed the actual dimension of $DR^{(1)}(W)$ for the reflection groups B_4, B_5 , and D_4 . The results can be found in Fig. 2. These “counterexamples” led him to the following conjecture [25, Conjecture 7.1.2]:

Conjecture 1 (M. Haiman) *For any real (or eventually crystallographic) reflection group W , there exists a “natural” quotient ring R_W of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ by some homogeneous ideal containing \mathcal{I} such that*

$$\dim R_W = (h + 1)^\ell.$$

This conjecture was proved by Gordon [19] in the context of *rational Cherednik algebras*:

Theorem 1 (I. Gordon) *Let W be a real reflection group. There exists a graded W -stable quotient ring R_W of $DR^{(1)}(W)$ such that*

- (i) $\dim(R_W) = (h + 1)^\ell$, and moreover,
- (ii) $q^N \mathcal{H}(R_W; q) = [h + 1]_q^\ell$.

In Sect. 4.2.3, we will slightly generalize this theorem to $DR^{(m)}(W)$ for arbitrary $m \geq 1$.

3.2 q, t -Fuß–Catalan numbers for complex reflection groups

Haiman’s computations of the dimension of the diagonal coinvariants in types B_4, B_5 , and D_4 seemed to be the end of the story, but computations of the dimension of the determinantal component of the generalized diagonal coinvariants $DR^{(m)}(W)$ suggest the following conjecture:

Conjecture 2 *Let W be a well-generated complex reflection group. Then*

$$\dim e_\epsilon(DR^{(m)}(W)) = \text{Cat}^{(m)}(W).$$

We used the computer algebra system Singular [40] for the aforementioned computations for several classical groups including types B_4 and D_4 . The computations are listed in Table 1 in the Appendix.

For the computations, we used the following isomorphism which was mentioned in type A in Sect. 2:

Theorem 2 *Let W be a complex reflection group. The determinantal component of $DR^{(m)}(W)$ is, except for a determinantal factor, naturally isomorphic (as a bigraded W -module) to the minimal generating space of the ideal \mathcal{A}^m in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$,*

$$e_\epsilon(DR^{(m)}(W)) \cong (\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m) \otimes \epsilon^{\otimes(1-m)}. \tag{9}$$

We will prove the theorem using Lemma 1 (Nakayama’s Lemma).¹ We also need the following simple equivalences concerning invariant and determinantal polynomials:

Lemma 2 *Let $k \in \mathbb{N}$, and let W be a complex reflection group acting on $S := \mathbb{C}[\mathbf{x}, \mathbf{y}] \otimes \epsilon^{\otimes k}$. Let $p_i \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$, let $\text{inv}_i \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ be invariant in S , and let $\text{alt}_i \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ be determinantal in S . Set $p := \sum_i p_i \text{alt}_i$ and $q := \sum_i p_i \text{inv}_i$. Then*

$$\begin{aligned} p \text{ determinantal in } S &\Leftrightarrow p = \sum e_\epsilon(p_i) \text{alt}_i, \\ q \text{ determinantal in } S &\Leftrightarrow q = \sum e_\epsilon(p_i) \text{inv}_i. \end{aligned}$$

Proof We prove the first statement; the proof of the second is analogous. p is determinantal in S if and only if

$$\begin{aligned} p &= e_\epsilon(p) = \frac{1}{|W|} \sum_\omega \det^{-1}(\omega) \omega \left(\sum_i p_i \text{alt}_i \right) \\ &\stackrel{(8)}{=} \frac{1}{|W|} \sum_\omega \sum_i \det^{-1}(\omega) \omega(p_i) \det(\omega) \text{alt}_i \\ &= \sum_i \left(\frac{1}{|W|} \sum_\omega \omega(p_i) \right) \text{alt}_i = \sum_i e_\epsilon(p_i) \text{alt}_i. \quad \square \end{aligned}$$

Proof of Theorem 2 Using complete reducibility, one can rewrite the left-hand side of (9) as

$$e_\epsilon(\mathcal{A}^{m-1} \otimes \epsilon^{\otimes(1-m)}) / e_\epsilon(\mathcal{I}\mathcal{A}^{m-1} \otimes \epsilon^{\otimes(1-m)}). \tag{10}$$

Let $p \in \mathcal{A}^{m-1}$, that is, p can be written as

$$p = \sum p_i \text{alt}_{(i,1)} \cdots \text{alt}_{(i,m-1)},$$

where $p_i \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ and where $\text{alt}_{(i,j)}$ is determinantal in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. By definition,

$$\omega(\text{alt}_{(i,1)} \cdots \text{alt}_{(i,m-1)}) = \det^{m-1}(\omega) \text{alt}_{(i,1)} \cdots \text{alt}_{(i,m-1)}$$

¹We thank Vic Reiner for improving several arguments in the proof of Theorem 2.

for $\omega \in W$, or equivalently, $\text{alt}_{(i,1)} \cdots \text{alt}_{(i,m-1)}$ is invariant in $\mathbb{C}[\mathbf{x}, \mathbf{y}] \otimes \epsilon^{\otimes(1-m)}$. Lemma 2 now implies that p is determinantal in $\mathbb{C}[\mathbf{x}, \mathbf{y}] \otimes \epsilon^{\otimes(1-m)}$ if and only if

$$p = \sum \mathbf{e}_\epsilon(p_i) \text{alt}_{(i,1)} \cdots \text{alt}_{(i,m-1)}.$$

Let $\tilde{\mathcal{A}}^{(m)}$ be the complex vector space of all linear combinations of products of m determinantal polynomials. The product of an invariant and a determinantal polynomial is again determinantal; this turns $\tilde{\mathcal{A}}^{(m)}$ into a $\mathbb{C}[\mathbf{x}, \mathbf{y}]^W$ -module.

As $\mathbf{e}_\epsilon(p_i)$ is determinantal in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$, we get that p is determinantal in $\mathcal{A}^{m-1} \otimes \epsilon^{\otimes(1-m)}$ if and only if $p \in \tilde{\mathcal{A}}^{(m)}$. By the same argument, $p \in \mathcal{I}\mathcal{A}^{m-1}$ is determinantal in $\mathcal{I}\mathcal{A}^{m-1} \otimes \epsilon^{\otimes(1-m)}$ if and only if $p \in \mathbb{C}[\mathbf{x}, \mathbf{y}]_+^W \tilde{\mathcal{A}}^{(m)} \subseteq \tilde{\mathcal{A}}^{(m)}$.

Together with (10), we get

$$\mathbf{e}_\epsilon(DR^{(m)}(W)) = (\tilde{\mathcal{A}}^{(m)} / \mathbb{C}[\mathbf{x}, \mathbf{y}]_+^W \tilde{\mathcal{A}}^{(m)}) \otimes \epsilon^{\otimes(1-m)}. \tag{11}$$

By Nakayama’s Lemma, the right-hand side of (11) has a vector space basis given by (the images of) a minimal generating set of $\tilde{\mathcal{A}}^{(m)}$ as a $\mathbb{C}[\mathbf{x}, \mathbf{y}]^W$ -module. On the other hand, $\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m$ has a vector space basis given by (the images of) a minimal generating set of \mathcal{A}^m considered as a $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ -module.

\mathcal{A}^m is generated as a $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ -module by all products of m determinantal polynomials. Therefore, it has a minimal generating set \mathcal{S} that is also contained in $\tilde{\mathcal{A}}^{(m)}$. Using again Lemma 2, \mathcal{S} minimally generates $\tilde{\mathcal{A}}^{(m)}$ as a $\mathbb{C}[\mathbf{x}, \mathbf{y}]^W$ -module. Thus, the map

$$s + \mathbb{C}[\mathbf{x}, \mathbf{y}]_+^W \tilde{\mathcal{A}}^{(m)} \mapsto s + \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m$$

for $s \in \mathcal{S}$ extends to a bigraded vector space isomorphism. As both W -actions coincide, this completes the proof. □

Definition 2 Define the W -module $M^{(m)}(W)$ to be the determinantal component of $DR^{(m)}(W)$,

$$M^{(m)}(W) := \mathbf{e}_\epsilon(DR^{(m)}(W)) \cong (\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m) \otimes \epsilon^{\otimes(1-m)}.$$

As we have seen in (4) for the symmetric group, the space $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ has a well-known basis given by

$$\mathcal{B}_W := \{ \mathbf{e}_\epsilon(m(\mathbf{x}, \mathbf{y})) : m(\mathbf{x}, \mathbf{y}) \text{ monomial in } \mathbf{x}, \mathbf{y} \text{ with } \mathbf{e}_\epsilon(m(\mathbf{x}, \mathbf{y})) \neq 0 \},$$

and the ideal $\mathcal{A} \subseteq \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is generated by \mathcal{B}_W . Thus, finding a minimal generating set for \mathcal{A} as a $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ -module is equivalent to finding a maximal linearly independent subset of \mathcal{B}_W with coefficients in $\mathbb{C}[\mathbf{x}, \mathbf{y}]_+^W$. As for the symmetric group, it is an open problem to construct such a maximal linearly independent subset of \mathcal{B}_W .

