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Tetravalent half-arc-transitive graphs of order $2pq$

Yan-Quan Feng · Jin Ho Kwak · Xiuyun Wang · Jin-Xin Zhou

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Abstract A graph is *half-arc-transitive* if its automorphism group acts transitively on its vertex set, edge set, but not arc set. Let p and q be primes. It is known that no tetravalent half-arc-transitive graphs of order $2p^2$ exist and a tetravalent half-arc-transitive graph of order $4p$ must be non-Cayley; such a non-Cayley graph exists if and only if $p-1$ is divisible by 8 and it is unique for a given order. Based on the constructions of tetravalent half-arc-transitive graphs given by Marušič (J. Comb. Theory B 73:41–76, 1998), in this paper the connected tetravalent half-arc-transitive graphs of order $2pq$ are classified for distinct odd primes p and q.

Keywords Cayley graph · Vertex-transitive graph · Half-arc-transitive graph

1 Introduction

All graphs considered in this paper are finite, connected, undirected and simple, but with an implicit orientation of the edges when appropriate. Given a graph X, denote by $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ the vertex set, edge set, arc set and automorphism group of X, respectively. A graph X is said to be *vertex-transitive*, *edge-transitive* and

Y.-Q. Feng (✉) · X. Wang · J.-X. Zhou
Mathematics, Beijing Jiaotong University, Beijing 100044, P.R. China
e-mail: yqfeng@bjtu.edu.cn

X. Wang
e-mail: 06118308@bjtu.edu.cn

J.-X. Zhou
e-mail: jxzhou@bjtu.edu.cn

J.H. Kwak
Mathematics, Pohang University of Science and Technology, Pohang 790-784, Korea
e-mail: jinkwak@postech.ac.kr

arc-transitive if $\mathrm{Aut}(X)$ acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively. The graph X is said to be *half-arc-transitive* provided that it is vertex- and edge- but not arc-transitive. More generally, by a *half-arc-transitive* action of a subgroup G of $\mathrm{Aut}(X)$ on X we shall mean a vertex- and edge-, but not arc-transitive action of G on X. In this case we say that the graph X is *G-half-arc-transitive*.

In 1947, Tutte [31] initiated an investigation of half-arc-transitive graphs by showing that a vertex- and edge-transitive graph with odd valency must be arc-transitive. A few years later, in order to answer Tutte's question of the existence of half-arc-transitive graphs of even valency, Bouwer [5] gave a construction of $2k$-valent half-arc-transitive graph for every $k \geq 2$. Following these two classical articles, half-arc-transitive graphs have been extensively studied from different perspectives over decades by many authors. See, for example, [2, 9, 15, 16, 18, 32, 33].

One of the standard problems in the study of half-arc-transitive graphs is to classify such graphs of certain orders. Let p be a prime. It is well-known that there are no half-arc-transitive graphs of order p or p^2 [6], and by Cheng and Oxley [7], there are no half-arc-transitive graphs of order $2p$. Alspach and Xu [2] classified the half-arc-transitive graphs of order $3p$ and Wang [33] classified the half-arc-transitive graphs of order a product of two distinct primes. Despite all of these efforts, however, further classifications of half-arc-transitive graphs with general valencies seem to be very difficult. For example, the classification of half-arc-transitive graphs of order $4p$ has been considered for many years, but it still has not been achieved.

In view of the fact that 4 is the smallest admissible valency for a half-arc-transitive graph, special attention has rightly been given to the study of tetravalent half-arc-transitive graphs. In particular, constructing and classifying the tetravalent half-arc-transitive graphs is currently an active topic in algebraic graph theory (for example, see [1, 8, 10–13, 17–28] and [30, 34, 35, 37, 38]). For tetravalent half-arc-transitive graphs of given orders, in 1992 Xu [37] classified the tetravalent half-arc-transitive graphs of order p^3 for each prime p, and recently, it was extended to the case of p^4 by Feng et al. [11]. Also, Feng el al. [13] classified the tetravalent half-arc-transitive graphs of order $4p$, and such a graph exists if and only if $p-1$ is divisible by 8. It follows from [34] that no half-arc-transitive graphs of order $2p^2$ exist for each prime p. In this paper we classify connected tetravalent half-arc-transitive graphs of order $2pq$ for odd primes $q<p$. There are two infinite families of connected tetravalent half-arc-transitive graphs of order $2pq$ with one family Cayley and the other non-Cayley; the family of Cayley ones exists if and only if $(p,q) \neq (7,3)$ and $p \equiv 1 \pmod q$, and the family of non-Cayley ones exists if and only if $p \equiv 1 \pmod{4q}$. For each family there are exactly $\frac{1}{2}(q-1)$ non-isomorphic connected tetravalent half-arc-transitive graphs for a given order.

2 Preliminary results

We start by some notational conventions used throughout this paper. Let X be a graph. For $u, v \in V(X)$, denote by $\{u, v\}$ the edge incident to u and v in X. Let B be a subset of $V(X)$. The subgraph of X induced by B will be denoted by $X[B]$. Let n be a non-negative integer. By C_n and K_n, we denote the cycle and the complete graph of order n, respectively. Let D_{2n} represent the dihedral group of order $2n$, and $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n. Denote by $\mathbb{Z}_n^*$ the multiplicative group of the ring $\mathbb{Z}_n$ consisting of integers coprime to n.

