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A relation between the Laplacian and signless Laplacian eigenvalues of a graph

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Abstract Let G be a graph of order n such that $\sum_{i=0}^{n}(-1)^i a_i \lambda^{n-i}$ and $\sum_{i=0}^{n}(-1)^i b_i \lambda^{n-i}$ are the characteristic polynomials of the signless Laplacian and the Laplacian matrices of G, respectively. We show that $a_i \geq b_i$ for $i = 0, 1, \ldots, n$. As a consequence, we prove that for any α, $0 < \alpha \leq 1$, if $q_1, \ldots, q_n$ and $\mu_1, \ldots, \mu_n$ are the signless Laplacian and the Laplacian eigenvalues of G, respectively, then $q_1^\alpha + \cdots + q_n^\alpha \geq \mu_1^\alpha + \cdots + \mu_n^\alpha$.

Keywords Laplacian · Signless Laplacian · Incidence energy · Laplacian-like energy

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1 Introduction

Let G be a graph with vertex set $V(G)=\{v_1,\dots,v_n\}$ and edge set $E(G)=\{e_1,\dots,e_m\}$. The adjacency matrix of G, $A=(a_{ij})$, is an $n\times n$ matrix such that $a_{ij}=1$ if v_i and v_j are adjacent, and otherwise $a_{ij}=0$. The *incidence matrix* of G, denoted by $X=(x_{ij})$, is an $n\times m$ matrix whose rows are indexed by the set of vertices of G and columns are indexed by the set of edges of G defined by

$$x_{ij} := \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i; \\ 0 & \text{otherwise.} \end{cases}$$

If we consider an orientation for G, then in a similar manner as the incidence matrix, the *directed incidence matrix* of the (oriented) graph G, denoted by $D=(d_{ij})$, is defined as

$$d_{ij} := \begin{cases} +1 & \text{if } e_j \text{ is an incoming edge to } v_i; \\ -1 & \text{if } e_j \text{ is an outgoing edge from } v_i; \\ 0 & \text{otherwise.} \end{cases}$$

Let Δ be the diagonal matrix whose entries are the degrees of vertices of G. It is well known that $DD^\top=\Delta-A$ is the *Laplacian matrix* of G denoted by L and $XX^\top=\Delta+A$ is the *signless Laplacian* G denoted by Q. Since L and Q are symmetric matrices, their eigenvalues are real. We denote the eigenvalues of L and Q by $\mu_1(G)\geq\cdots\geq\mu_n(G)$ and $q_1(G)\geq\cdots\geq q_n(G)$, respectively (we drop G when it is clear from the context). The matrices L and Q are similar if and only if G is bipartite (see [4]). The *incidence energy* $\mathrm{IE}(G)$ of the graph G is defined as the sum of singular values of the incidence matrix [8]. The *Laplacian-energy like invariant* $\mathrm{LEL}(G)$ is defined as the sum of square roots of the Laplacian eigenvalues [9]. In other words,

$$\mathrm{IE}(G)=\sum_{i=1}^{n}\sqrt{q_i(G)}, \quad \text{and} \quad \mathrm{LEL}(G)=\sum_{i=1}^{n}\sqrt{\mu_i(G)}.$$

The connection between IE and Laplacian eigenvalues was first pointed out in [5]. For more information on IE and LEL, see [6] and the references therein.

Our motivation of this work is the following conjecture.

Conjecture 1 ([1]) *For any graph G of order n, we have the following*:

$$\sqrt{q_1}+\cdots+\sqrt{q_n}\geq\sqrt{\mu_1}+\cdots+\sqrt{\mu_n}.$$

The conjecture is equivalent to say that, for any graph G,

$$\mathrm{IE}(G)\geq\mathrm{LEL}(G).$$

In this paper we confirm this conjecture by proving the more general statement that for any $0<\alpha\leq 1$,

$$q_1^\alpha+\cdots+q_n^\alpha\geq\mu_1^\alpha+\cdots+\mu_n^\alpha. \tag{1}$$

To settle this result, we make use of a property of elementary symmetric functions as described below. We remark that the quantity $\mu_1^\alpha + \cdots + \mu_n^\alpha$ has been studied recently (see [10, 11]) and is denoted by $S_\alpha(G)$.