For the other classical types, \mathcal{B}_W can also be described using the bivariate Vandermonde determinant. In type B , it reduces to

$$\mathcal{B}_{B_n} = \{ \Delta_X : X = \{ (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n) \} \subseteq \mathbb{N} \times \mathbb{N}, |X| = n, \alpha_i + \beta_i \equiv 1 \pmod{2} \},$$

and in type D , it reduces to

$$\mathcal{B}_{D_n} = \{ \Delta_X : X = \{(\alpha_i, \beta_i), \dots, (\alpha_n, \beta_n)\} \subseteq \mathbb{N} \times \mathbb{N}, \\ |X| = n, \alpha_i + \beta_i \equiv \alpha_j + \beta_j \pmod{2} \}.$$

Conjecture 2 leads to the following definition:

Definition 3 Let W be a complex reflection group, let $DR^{(m)}(W)$ be the space of generalized diagonal coinvariants, and let $M^{(m)}(W)$ be its alternating component. Define the q, t -Fuß–Catalan numbers $\text{Cat}^{(m)}(W; q, t)$ as

$$\text{Cat}^{(m)}(W; q, t) := \mathcal{H}(M^{(m)}(W); q, t) = \mathcal{H}(\mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m; q, t).$$

By definition, the q, t -Fuß–Catalan numbers $\text{Cat}^{(m)}(W; q, t)$ are polynomials in q and t with nonnegative integer coefficients; moreover, they are, for real reflection groups, symmetric in q and t . Conjecture 2 would imply that

$$\text{Cat}^{(m)}(W; 1, 1) = \text{Cat}^{(m)}(W).$$

In the following section, we present conjectured properties of those polynomials which are based on computer experiments and which will be later supported by proving several special cases.

3.3 Conjectured properties of the q, t -Fuß–Catalan numbers

In addition to the computations of the dimension of $M^{(m)}(W)$, we computed its bigraded Hilbert series $\text{Cat}^{(m)}(W; q, t)$ using the computer algebra system Macaulay 2 [33]. The computations are as well listed in Tables 2–4 in the Appendix. All further conjectures are based on these computations.

3.3.1 The specializations $t^{\pm 1} = q^{\mp 1}$

The following conjecture, which is obviously stronger than Conjecture 2, would generalize (7) and would thereby answer a question of Kriloff and Reiner [2, Problem 2.2]:

Conjecture 3 Let W be a well-generated complex reflection group acting on $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[V \oplus V^*]$ as described above. Set $N = \sum (d_i - 1)$ to be the number of reflections in W , and set $N^* = \sum (d_i^* + 1)$ to be the number of reflecting hyperplanes. Then

$$q^{mN} \text{Cat}^{(m)}(W; q, q^{-1}) = q^{mN^*} \text{Cat}^{(m)}(W; q^{-1}, q) = \prod_{i=1}^{\ell} \frac{[d_i + mh]_q}{[d_i]_q}.$$

For W being a real reflection group acting on a real vector space V , there is a one-to-one correspondence between reflections in W and reflecting hyperplanes for W in V .

Thus, the conjecture is consistent with the fact that the q, t -Fuß–Catalan numbers are symmetric in q and t in this case.

This q -extension of the Fuß–Catalan numbers seems to have first appeared for real reflection groups in a paper by Berest, Etingof, and Ginzburg [8], where it is obtained as a certain Hilbert series. Their work implies that in this case, the extension is in fact a polynomial with nonnegative integer coefficients. For well-generated complex reflection groups, this is still true, but so far it has only been verified by appeal to the classification. In Sect. 4, we will exhibit the connection of the presented conjecture to the work of Berest, Etingof, and Ginzburg.

Corollary 1 *Conjecture 3 would imply that the q -degree of $\text{Cat}^{(m)}(W; q, t)$ is given by mN^* and the t -degree is given by mN .*

3.3.2 The specialization $t = 1$

By definition, $\text{Cat}^{(m)}(W; q, t)$ is a polynomial in $\mathbb{N}[q, t]$. As for type A , this leads to the natural question of a combinatorial description of $\text{Cat}^{(m)}(W; q, t)$: are there statistics $q\text{stat}$ and $t\text{stat}$ on objects counted by $\text{Cat}^{(m)}(W)$ which generalize the *area* and the *bounce* statistics on m -Dyck paths $\mathcal{D}_n^{(m)}$ such that

$$\text{Cat}^{(m)}(W; q, t) = \sum_D q^{\text{qstat}(D)} t^{\text{tstat}(D)} ?$$

The conjecture we want to present in this section concerns the case of crystallographic reflection groups and the specialization $t = 1$ of this open problem.

The following definition is due to Athanasiadis [5] and generalizes a construction of Shi [39]:

Definition 4 Let W be a crystallographic reflection group acting on a real vector space V . The extended Shi arrangement $\text{Shi}^{(m)}(W)$ is the collection of hyperplanes in V given by

$$H_\alpha^{(k)} := \{x : (\alpha, x) = k\} \quad \text{for } \alpha \in \Phi^+ \text{ and } -m < k \leq m,$$

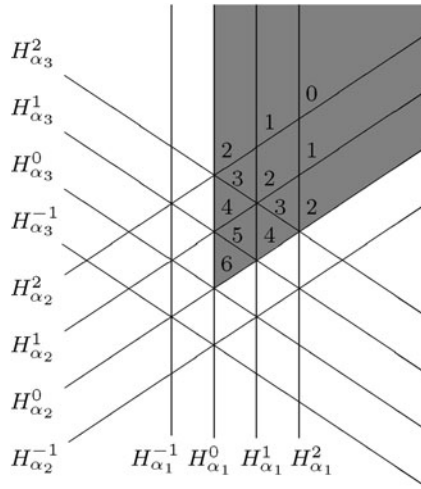
where $\Phi^+ \subset V$ is a set of positive roots associated to W .

A connected component of the complement of the hyperplanes in $\text{Shi}^{(m)}(W)$ is called a region of $\text{Shi}^{(m)}(W)$, and a positive region is a region which lies in the fundamental chamber of the associated Coxeter arrangement, see Fig. 3 for an example.

The following result concerning the total number of regions of $\text{Shi}^{(m)}(W)$ had been conjectured by Edelman and Reiner [13, Conjecture 3.3] and by Athanasiadis [4, Question 6.2] and was proved uniformly by Yoshinaga [44, Theorem 1.2]:

Theorem 3 (M. Yoshinaga) *Let W be a crystallographic reflection group, and let m be a positive integer. Then the number of regions of $\text{Shi}^{(m)}(W)$ is equal to $(mh + 1)^\ell$, where ℓ is the rank of W and where h is its Coxeter number.*

Fig. 3 The extended Shi arrangement $\text{Shi}^{(2)}(A_2)$



In [6], Athanasiadis counted the number of positive regions of $\text{Shi}^{(m)}(W)$:

Theorem 4 (C.A. Athanasiadis) *Let W be a crystallographic reflection group, and let m be a positive integer. Then*

$$|\{\text{positive regions of } \text{Shi}^{(m)}(W)\}| = \text{Cat}^{(m)}(W).$$

Let W be a crystallographic reflection group. Fix the positive region R^0 to be the region given by $\{x : 0 < (\alpha, x) < 1 \text{ for all } \alpha \in \Phi^+\}$, and the positive region R^∞ to be the region given by $\{x : (\alpha, x) > m \text{ for all } \alpha \in \Phi^+\}$. In terms of affine reflection groups, R^0 is called fundamental alcove. The height of a region is defined to be the number of hyperplanes in $\text{Shi}^{(m)}(W)$ that separate R from R^0 , and the coheight of a region R , denoted by $\text{coh}(R)$, is defined by

$$\text{coh}(R) := mN - \text{height}(R),$$

where N denotes the number of positive roots and of reflecting hyperplanes. Observe that the coheight counts, for a positive region R , the number of hyperplanes separating R from R^∞ .

Conjecture 4 *Let W be a crystallographic reflection group. Then the q, t -Fuß-Catalan numbers reduce for the specialization $t = 1$ to*

$$\text{Cat}^{(m)}(W; q, 1) = \text{Cat}^{(m)}(W; q) := \sum_R q^{\text{coh}(R)},$$

where the sum ranges over all positive regions of $\text{Shi}^{(m)}(W)$.

For example, it can be seen in Fig. 3 that

$$\text{Cat}^{(2)}(A_2; q) = 1 + 2q + 3q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

This is equal to the specialization $t = 1$ in

$$\text{Cat}^{(2)}(A_2; q, t) = q^6 + q^5t + \cdots + qt^5 + t^6 + q^4t + \cdots + qt^4 + q^2t^2.$$

Proposition 1 *Let $W = A_{n-1}$. Then $\text{Cat}^{(m)}(W; q)$ is equal to the area generating function on m -Dyck paths.*

To prove the proposition, we define filtered chains in the root poset associated to W . They were introduced by Athanasiadis [6]. Define a partial order on a set of positive roots Φ^+ associated to W by letting $\alpha < \beta$ if $\beta - \alpha$ is a nonnegative linear combination of simple roots. Equipped with this partial order, Φ^+ is called the root poset associated to W ; it does not depend on the specific choice of positive roots. Let $\mathcal{I} = \{I_1 \subseteq \cdots \subseteq I_m\}$ be an increasing chain of order ideals in Φ^+ (i.e., $\alpha \leq \beta \in I_i \Rightarrow \alpha \in I_i$). \mathcal{I} is called a filtered chain of length m if

$$(I_i + I_j) \cap \Phi^+ \subseteq I_{i+j}$$

for all $i, j \geq 1$ with $i + j \leq m$ and

$$(J_i + J_j) \cap \Phi^+ \subseteq J_{i+j}$$

for all $i, j \geq 1$, where $J_i := \Phi^+ \setminus I_i$ and $J_i = J_m$ for $i > m$. Athanasiadis constructed an explicit bijection ψ between positive regions of $\text{Shi}^{(m)}(W)$ and filtered chains of length m in Φ^+ , such that

$$\text{coh}(R) = |\psi(R)|,$$

where $|\{I_1 \subseteq \cdots \subseteq I_m\}| := |I_1| + \cdots + |I_m|$. In particular, this implies that

$$\text{Cat}^{(m)}(W; q) = \sum_{\mathcal{I}} q^{|\mathcal{I}|},$$

where the sum ranges over all filtered chains of length m in Φ^+ .