Let X be a tetravalent G-half-arc-transitive graph for a subgroup G of $\mathrm{Aut}(X)$. Then under the natural G-action on $V(X) \times V(X)$, the arc set $A(X)$ is partitioned into two G-orbits, say A_1 and A_2, which are paired with each other, that is, $A_2 = \{(v, u) \mid (u, v) \in A_1\}$. Each of two corresponding oriented graphs $(V(X), A_1)$ and $(V(X), A_2)$ has out-valency and in-valency which are equal to 2, and admits G as a vertex- and arc-transitive group of automorphisms. Moreover, each of them has X as its underlying graph. Let $D_G(X)$ be one of these two oriented graphs, fixed from now on. For an arc (u, v) in $D_G(X)$, we say that u and v are the tail and the head of the arc (u, v), respectively. An even length cycle C in X is called a *G-alternating cycle* if the vertices of C are alternatively the tail or the head in $D_G(X)$ of their two incident edges in C. It was shown in [21, Proposition 2.4(i)] that, first, all G-alternating cycles in X have the same length—half of this length is called the *G-radius* of X—and second, that any two adjacent G-alternating cycles in X intersect in the same number of vertices, called the *G-attachment number* of X. The intersection of two adjacent G-alternating cycles is called a *G-attachment set*. We say that X is *tightly G-attached* if its G-attachment number coincides with G-radius. If X is half-arc-transitive, the terms $\mathrm{Aut}(X)$-alternating cycle, $\mathrm{Aut}(X)$-radius, and $\mathrm{Aut}(X)$-attachment number are referred to as an alternating cycle of X, radius of X and attachment number of X, respectively. Similarly, if X is tightly $\mathrm{Aut}(X)$-attached, we say that X is *tightly attached*. Tightly attached tetravalent graphs with odd radius and even radius have been completely classified by Marušič [21] and Šparl [30], respectively. For the purpose of this paper, we introduce a result due to Marušič.

Let $m \geq 3$ be an integer, $n \geq 3$ an odd integer and let $r \in \mathbb{Z}_n^*$ satisfy $r^m = \pm 1$. The graph $X(r; m, n)$ is defined to have vertex set $V = \{u_i^j \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$ and edge set $E = \{\{u_i^j, u_{i+1}^{j \pm r^i}\} \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$.

Proposition 2.1 [21, Theorem 3.4] *A connected tetravalent graph X is a tightly attached half-arc-transitive graph of odd radius n if and only if $X \cong X(r; m, n)$, where $m \geq 3$, and $r \in \mathbb{Z}_n^*$ satisfying $r^m = \pm 1$, and moreover none of the following conditions is fulfilled*:

(1) $r^2 = \pm 1$;
(2) $(r; m, n) = (2; 3, 7)$;
(3) $(r; m, n) = (r; 6, 7k)$, *where $k \geq 1$ is odd*, $(7, k) = 1$, $r^6 = 1$, *and there exists a unique solution $q \in \{r, -r, r^{-1}, -r^{-1}\}$ of the equation $x^2 + x - 2 = 0$ such that* $7(q - 1) = 0$ *and* $q \equiv 5 \pmod 7$.

The following proposition is due to Marušič and Praeger [25].

Proposition 2.2 [25, Lemma 3.5] *Let X be a connected tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(X)$, and let A be a G-attachment set of X. If $|A| \geq 3$, then the vertex-stabilizer of $v \in V(X)$ in G is of order* 2.

Given a finite group G, an inverse closed subset $S \subseteq G \setminus \{1\}$ is called a *Cayley subset* of G. The *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to a Cayley subset S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. The automorphism group $\mathrm{Aut}(X)$ of X contains the right regular representation $R(G)$ of G, the acting group of G by right multiplication, as a subgroup. Thus, Cayley graphs are vertex-transitive. In general, we have the following result.

Proposition 2.3 [4, Lemma 16.3] *A graph X is isomorphic to a Cayley graph on G if and only if its automorphism group* $\mathrm{Aut}(X)$ *has a subgroup isomorphic to G, acting regularly on vertices.*

Let S be a Cayley subset of a finite group G. We call S a *CI-subset*, if for any Cayley subset T of G, $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ implies that there is $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$. The following result is a well-known criterion for CI-subset due to Babai [3].

Proposition 2.4 *Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on a finite group G with respect to S. Then S is a CI-subset of G if and only if for any $\sigma \in S_G$ with $\sigma^{-1}R(G)\sigma \le \mathrm{Aut}(X)$, there exists an $\alpha \in \mathrm{Aut}(X)$ such that $\sigma^{-1}R(G)\sigma = \alpha^{-1}R(G)\alpha$, where S_G denotes the symmetric group on G.*

Now we state two simple observations about half-arc-transitive graphs.