The elementary symmetric functions of real numbers $x_1, \ldots, x_n$ are the values

$$c_k(x_1, \ldots, x_n) = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} \prod_{i \in S} x_i,$$

for $k = 1, \ldots, n$. We define the ordering $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$ if

$$c_k(x_1, \ldots, x_n) \le c_k(y_1, \ldots, y_n), \quad \text{for all } k = 1, \ldots, n.$$

Efroymson, Swartz, and Wendroff [3] gave a necessary and sufficient condition for a real continuously differentiable function f to satisfy the following property:

$$(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n) \Longrightarrow \big(f(x_1), \ldots, f(x_n)\big) \preceq \big(f(y_1), \ldots, f(y_n)\big). \tag{2}$$

They showed that for any real α with $0 < \alpha \le 1$, the function $f(x) = x^\alpha$ satisfies (2). In particular:

Theorem 2 ([3]) *If* $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$, *then, for any* $0 < \alpha \le 1$,

$$x_1^\alpha + \cdots + x_n^\alpha \le y_1^\alpha + \cdots + y_n^\alpha.$$

Theorem 2 suggests that in order to establish Inequality (1), it is sufficient to prove that

$$(\mu_1, \ldots, \mu_n) \preceq (q_1, \ldots, q_n).$$

We will do this in Sect. 2. In Sect. 3, two bounds are obtained for the sum of powers of the Laplacian eigenvalues of graphs.

2 The coefficients of the Laplacian and signless Laplacian polynomials

Let M be a matrix and R and S be two subsets of the sets of rows and columns of M, respectively. By $M(R, S)$ we mean the submatrix of M defined by the rows corresponding to the rows in R and the columns corresponding to the columns in S.

Let G be a graph and $S \subseteq E(G)$. By $\langle S \rangle$ we denote the spanning subgraph of G whose edge set is S.

Lemma 3 ([2, Lemma 7.4]) *Let G be a graph and R and S be non-empty subsets of the vertex set and edge set of G with $|R| = |S|$. Let V_0 denote the set of vertices of the induced subgraph $\langle S \rangle$. Then $D(R, S)$ is invertible if and only if the following conditions are satisfied*:

(i) *R is a subset of V_0.*
(ii) *$\langle S \rangle$ is a forest.*
(iii) *$V_0 \setminus R$ contains precisely one vertex from each connected component of $\langle S \rangle$.*

Lemma 4 *Let G be a connected graph of order n with the incidence matrix X. Then any set of $n-1$ rows of X is linearly independent.*

Proof Let $\{v_1, \ldots, v_n\}$ be the vertex set of G and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be the rows of X, where $\mathbf{x}_i$ corresponds to v_i, for $i = 1, \ldots, n$. The case $n = 1$ is trivial. Let $n \geq 2$. If $\operatorname{rank} X = n$, then we are done. So, suppose that $\operatorname{rank} X \leq n-1$. Let $(w_1, \ldots, w_n)$ be a non-zero left null-vector of X, that is $w_1\mathbf{x}_1 + \cdots + w_n\mathbf{x}_n = \mathbf{0}$. Suppose that $w_j \neq 0$. If v_r, for some r, is adjacent to v_j, then $w_r = -w_j$ because in each column of X there are exactly two non-zero entries. Since G is connected, it follows that for every i, $i = 1, \ldots, n$, $w_i = \pm w_j$. This means that any non-zero left null-vector of X has no zero entry. Thus, no set of $n-1$ rows of X can be linearly dependent. □

Theorem 5 *Let G be a graph of order n. Let $\sum_{i=0}^{n}(-1)^i a_i \lambda^{n-i}$ and $\sum_{i=0}^{n}(-1)^i b_i \lambda^{n-i}$ be the characteristic polynomials of the signless Laplacian and the Laplacian matrices of G, respectively. Then*

$$a_i \geq b_i, \quad \textit{for } i = 0, 1, \ldots, n. \tag{3}$$

Equalities hold in (3) for all $i = 0, 1, \ldots, n$ if and only if G is bipartite.