Proof of Proposition 1 To prove the proposition, we construct a bijection between m -Dyck paths of semilength n and filtered chains of length m in the root poset $\Phi^+ = \{\epsilon_j - \epsilon_i : 1 \leq i < j \leq n\}$ of type A_{n-1} . An m -Dyck path of semilength n can be encoded as a sequence (a_1, \dots, a_n) of integers such that $a_1 = 0$ and $a_{i+1} \leq a_i + m$. Define the sum of an m -Dyck path and an m' -Dyck path, both of semilength n , to be the $(m + m')$ -Dyck path of semilength n obtained by adding the associated sequences componentwise,

$$(a_1, \dots, a_n) + (a'_1, \dots, a'_n) := (a_1 + a'_1, \dots, a_n + a'_n).$$

Together with the well-known bijection between order ideals in Φ^+ and 1-Dyck paths, this yields a map from filtered chains of length m in Φ^+ to m -Dyck paths by summing the 1-Dyck paths associated to the ideals in the filtered chain; see Fig. 4 for an example. Observe that this map sends the coheight of a given filtered chain to the area of the associated m -Dyck path.

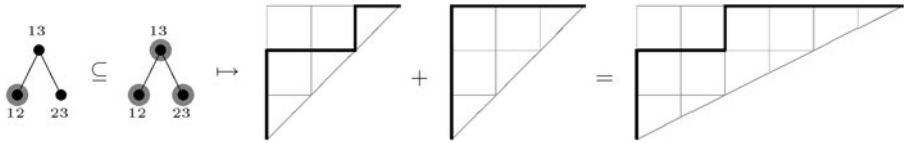


Fig. 4 A filtered chain of order ideals in the root poset of type A_2 , where the root $e_j - e_i$ for $1 \leq i < j \leq 3$ is denoted by ij , and the associated 2-Dyck path is $(0, 1, 0) + (0, 1, 2) = (0, 2, 2)$

To show that this map is in fact a bijection, let $\{I_1 \subseteq \dots \subseteq I_m\}, \{I'_1 \subseteq \dots \subseteq I'_m\}$ be two filtered chains which map to the same path. Assume that they are not equal, i.e., there exists $\epsilon_j - \epsilon_i \in I_\ell \setminus I'_\ell$ for some ℓ . As both chains map to the same path, there exists $k < i, \ell' > 0$, such that $\epsilon_j - \epsilon_k \in I'_{\ell+\ell'} \setminus I_{\ell+\ell'}$. As both chains are filtered, this gives $\epsilon_i - \epsilon_k \in I'_{\ell'} \setminus I_{\ell'}$. This gives rise to an infinite sequence $(\epsilon_j - \epsilon_i, \epsilon_i - \epsilon_k, \dots)$ of pairwise different positive roots, which is a contradiction. As it is known that both sets have the same cardinality, the statement follows. \square

3.4 The dihedral groups

In an unpublished work in the context of their PhD theses [1, 35], E. Alfano and J. Reiner were able to describe uniformly the diagonal coinvariant ring $DR^{(1)}(W)$ for W being a dihedral group. For the sake of readability, we introduce the q, t -extension $[n]_{q,t}$ of an integer n which we define by

$$[n]_{q,t} := \frac{q^n - t^n}{q - t} = q^{n-1} + q^{n-2}t + \dots + qt^{n-2} + t^{n-1}.$$

Then $[n]_{q,1} = [n]_{1,q} = [n]_q$ is the well-known q -extension of an integer n . The following description is taken from [25, Sect. 7.5]:

Theorem 5 (J. Alfano, E. Reiner) *Let $W = I_2(k)$ be the dihedral group of order $2k$. Then*

$$\mathcal{H}(DR^{(1)}(W); q, t) = 1 + [k + 1]_{q,t} + qt + 2 \sum_{i=1}^{k-1} [i + 1]_{q,t}.$$

By a simple computation, we get the following corollary.

Corollary 2 *Let $N = k$ be the number of reflections in $W = I_2(k)$, and let $h = k$ be its Coxeter number. Then*

$$q^N \mathcal{H}(DR^{(1)}(W); q, q^{-1}) = [h + 1]_q^2.$$

In particular, the quotient in Theorem 1 is trivial for the dihedral groups, $R_W = DR^{(1)}(W)$.

Alfano and Reiner obtained Theorem 5 by providing an explicit description of $DR^{(1)}(W)$:

- (i) The first 1 belongs to the unique copy of the trivial representation in bidegree $(0, 0)$,
- (ii) The string $[k + 1]_{q,t} + qt$ belongs to copies of the determinantal representation which are generated by

$$D, \Delta(D), \dots, \Delta^k(D) \quad \text{and} \quad x_1y_2 - x_2y_1,$$

where

$$D(x_1, x_2) := 2k \prod_{i=0}^{k-1} (\sin(\pi i/k)x_1 + \cos(\pi i/k)x_2)$$

is the discriminant of W , Δ is the operator defined by $\Delta := \partial_{x_1} \cdot y_1 + \partial_{x_2} \cdot y_2$, and where $\Delta^\ell(D)$ has bidegree $(k - \ell, \ell)$.

- (iii) The later sum belongs to \mathfrak{sl}_2 -strings.

By Theorem 5 and the following discussion, we can immediately compute the q, t -Fuß–Catalan numbers for the dihedral groups:

Corollary 3 *Let $W = I_2(k)$. Then*

$$\text{Cat}^{(1)}(W; q, t) = [k + 1]_{q,t} + qt = q^k + q^{k-1}t + \dots + qt^{k-1} + t^k + qt.$$

In [3, Chap. 5.4.1], Armstrong suggests how the “root poset” for the dihedral group $I_2(k)$ should look like for any k . For the crystallographic dihedral groups, it reduces to the root poset introduced in Sect. 3.3.2. We reproduce his suggestion in Fig. 5.

Corollary 4 *Let Φ^+ be the “root poset” associated to the dihedral group $I_2(k)$ as shown in Fig. 5. Then*

$$\text{Cat}^{(1)}(I_2(k); q, 1) = \sum_I q^{\text{coh}(I)},$$

where the sum ranges over all order ideals in Φ^+ . In particular, Conjecture 4 holds for the crystallographic dihedral groups $I_2(k)$ with $k \in \{2, 3, 4, 6\}$ and $m = 1$.

From Theorem 5 one can also deduce the q, t -Fuß–Catalan numbers $\text{Cat}^{(m)}(W; q, t)$.

Fig. 5 Armstrong’s suggestion for a “root poset” of type $I_2(k)$



Theorem 6 *The q, t -Fuß–Catalan numbers for the dihedral group $I_2(k)$ are given by*

$$\text{Cat}^{(m)}(I_2(k); q, t) = \sum_{j=0}^m q^{m-j} t^{m-j} [jk + 1]_{q,t}.$$

To prove the theorem, we need the following lemma:

Lemma 3 *Let k be a positive integer, and let $p \in \mathbb{C}[x_1, x_2]$ be homogeneous of degree $a + b$. Let $0 \leq i_1, \dots, i_k, j_1, \dots, j_k \leq a + b$ be two sequences such that $\sum i_\ell = \sum j_\ell$. Then*

$$\Delta^{j_1}(p) \cdots \Delta^{j_k}(p) \in \langle \Delta^{i_1}(p) \cdots \Delta^{i_k}(p), B^{k-1} \cdot (x_1 y_2 - x_2 y_1) \rangle,$$

where $B := \{p, \Delta(p), \dots, \Delta^{a+b}(p), x_1 y_2 - x_2 y_1\}$.

Proof Let $\mathbf{m} = x_1^a x_2^b$ be a monomial in p , and let $i \leq a + b$. By definition,

$$\Delta^i(\mathbf{m}) = i! \sum_{\ell=0}^i \binom{a}{\ell} \binom{b}{i-\ell} x_1^{a-\ell} y_1^\ell x_2^{b-i+\ell} y_2^{i-\ell},$$

and therefore,

$$\Delta^i(\mathbf{m}) \equiv \Delta_{\text{mod}}^i(\mathbf{m}) := c_i x_2^{-i} y_2^i \mathbf{m} \pmod{\langle x_1 y_2 - x_2 y_1 \rangle},$$

where $c_i := i! \sum_{\ell=0}^i \binom{a}{\ell} \binom{b}{i-\ell} = i! \binom{a+b}{i}$. Observe that c_i does only depend on i and on $a + b$ and that $c_i > 0$. Hence, as p is homogeneous of degree $a + b$, the linearity of Δ implies

$$\Delta^i(p) \equiv \Delta_{\text{mod}}^i(p) := c_i x_2^{-i} y_2^i p \pmod{\langle x_1 y_2 - x_2 y_1 \rangle}.$$

Moreover, we have

$$\Delta^{i_1}(p) \cdots \Delta^{i_k}(p) \equiv \Delta^{i_1}(p) \cdots \Delta^{i_{k-1}}(p) \Delta_{\text{mod}}^{i_k}(p) \pmod{\langle B^{k-1} \cdot (x_1 y_2 - x_2 y_1) \rangle}.$$