Proposition 2.5 [35, Proposition 2.6] *Let X be a connected half-arc-transitive graph of valency $2n$. Let $A = \mathrm{Aut}(X)$ and let A_u be the stabilizer of $u \in V(X)$ in A. Then each prime divisor of $|A_u|$ is a divisor of $n!$.*

Proposition 2.6 [13, Propositions 2.1 and 2.2] *Let $X = \mathrm{Cay}(G, S)$ be half-arc-transitive. Then S contains no involutions, and there is no $\alpha \in \mathrm{Aut}(G, S)$ such that $s^\alpha = s^{-1}$ for some $s \in S$.*

Finally, we give two group-theoretic propositions. Let H be a subgroup of a finite group G. Denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$.

Proposition 2.7 [29, Theorem 1.6.3] *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

As a result of the well-known classification of finite simple groups, we have the following proposition.

Proposition 2.8 [14, pp. 12–14] *A non-abelian simple group whose order has at most three prime divisors is isomorphic to one of the following groups*:

$$A_5, A_6, \mathrm{PSL}(2,7), \mathrm{PSL}(2,8), \mathrm{PSL}(2,17), \mathrm{PSL}(3,3), \mathrm{PSU}(3,3), \mathrm{PSU}(4,2),$$

whose orders are $2^2 \cdot 3 \cdot 5$, $2^3 \cdot 3^2 \cdot 5$, $2^3 \cdot 3 \cdot 7$, $2^3 \cdot 3^2 \cdot 7$, $2^4 \cdot 3^2 \cdot 17$, $2^4 \cdot 3^3 \cdot 13$, $2^5 \cdot 3^3 \cdot 7$, $2^6 \cdot 3^4 \cdot 5$, *respectively.*

3 Constructions

In this section, we introduce two infinite families of tetravalent half-arc-transitive graphs of order $2pq$, where $p > q$ are odd primes.

Construction of a Cayley model Let p, q be odd primes such that $(p, q) \neq (7, 3)$ and $q \mid (p-1)$. It is well-known that there is a unique non-abelian group of order pq, which is the Frobenius group $F_{pq} = \langle a, b \mid a^p = b^q = 1, b^{-1}ab = a^r\rangle$, where r is an element of order q in $\mathbb{Z}_p^*$. Let $G = \langle a, b, c \mid a^p = b^q = c^2 = 1, b^{-1}ab = a^r, ac = ca, cb = bc\rangle \cong F_{pq} \times \mathbb{Z}_2$. Then G is independent of the choice of r and a non-abelian group of order $2pq$. For $k \in \mathbb{Z}_q^*$, define

$$\mathcal{C}_{2pq}^k := \mathrm{Cay}\big(G, \big\{cb^k, cb^{-k}, cb^k a, \big(cb^k a\big)^{-1}\big\}\big).$$

Lemma 3.1 *Let p, q and r be given as above. Then for each $k \in \mathbb{Z}_q^*$, $\mathcal{C}_{2pq}^k \cong X(r^k; 2q, p)$. Thus, $\mathcal{C}_{2pq}^k$ is a connected tetravalent half-arc-transitive graph of order $2pq$, and there are exactly $\frac{1}{2}(q-1)$ non-isomorphic such graphs, that are $\mathcal{C}_{2pq}^k$ for $k = 1, 2, \ldots, \frac{1}{2}(q-1)$.*

Proof For each $k \in \mathbb{Z}_q^*$, set $T_k = \{cb^k, cb^{-k}, cb^k a, (cb^k a)^{-1}\}$. Recall that $X(r^k; 2q, p)$ has vertex set $V = \{u_i^j \mid i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p\}$ and edge set $E = \{\{u_i^j, u_{i+1}^{j \pm r^{ki}}\} \mid i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p\}$. It is easy to see that $a^s b^t = b^t a^{sr^t}$ for all integers s and t. Also, one may easily check that the map $\phi : u_i^j \mapsto (cb^k)^i a^j$ $(i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p)$ is an isomorphism from $X(r^k; 2q, p)$ to $\mathrm{Cay}(G, T)$, where $T = \{cb^k a^{-1}, (cb^k a^{-1})^{-1}, cb^k a, (cb^k a)^{-1}\}$.

For any $\ell \in \mathbb{Z}_q^*$, the map $a \mapsto a^\ell$, $b \mapsto b$, $c \mapsto c$ induces an automorphism of G. This implies that $\mathrm{Aut}(G)$ is 2-transitive on the set $\{b^i a^j \mid j \in \mathbb{Z}_p\}$ for a given $i \in \mathbb{Z}_q^*$ because the Sylow q-subgroups of G are conjugate. It follows that G has an automorphism φ such that $(b^k a)^\varphi = b^k a$ and $(b^k a^{-1})^\varphi = b^k$. Since the automorphism group $\mathrm{Aut}(G)$ of G fixes c (G has the center $\langle c\rangle$), one has $T^\varphi = T_k$, and hence φ is an isomorphism from $\mathrm{Cay}(G, T)$ to $\mathcal{C}_{2pq}^k$. Consequently, $\mathcal{C}_{2pq}^k \cong X(r^k; 2q, p)$. By hypothesis, we have $p \geq 11$ and $q \geq 3$, and since T_k generates G, $\mathcal{C}_{2pq}^k$ is a connected tetravalent tightly attached half-arc-transitive graph of order $2pq$ by Proposition 2.1.

