On vertex operator realizations of Jack functions

Wuxing Cai · Naihuan Jing

Abstract On the vertex operator algebra associated with a rank one lattice we derive a general formula for products of vertex operators in terms of generalized homogeneous symmetric functions. As an application we realize Jack symmetric functions of rectangular shapes as well as marked rectangular shapes.

Keywords Symmetric functions · Jack polynomials · Vertex operators

1 Introduction

Classical symmetric functions play important roles in many areas of mathematics and physics, and in particular they have close ties with representation theory of both finite and infinite dimensional Lie algebras. Starting with Bernstein's work (see [15]), vertex operators have been used in constructing several families of symmetric functions such as Schur and Schur's Q-functions [4] as well as Hall–Littlewood symmetric functions [5]. Though one can still define a certain family of vertex operators associated with Macdonald polynomials similar to the Schur case, the products of the vertex operators are in general no longer equal to the Macdonald polynomials. At least in the case of two rows the transition function from the basis of generalized homogeneous

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symmetric functions (product of one-row Mcdonald polynomials) involves with hypergeometric series of type $_4\phi_3$. Except for a few cases [6, 14], it has been an open problem to find the transition function from the generalized homogeneous functions to Macdonald symmetric functions [2, 6].

On the other hand, in the study of vertex representations of affine Lie algebras, Lepowsky and Wilson [9] have long posted the important problem on whether certain products in the representation space are linearly independent. In the special case of level three representations, this problem can be solved using Rogers–Ramanujan identities (see also [8] for the homogeneous case). Later it was realized [6] that the vertex operators at level three are actually related with vertex operators associated to certain Jack polynomials, namely, the half vertex operators are actually the generating function of the one-row Jack functions. Thus it is also an interesting question to study the linear independence problem for those vertex operators associated to Jack functions.

Motivated by [1] we define the vertex operator associated with Jack functions in the vertex operator algebra of rank one lattice. They are called Jack vertex operator since the product of identical modes of the vertex operator will be shown to be Jack functions of rectangular shapes. It is interesting that the contraction functions for products of the new vertex operator are of the form $\prod_{i<j}(z_i - z_j)^{2\alpha}$ instead of $\prod_{i<j}(z_i - z_j)^{\alpha}$ for the Jack parameter α^{-1} (which are for $Y_1(z)$ in Sect. 3.2), as expected from experience with the Schur and Hall–Littlewood cases. It turns out that one really needs this new form of vertex operators (vertex operator $X(z)$ in Sect. 3.2) to generate rectangular Jack functions. At the special case of Schur functions ($\alpha = 1$), our new vertex operators provide another formula for the rectangular shapes.

We also study the problem of linear independence for the new vertex operators in the case of Jack functions. We show that under certain conditions the set of vertex operator products is indeed a basis for the representation space (see [1] for a similar statement). We achieve this by deriving a Jacobi–Trudi like formula for the Jack vertex operator, and then we reprove Mimachi–Yamada's theorem [11] that the product of the vertex operators are Jack functions for the rectangular shapes, and then we further generalize this formula to the case of marked rectangular shapes, i.e., rectangular shapes minus part of the last column. This general case contains part of [7] as a special case.

This paper is organized as follows. In Sect. 2 we recall some necessary notions of symmetric functions. In Sect. 3 we first review the vertex operator approach based on the second author's work on Schur functions, then we define the Jack vertex operators and give an explicit formula of the vertex operator products and a Jacobi–Trudi like formula in terms of tableaux. In Sect. 4 we provide a detailed analysis of certain matrix coefficients of vertex operators and prove the theorem of realizing Jack functions of rectangular and marked rectangular shapes.

2 Jack functions

We recall some basic notions about symmetric functions following the standard reference [10]. A partition λ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of nonnegative integer such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$; the set of all partitions is denoted by $\mathcal{P}$; we

sometimes write λ as $(1^{m_1} 2^{m_2} \cdots)$, where m_i is the multiplicity of i occurring in the parts of λ. The number of non-zero λ_i's is called the length of λ, denoted by $l(\lambda)$, and the weight $|\lambda|$ is defined as $\lambda_1 + \cdots + \lambda_s$. A partition λ of weight n is usually denoted by $\lambda \vdash n$. We also recall that the dominance order is defined by comparing the partial sums of the parts. For two partitions λ and μ of the same weight, if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i, one says that λ is greater than μ and denoted as $\lambda \geq \mu$; conventionally, $\lambda > \mu$ means $\lambda \geq \mu$ but $\lambda \neq \mu$. For $\lambda = (\lambda_1, \lambda_2, \ldots) = (1^{m_1}, 2^{m_2}, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots) = (1^{n_1}, 2^{n_2}, \ldots)$ the notation $\lambda - \mu$ means $(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots)$, $\mu \subset' \lambda$ means that $n_1 \leq m_1, n_2 \leq m_2, \ldots$, and $\lambda \setminus \mu$ denotes the partition $(1^{m_1-n_1} 2^{m_2-n_2} \cdots)$. We also define $\binom{m(\lambda)}{m(\mu)} = \binom{m_1}{n_1}\binom{m_2}{n_2}\cdots$, and $\lambda \cup \mu = (1^{m_1+n_1} 2^{m_2+n_2} \cdots)$.

The ring Λ of symmetric functions over $\mathbb{Z}$ has various linear $\mathbb{Z}$-bases indexed by partitions: the monomial symmetric functions $m_\lambda = \sum x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$, the elementary symmetric functions $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$ with $e_n = m_{(1^n)}$, and the Schur symmetric functions s_λ. The power sum symmetric functions $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$ form a $\mathbb{Q}$-basis.

Let $F = \mathbb{Q}(\alpha)$ be the field of rational functions in indeterminate α. The Jack polynomial [3] is a special orthogonal symmetric function under the following inner product. For two partitions $\lambda, \mu \in \mathcal{P}$ the scalar product on Λ_F is given by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \alpha^{-l(\lambda)} z_\lambda, \tag{2.1}$$

where $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$, m_i is the occurrence of integer i in the partition λ, and δ is the Kronecker symbol. Here our parameter α is chosen as the reciprocal to the usual convention in view of our vertex operator realization.