With $\Delta^i(p)$ and $x_1 y_2 - x_2 y_1$, $\Delta_{\text{mod}}^i(p)$ is as well contained in $\langle B \rangle$, and therefore,

$$\Delta^{i_1}(p) \cdots \Delta^{i_k}(p) \equiv \Delta_{\text{mod}}^{i_1}(p) \cdots \Delta_{\text{mod}}^{i_k}(p) \pmod{\langle B^{k-1} \cdot (x_1 y_2 - x_2 y_1) \rangle}.$$

Setting $d := c_{i_1} \cdots c_{i_k}$, the right-hand side equals $d x_2^{-\sum i_\ell} y_2^{\sum i_\ell} p^k$. By the same argument,

$$\Delta^{j_1}(p) \cdots \Delta^{j_k}(p) \equiv c x_2^{-\sum j_\ell} y_2^{\sum j_\ell} p^k \pmod{\langle B^{k-1} \cdot (x_1 y_2 - x_2 y_1) \rangle},$$

where $c := c_{j_1} \cdots c_{j_k}$. As $\sum i_\ell = \sum j_\ell$, we obtain

$$d \Delta^{j_1}(p) \cdots \Delta^{j_k}(p) - c \Delta^{i_1}(p) \cdots \Delta^{i_k}(p) \in \langle B^{k-1} \cdot (x_1 y_2 - x_2 y_1) \rangle.$$

As $c, d \neq 0$, the lemma follows. □

Proof of Theorem 6 Recall that the ideal \mathcal{A}^m is generated by all products of m generators of determinantal representations in the various bidegrees. We have seen above that \mathcal{A} is minimally generated by

$$B := \{D, \Delta(D), \dots, \Delta^k(D), x_1y_2 - x_2y_1\}$$

and that $\Delta^\ell(D)$ has bidegree $(k - \ell, \ell)$ and $x_1y_2 - x_2y_1$ has bidegree $(1, 1)$. For given $0 \leq j \leq m$ and $0 \leq i \leq k(m - j)$, the generators of \mathcal{A}^m having bidegree $((m - j)k - i + j, i + j)$ are of the form

$$\Delta^{i_1}(D) \dots \Delta^{i_{m-j}}(D) \cdot (x_1y_2 - x_2y_1)^j, \tag{12}$$

where $\sum i_\ell = i$. As D is homogeneous in \mathbf{x} of degree k , the previous lemma implies that any subset of B^m which minimally generates \mathcal{A}^m contains one and only one generator of the form (12) for each $0 \leq j \leq m$ and each $0 \leq i \leq k(m - j)$. Thus,

$$\begin{aligned} \text{Cat}^{(m)}(I_2(k); q, t) &= \sum_{j=0}^m \sum_{i=0}^{k(m-j)} q^{(m-j)k-i+j} t^{i+j} = \sum_{j=0}^m \sum_{i=0}^{kj} q^{jk-i+(m-j)t} t^{i+(m-j)} \\ &= \sum_{j=0}^m q^{m-j} t^{m-j} [kj + 1]_{q,t}. \end{aligned} \tag{12}$$

From this theorem we can immediately deduce the following recurrence relation:

Corollary 5 *The q, t -Fuß–Catalan numbers for the dihedral group $I_2(k)$ satisfy the recurrence relation*

$$\text{Cat}^{(m)}(W; q, t) = [mk + 1]_{q,t} + qt \text{Cat}^{(m-1)}(W; q, t).$$

We can also deduce Conjectures 2 and 3 for the dihedral groups:

Corollary 6 *Conjectures 2 and 3 hold for the dihedral groups: let $W = I_2(k)$; then*

$$q^{mk} \text{Cat}^{(m)}(W; q, q^{-1}) = \frac{[2 + mk]_q [k + mk]_q}{[2]_q [k]_q}.$$

Proof By Theorem 6, we have

$$q^{mk} \text{Cat}^{(m)}(W; q, q^{-1}) = q^{mk} \sum_{j=0}^m [jk + 1]_{q,q^{-1}} = \sum_{j=0}^m q^{(m-j)k} \frac{[2jk + 2]_q}{[2]_q}.$$

Therefore, it remains to show that

$$\sum_{j=0}^m q^{(m-j)k} [2jk + 2]_q = \frac{[2 + mk]_q [k + mk]_q}{[k]_q}. \tag{13}$$

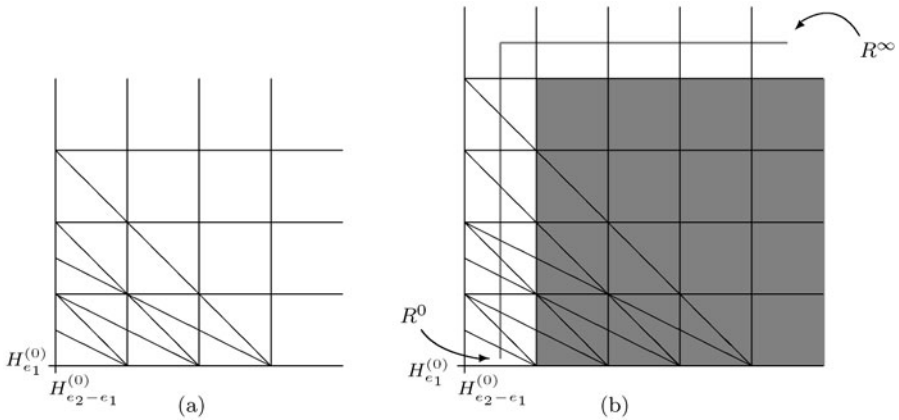


Fig. 6 (a) The positive regions of $\text{Shi}^{(3)}(B_2)$. (b) Their embedding into the positive regions of $\text{Shi}^{(4)}(B_2)$; the remaining region walk from R^∞ to R^0

To prove this equality, observe that on the left-hand side of (13), the terms for j and $m - j$ sum up to $[mk + 2]_q(q^{jk} + q^{(m-j)k})$. This gives

$$\sum_{j=0}^m q^{(m-j)k} [2jk + 2]_q = [mk + 2]_q (1 + q^k + \dots + q^{mk}). \quad \square$$

Our next goal is to generalize Corollary 4 for the crystallographic dihedral groups.

Theorem 7 *Conjecture 4 holds for the dihedral group $I_2(k)$ with $k \in \{2, 3, 4, 6\}$.*

To prove the theorem, we define a region walk from R^∞ to R^0 to be a sequence of consecutive regions $R^\infty = R_0, \dots, R_i = R^0$ of $\text{Shi}^{(m)}(W)$ such that $\text{coh}(R_\ell) = \ell$ and R_ℓ and $R_{\ell+1}$ are separated by exactly one hyperplane.

First, we prove the case $B_2 = I_2(4)$; the cases $k \in \{2, 3\}$ are analogous.

Proof for $k = 4$ Our goal is to show that $\text{Cat}^{(m)}(B_2, q)$ satisfy the recurrence relation in Corollary 5 for $t = 1$,

$$\text{Cat}^{(m)}(B_2; q) = [4m + 1]_q + q \text{Cat}^{(m-1)}(B_2; q);$$

the theorem then follows from Corollary 3.

As indicated in Fig. 6, it is immediate how we can embed the positive regions of $\text{Shi}^{(m-1)}(B_2)$ into the positive regions of $\text{Shi}^{(m)}(B_2)$; note that this embedding increases the coheight by 1. The remaining regions form a region walk from R^∞ to R^0 and $\sum_R q^{\text{coh}(R)} = [4m + 1]_q$, where the sum ranges over all regions in the region walk. This gives the proposed recurrence relation. \square

The proof for $G_2 = I_2(6)$ is more involved. We again want to embed the positive regions of $\text{Shi}^{(m-1)}(G_2)$ into the positive regions of $\text{Shi}^{(m)}(G_2)$. To do this, we first

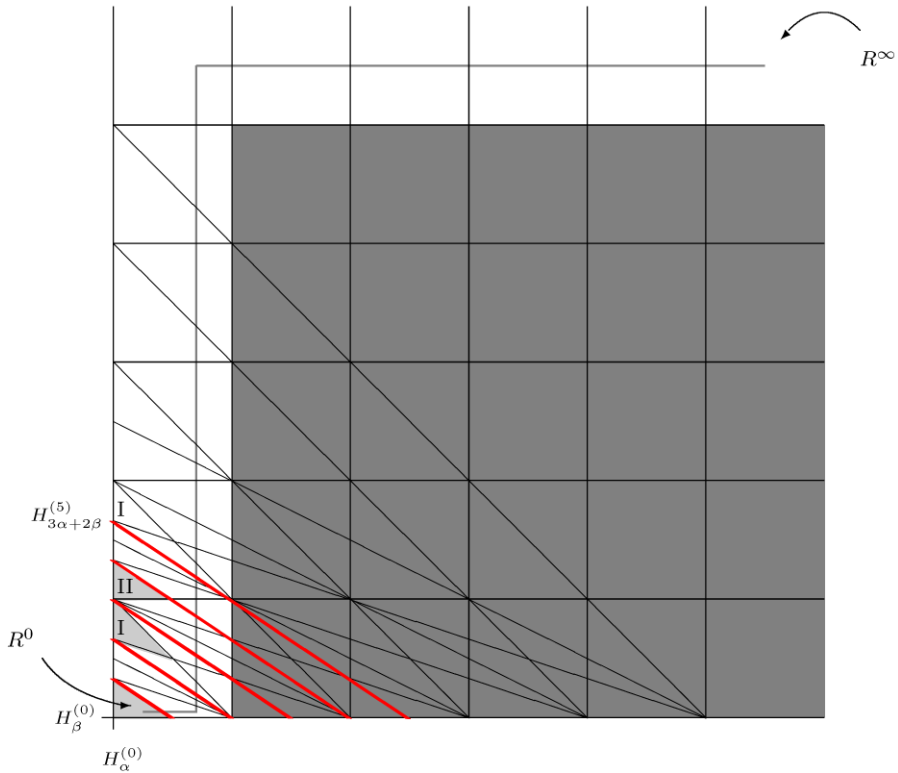


Fig. 7 The embedding of the positive regions of $\text{Shi}^{(4,3)}(G_2)$ into the positive regions of $\text{Shi}^{(5,5)}(G_2)$; the remaining region walk from R^∞ to R^0 and the “additional regions”