Let $k \in \mathbb{Z}_q^*$. Note that $a^{-1}b^k = b^k a^{-r^k}$. The automorphism of G induced by $a \mapsto a^{-r^k}$, $b \mapsto b$ and $c \mapsto c$, maps T_k to $\{cb^{q-k}, (cb^{q-k})^{-1}, cb^{q-k}a, (cb^{q-k}a)^{-1}\}$. This implies that $\mathcal{C}_{2pq}^k \cong \mathcal{C}_{2pq}^{q-k}$. To complete the proof, it suffices to show that $\mathcal{C}_{2pq}^k$, $1 \leq k \leq \frac{1}{2}(q-1)$, are pair-wise non-isomorphic.

Set $A = \mathrm{Aut}(\mathcal{C}_{2pq}^k)$. By Proposition 2.2, $|A| = 4pq$ and $A_u \cong \mathbb{Z}_2$ for $u \in V(\mathcal{C}_{2pq}^k)$. It follows that $R(G) \trianglelefteq A$. Note that $G = \langle a, b\rangle \times \langle c\rangle$. Then the subgroup H of $R(G)$ of order pq is also the unique subgroup of A of order pq, and $R(c) \in C_A(H)$, the centralizer of H in A. Clearly, $C_A(H)$ is a 2-group. Suppose $C_A(H)$ has order 4. Then $C_A(H)$ is a Sylow 4-subgroup of A. This implies that $A_u \leq C_A(H)$ and hence

$A_u \le C_A(R(G))$, which forces that $A_u = 1$, a contradiction. Thus, $C_A(H) = \langle R(c)\rangle$ and $R(G) = H \times C_A(H)$. Take $\sigma \in S_G$ such that $\sigma^{-1}R(G)\sigma \le A$. Then $R(G)^\sigma = H^\sigma \times C_A(H^\sigma)$. By the uniqueness of H in A, one has $R(G)^\sigma = R(G)$, and by Proposition 2.4, T_k is a CI-subset of G.

Let $1 \le k_1, k_2 \le \frac{1}{2}(q-1)$ with $k_1 \ne k_2$. Suppose that $\mathcal{C}_{2pq}^{k_1} \cong \mathcal{C}_{2pq}^{k_2}$. Since $T_{k_i} = \{cb^{k_i}, (cb^{k_i})^{-1}, cab^{k_i}, (cab^{k_i})^{-1}\}$ $(i = 1, 2)$ are CI-subsets of G, $\mathcal{C}_{2pq}^{k_1} \cong \mathcal{C}_{2pq}^{k_2}$ implies that there is a $\beta \in \mathrm{Aut}(G)$ such that $T_{k_1}^\beta = T_{k_2}$. Note that β must map c to c and b to $a^m b$ for some $m \in \mathbb{Z}_p$. Thus, $(cb^{k_1})^\beta = ca^\ell b^{k_1} \in T_{k_2}$ for some $\ell \in \mathbb{Z}_p$. This means that $ca^\ell b^{k_1} = cb^{k_2}$, $(cb^{k_2})^{-1}$, cab^{k_2} or $(cab^{k_2})^{-1}$, each of which is impossible because $1 \le k_1, k_2 \le \frac{1}{2}(q-1)$. Thus, $\mathcal{C}_{2pq}^{k_1} \ncong \mathcal{C}_{2pq}^{k_2}$. □

Construction of a non-Cayley model Let p, q be odd primes such that $4q \mid (p-1)$, and let r be an element of order $4q$ in $\mathbb{Z}_p^*$. Let $K = \{k \mid k$ is an odd integer and $1 \le k \le q-1\}$. For any $k \in K$, define

$$\mathcal{NC}_{2pq}^{r^k} := X\big(r^k; 2q, p\big).$$

Lemma 3.2 *Let p, q, r and K be given as above. Then $\mathcal{NC}_{2pq}^{r^k}$, $k \in K$, are pair-wise non-isomorphic connected tetravalent tightly attached half-arc-transitive non-Cayley graphs of order $2pq$.*

Proof Since r is assumed to have order $4q$ in $\mathbb{Z}_p^*$, r^k has order $4q$ in $\mathbb{Z}_p^*$ for any $k \in K$. It follows that $(r^k)^{2q} = -1$ and $(r^k)^2 \ne \pm 1$ in $\mathbb{Z}_p^*$. By Proposition 2.1, $\mathcal{NC}_{2pq}^{r^k}$ is a connected tetravalent tightly attached half-arc-transitive graph of order $2pq$. Let $\rho : u_i^j \mapsto u_i^{j+1}$ $(i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p)$ and $\sigma : u_i^j \mapsto u_{i+1}^{r^k j}$ $(i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p)$ be defined as permutations on $V(\mathcal{NC}_{2pq}^{r^k})$. It is easy to see that ρ, σ are automorphisms of $\mathcal{NC}_{2pq}^{r^k}$, and that $\sigma^{-1}\rho\sigma = \rho^{r^k}$. Moreover, $\langle\rho, \sigma\rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{4q}$ is half-arc-transitive on $\mathcal{NC}_{2pq}^{r^k}$. Set $A = \mathrm{Aut}(\mathcal{NC}_{2pq}^{r^k})$. By Proposition 2.2, $|A| = 4pq$ and hence $A = \langle\rho, \sigma\rangle$. Clearly, every Sylow 2-subgroup of A is cyclic. If $\mathcal{NC}_{2pq}^{r^k}$ is a Cayley graph, then A has a subgroup, say G, acting regularly on $V(\mathcal{NC}_{2pq}^{r^k})$. Then necessarily $|G| = 2pq$ and $G \trianglelefteq A$. Moreover, $A = GA_v$ for some $v \in V(\mathcal{NC}_{2pq}^{r^k})$. Since $A_v \cong \mathbb{Z}_2$, A has a Sylow 2-subgroup P such that $A_v \le P$. Then $P = P \cap A = (P \cap G) \times A_v \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, contrary to the fact that every Sylow 2-subgroup of A is cyclic. Thus, $\mathcal{NC}_{2pq}^{r^k}$ is a non-Cayley graph.