In [10] Macdonald proved the existence and uniqueness of what is called the Jack function as a distinguished family of orthogonal symmetric functions $P_\lambda(\alpha^{-1})$ with respect to the scalar product (2.1) in the following sense:

$$P_\lambda(\alpha^{-1}) = \sum_{\lambda \geq \mu} c_{\lambda\mu}(\alpha^{-1}) m_\mu$$

in which $c_{\lambda\mu}(\alpha^{-1}) \in F$, $\lambda, \mu \in \mathcal{P}$, and $c_{\lambda\lambda}(\alpha^{-1}) = 1$. Let

$$b_\lambda(\alpha^{-1}) = \langle P_\lambda, P_\lambda \rangle^{-1},$$

then $Q_\lambda(\alpha^{-1}) = b_\lambda(\alpha^{-1}) P_\lambda(\alpha^{-1})$ is the dual Jack function.

It is known that the special case $Q_{(n)}(\alpha^{-1})$, simplified as $Q_n(\alpha^{-1})$, can be written explicitly:

$$Q_n(\alpha^{-1}) = \sum_{\lambda \vdash n} \alpha^{l(\lambda)} z_\lambda^{-1} p_\lambda. \tag{2.2}$$

For a partition λ, we will denote $q_\lambda(\alpha^{-1}) = Q_{\lambda_1}(\alpha^{-1}) Q_{\lambda_2}(\alpha^{-1}) \cdots Q_{\lambda_l}(\alpha^{-1})$.

According to Stanley [13] the q_λ's also form another basis of Λ_F, and they are dual to that of m_λ. Hence the transition matrix from Q_λ's to q_λ's is the transpose of that from m_λ's to P_λ's. Explicitly, we have

Lemma 2.1 *For any partition* λ, *one has*

$$q_\lambda(\alpha^{-1}) = \sum_{\mu \geq \lambda} c'_{\lambda\mu} Q_\mu(\alpha^{-1}),$$

$$Q_\lambda(\alpha^{-1}) = \sum_{\mu \geq \lambda} d'_{\lambda\mu} q_\mu(\alpha^{-1}),$$

where $d'_{\lambda,\mu} \in F$, *with* $c'_{\lambda\mu} = c_{\mu\lambda}$ *and* $c'_{\lambda\lambda} = d'_{\lambda\lambda} = 1$.

3 Vertex operators and symmetric functions

Vertex operators can be used to realize several classical types of symmetric functions such as Schur and Hall–Littlewood polynomials [4, 5]. There are some partial progress towards realizations of Macdonald polynomials [4, 14]. In order to discuss the Jack case, we will use the standard vertex algebra technique and recall the construction of lattice vertex operator algebra for rank one case.

3.1 Representation space V and transformation to Λ_C

For a positive integer α, the complex Heisenberg algebra $H_\alpha = \bigoplus_{n \neq 0} \mathbb{C} \, h_n + \mathbb{C} \, c$ is the infinite dimensional Lie algebra generated by h_n and c subject to the following defining relations:

$$[h_m, h_n] = \delta_{m,-n}\alpha^{-1}mc, \qquad [h_m, c] = 0.$$

We remark that the integer α is included for identification with Jack inner product. If it is clear from the context, we will omit the subscript α in H_α and simply refer it as H.

It is well known that H has a unique canonical representation given as follows. The representation space can be realized as the infinite dimensional vector space $V_0 = Sym(h_{-1}, h_{-2}, \ldots)$, the symmetric algebra over $\mathbb{C}$, generated by $h_{-1}, h_{-2}, \ldots$. The action of H is given by

$$h_n.v = \alpha^{-1}n\frac{\partial}{\partial h_{-n}}v \quad \text{if } n > 0,$$

$$h_n.v = h_n v \quad \text{if } n < 0,$$

$$c.v = v.$$

To simplify the indices we enlarge the space V_0 by the group algebra of $\frac{1}{2}\mathbb{Z}$. Let $V = V_0 \otimes \mathbb{C}[\frac{1}{2}\mathbb{Z}]$, where $\mathbb{C}[\frac{1}{2}\mathbb{Z}]$ is the group algebra of $\frac{1}{2}\mathbb{Z}$ with generators $\{e^{nh} \mid n \in \frac{1}{2}\mathbb{Z}\}$. We define the action of the group algebra as usual with the multiplication given by $e^{mh}e^{nh} = e^{(m+n)h}$. We also define the action of $\partial = \partial_h$ on $\mathbb{C}[Z]$ by $\partial.e^{mh} = me^{mh}$. The space V_0 is $\mathbb{Z}$-graded. The enlarged space V is $\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$-graded as

follows. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition, for $v = h_{-\lambda} \otimes e^{nh} \in V$, define the degree of v by $d(v) = (|\lambda|, n)$, where we have used the usual notation $h_{-\lambda} = h_{-\lambda_1} h_{-\lambda_2} \cdots$. For convenience, we consider the degree of zero element to be of any value.

The vertex operator space V has a canonical scalar product. For any polynomials P, Q in the h_i's we have

$$\langle h_n P, Q \rangle = \langle P, h_{-n} Q \rangle,$$

$$\langle 1, 1 \rangle = 1,$$

$$\langle e^{mh}, e^{nh} \rangle = \delta_{m,n}.$$

Thus, for partitions λ, μ we have

$$\langle h_{-\lambda} \otimes e^{mh}, h_{-\mu} \otimes e^{nh} \rangle = z_\lambda \alpha^{-l(\lambda)} \delta_{\lambda,\mu} \delta_{m,n}.$$

We define a linear map $T : V = \sum_{s \in \frac{1}{2}\mathbb{Z}} V_s \mapsto \Lambda_{\mathbb{Q}}$ by:

$$T : h_{-\lambda} \otimes e^{sh} \mapsto p_\lambda.$$

We remark that the restriction of T on $V_s = V_0 \otimes e^{sh}$ is a bijection preserving the products.