have to generalize the notion. The positive roots are given by

$$\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\},$$

where one possible choice is $\alpha = e_2 - e_1, \beta = 2e_2 - e_2 - e_1$, compare, e.g., [31, Sect. 2.10]. For $k \leq m$, define $\text{Shi}^{(m,k)}(G_2)$ to be the collection of the hyperplanes in $\text{Shi}^{(m)}(G_2)$ other than $H_{3\alpha+2\beta}^{(i)}$ for $i > k$, see the shaded part on the right in Fig. 7 for an example. Furthermore, define $\text{Cat}^{(m,k)}(G_2; q) := \sum_R q^{\text{coh}(R)}$, where the sum ranges over all positive regions of $\text{Shi}^{(m,k)}(G_2)$. In particular, $\text{Cat}^{(m,m)}(G_2; q) = \text{Cat}^{(m)}(G_2; q)$.

Proof for $k = 6$ For $k \geq 2$, it is again immediate how we can embed the positive regions of $\text{Shi}^{(m-1,k-2)}(G_2)$ into the positive regions of $\text{Shi}^{(m,k)}(G_2)$ as indicated in Fig. 7; moreover, this embedding increases the coheight by 1. As in the proof for $k = 4$, we want to sum over all positive regions of $\text{Shi}^{(m,k)}(G_2)$ which do not come from positive regions of $\text{Shi}^{(m-1,k-2)}(G_2)$. We obtain as well a region walk from R^∞ to R^0 with $\sum_R q^{\text{coh}(R)} = [5m + k + 1]_q$, where the sum ranges over all regions in the region walk. Now, there are two types of “additional regions” remaining neither

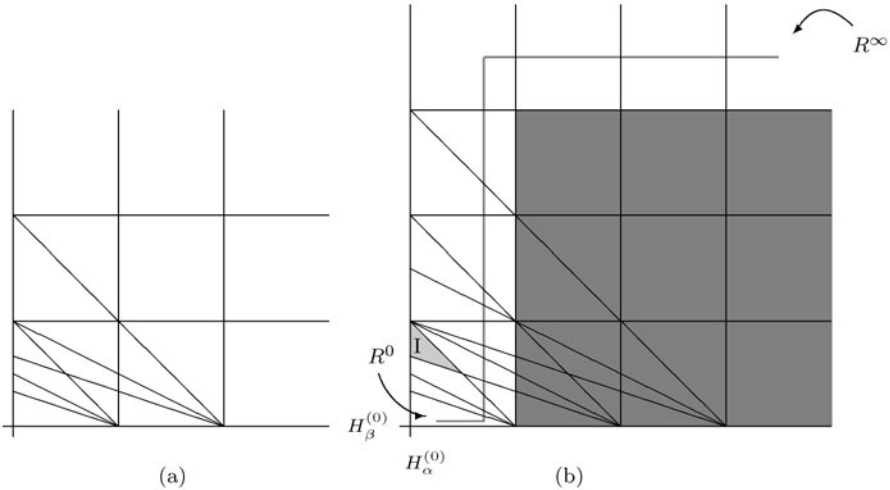


Fig. 8 (a) The positive regions of $\text{Shi}^{(2,0)}(G_2)$. (b) Their embedding into the positive regions of $\text{Shi}^{(3,0)}(G_2)$; the remaining region walk from R^∞ to R^0 and the “additional region”

of which lie in the region walk. These are confined by the hyperplane $H_\alpha^{(0)}$ and either hyperplanes of the form $H_{\alpha+\beta}^{(i)}, H_{3\alpha+\beta}^{(3i-1)}$ for $1 \leq i \leq \lceil \frac{m-1}{3} \rceil$ or hyperplanes of the form $H_\beta^{(i)}, H_{3\alpha+2\beta}^{(3i+3)}$ for $1 \leq i \leq \lfloor \frac{k-1}{3} \rfloor$. The former are labeled in the picture by I, and the latter by II. In total, we obtain

$$\begin{aligned} \text{Cat}^{(m,k)}(G_2; q) &= q \text{Cat}^{(m-1,k-2)}(G_2; q) + [5m + k + 1]_q \\ &\quad + \sum_{\ell=1}^{\lfloor \frac{k}{3} \rfloor + \lfloor \frac{k-1}{3} \rfloor} q^{5m+k-5\ell} + \sum_{\ell=\lfloor \frac{k}{3} \rfloor + 1}^{\lceil \frac{m-1}{3} \rceil} q^{5m+4-7\ell}. \end{aligned} \tag{14}$$

Here, the q in front of $\text{Cat}^{(m-1,k-2)}(G_2; q)$ comes from the fact that the embedding increases the coheight by 1; $[5m + k + 1]_q$ is obtained from the region walk from R^∞ to R^0 ; the first sum is obtained from the “additional regions” below the hyperplane $H_{3\alpha+2\beta}^{(k)}$; and the second sum is obtained from the “additional regions” above this hyperplane.

The next step is to embed the positive regions of $\text{Shi}^{(m-1,0)}(G_2)$ into the positive regions of $\text{Shi}^{(m,0)}(G_2)$ in the same sense as above, compare Fig. 8. By taking the region walk from R^∞ to R^0 and the “additional regions” into account, we obtain

$$\text{Cat}^{(m,0)}(G_2; q) = q \text{Cat}^{(m-1,0)}(G_2; q) + [5m + 1]_q + \sum_{\ell=1}^{\lceil \frac{m-1}{3} \rceil} q^{5m+4-7\ell}. \tag{15}$$

The q in front of $\text{Cat}^{(m-1,0)}(G_2; q)$ comes again from the fact that the embedding increases the coheight by 1; $[5m + 1]_q$ is obtained from the region walk from R^∞ to R^0 ; and the sum is obtained from the “additional regions” confined by the hyperplane

$H_\alpha^{(0)}$ and hyperplanes of the form $H_{\alpha+\beta}^{(i)}, H_{3\alpha+\beta}^{(3i-1)}$ for $1 \leq i \leq \lceil \frac{m-1}{3} \rceil$. The additional region in the figure is labeled by I.

Moreover, one has the obvious relations

$$\text{Cat}^{(0,0)}(G_2; q) = 1, \tag{16}$$

$$\text{Cat}^{(m,1)}(G_2; q) = \text{Cat}^{(m,0)}(G_2; q) + q^{5m+1}. \tag{17}$$

The relations (14)–(17) uniquely determine $\text{Cat}^{(m,k)}(G_2; q)$. In particular, we get

$$\text{Cat}^{(1,1)}(G_2; q) \stackrel{(17)}{=} q^6 + \text{Cat}^{(1,0)}(G_2; q) \stackrel{(15)}{=} q^6 + [6]_q + q \text{Cat}^{(0,0)}(G_2; q) \stackrel{(16)}{=} [7]_q + q,$$

which again proves Corollary 4 in this case. We want to obtain the recurrence relation

$$\begin{aligned} \text{Cat}^{(m,m)}(G_2; q) &= [6m + 1]_q + q \text{Cat}^{(m-1,m-1)}(G_2; q) \\ &\stackrel{(14)}{=} q \text{Cat}^{(m-1,m-2)}(G_2; q) + [6m + 1]_q + \sum_{\ell=1}^{\lceil \frac{m-1}{3} \rceil + \lfloor \frac{m-1}{3} \rfloor} q^{6m-5\ell} \end{aligned}$$

for $m \geq 2$. Thus, after a shift in m , it remains to show that the difference $\text{Cat}^{(m,m)}(G_2; q) - \text{Cat}^{(m,m-1)}(G_2; q)$ is given by $\sum_{\ell=1}^{\lceil \frac{m}{3} \rceil + \lfloor \frac{m}{3} \rfloor} q^{6m+5-5\ell} = q^{6m} \lceil \frac{m}{3} \rceil + \lfloor \frac{m}{3} \rfloor q^{-5}$.

From the discussion after (14), this is equivalent to show that

$$\text{Cat}^{(m,m)}(G_2; q) - \text{Cat}^{(m,m-1)}(G_2; q) = \sum_R q^{\text{coh}(R)},$$

where the sum ranges over the fundamental region R^0 and the ‘‘additional regions’’ below the hyperplane $H_{3\alpha+2\beta}^{(m)}$; those regions are indicated in Fig. 7 in light grey.