To complete the proof, it suffices to show that $\mathcal{NC}_{2pq}^{r^k}$ $(k \in K)$ are pair-wise non-isomorphic. Suppose on the contrary that $\mathcal{NC}_{2pq}^{r^m} \cong \mathcal{NC}_{2pq}^{r^n}$, where $m, n \in K$ are distinct. Then $|m-n|$, $2q+m-n$, $m+n$ and $2q+m+n$ are integers between 1 and $4q-1$. Since r is an element of order $4q$ in $\mathbb{Z}_p^*$, we have $r^{m-n} \ne 1$, $r^{2q+m-n} \ne 1$, $r^{m+n} \ne 1$ and $r^{2q+m+n} \ne 1$ in $\mathbb{Z}_p^*$.

Let $V_i = \{v_i^j \mid j \in \mathbb{Z}_p\}$ for each $i \in \mathbb{Z}_{2q}$. Then $V(\mathcal{NC}_{2pq}^{r^m}) = V(\mathcal{NC}_{2pq}^{r^n}) = \bigcup_{i\in\mathbb{Z}_{2q}} V_i$. Note that $\mathcal{NC}_{2pq}^{r^n}$ has an automorphism which fixes v_0^0 and interchanges

v_1^1 and v_1^{-1}, and $v_{2q-1}^{r^{-n}}$ and $v_{2q-1}^{-r^{-n}}$. Thus, $\mathcal{NC}_{2pq}^{r^m} \cong \mathcal{NC}_{2pq}^{r^n}$ implies that there is an isomorphism α from $\mathcal{NC}_{2pq}^{r^m}$ to $\mathcal{NC}_{2pq}^{r^n}$ such that $(v_0^0)^\alpha = v_0^0$ and either $(v_1^1)^\alpha = v_1^1$ or $(v_1^1)^\alpha = v_{2q-1}^{r^{-n}}$. Note that V_i ($i \in \mathbb{Z}_{2q}$) are orbits of the unique normal Sylow p-subgroup of $\mathrm{Aut}(\mathcal{NC}_{2pq}^{r^m})$ and $\mathrm{Aut}(\mathcal{NC}_{2pq}^{r^n})$, respectively. This implies that α maps each V_i to some V_j. Thus, $V_0^\alpha = V_0$ and $V_1^\alpha = V_1$ or V_{2q-1}.

Let $V_1^\alpha = V_1$. Then $(v_1^1)^\alpha = v_1^1$ and $V_\ell^\alpha = V_\ell$ for any $\ell \in \mathbb{Z}_{2q}$. Since the subgraphs induced by $V_0 \cup V_1$ in $\mathcal{NC}_{2pq}^{r^m}$ and also in $\mathcal{NC}_{2pq}^{r^n}$ are cycles of length $2p$, it is easy to see that $(v_0^\ell)^\alpha = v_0^\ell$ and $(v_1^\ell)^\alpha = v_1^\ell$ for any $\ell \in \mathbb{Z}_p$. Similarly, since the subgraphs induced by $V_1 \cup V_2$ in $\mathcal{NC}_{2pq}^{r^m}$ and in $\mathcal{NC}_{2pq}^{r^n}$ are cycles of length $2p$, one has $(v_2^{r^m})^\alpha = v_2^{r^n}$ or $v_2^{-r^n}$ because $(v_1^0)^\alpha = v_1^0$. If $(v_2^{r^m})^\alpha = v_2^{r^n}$ then $(v_1^{2r^m})^\alpha = v_1^{2r^n}$. Note that $(v_1^{2r^m})^\alpha = v_1^{2r^m}$. Thus, $2r^m = 2r^n$ in $\mathbb{Z}_p^*$, that is $r^{m-n} = 1$ in $\mathbb{Z}_p^*$, a contradiction. Similarly, if $(v_2^{r^m})^\alpha = v_2^{-r^n}$ then $(v_1^{2r^m})^\alpha = v_1^{-2r^n}$. Thus, $2r^m = -2r^n$ in $\mathbb{Z}_p^*$, that is $r^{2q+m-n} = 1$ in $\mathbb{Z}_p^*$, also a contradiction.