3.2 Jack vertex operator on V and an explicit formula

For a complex parameter a we let the vertex operator $Y_a(z)$ acts on V via the generating series:

$$Y_a(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \alpha h_{-n} \right) \exp\left(\sum_{n=1}^{\infty} \frac{z^{-n}}{-n} a\alpha h_n \right) = \sum_n Y_a(n) z^{-n}.$$

For $\alpha \in \frac{1}{2}\mathbb{Z}$, we define $X(z) = Y_2(z) \exp(2\alpha \ln z \partial_h + h)$, i.e.

$$X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \alpha h_{-n} \right) \exp(2\alpha \ln z \partial_h + h) \exp\left(\sum_{n=1}^{\infty} \frac{z^{-n}}{-n} 2\alpha h_n \right),$$

where the middle term acts as follows:

$$\exp(2\alpha \ln z \partial_h + h).e^{sh} = z^{(s+\frac{1}{2})2\alpha} e^{(s+1)h}. \tag{3.1}$$

The operator X_n on V is defined as the component of $X(z)$:

$$X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n}.$$

Clearly for a vector v of degree (k, m), one has

$$d(X_n v) = \big(k - n - (2m + 1)\alpha, m + 1\big).$$

For simplicity we consider a special case of the vertex operator $Y_a(z)$, and let

$$Y(z) = Y_0(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \alpha h_{-n}\right) = \sum Y_{-n} z^n,$$

$$Y^*(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \alpha h_n\right) = \sum Y_n^* z^n.$$

We remark that one can also use the operator $Y_a(z), (a \neq 0)$ in place of $Y(z)$, and most proofs will remain the same.

We note that when $\alpha = 1$ the vertex operator $X(z)$ differs from the Schur vertex operator [4] or its truncated form Bernstein operator for Schur functions.

To simplify the notations, for partition $\lambda = (\lambda_1, \ldots, \lambda_s)$, we denote the product $X_{-\lambda_1} \cdots X_{-\lambda_s}$ simply as $X_{-\lambda}$, and similarly for $Y_{-\lambda}$.

We will first give an explicit formula for the vertex operator products. For this purpose, we need the following

Lemma 3.1 *For the creation part and annihilation part of $X(z)$ we have*

$$\exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \alpha h_{-n}\right) = \sum_{\lambda \in \mathcal{P}} z_\lambda^{-1} \alpha^{l(\lambda)} h_{-\lambda} z^{|\lambda|} \tag{3.2}$$

and

$$\exp\left(\sum_{n=1}^{\infty} \frac{z^{-n}}{-n} 2\alpha h_n\right).h_{-\lambda} = \sum_{\mu \subset' \lambda} \binom{m(\lambda)}{m(\mu)} (-2)^{l(\mu)} z^{-|\mu|} h_{-\lambda \backslash \mu} \tag{3.3}$$

for any partition λ,

Proof The first one is a direct computation:

$$\exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \alpha h_{-n}\right) = \prod_{n \geq 1} \exp\left(\frac{z^n}{n} \alpha h_{-n}\right) = \prod_{n \geq 1} \sum_{i \geq 0} \frac{1}{i!} \frac{z^{ni}}{n^i} \alpha^i h_{-n}^i$$

$$= \sum_{\lambda \in \mathcal{P}} z_\lambda^{-1} \alpha^{l(\lambda)} h_{-\lambda} z^{|\lambda|}.$$

For the second one, let $h_n^{(i)} = h_n^i / i!$. The Heisenberg canonical commutation relation implies that

$$h_n^{(i)}.h_{-n}^m = \left(\frac{n}{\alpha}\right)^i \binom{m}{i} h_{-n}^{m-i}. \tag{3.4}$$

Using this we have

$$\exp\left(\frac{z^{-n}}{-n}2\alpha h_n\right).h^m_{-n} = \sum_{i\geq 0}\left(\frac{2\alpha z^{-n}}{-n}\right)^i h_n^{(i)}.h^m_{-n}$$

$$= \sum_{i\geq 0} z^{-ni}(-2)^i \binom{m}{i}h^{m-i}_{-n}.$$

For $\lambda = (1^{m_1}2^{m_2}\cdots)$, we have

$$\exp\left(\sum_{n=1}^{\infty}\frac{z^{-n}}{-n}2\alpha h_n\right).h_{-\lambda}$$

$$= \prod_{n\geq 1}\exp\left(\frac{z^{-n}}{-n}2\alpha h_n\right).h^{m_1}_{-1}h^{m_2}_{-2}\cdots$$

$$= \prod_{n\geq 1}\left(\sum_{i_n\geq 0}z^{-ni_n}(-2)^{i_n}\binom{m_n}{i_n}h^{m_n-i_n}_{-n}\right) = \sum_{\mu\in\mathcal{P}}\binom{m(\lambda)}{m(\mu)}(-2)^{l(\mu)}z^{-|\mu|}h_{-\lambda\backslash\mu},$$

where the sum runs through all partitions $\mu \subset' \lambda$. $\qquad\square$

In the following we let $\underline{\lambda} = (\lambda^1, \lambda^2, \ldots, \lambda^s)$ denote a sequence of partitions $\lambda^1, \lambda^2, \ldots, \lambda^s$.

Theorem 3.2 *For integer $s \geq 1$, we have*

$$X_{-\lambda_s}\cdots X_{-\lambda_1}.e^{nh}$$

$$= \sum_{\underline{\mu},\underline{\nu}}\prod_{i=1}^{s}\frac{(-2\alpha)^{l(\nu^i)}}{z_{\nu^i}}\binom{m(\mu^{i-1})}{m(\mu^i\backslash\nu^i)}\frac{h_{-\mu^s}}{(-2)^{l(\mu^s)}}\otimes e^{(n+s)h}, \qquad (3.5)$$

where the sum is over $\underline{\mu} = (\mu^1, \mu^2, \ldots, \mu^s)$ and $\underline{\nu} = (\nu^1, \nu^2, \ldots, \nu^s)$ such that $\nu^i \subset' \mu^i$, $|\mu^i| = \lambda_1 + \cdots + \lambda_i - i(2n+i)\alpha$, $\mu^i\backslash\nu^i \subset' \mu^{i-1}$, $\mu^0 = (0)$.