Obviously, $\text{Shi}^{(m,m)}(G_2)$ is obtained from $\text{Shi}^{(m,m-1)}(G_2)$ by introducing the hyperplane $H_{3\alpha+2\beta}^{(m)}$. Thus,

$$\text{Cat}^{(m,m)}(G_2; q) - \text{Cat}^{(m,m-1)}(G_2; q) = \sum_R q^{\text{coh}(R)} - \sum_{R'} q^{\text{coh}(R')-1},$$

where the first sum ranges over all ‘‘shifted additional regions,’’ labeled in Fig. 9 by i_j for $1 \leq j \leq i \leq \lceil \frac{m}{3} \rceil + \lfloor \frac{m}{3} \rfloor$; the second sum ranges over all ‘‘shifted additional regions,’’ labeled in the figure by i_j for $j < i$; and where the coheight is considered in $\text{Shi}^{(m,m)}(G_2)$. Here, shifted means that the confining hyperplane $H_\alpha^{(0)}$ is replaced by $H_\alpha^{(j-1)}$ for $1 \leq j \leq \lceil \frac{m}{3} \rceil + \lfloor \frac{m}{3} \rfloor$. The term in the second sum labeled by i_j for $j < i$ cancels the term in the first labeled by i_{j+1} . In total, we obtain $\sum_R q^{\text{coh}(R)}$, where the sum ranges over all ‘‘additional regions’’ labeled by i_1 for $1 \leq j \leq i \leq \lceil \frac{m}{3} \rceil + \lfloor \frac{m}{3} \rfloor$. Those regions are exactly the fundamental region R^0 and the ‘‘additional regions’’ below the hyperplane $H_{3\alpha+2\beta}^{(m)}$. This completes the proof. \square

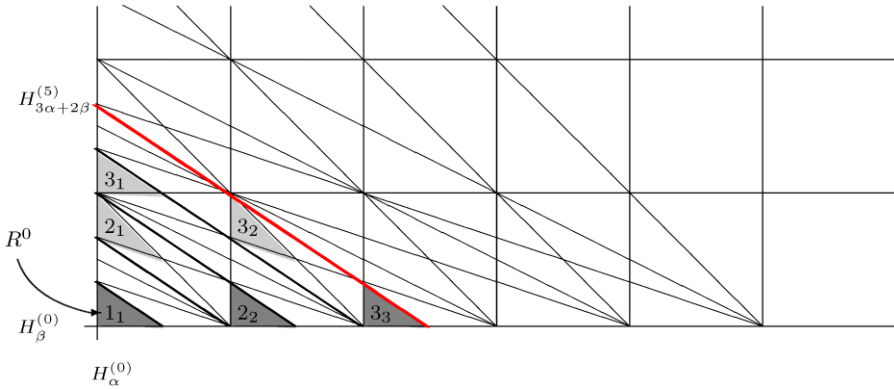


Fig. 9 Introducing the hyperplane $H_{3\alpha+2\beta}^{(5)}$ into $\text{Shi}^{(5,4)}(G_2)$

Remark 2 It follows from (15) that we cannot generalize Corollary 4 to $m \geq 2$ for noncrystallographic dihedral groups in terms of the extended Shi arrangement or, equivalently, in terms of filtered chains in Armstrong’s suggested “root poset.”

3.5 The cyclic groups

We can simply compute the q, t -Fuß–Catalan numbers for the cyclic groups: the described action of $C_k = G(k, 1, 1)$ on $\mathbb{C}[x, y]$ is given by $\zeta_k(x^a y^b) = \zeta_k^{a-b} x^a y^b$, where ζ_k is a k th root of unity. This gives

$$\begin{aligned} \mathbb{C}[x, y]^{C_k} &= \text{span} \{x^a y^b : a \equiv b \pmod k\}, \\ \mathbb{C}[x, y]^\epsilon &= \text{span} \{x^a y^b : a \equiv b + 1 \pmod k\}, \end{aligned}$$

and therefore, $\mathbb{C}[x, y]^\epsilon = x\mathbb{C}[x, y]^W + y^{k-1}\mathbb{C}[x, y]^W$. Thus, $\text{Cat}^{(1)}(C_k; q, t) = q + t^{k-1}$, and more generally,

$$\text{Cat}^{(m)}(C_k; q, t) = \sum_{i=0}^m q^i t^{(m-i)(k-1)}.$$

Corollary 7 *Conjectures 2 and 3 hold for the cyclic groups: let $C_k = G(k, 1, 1)$ be the cyclic group of order k acting diagonally on $\mathbb{C}[x, y]$ as described above. Let $N = k - 1$ be the number of reflections in C_k , and let $N^* = 1$ be the number of reflecting hyperplanes. Then*

$$\begin{aligned} q^{mN} \text{Cat}^{(m)}(C_k; q, q^{-1}) &= q^{mN^*} \text{Cat}^{(m)}(C_k; q^{-1}, q) \\ &= 1 + q^k + \dots + q^{mk}. \end{aligned}$$

As the cyclic group C_k is not crystallographic for $k \geq 3$, Conjecture 4 is not relevant in this case.

4 Connections to rational Cherednik algebras

In this section, we investigate (conjectured) connections between the generalized diagonal coinvariants $DR^{(m)}(W)$ and a module that naturally arises in the context of rational Cherednik algebras. These algebras were introduced by Etingof and Ginzburg [14] and were further studied by Berest, Etingof, and Ginzburg [7, 8]. Most of the facts about rational Cherednik algebras are taken from these references.

The work by Berest, Etingof, and Ginzburg deals only with real reflection groups, but in [21], Griffeth partially generalized the work to complex reflection groups. We expect that the conjectured connection of the q, t -Fuß–Catalan numbers can also be transferred to this generalized context.

In this section, fix W to be a real reflection group acting on a complex vector space V . Note that $V = V' \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of the real vector space V' on which W naturally acts. Furthermore, let $T \subseteq W$ be the set of reflections in W , and let W act on T by conjugation (note in particular that $\omega t \omega^{-1} \in T$).

4.1 The rational Cherednik algebra

Etingof and Ginzburg [14] defined the rational Cherednik algebra as follows:

Definition 5 Let $c : T \rightarrow \mathbb{C}, t \mapsto c_t$ be a W -invariant function on the set of reflections. The rational Cherednik algebra $H_c = H_c(W)$ is the associative algebra generated by the vector spaces V, V^* and the reflection group W , subject to the defining relations

$$\begin{aligned} \omega x - \omega(x)\omega &= \omega y - \omega(y)\omega = 0 \quad \text{for all } y \in V, x \in V^*, \omega \in W, \\ [x_1, x_2] &= [y_1, y_2] = 0 \quad \text{for all } y_1, y_2 \in V, x_1, x_2 \in V^*, \\ [y, x] &= \langle y, x \rangle - \sum_{t \in T} c_t \langle y, \alpha_t \rangle \langle \alpha_t^\vee, x \rangle t \quad \text{for all } y \in V, x \in V^*. \end{aligned} \tag{18}$$

Here, $[a, b] := ab - ba$ denotes the commutator of a and b , $\langle \cdot, \cdot \rangle$ denotes the pairing on $V \times V^*$, and $\alpha_t \in V^*$ (resp. $\alpha_t^\vee \in V$) denotes the positive root (resp. positive coroot) associated to $t \in T$.

The polynomial ring $\mathbb{C}[V]$ sits inside H_c as the subalgebra generated by V^* , and the polynomial ring $\mathbb{C}[V^*]$ as the subalgebra generated by V . Furthermore, the elements in W span a copy of the group algebra $\mathbb{C}[W]$ sitting naturally inside H_c .

The spherical subalgebra of H_c is defined as $eH_c e \subseteq H_c$, where e is the trivial idempotent viewed as an element in H_c .

4.1.1 An induced grading

Define the degree operator $\mathbf{h} \in H_c$ by $\mathbf{h} := \frac{1}{2} \sum_i (x_i y_i + y_i x_i)$, where $\{x_i\}$ and $\{y_i\}$ are dual bases of V^* and of V , respectively. In [7, Eq. (2.6)], it is shown how the generators of H_c behave under the adjoint action $\text{ad } \mathbf{h} : u \mapsto [\mathbf{h}, u]$,

$$\text{ad } \mathbf{h}(x) = x, \quad \text{ad } \mathbf{h}(y) = -y, \quad \text{ad } \mathbf{h}(\omega) = 0 \quad \text{for } x \in V^*, y \in V, \omega \in W. \tag{19}$$

As the defining relations for H_c become homogeneous, this gives the induced grading

$$H_c = \bigoplus_{\lambda \in \mathbb{Z}} \{a \in H_c : [\mathbf{h}, a] = \lambda a\}.$$

We denote the degree with respect to the $\text{ad } \mathbf{h}$ -grading by $\text{deg}_{\mathbf{h}}$. From (19) it follows immediately that

$$\text{deg}_{\mathbf{h}} x = 1, \quad \text{deg}_{\mathbf{h}} y = -1, \quad \text{deg}_{\mathbf{h}} \omega = 0 \quad \text{for } x \in V^*, y \in V, \omega \in W. \tag{20}$$

4.1.2 A natural filtration

Assign a total degree on H_c by

$$\text{deg}_t x = \text{deg}_t y = 1, \quad \text{deg}_t \omega = 0 \quad \text{for } x \in V^*, y \in V, \omega \in W.$$

For $c = c_t \neq 0$, the defining relation (18) now becomes inhomogeneous. Thus, one only gets an (increasing) filtration $F_{\bullet}(H_c)$, where $F_k(H_c)$ denotes all elements in H_c of total degree less than or equal to k (in particular, $F_k(H_c) = \{0\}$ for negative k). This filtration is stable under the adjoint action; hence, it induces an $\text{ad } \mathbf{h}$ -action on the associated graded algebra

$$\text{gr}(H_c) := \bigoplus_{k \geq 0} F_k(H_c) / F_{k-1}(H_c).$$

Observe that the commutator $[y, x]$ is annihilated in $\text{gr}(H_c)$, i.e., $yx - xy = 0 \in \text{gr}(H_c)$. By [14, Theorem 1.3], one has the algebra isomorphism

$$\text{gr}(H_c) \cong \mathbb{C}[V \oplus V^*] \rtimes W, \tag{21}$$

which is graded with respect to the grading induced by $\text{ad } \mathbf{h}$ and also with respect to the grading induced by the filtration. As the degree operator commutes with the trivial idempotent $\mathbf{e} \in H_c$ and with the determinantal idempotent $\mathbf{e}_{\epsilon} \in H_c$, the filtration on H_c carries over to the spherical subalgebra $\mathbf{e}H_c\mathbf{e}$ and to $\mathbf{e}H_c\mathbf{e}_{\epsilon}$, and one gets the graded isomorphisms

$$\text{gr}(\mathbf{e}H_c\mathbf{e}) \cong \mathbf{e}\mathbb{C}[V \oplus V^*], \quad \text{gr}(\mathbf{e}H_c\mathbf{e}_{\epsilon}) \cong \mathbf{e}_{\epsilon}\mathbb{C}[V \oplus V^*], \tag{22}$$

see (7.8) in [8].