Proof It is well known that for every i, $1 \leq i \leq n$, $b_i = \sum_R \det L(R, R)$, where the summation is taken over all set of vertices R of cardinality i. By the Binet–Cauchy Theorem (see [7, p. 22]),

$$b_i = \sum_{R,S} \det D(R, S)^2,$$

where the summation is taken over all subsets R of the vertices and all subsets S of the edges of G with $|R| = |S| = i$. Similarly, one has

$$a_i = \sum_{R,S} \det X(R, S)^2,$$

with the summation taken over the same set.

We claim that $\det X(R, S)^2 \geq \det D(R, S)^2$, for every subset R of the vertices and every subset S of the edges of G with $|R| = |S|$. By a result due to Poincaré (see [2, Proposition 5.3]), $\det D(R, S) \in \{-1, 0, 1\}$. Thus it suffices to prove that $X(R, S)$ is invertible whenever $D(R, S)$ is so. Suppose that $D(R, S)$ is invertible. Let $T_1, \ldots, T_k$ be all connected components of $\langle S \rangle$. By Lemma 3(i), $R \subseteq V_0$ where V_0 is the set of vertices of the subgraph $\langle S \rangle$. Also, by Lemma 3(iii), for $i = 1, \ldots, k$, $V(T_i) \setminus R$ has exactly one vertex. Therefore, $X(R, S)$ is a block diagonal matrix such that each block of $X(R, S)$ is the incidence matrix of a tree from which one row is omitted. Thus, by Lemma 4, $X(R, S)$ is invertible.

Equalities hold in (3) for all $i = 0, 1, \ldots, n$ if and only if L and Q are similar and this happens if and only if G is a bipartite graph. □

Now, our main result follows immediately from Theorem 2 and Theorem 5.

Theorem 6 *Let G be a graph of order n. Then for every α, $0 < \alpha \leq 1$,*

$$q_1^\alpha + \cdots + q_n^\alpha \geq \mu_1^\alpha + \cdots + \mu_n^\alpha.$$

We conjecture that the equality holds if and only if G is bipartite.

3 Sum of powers of the Laplacian eigenvalues of graphs

Let G be a graph and $\mu_1, \ldots, \mu_k$ be the non-zero Laplacian eigenvalues of G. In [10, 11], the sum of powers of the Laplacian eigenvalues of a graph G,

$$S_\alpha(G) := \mu_1^\alpha + \cdots + \mu_k^\alpha,$$

for $\alpha \neq 0$ was studied. In this section we provide new bounds for $S_\alpha(G)$ when $0 < \alpha \leq 1$. We denote by S_n and P_n the star and the path on n vertices, respectively. Let T be a tree of order n and $\sum_{i=0}^{n}(-1)^i b_i(T)\lambda^{n-i}$ be the characteristic polynomial of its Laplacian. It is known that [12]

$$b_i(S_n) \leq b_i(T) \leq b_i(P_n), \quad \text{for } i = 0, \ldots, n.$$

Therefore, we have the following corollary from Theorem 2:

Corollary 7 *Let α, $0 < \alpha \leq 1$ be a real number. Then, for any tree of order n,*

$$S_\alpha(S_n) \leq S_\alpha(T) \leq S_\alpha(P_n).$$

The following follows from the Courant–Weyl inequalities (see, e.g., [7, Theorem 4.3.7]).

Lemma 8 *Let G be a graph of order n and e be an edge of G. Then the Laplacian eigenvalues of G and $G' = G - e$ interlace*:

$$\mu_1(G) \geq \mu_1(G') \geq \mu_2(G) \geq \mu_2(G') \geq \cdots \geq \mu_n(G) = \mu_n(G') = 0.$$

To prove our next result, we recall that the Laplacian eigenvalues of S_n are 0 with multiplicity 1, 1 with multiplicity $n-2$, and n with multiplicity 1.

Corollary 9 *For any connected graph G of order n and $0 < \alpha \leq 1$,*

$$S_\alpha(G) \geq n + n^\alpha - 2.$$

Proof Let T be a spanning tree of G, then by Lemma 8 and Corollary 7,

$$S_\alpha(G) \geq S_\alpha(T) \geq S_\alpha(S_n) = n + n^\alpha - 2. \qquad \square$$

The corollary extends a result of [6] in which the special case of the inequality for $\alpha = \frac{1}{2}$ was established.

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