Proof We use induction on s. Note that $\nu^1 = \mu^1$, it is trivial for the case of $s = 1$. Applying the annihilation part and the middle term of $X(z)$ to (3.5), we find the right side

$$\sum_{\mu\subset'\mu^s}\sum_{\underline{\mu},\underline{\nu}}\prod_{i=1}^{s}\frac{(-2\alpha)^{l(\nu^i)}}{z_{\nu^i}}\binom{m(\mu^{i-1})}{m(\mu^i\backslash\nu^i)}$$

$$\cdot\frac{(-2)^{l(\mu)}}{(-2)^{l(\mu^s)}}\binom{m(\mu^s)}{m(\mu)}\frac{z^{(2n+2s+1)\alpha}}{z^{|\mu|}}h_{-\mu^s\backslash\mu}e^{(n+s+1)h}.$$

Replacing μ with $\mu_s \backslash \mu$, it becomes

$$\sum_{\mu \subset' \mu^s} \sum_{\underline{\mu}, \underline{\nu}} \prod_{i=1}^{s} \frac{(-2\alpha)^{l(\nu^i)}}{z_{\nu^i}} \binom{m(\mu^{i-1})}{m(\mu^i \backslash \nu^i)}$$
$$\cdot (-2)^{-l(\mu)} \binom{m(\mu^s)}{m(\mu)} \frac{z^{(2n+2s+1)\alpha}}{z^{|\mu^s \backslash \mu|}} h_{-\mu} e^{(n+s+1)h}.$$

Thus, we have

$$X_{-\lambda_{s+1}} X_{-\lambda_s} \cdots X_{-\lambda_1} . e^{nh} = \sum_{\nu} \sum_{\mu \subset' \mu^s} \sum_{\underline{\mu}, \underline{\nu}} \prod_{i=1}^{s} \frac{(-2\alpha)^{l(\nu^i)}}{z_{\nu^i}} \binom{m(\mu^{i-1})}{m(\mu^i \backslash \nu^i)}$$
$$\cdot (-2)^{-l(\mu)} \binom{m(\mu^s)}{m(\mu)} \frac{\alpha^{l(\nu)}}{z_\nu} h_{-\mu \cup \nu} e^{(n+s+1)h},$$

where in the first sum ν is restricted to: $|\nu| - |\mu^s \backslash \mu| = \lambda_{s+1} - (2n + 2s + 1)\alpha$. Changing variables $\nu = \nu^{s+1}$, $\mu = \mu^{s+1} \backslash \nu^{s+1}$, we finish the proof.   □

Note that a necessary condition for $X_{-\lambda_s} \cdots X_{-\lambda_1} . e^{nh} \neq 0$ is that $k_i = \lambda_1 + \cdots + \lambda_i - i(2n + i)\alpha \geq 0$ for $i = 1, 2, \ldots, s$.

3.3 Generalized Jacobi–Trudi theorem

The generalized homogeneous symmetric function with parameter α can be written as follows. For $n \in \mathbb{Z}$,

$$H_n(\alpha^{-1}) = \sum_{\lambda \vdash n} \alpha^{(\lambda)} z_\lambda^{-1} h_{-\lambda}. \tag{3.6}$$

Its image under T is $Q_n(\alpha)$ by (2.2). Clearly one has $Y_a(-n).1 = H_n(\alpha^{-1})$. Notice that by definition $H_n(\alpha^{-1}) = 0$ for $n < 0$. Combining these, we have the following statement after a simple computation (see also Lemma 2.1).

Lemma 3.3 *For any positive integer s, we have*

$$Y(z_1) \cdots Y(z_s) = \sum_{n_1 \geq 0, \ldots, n_s \geq 0} H_{n_1}(\alpha^{-1}) \cdots H_{n_s}(\alpha^{-1}) z_1^{n_1} \cdots z_s^{n_s}.$$

In particular, $Y_{-r_1} \cdots Y_{-r_s} \cdot 1 = H_{r_1}(\alpha^{-1}) \cdots H_{r_s}(\alpha^{-1})$ for any integers $r_1, \ldots, r_s$. Moreover, for partition λ

$$T(Y_{-\lambda} \cdot 1) = q_\lambda(\alpha^{-1}).$$

Thus the vectors $\{Y_{-\lambda} e^{mh}\}$ ($\lambda \in \mathcal{P}$ and $m \in \frac{1}{2}\mathbb{Z}$) form a linear basis in the representation space V, corresponding to the basis of generalized homogeneous polynomials.

To proceed further we define the normalization of vertex operators, which helps to separate the singular part. The normalization of $X(z_1) \cdots X(z_s)$ is defined as

$$: X(z_1) \cdots X(z_s):$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{z_1^n + \cdots + z_s^n}{n} \alpha h_{-n} \right) \exp\left(\sum_{n=1}^{\infty} \frac{z_1^{-n} + \cdots + z_s^{-n}}{-n} 2\alpha h_n \right) A,$$

where $A = e^{sh} \exp(2\alpha(\ln z_1 + \cdots + \ln z_s)\partial_h)(z_1 \cdots z_s)^{\alpha}$. Similarly when the normal product is taken on mixed product of $X(z)$, $Y(z)$, and $Y^*(z)$, one always moves the annihilation operators to the right.

We define the lowering operator D_i on the bases of symmetric functions by

$$D_i(H_\lambda) = H_{\lambda_1} \cdots H_{\lambda_{i-1}} H_{\lambda_i - 1} H_{\lambda_{i+1}} \cdots H_{\lambda_l}.$$

The rising operator is defined by $R_{ij} = D_i^{-1} D_j$. Like the raising operator R_{ij} the lowering operator D_i is not always invertible, one needs to make sure that each application of D_i is non-zero.