4.2 A module over the rational Cherednik algebra

For any H_c -representation τ , define an H_c -module $\mathcal{M}(\tau)$ to be the induced module

$$\mathcal{M}_c(\tau) := H_c \otimes_{\mathbb{C}[V] \rtimes W} \tau,$$

where $\mathbb{C}[V] \rtimes W$ acts on τ by $p\omega \cdot a := p(0)(\omega(a))$ for $p \in \mathbb{C}[V]$, $\omega \in W$, and $a \in \tau$.

For our purposes, it is enough to restrict to the case where τ is the trivial representation and where the parameter $c = c_t = \frac{1}{\hbar} + m$ is a rational constant (as above, h

denotes the Coxeter number associated to W). Set $H^{(m)} := H_c$ and denote this $H^{(m)}$ -module by $\mathcal{M}^{(m)} := \mathcal{M}_c(\mathbb{C})$. Many properties even hold in the more general context of τ being any H_c -representation and for arbitrary W -invariant parameter c .

Dunkl and Opdam [12] showed that $\mathcal{M}^{(m)}$ has a unique simple quotient, which we denote by $L^{(m)} = L_c(\mathbb{C})$ for $c = \frac{1}{h} + m$. Berest, Etingof, and Ginzburg [8] investigated this $H^{(m)}$ -module and showed in [8, Theorem 1.4] that $L^{(m)}$ is the only finite-dimensional irreducible $H^{(m)}$ -module. Moreover, they computed the graded Hilbert series of $L^{(m)}$ with respect to the grading induced by the adjoint action, see [8, Theorem 1.6].

Theorem 8 (Y. Berest, P. Etingof, V. Ginzburg) *Let $L^{(m)}$ be the unique simple $H^{(m)}$ -module. The Hilbert series of $L^{(m)}$ with respect to the grading induced by $\text{ad } \mathfrak{h}$ is given by*

$$\mathcal{H}(L^{(m)}; q) = q^{-mN} ([mh + 1]_q)^\ell,$$

where ℓ is the rank of W , and N is the number of positive roots. In particular,

$$\dim L^{(m)} = (mh + 1)^\ell.$$

Observe that $\mathfrak{e}L^{(m)} \subseteq L^{(m)}$ has a natural $\mathfrak{e}H^{(m)}\mathfrak{e}$ -module structure and that the degree operator \mathfrak{h} preserves $\mathfrak{e}L^{(m)}$. Hence, $\text{ad } \mathfrak{h}$ also induces a grading on $\mathfrak{e}L^{(m)}$. For the following theorem, see [8, Theorem 1.10].

Theorem 9 (Y. Berest, P. Etingof, V. Ginzburg) *$\mathfrak{e}L^{(m)}$ is the only finite-dimensional simple $\mathfrak{e}H^{(m)}\mathfrak{e}$ -module. The Hilbert series of $\mathfrak{e}L^{(m)}$ with respect to the grading induced by $\text{ad } \mathfrak{h}$ is given by*

$$\mathcal{H}(\mathfrak{e}L^{(m)}; q) = q^{-mN} \prod_{i=1}^{\ell} \frac{[d_i + mh]_q}{[d_i]_q},$$

where ℓ is the rank of W , $d_1 \leq \dots \leq d_\ell$ are the degrees, and h is the Coxeter number.

4.2.1 Filtrations on $L^{(m)}$ and on $\mathfrak{e}L^{(m)}$ and the induced graded modules

There exist nice and important decompositions of the modules in question. These are based on the algebra isomorphism

$$\mathfrak{e}H^{(m)}\mathfrak{e} \xrightarrow{\sim} \mathfrak{e}_\epsilon H^{(m+1)}\mathfrak{e}_\epsilon,$$

which was discovered in [8, Proposition 4.6] and in [19, Proposition 4.8]. Berest, Etingof, and Ginzburg [8, Lemma 4.7, Proposition 4.8] used this isomorphism to connect the modules $L^{(m+1)}$ and $L^{(m)}$ by the isomorphism

$$L^{(m+1)} \cong H^{(m+1)}\mathfrak{e}_\epsilon \otimes_{\mathfrak{e}H^{(m)}\mathfrak{e}} \mathfrak{e}L^{(m)}. \tag{23}$$

For $m = 1$, this isomorphism was studied by Gordon [19, Theorem 4.9]. In [19, Lemma 4.6], he moreover shows that $L^{(0)} \cong \mathbb{C}$ carries the trivial representation. Applying (23) iteratively gives rise to a decomposition of the $H^{(m)}$ -module $L^{(m)}$, which is taken from [8, (7.6)] as

$$L^{(m)} \cong H^{(m)} \mathbf{e}_\epsilon \otimes_{\mathbf{e}H^{(m-1)}\mathbf{e}} \mathbf{e}H^{(m-1)} \mathbf{e}_\epsilon \otimes_{\mathbf{e}H^{(m-2)}\mathbf{e}} \cdots \otimes_{\mathbf{e}H^{(1)}\mathbf{e}} \mathbf{e}H^{(1)} \mathbf{e}_\epsilon \otimes_{\mathbf{e}H^{(0)}\mathbf{e}} \mathbb{C}, \tag{24}$$

and also a decomposition of the $\mathbf{e}H^{(m)}\mathbf{e}$ -module $\mathbf{e}L^{(m)}$, see [8, (7.7)], as

$$\mathbf{e}L^{(m)} \cong \mathbf{e}H^{(m)} \mathbf{e}_\epsilon \otimes_{\mathbf{e}H^{(m-1)}\mathbf{e}} \mathbf{e}H^{(m-1)} \mathbf{e}_\epsilon \otimes_{\mathbf{e}H^{(m-2)}\mathbf{e}} \cdots \otimes_{\mathbf{e}H^{(1)}\mathbf{e}} \mathbf{e}H^{(1)} \mathbf{e}_\epsilon \otimes_{\mathbf{e}H^{(0)}\mathbf{e}} \mathbb{C}.$$

From those decompositions one can see that the filtration on H_c described in Sect. 4.1.2 carries over to $L^{(m)}$ and to $\mathbf{e}L^{(m)}$ via the tensor product filtration: let R be a filtered \mathbb{C} -algebra, A a filtered right R -module, and B a filtered left R -module. The tensor product filtration $F_\bullet(A \otimes B)$ is then defined by

$$F_k(A \otimes_R B) = \sum_j F_j(A) \otimes_R F_{k-j}(B).$$

With $L^{(0)} \cong \mathbb{C}$ sitting in degree 0, the iterative application of the tensor product filtration to the decompositions of $L^{(m)}$ and of $\mathbf{e}L^{(m)}$ defines a filtration on $L^{(m)}$ and a filtration on $\mathbf{e}L^{(m)}$. We denote the associated graded modules by $\text{gr}(L^{(m)})$ and by $\text{gr}(\mathbf{e}L^{(m)})$, respectively.

We have already seen that the degree operator \mathbf{h} acts on $L^{(m)}$ and on $\mathbf{e}L^{(m)}$. The filtrations on $L^{(m)}$ and on $\mathbf{e}L^{(m)}$ are stable under the adjoint action; thus, $\text{ad } \mathbf{h}$ induces a grading on $\text{gr}(L^{(m)})$ and on $\text{gr}(\mathbf{e}L^{(m)})$. Moreover, with respect to this grading, (19) and (20) still hold (compare [19, Proof of Theorem 5]); thus,

$$\mathcal{H}(\text{gr}(L^{(m)}); q) = \mathcal{H}(L^{(m)}; q), \quad \mathcal{H}(\text{gr}(\mathbf{e}L^{(m)}); q) = \mathcal{H}(\mathbf{e}L^{(m)}; q).$$

4.2.2 A bigrading on the associated graded modules

The two different gradings described above allow one to define bigradings on the various objects as follows, see [8, Sect. 7.2]:

$$\text{gr}(H_c) = \bigoplus_{p,q \geq 0} (\text{gr}(H_c))_{p,q},$$

where $(\text{gr}(H_c))_{p,q}$ consists of all elements in $\text{gr}(H_c)$ that are homogeneous with respect to both gradings and which have total degree $p + q$ and $\text{ad } \mathbf{h}$ -degree $p - q$. Observe that the isomorphisms (21) and (22) become bigraded, and one gets

$$\begin{aligned} \text{deg}(x) &= (1, 0), & \text{deg}(y) &= (0, 1), \\ \text{deg}(\omega) &= (0, 0) & \text{for } x \in V^*, y \in V, \omega \in W, \end{aligned}$$

where deg denotes the bidegree with respect to this bigrading.