Now let $V_1^\alpha = V_{2q-1}$. Then $(v_1^1)^\alpha = v_{2q-1}^{r^{-n}}$ and $V_\ell^\alpha = V_{2q-\ell}$ for any $\ell \in \mathbb{Z}_{2q}$. Since the subgraphs induced by $V_0 \cup V_{2q-1}$ in $\mathcal{NC}_{2pq}^{r^m}$ and in $\mathcal{NC}_{2pq}^{r^n}$ are cycles of length $2p$, one has $(v_0^j)^\alpha = v_0^{jr^{-n}}$ and $(v_1^j)^\alpha = v_{2q-1}^{jr^{-n}}$ for any $j \in \mathbb{Z}_p$. In particular, $(v_1^0)^\alpha = v_{2q-1}^0$ and $(v_1^{2r^m})^\alpha = v_{2q-1}^{2r^{m-n}}$. It follows that $(v_2^{r^m})^\alpha = v_{2q-2}^{r^{-2n}}$ or $v_{2q-2}^{-r^{-2n}}$. If $(v_2^{r^m})^\alpha = v_{2q-2}^{r^{-2n}}$ then $(v_1^{2r^m})^\alpha = v_{2q-1}^{2r^{-2n}}$; thus $v_{2q-1}^{2r^{m-n}} = v_{2q-1}^{2r^{-2n}}$, implying $r^{m+n} = 1$ in $\mathbb{Z}_p^*$, a contradiction. One may assume that $(v_2^{r^m})^\alpha = v_{2q-2}^{-r^{-2n}}$ and hence $(v_1^{2r^m})^\alpha = v_{2q-1}^{-2r^{-2n}}$; thus $v_{2q-1}^{2r^{m-n}} = v_{2q-1}^{-2r^{-2n}}$, implying $r^{2q+m+n} = 1$ in $\mathbb{Z}_p^*$, a contradiction. It follows that all cases are impossible. □

4 A classification

Now, we classify the tetravalent half-arc-transitive graphs of order $2pq$ for $q < p$ odd primes. We first introduce two concepts which will be used later. Let X and Y be two graphs. The *lexicographic product* $X[Y]$ is defined as the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(X[Y])$, u is adjacent to v in $X[Y]$ whenever either $\{x_1, x_2\} \in E(X)$ or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$. It is easy so see that if X and Y are symmetric graphs then so is $X[Y]$. Let N be a normal subgroup of $\mathrm{Aut}(X)$. The *quotient graph* X_N of X relative to N is defined as the graph whose vertices are the orbits of N in $V(X)$ and two orbits are adjacent if there is an edge in X between vertices lying in these two orbits.

The following theorem is the main result of this paper.

Theorem 4.1 *Let $q < p$ be odd primes and let X be a connected tetravalent graph of order $2pq$. Then, X is half-arc-transitive if and only if either $(p,q) \neq (7,3)$, $q \mid (p-1)$ and $X \cong \mathcal{C}_{2pq}^\ell$ for $1 \le \ell \le \frac{1}{2}(q-1)$ or $4q \mid (p-1)$ and $X \cong \mathcal{NC}_{2pq}^{r^k}$ where r is an element of order $4q$ in $\mathbb{Z}_p^*$ and k is an odd integer satisfying $1 \le k \le q-1$.*

Furthermore, the number of non-isomorphic connected tetravalent half-arc-transitive graphs of order $2pq$ is equal to

$$\begin{cases} 0 & \text{if } q \nmid (p-1) \text{ or } (p,q)=(7,3), \\ q-1 & \text{if } q \mid (p-1) \text{ and } 4 \mid (p-1), \\ \frac{1}{2}(q-1) & \text{if } q \mid (p-1),\ 4 \nmid (p-1) \text{ and } (p,q) \neq (7,3). \end{cases}$$

Proof By Lemmas 3.1 and 3.2, we only need to show the necessity of the first part. Let X be a connected tetravalent half-arc-transitive graph of order $2pq$. By Wilson and Potočnik [36], no tetravalent half-arc-transitive graphs of order 30 or 42 exist. In what follows, assume that $(p,q) \neq (5,3)$ or $(7,3)$. Let $A = \mathrm{Aut}(X)$ and $u \in V(X)$. By Proposition 2.5, the stabilizer A_u of u in A is a 2-group. Thus, $|A| = 2^{\ell+1}pq$ for some positive integer ℓ. In particular, $4pq \mid |A|$. Let B be a normal subgroup of A. First we prove three claims.

Claim 1: $B \ncong \mathbb{Z}_{pq}$.