Lemma 3.4 *For $s \geq 1$, we have*

$$X_{-\lambda_1} X_{-\lambda_2} \cdots X_{-\lambda_s}.e^{nh} = \left(\prod_{1 \leq i < j \leq s} (D_i - D_j)^{2\alpha} \cdot \prod_{1 \leq i \leq s} D_i^{(n+\frac{1}{2})2\alpha} \right)$$

$$\cdot H_{\lambda_1}(\alpha^{-1}) \cdots H_{\lambda_s}(\alpha^{-1}) \otimes e^{(n+s)h}.$$

Proof Observe that for $|w| < |z|$

$$\exp\left(\sum_{n=1}^{\infty} \frac{z^{-n}}{-n} 2\alpha h_n \right) \exp\left(\sum_{n=1}^{\infty} \frac{w^n}{n} \alpha h_{-n} \right)$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{w^n}{n} \alpha h_{-n} \right) \exp\left(\sum_{n=1}^{\infty} \frac{z^{-n}}{-n} 2\alpha h_n \right) \left(1 - \frac{w}{z} \right)^{2\alpha}.$$

Using induction on s and the normalization we have

$$X(z_1) \cdots X(z_s) = \prod_{1 \leq i < j \leq s} (z_i - z_j)^{2\alpha} : X(z_1) \cdots X(z_s): .$$

Applying the action on e^{nh}, we have

$$X(z_1) \cdots X(z_s).e^{nh} = \prod_{1 \leq i < j \leq s} (z_i - z_j)^{2\alpha} \prod_{i=1}^{s} z_i^{(n+\frac{1}{2})2\alpha}$$

$$\cdot \prod_{i=1}^{s} \left(\sum_{n \geq 0} H_n(\alpha^{-1}) z_i^n \right) \otimes e^{(n+s)h},$$

where $(z_i - z_j)^{2\alpha} = z_i^{2\alpha}(1 - z_j/z_i)^{2\alpha}$ if there is an expansion. Taking the coefficient of z^λ we obtain the statement.   □

In concern with the operator in Lemma 3.4, we have the following result on the square of Vandermonde determinant.

Lemma 3.5 *Let* $V(X_s) = V(x_1, \ldots, x_s) = \prod_{1 \le i < j \le s}(x_i - x_j)$, $s \ge 2$. *For* $V(X_s)^2$, *the coefficient of the term* $\prod_{i=1}^s x_i^{s-1}$ *is* $(-1)^{s(s-1)/2}s!$, *and the coefficient of* $x_k x_s^{-1} \prod_{i=1}^s x_i^{s-1}$ $(k = 1, \ldots, s-1)$ *is* $-(-1)^{s(s-1)/2}(s-1)!$.

Proof The Vandermonde determinant $V(X_s)$ is the determinant of $M = (x_j^{i-1})_{s \times s}$. Then $V(X_s)^2 = \det(MM^T) = \det(p_{i+j-2})_{s \times s}$, where $p_n = x_1^n + \cdots + x_s^n$. The product $\prod_{1 \le i \le s} x_i^{s-1}$ only appears in the minor-diagonal term $p_{s-1}p_{s-1} \cdots p_{s-1} = p_{s-1}^s$ of the determinant, thus the coefficient is $(-1)^{s(s-1)/2}s!$. Similarly the term $x_k x_s^{-1} \prod_{1 \le i \le s} x_i^{s-1}$ $(k = 1, \ldots, s-1)$ only appears in the term of the form $p_{s-2}p_s p_{s-1}^{s-2}$ of the determinant, so the coefficient is $-(-1)^{s(s-1)/2}(s-1)(s-2)!$. □

For any partition λ and a fixed parameter α, we set

$$H_\lambda(\alpha) = H_{\lambda_1}(\alpha) \cdots H_{\lambda_l}(\alpha). \tag{3.7}$$

Clearly the set of vectors $H_\lambda(\alpha^{-1})e^{mh}$ forms an $\mathbb{C}$-basis of the vertex operator space V. Under the map T, the vector $H_\lambda(\alpha^{-1})$ is the symmetric function $q_\lambda(\alpha^{-1})$. For fixed $\alpha \in \mathbb{N}$ and $m \in \frac{1}{2}\mathbb{Z}$, we define $\mathcal{P}_{\alpha,m}$ to be the set of partitions λ such that $\lambda_i - \lambda_{i+1} \ge 2\alpha$ and $\lambda_{l(\lambda)} \ge (2m+1)\alpha$. The following result is a generalization of Jacobi–Trudi theorem for our vertex operator basis.

Theorem 3.6 *Let* $\alpha \in \mathbb{N}$, $m \in \frac{1}{2}\mathbb{Z}$. *The set of products* $X_{-\lambda}e^{mh}$ $(\lambda \in \mathcal{P}_{\alpha,m}, m \in \mathbb{Z})$ *forms a* $\mathbb{C}$*-basis in the vertex algebra* V. *Moreover one has, for a partition* λ *of length* l *and* $\lambda_l \ge (2m+1)\alpha$,

$$X_{-\lambda}e^{mh} = \sum_{\mu \ge \lambda} a_{\lambda\mu}(\alpha^{-1})H_{\mu-(2m+1)\alpha\mathbf{1}-2\alpha\delta}e^{(m+l(\lambda))h}, \tag{3.8}$$

where μ *runs through the* l*-tuple compositions such that* $|\mu| = |\lambda|$, *and* $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^l$ *and* $\delta = (l-1, l-2, \ldots, 1, 0)$. *If* $\lambda \in \mathcal{P}_{\alpha,m}$, *the leading coefficient* $a_{\lambda\lambda}(\alpha^{-1}) = 1$, *and the degree of* $X_{-\lambda}e^{mh}$ *is* $(|\lambda| - \alpha(2m+l)l, m+l)$.

Proof For any partition λ of length l, we can rewrite Lemma 3.4 in terms of raising operators.

$$X_{-\lambda_1}X_{-\lambda_2} \cdots X_{-\lambda_l} \cdot e^{mh} = \left(\prod_{1 \le i < j \le l}(1 - R_{ij})^{2\alpha} \cdot \prod_{1 \le i \le l} D_i^{(m+l-i+\frac{1}{2})2\alpha} \right)$$
$$\cdot H_{\lambda_1}(\alpha^{-1}) \cdots H_{\lambda_l}(\alpha^{-1}) \otimes e^{(m+l)h}. \tag{3.9}$$