Similar considerations apply to the $H^{(m)}$ -module $L^{(m)}$ and to the $\mathbf{e}H^{(m)}\mathbf{e}$ -module $\mathbf{e}L^{(m)}$, turning $\text{gr}(L^{(m)})$ and $\text{gr}(\mathbf{e}L^{(m)})$ into bigraded modules.

4.2.3 $\text{gr}(L^{(m)})$ and the generalized diagonal coinvariants

As indicated in Theorem 1, Gordon connected the diagonal coinvariant ring with the $H^{(1)}$ -module $L^{(1)}$. Using the above decompositions, we prove his theorem in a slightly more general context using the same argument. The following well-known lemma is taken from [20, Lemma 6.7(2)]:

Lemma 4 *Let R be a filtered \mathbb{C} -algebra, A a filtered right R -module, and B a filtered left R -module. Then there is a natural surjection*

$$\text{gr } A \otimes_{\text{gr}(R)} \text{gr } B \rightarrow \text{gr}(A \otimes_R B).$$

Theorem 10 *Let W be a real reflection group, let $DR^{(m)}(W)$ be the space of generalized diagonal coinvariants, and let $\text{gr}(L^{(m)})$ be bigraded as described in Sect. 4.2.2. Then there exists a natural surjection of bigraded W -modules,*

$$DR^{(m)}(W) \otimes \epsilon \rightarrow \text{gr}(L^{(m)}).$$

Proof Consider the decomposition (24),

$$L^{(m)} \cong H^{(m)} \mathbf{e}_\epsilon \otimes_{eH^{(m-1)}\mathbf{e}} eH^{(m-1)} \mathbf{e}_\epsilon \otimes_{eH^{(m-2)}\mathbf{e}} \cdots \otimes_{eH^{(1)}\mathbf{e}} eH^{(1)} \mathbf{e}_\epsilon \otimes_{eH^{(0)}\mathbf{e}} \mathbb{C}.$$

By Lemma 4, (21), and (22), we get the surjection

$$(S \otimes \epsilon) \otimes_{eS} \mathbf{e}_\epsilon S \otimes_{eS} \cdots \otimes_{eS} \mathbf{e}_\epsilon S \otimes_{eS} \mathbb{C} \rightarrow \text{gr}(L^{(m)}),$$

where we write S for $\mathbb{C}[V \oplus V^*]$. As the left-hand side equals $DR^{(m)}(W) \otimes \epsilon$, the theorem follows. □

Theorem 10 partially generalizes Theorem 1:

Corollary 8 *Let W be a real reflection group. There exists a graded W -stable quotient ring R_W of $DR^{(m)}(W)$ such that*

- (i) $\dim(R_W) = (mh + 1)^\ell$, and
- (ii) $q^{mN} \mathcal{H}(R_W; q) = [mh + 1]_q^\ell$.

For real reflection groups, Theorems 9 and 10 show that Conjectures 2 and 3 would be implied by the following conjecture, which is, for $m = 1$, due to Haiman [27, Conjecture 7.2.5]:

Conjecture 5 (M. Haiman) *The kernel of the surjection defined in Theorem 10 does not contain a copy of the trivial representation.*

This conjecture would show that the module $M^{(m)}(W)$, and thereby the q, t -Fuß–Catalan numbers $\text{Cat}^{(m)}(W; q, t)$, can be described in terms of the $H^{(m)}$ -module $L^{(m)}$:

Corollary 9 *Let W be a real reflection group, and let $M^{(m)}(W)$ be the bigraded W -module defined in Sect. 3.2. If Conjecture 5 holds, then*

$$M^{(m)}(W) \otimes \epsilon \cong \mathbf{e}(\text{gr}(L)).$$

In particular, Conjecture 5 implies Conjectures 2 and 3 for real reflection groups.

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Appendix: The computations with Singular and Macaulay 2

To compute the dimensions of the module $M^{(m)}(W)$ defined in Sect. 3.2 for the classical reflection groups, we used the computer algebra system Singular [40].

For the computations of the bigraded Hilbert series of $M^{(m)}$, we used the computer algebra system Macaulay 2 [33]. We write $[n]$ for $[n]_{q,t}$.

Table 1 $\dim M^{(m)}(W)$ for types B and D

| m | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
|---------|----|-----|----|----|----|-----|-----|-----|
| $n = 1$ | 2 | 3 | 4 | 5 | 1 | 1 | 1 | 1 |
| $n = 2$ | 6 | 15 | 28 | 45 | 4 | 9 | 16 | 25 |
| $n = 3$ | 20 | 84 | | | 14 | 55 | 140 | 285 |
| $n = 4$ | 70 | 495 | | | 50 | 336 | | |

Table 2 $\text{Cat}^{(m)}(B_n, q, t)$

| | |
|----------------|--|
| $n = 2, m = 1$ | $[5] + qt[1]$ |
| $m = 2$ | $[9] + qt[5] + q^2t^2[1]$ |
| $m = 3$ | $[13] + qt[9] + q^2t^2[5] + q^3t^3[1]$ |
| $n = 3, m = 1$ | $[10] + qt[6] + qt[4]$ |
| $m = 2$ | $[19] + qt[15] + qt[13] + q^2t^2[11] + q^2t^2[9] + q^3t^3[7] + q^2t^2[7] + q^4t^4[3]$ |
| $m = 3$ | $[28] + qt[24] + qt[22] + q^2t^2[20] + q^2t^2[18] + q^3t^3[16] + q^2t^2[16] + q^3t^3[14] + q^4t^4[12] + q^3t^3[12] + q^4t^4[10] + q^5t^5[8] + q^3t^3[10] + q^5t^5[6] + q^6t^6[4]$ |
| $n = 4, m = 1$ | $[17] + qt[13] + qt[11] + q^2t^2[9] + qt[9] + q^3t^3[5] + q^2t^2[5] + q^4t^4[1]$ |
| $m = 2$ | $[33] + qt[29] + qt[27] + q^2t^2[25] + qt[25] + q^2t^2[23] + q^3t^3[21] + 2q^2t^2[21] + q^3t^2[19] + q^4t^4[17] + q^2t^2[19] + 2q^3t^3[17] + q^4t^4[15] + q^5t^5[13] + q^2t^2[17] + q^3t^3[15] + 2q^4t^4[13] + q^5t^5[11] + q^6t^6[9] + q^3t^3[13] + q^4t^4[11] + 2q^5t^5[9] + q^6t^6[7] + q^7t^7[5] + q^9t^9[1] + q^4t^4[9] + 2q^6t^6[5] + q^8t^8[1]$ |

Table 3 $\text{Cat}^{(m)}(D_n, q, t)$

| | |
|----------------|---|
| $n = 2, m = 1$ | $[3] + qt[1]$ |
| $m = 2$ | $[5] + qt[3] + q^2t^2[1]$ |
| $m = 3$ | $[7] + qt[5] + q^2t^2[3] + q^3t^3[1]$ |
| $n = 3, m = 1$ | $[7] + qt[4] + qt[3]$ |
| $m = 2$ | $[13] + qt[10] + qt[9] + q^2t^2[7] + q^2t^2[6] + q^2t^2[5] + q^3t^3[4] + q^4t^4[1]$ |
| $m = 3$ | $[19] + qt[16] + qt[15] + q^2t^2[13] + q^2t^2[12] + q^3t^3[10] + q^2t^2[11] + q^3t^3[9] + q^4t^4[7] + q^3t^3[8] + q^4t^4[6] + q^5t^5[4] + q^3t^3[7] + q^5t^5[3]$ |
| $n = 4, m = 1$ | $[13] + 2qt[9] + qt[7] + 2q^2t^2[5] + q^4t^4[1] + q^3t^3[1]$ |
| $m = 2$ | $[25] + 2qt[21] + qt[19] + 3q^2t^2[17] + 2q^2t^2[15] + 4q^3t^3[13] + q^2t^2[13] + 2q^3t^3[11] + 5q^4t^4[9] + q^5t^5[7] + 2q^6t^6[5] + q^8t^8[1] + q^4t^4[7] + 2q^5t^5[5] + q^7t^7[1] + q^6t^6[1]$ |

Table 4 $\text{Cat}^{(1)}(W, q, t)$ for several complex reflection groups of rank 2

| | |
|--------------|--|
| $G(3, 1, 2)$ | $q^7 + q^5t + q^3t^2 + q^2t + qt^3 + t^5$ |
| $G(4, 1, 2)$ | $q^{10} + q^7t + q^4t^2 + q^3t + qt^3 + t^6$ |
| $G(6, 1, 2)$ | $q^{16} + q^{11}t + q^6t^2 + q^5t + qt^3 + t^8$ |
| $G(3, 2, 2)$ | $q^7 + q^5t + q^3t^2 + q^2t + qt^3 + t^5$ |
| $G(4, 2, 2)$ | $q^6 + q^4t^2 + 2q^3t + q^2t^4 + 2qt^3 + t^6$ |
| $G(6, 2, 2)$ | $q^{10} + q^6t^2 + 2q^5t + q^3t^5 + q^2t^4 + qt^3 + t^8$ |

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