Suppose to the contrary that $B \cong \mathbb{Z}_{pq}$. Clearly, B acts semiregularly on $V(X)$ with two orbits, say Δ and Δ'. Let us write $\Delta = \{\Delta(b) \mid b \in B\}$ and $\Delta' = \{\Delta'(b) \mid b \in B\}$. One may assume that the actions of B on Δ and Δ' are just by right multiplication, that is, $\Delta(b)^g = \Delta(bg)$ and $\Delta'(b)^g = \Delta'(bg)$ for any $b, g \in B$. By half-arc-transitivity of X, the blocks Δ and Δ' have no edge, implying that X is bipartite. Let the neighbors of $\Delta(1)$ be $\Delta'(b_1)$, $\Delta'(b_2)$, $\Delta'(b_3)$ and $\Delta'(b_4)$, where $b_1, b_2, b_3, b_4 \in B$. Note that B is abelian. For any $b \in B$, the neighbors of $\Delta(b)$ are $\Delta'(bb_1)$, $\Delta'(bb_2)$, $\Delta'(bb_3)$ and $\Delta'(bb_4)$, and furthermore, the neighbors of $\Delta'(b)$ are $\Delta(bb_1^{-1})$, $\Delta(bb_2^{-1})$, $\Delta(bb_3^{-1})$ and $\Delta(bb_4^{-1})$. The map α defined by $\Delta(b) \mapsto \Delta'(b^{-1})$, $\Delta'(b) \mapsto \Delta(b^{-1})$ for any $b \in B$, is an automorphism of X of order 2. For any $b', b \in B$, one has $\Delta(b')^{\alpha b \alpha} = \Delta(b'b^{-1}) = \Delta(b')^{b^{-1}}$ and $\Delta'(b')^{\alpha b \alpha} = \Delta'(b'b^{-1}) = \Delta'(b')^{b^{-1}}$, implying that $b^\alpha = b^{-1}$. Set $G = \langle B, \alpha \rangle$. Since $B \cong \mathbb{Z}_{pq}$, one has $G \cong D_{2pq}$ and hence G acts regularly on $V(X)$. It follows that X is a Cayley graph on G, say $X = \mathrm{Cay}(G, S)$. Since X is connected, S generates G. This forces S to contain an involution, contrary to Proposition 2.6.

Claim 2: If B is a 2-subgroup, then $B \cong \mathbb{Z}_2$.

Consider the quotient graph X_B of X relative to B, and let K be the kernel of A acting on $V(X_B)$. Then each orbit of B in $V(X)$ has length 2 and $|V(X_B)| = pq > 2$. By half-arc-transitivity of X, the subgraph of X induced by each orbit of B has no edges. It follows that X_B has valency 2 or 4. If X_B has valency 2, then X is isomorphic to $C_n[2K_1]$ which is symmetric, a contradiction. Thus, X_B has valency 4, and consequently, $K_u = 1$. Therefore, $K = BK_u = B \cong \mathbb{Z}_2$.

Claim 3: A is solvable with a normal Sylow p-subgroup.

Suppose that A is non-solvable. Then A has a non-abelian simple composite factor T_1/T_2 whose order divides $2^{n+1}pq$. Since $p > q$ are odd primes, by Proposition 2.8, $T_1/T_2 \cong A_5$ or $\mathrm{PSL}(2, 7)$, forcing $(p, q) = (5, 3)$ or $(7, 3)$, a contradiction. Thus, A is solvable.

Let T be a minimal normal subgroup of A. By solvability of A, T must be an elementary abelian group, and by Claim 2, $T \cong \mathbb{Z}_2$, $\mathbb{Z}_p$ or $\mathbb{Z}_q$. If $T \cong \mathbb{Z}_2$ then, by Claim 2 again, T is a maximal normal 2-subgroup of A. Let L/T be a minimal normal subgroup of A/T. Then $L/T \cong \mathbb{Z}_p$ or $\mathbb{Z}_q$. Thus, L has normal Sylow p- and q-subgroups, which are characteristic in L. By normality of L in A, A has a normal subgroup of order p or q. Thus, A always has a normal subgroup of order p or q, say N.

Suppose that $|N| = q$. Set $C = C_A(N)$. Clearly, $N \le C$ and by Proposition 2.7, $A/C \le \mathrm{Aut}(N) \cong \mathbb{Z}_{q-1}$. Since $p > q$, one has $p \mid |C|$ and hence $N \ne C$. Let M/N be a minimal normal subgroup of A/N contained in C/N. Then $M \trianglelefteq A$ and M/N is an elementary abelian r-group for $r = 2$ or p. Furthermore, $M = N \times R$, where R is a Sylow r-subgroup of M. Clearly, R is characteristic in M and so normal in A. If $r = p$ then $M \cong \mathbb{Z}_{pq}$, contrary to Claim 1. Thus, $r = 2$. By Claim 2, $R \cong \mathbb{Z}_2$, and hence $M \cong \mathbb{Z}_{2q}$. Then $M \le C_A(M)$ and again by Proposition 2.7, $A/C_A(M) \le \mathrm{Aut}(M) \cong \mathbb{Z}_{q-1}$. Also, since $p > q$, one has $p \mid |C_A(M)|$, and consequently, $M \ne C_A(M)$. Let H/M be a minimal normal subgroup of A/M contained in $C_A(M)/M$. Then $H \trianglelefteq A$ and H/M is an elementary abelian 2- or p-group. For the former case, the Sylow 2-subgroup of H would be a normal subgroup of A of order at least 4, contrary to Claim 2. For the latter case, $H \cong \mathbb{Z}_{2pq}$. In this case, the subgroup of H of order pq is a normal cyclic subgroup of A, contrary to Claim 1. Thus, $|N| = p$, and hence N is a normal Sylow p-subgroup of A, as claimed.