The raising operators map H_λ into H_μ with $\mu \geq \lambda$, and the product

$$\prod_{1 \leq i < j \leq l} (1 - R_{ij})^{2\alpha} = 1 + \sum_{e \neq 0} \pm \prod_{i < j} R_{ij}^{e_{ij}},$$

where e_{ij} are non-negative exponents, so one obtains the expansion of $X_{-\lambda} e^{mh}$ into the right-hand side of (3.8). Note that when $\lambda_l \geq (2m+1)\alpha$ and $\lambda_i - \lambda_{i+1} \geq 2\alpha$ ($i = 0, \ldots, l-1$), the leading term $H_{\lambda-(2m+1)\alpha\mathbf{1}-2\alpha\delta} e^{(m+l(\lambda))h}$ is non-zero. As $e \neq 0$, all the remaining terms in the expansion differ from the leading term, which shows that the representation matrix of the set $X_{-\lambda} e^{mh}$ ($\lambda \in \mathcal{P}_{\alpha,m}$) in terms of the basis $H_\mu e^{(m+l(\lambda))h}$ (of fixed degree and length) is triangular and has ones on the diagonal. This implies that the basis element $H_\mu e^{nh}$ (of fixed degree and length l) can be expressed as a linear combination of $X_{-\lambda} e^{(n-l)h}$, where $\lambda \geq \mu + (2n-2l+1)\alpha\mathbf{1} + 2\alpha\delta$, i.e. $\lambda \in \mathcal{P}_{\alpha,n-l}$. Hence the set $X_{-\lambda} e^{mh}$ ($\lambda \in \mathcal{P}_{\alpha,m}$) forms a basis of the vertex operator algebra. $\qquad\square$

We will see that in certain cases the vectors of type $X_{-\lambda} e^{mh}$ are actually Jack symmetric functions.

3.4 Jack functions of rectangular shapes

We first look at combinatorial properties of rectangular and marked rectangular partitions.

Lemma 3.7 *Let $\lambda = ((k+1)^s, (k)^t)$ be a partition with $t \in \mathbb{Z}_{>0}$, $k, s \in \mathbb{Z}_{\geq 0}$. Then for any partition μ satisfying $|\mu| = |\lambda|$ and $l(\mu) \leq l(\lambda)$, we have $\mu \geq \lambda$.*

Proof Let $\lambda = (\lambda_1, \ldots, \lambda_{s+t})$. If $s = 0$ it is obviously true. So we can assume that $s \geq 1$. Let $\mu = (\mu_1, \ldots, \mu_{s+t})$ be another partition with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{s+t} \geq 0$. The assumption says that $\mu_1 \geq k+1$. If we do not have $\mu \geq \lambda$, there should be an r, $1 \leq r < s+t$ such that

$$\sum_{i=1}^{r} \mu_i \geq \sum_{i=1}^{r} \lambda_i$$

and

$$\sum_{i=1}^{r+1} \mu_i < \sum_{i=1}^{r+1} \lambda_i.$$

It then follows that $k \geq \mu_{r+2}, \mu_{r+3}, \ldots$, but $k \leq \lambda_{r+2}, \lambda_{r+3}, \ldots$. Subsequently

$$\sum_{i=1}^{s+t} \mu_i < \sum_{i=1}^{s+t} \lambda_i,$$

a contradiction with $|\mu| = |\lambda|$. $\qquad\square$

Next we consider the mixed products of vertex operators. The following result is an easy computation by vertex operator calculus.

Lemma 3.8 *The operator product expansion of mixed products is given by*

$$Y^*(w_1) \cdots Y^*(w_t) X(z_1) \cdots X(z_s)$$

$$=: - : \prod_{1 \leq i < j \leq s} (z_i - z_j)^{2\alpha} \prod_{j=1}^{s} \prod_{i=1}^{t} (1 - z_j w_i)^{-\alpha},$$

where $: - :$ *denotes* $: Y^*(w_1) \cdots Y^*(w_t) X(z_1) \cdots X(z_s) :$.

For any partition λ, we observe that the degree of the vector $X_{-\lambda} e^{-\frac{l(\lambda)}{2}h}$ is $(|\lambda|, l(\lambda)/2)$ (see Theorem 3.6). Now we can prove the main theorem.

Theorem 3.9 *For partition* $\lambda = ((k+1)^s, (k)^t)$ *with* $k \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{>0}, t \in \{0, 1\}$, *we have*

$$T\left(X_{-\lambda} e^{-(s+t)h/2}\right) = c(\alpha) Q_\lambda\left(\alpha^{-1}\right),$$

where $c(\alpha)$ *is a rational function of* α, *and* $c(1) = (-1)^{s(s-1+2t)/2} s!$.

We remark that when λ is a rectangular Young diagram (i.e. $t = 0$) the result was first proved by Mimachi and Yamada [11] using differential operators. When $s = t = 1$, it was proved in [7]. Another interesting phenomenon is that when $\alpha = 1$, we obtain a new vertex operator formula for the rectangular shapes and marked rectangular shapes. In [12] Schur functions of rectangular shapes are proved to be singular vectors for the Virasoro algebra ($\alpha = 1$).

Proof For $\lambda = ((k+1)^s k^t)$, let u denote $T(X_{-\lambda}.e^{-(s+t)h/2})$. Note that u is a linear combination of $Q_\mu(\alpha^{-1})$'s with $\mu \geq \lambda$ by Lemmas 3.4 and 2.1. By Lemma 3.7, we need to show that u is orthogonal to $Q_\mu(\alpha^{-1})$, for all μ such that $\mu \vdash kt + s(k+1)$ and $\mu \neq \lambda$. This will be done in the following. As for the coefficient, we have applied Lemmas 3.5 and 2.1 as well as Lemma 3.4, which confirms that u is non-zero and the coefficient $c(\alpha)$ satisfies the given formula.

To prove the orthogonality, consider μ in two cases:

(1) By Corollary 3.7 and Lemma 2.1, for $\mu < \lambda$, or μ is incomparable with λ, it follows easily that u is orthogonal to $Q_\mu(\alpha^{-1})$.

(2) If partition $\mu = (m_1, \ldots, m_r)$ and $\mu > \lambda$ (or μ are incomparable with λ), then it follows that $m_1 > k + 1$. By Lemmas 2.1 and 3.3, to prove that u is orthogonal to $Q_\mu(\alpha^{-1})$, we only need to see that the following product is zero:

$$\left\langle (X_{-(k+1)})^s (X_{-k})^t . e^{-(s+t)h/2}, Y_{-m_r} \cdots Y_{-m_1} . e^{(s+t)h/2} \right\rangle$$

$$= \left\langle Y^*_{-m_1} \cdots Y^*_{-m_t} (X_{-(k+1)})^s (X_{-k})^t, e^{(s+t)h/2} \right\rangle,$$

which equals to the coefficient of $w_1^{-m_1} \cdots w_t^{-m_t}(z_1 \cdots z_s)^{k+1}(z_{s+1} \cdots z_{s+t})^k$ in the following expression

$$
\begin{aligned}
&\left\langle Y^*(w_1) \cdots Y^*(w_r) X(z_1) \cdots X(z_{s+t}) . e^{-(s+t)h/2}, e^{(s+t)h/2} \right\rangle \\
&= z_{s+t}^{(-\frac{s+t}{2}+\frac{1}{2})2\alpha} z_{s+t-1}^{(-\frac{s+t}{2}+1+\frac{1}{2})2\alpha} \cdots z_1^{(-\frac{s+t}{2}+s+t-1+\frac{1}{2})2\alpha} \\
&\quad \cdot \prod_{s+t \geq j > i \geq 1} \left(1 - z_j z_i^{-1}\right)^{2\alpha} \prod_{i=1}^{s+t} (1 - z_j w_i^{-1})^{-\alpha} \\
&= \pm \prod_{1 \leq i \neq j \leq s+t} \left(1 - z_j z_i^{-1}\right)^{\alpha} \prod_{j=1}^{s+t} \prod_{i=1}^{r} (1 - z_j w_i^{-1})^{-\alpha}, \quad (3.10)
\end{aligned}
$$

where we have used Lemma 3.8. In Lemma 3.11, to be proved in the next section, we will see that this coefficient is indeed zero. Hence the theorem is proved. $\qquad \square$

In general we have the following result.

Corollary 3.10 *For partition $\lambda = ((k+1)^s, (k)^t)$ with $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{>0}$, $t \in \{0, 1\}$, and $r \in \mathbb{Z}$, $\alpha \in \mathbb{Z}_{>0}$ such that $r\alpha \leq k + \delta_{t,0}$ we have*

$$
T\left(X_{-\lambda} e^{-\frac{s+t-r}{2}h}\right) = c Q_{\lambda - r\alpha \mathbf{1}}(\alpha^{-1}),
$$

where c is a non-zero constant, and $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^{s+t}$.

Proof The proof is essentially the same as that of Theorem 3.9. The condition $r\alpha \leq k + \delta_{t,0}$ is necessary to guarantee $T(X_{-\lambda} e^{-\frac{s+t-r}{2}h}) \neq 0$ (see Remark of Theorem 3.2). $\qquad \square$

Lemma 3.11 *The expansion of the contraction function*

$$
H_\alpha(Z_s, W_t) = \prod_{s \geq i \neq j \geq 1} \left(1 - z_j z_i^{-1}\right)^{\alpha} \prod_{j=1}^{s} \prod_{i=1}^{t} (1 - z_j w_i^{-1})^{-\alpha}
$$

in the region of $|z_j| < |w_i|$ does not contain terms of the form $w_1^{-m_1} \cdots w_t^{-m_t} z_1^{k_1} \cdots z_s^{k_s}$ where $m_1 > k_i$ $(i = 1, 2, \ldots, s)$.

4 Analysis of $H_\alpha(Z_s, W_t)$

We have the following lemma to *split* $H_\alpha(Z_s, W_t)$:

Lemma 4.1 *For positive integers r, s, $i \neq j$, there are non-negative integers f_m and g_n such that*:

$$
\left(\frac{1 - z_i z_j^{-1}}{1 - z_i w^{-1}}\right)^r \left(\frac{1 - z_j z_i^{-1}}{1 - z_j w^{-1}}\right)^s = \sum_{m=1}^{r} \left(\frac{1 - z_i z_j^{-1}}{1 - z_i w^{-1}}\right)^m f_m + \sum_{n=1}^{s} \left(\frac{1 - z_j z_i^{-1}}{1 - z_j w^{-1}}\right)^n g_n.
$$

Proof For simplicity we denote $a = \frac{1 - z_i z_j^{-1}}{1 - z_i w^{-1}}$, $b = \frac{1 - z_j z_i^{-1}}{1 - z_j w^{-1}}$, it can be verified directly that $ab = a + b$. Repeatedly using this, we can write $a^r b^s$ into the wanted form. $\quad\square$

Assume first that α is a positive integer. Consider

$$H_n(Z_s, w) = H_n(Z_s, W_1) = \prod_{s \geq i \neq j \geq 1} \left(1 - z_j z_i^{-1}\right)^n \prod_{j=1}^{s} \left(1 - z_j w^{-1}\right)^{-n},$$

where we identified w_1 with w for simplicity. Notice that

$$H_n(Z_s, W_t) = H_n(Z_s, w) \prod_{i=2}^{t} \prod_{j=1}^{s} \left(1 - z_j w_i^{-1}\right)^{-n}.$$

If the expansion of $H_n(Z_s, w)$ in the region $|z_j| < |w_i|$ does not contain terms of the form $w_1^{-1} \cdots w_t^{-1} z_1^{k_1} \cdots z_s^{k_s}$ with $m_1 > k_i$ $(i = 1, \ldots, s)$, neither does $H_n(Z_s, w_t)$. Thus the following result will imply Lemma 3.11.

Theorem 4.2 *For $n, s \in \mathbb{N}$, $s \geq 2$, there are polynomials $f_{i,j}$ in $z_k z_l^{-1}$'s $(1 \leq k \neq l \leq s)$ such that*:

$$H_n(Z_s, w) = \sum_{i=1}^{s} \sum_{j=1}^{n} \left(1 - z_i w^{-1}\right)^{-j} f_{i,j}. \tag{4.1}$$

Moreover for each i and j, $f_{i,j}$ is a polynomial in z_i.

Proof To prove the existence of $f_{i,j}$'s, we will use induction on s.

In the case of $s = 2$ it is true by Lemma 4.1. Assume that it holds true for s, we have

$$H_n(Z_{s+1}, w) = H_n(Z_s, w) A_{s+1} = \sum_{i=1}^{s} \sum_{j=1}^{n} \left(1 - z_i w^{-1}\right)^{-j} A_{s+1} f_{i,j}$$

$$= \sum_{i=1}^{s} \sum_{j=1}^{n} \left(1 - z_i z_{s+1}^{-1}\right)^{j} \left(1 - z_i w^{-1}\right)^{-j} \left(1 - z_{s+1} z_i^{-1}\right)^{n}$$

$$\cdot \left(1 - z_{s+1} w^{-1}\right)^{-n} B_{s+1,i,j} f_{i,j},$$

where

$$A_{s+1} = \left(1 - z_{s+1} w^{-1}\right)^{-n} \prod_{l=1}^{s} \left(1 - z_l z_{s+1}^{-1}\right)^{n} \left(1 - z_{s+1} z_l^{-1}\right)^{n}$$

$$= \left(1 - z_i z_{s+1}^{-1}\right)^{j} \left(1 - z_{s+1} z_i^{-1}\right)^{n} \left(1 - z_{s+1} w^{-1}\right)^{-n} B_{s+1,i,j}.$$

Notice that the term inside the sum can be split by Lemma 4.1, while $B_{s+1,i,j}$ is a product of $(1 - z_k z_l^{-1})$'s, and the existence follows. $\quad\square$

As for the second part, note that $H_n(Z, w)$ is symmetric about $z_1, \ldots, z_s$, we only need to prove that $f_{1,j}$ $(j = 1, \ldots, n)$ are polynomials of z_1. Multiplying two sides of (4.1) by $(1 - z_1 w^{-1})^n \cdots (1 - z_s w^{-1})^n$, we have

$$\prod_{1 \le i \ne j \le s} \left(1 - z_j z_i^{-1}\right)^n$$

$$= \sum_{i=1}^{s} \sum_{j=1}^{n} \left(1 - z_1 w^{-1}\right)^n \cdots \left(1 - z_i w^{-1}\right)^{n-j} \cdots \left(1 - z_s w^{-1}\right)^n f_{i,j}. \quad (4.2)$$

Using induction on $j' = n - j$: first, let $w = z_1$ in (4.2), we have

$$\prod_{1 \le i \ne j \le s} \left(1 - z_i z_j^{-1}\right)^n = f_{1,n} \prod_{i=2}^{s} \left(1 - z_i z_1^{-1}\right)^n.$$

Eliminating the common factor we find,

$$f_{1,n} = \prod_{i=2}^{s} \left(1 - z_1 z_i^{-1}\right)^n \prod_{2 \le i \ne j \le s} \left(1 - z_i z_j^{-1}\right)^n,$$

which implies the case $j' = 0$. Assume that it's true for $j' < r$. Let $j' = r \le n - 1$. Differentiating both sides of (4.2) with respect to z_1, and set $w = z_1$, we have:

$$\frac{\partial^r}{\partial z_1^r} \prod_{1 \le i \ne j \le s} \left(1 - z_i z_j^{-1}\right)^n = \prod_{i=2}^{s} (1 - z_i/z_1)^n \cdot \sum_{i=0}^{r} \binom{r}{i} i! (-z_1^{-1})^i \frac{\partial^{r-i}}{\partial z_1^{r-i}} f_{1,n-i}.$$

The term $i = r$ in the sum contains $f_{1,n-r}$ and one finds that,

$$f_{1,n-r} = (r!)^{-1} (-z_1)^r \prod_{i=2}^{s} (1 - z_i/z_1)^{-n} \frac{\partial^r}{\partial z_1^r} \prod_{1 \le i \ne j \le s} \left(1 - z_i z_j^{-1}\right)^n$$

$$- \sum_{i=0}^{r-1} ((r-i)!)^{-1} (-z_1)^{r-i} \frac{\partial^{r-i}}{\partial z_1^{r-i}} f_{1,n-i}.$$

Note that

$$\frac{\partial^r}{\partial z_1^r} \prod_{1 \le i \ne j \le s} \left(1 - z_i z_j^{-1}\right)^n$$

$$= \prod_{2 \le i \ne j \le s} \left(1 - z_i z_j^{-1}\right)^n \sum c(a_i, b_i) \prod_{i=2}^{s} \frac{\partial^{a_i}}{\partial z_1^{a_i}} \left((1 - z_1/z_i)^n\right) \frac{\partial^{b_i}}{\partial z_1^{b_i}} \left((1 - z_i/z_1)^n\right),$$

where the sum is over vectors $(a_2 \cdots a_s, b_2, \ldots, b_s)$ with nonnegative integer components which sum up to r, and $c(a_i, b_i) = r!/(a_2! \cdots a_s! b_2! \cdots b_s!)$. Now the first part

of $f_{1,n-r}$ is

$$
(r!)^{-1}(-1)^r z_1^{r-b_2-\cdots-b_s} \prod_{2 \le i \ne j \le s} \left(1 - z_i z_j^{-1}\right)^n
$$

$$
\cdot \sum c(a_i, b_i) \prod_{i=2}^{s} \frac{\partial^{a_i}}{\partial z_1^{a_i}} \left((1 - z_1/z_i)^n\right) \frac{\partial^{b_i}}{\partial z_1^{b_i}} \left((1 - z_i/z_1)^n\right) \left(1 - z_i z_1^{-1}\right)^{-n} z_1^{b_i}.
$$

By the following lemma and the assumption of induction, $\lim_{z_1 \to 0} f_{1,n-r}$ exists if $z_i \ne 0$. Observe that $f_{1,n-r}$ is polynomial of $z_k z_l^{-1}$'s, it should be a polynomial of z_1 as well.

Lemma 4.3 *For $a \ne 0$, let $g_{s,m}(z) = z^s (1 - a/z)^{-m} \frac{\partial^s}{\partial z^s} (1 - a/z)^m$, then $\lim_{z \to 0} g_{s,m}(z)$ exists for $0 \le s < m$.*

Proof We use induction on s again. The initial step is trivial. Consider $s + 1 < m$,

$$
g_{s+1,m}(z) = z^{s+1} (1 - a/z)^{-m} \frac{\partial^s}{\partial z^s} \left(m(1 - a/z)^{m-1} a z^{-2}\right)
$$

$$
= z^{s+1} (1 - a/z)^{-m} am \sum_{i=0}^{s} d_i z^{-(2+s-i)} \frac{\partial^i}{\partial z^i} (1 - a/z)^{m-1}
$$

$$
= \sum_{i=0}^{s} c_i g_{i,m-1}(z)(z - a)^{-1},
$$

where $c_i = am\binom{s}{i}(1 + s - i)!(-1)^{s-i} = am d_i$, and the lemma follows. $\qquad\square$

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