Now we are ready to complete the proof. Let P be the Sylow p-subgroup of A. Then $P \cong \mathbb{Z}_p$ and by Claim 3, $P \trianglelefteq A$. Consider the quotient graph X_P, and let K be the kernel of A acting on $V(X_P)$. Then X_P has order $2q$. Since X is half-arc-transitive, the subgraph of X induced by each orbit of P has no edges, and further, X_P has valency 4 or 2.

Suppose that X_P has valency 4. Then $K_u = 1$ and $P = K$. This implies that X_P is A/P-half-arc-transitive and hence A/P is non-abelian. Let $C = C_A(P)$. Then $P \le C$ and by Proposition 2.7, $A/C \le \mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$. Thus, $P \ne C$. Take a minimal normal subgroup, say M/P, of A/P contained in C/P. Then $M \trianglelefteq A$ and M/P is an elementary abelian r-subgroup with $r = q$ or 2. If $r = q$, then $M \cong \mathbb{Z}_{pq}$, contrary to Claim 1. Thus, $r = 2$, and by Claim 2, one has $M = P \times R$ with $R \cong \mathbb{Z}_2$, that is $M \cong \mathbb{Z}_{2p}$. Again by Proposition 2.7, $A/C_A(M) \le \mathrm{Aut}(M) \cong \mathbb{Z}_{p-1}$. Clearly, $M \le C_A(M)$. If $M = C_A(M)$, then $(A/P)/(M/P) \cong A/M$ is cyclic. Since $M/P \cong \mathbb{Z}_2$ is normal in A/P, M/P is contained in the center of A/P. It follows that A/P is abelian, a contradiction. Thus, $M \ne C_A(M)$. Take a minimal normal subgroup, say H/M, of A/M in $C_A(M)$. Then $H \trianglelefteq A$ and by Claim 2, $H/M \cong \mathbb{Z}_q$. It follows that $H \cong \mathbb{Z}_{2pq}$ and the subgroup of H of order pq is a normal cyclic subgroup of A, contrary to Claim 1.

As the remaining case, let X_P have valency 2, namely, $X_P \cong C_{2q}$. Suppose $K_u = 1$. Then $P = K$, and so $A/P \le \mathrm{Aut}(X_P) \cong D_{4q}$. Recall that $4pq \mid |A|$, one has $A/P = \mathrm{Aut}(X_P) \cong D_{4q}$. Then $QP/P \trianglelefteq A/P$, where Q is a Sylow q-subgroup of A. Since $A/C_A(P) \le \mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$, one has $P \ne C_A(P)$. If $q \mid |C_A(P)|$, then $PQ \cong \mathbb{Z}_{pq}$, contrary to Claim 1. Thus, $q \nmid |C_A(P)|$ and $C_A(P)/P$ is a 2-group. It follows that $C_A(P) = P \times R$, where $R \cong \mathbb{Z}_2$ by Claim 2. Then $C_A(P)/P$ is contained in the center of A/P, and since $(A/P)/(C_A(P)/P) \cong A/C_A(P)$ is cyclic, A/P is abelian, contrary to the fact that $A/P \cong D_{4q}$. Consequently, $K_u \ne 1$. Let $V(X_P) = \{B_i \mid i \in \mathbb{Z}_{2q}\}$ such that $B_i \sim B_{i+1}$. Then $X[B_i \cup B_{i+1}] \cong C_{2p}$ for each $i \in \mathbb{Z}_{2q}$. Let $D_A(X)$ be one of the two oriented graphs associated with the action of A on X. Since P is transitive on each B_i and $K_u \ne 1$, all edges in $X[B_i \cup B_{i+1}]$ have the same direction either from B_i to B_{i+1} or from B_{i+1} to B_i in the oriented graph $D_A(X)$. This implies that for each $i \in \mathbb{Z}_{2q}$, $X[B_i \cup B_{i+1}]$ is an alternating cycle of X with radius p. Clearly, $X[B_i \cup B_{i+1}]$ and $X[B_{i+1} \cup B_{i+2}]$ intersect in p vertices. It follows that the attachment number of X is also p. Thus, X is a tetravalent tightly attached half-arc-transitive graph of odd radius p. By Proposition 2.1, $X \cong X(r; 2q, p)$, where $r \in \mathbb{Z}_p^*$ such that $r^{2q} = \pm 1$, and $r^2 \ne \pm 1$ and $(2q, p) \ne (6, 7)$. In particular, $(p, q) \ne (7, 3)$. Recall that $X(r; 2q, p)$ has vertex set $V = \{u_i^j \mid i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p\}$ and edge set $E = \{\{u_i^j, u_{i+1}^{j \pm r^i}\} \mid i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p\}$.

Let $r^{2q} = 1$. Then r is an element of $\mathbb{Z}_p^*$ of order q or $2q$ because $r^2 \ne 1$. If r has order $2q$, then r^{q+1} has order q, and it is easy to see that $X(r; 2q, p) = X(r^{q+1}; 2q, p)$. Thus, we can always assume that r is of order q. By Lemma 3.1, X is isomorphic to one of $\mathcal{C}_{2pq}^{\ell}$ for some $1 \le \ell \le \frac{1}{2}(q-1)$.

Let $r^{2q} = -1$. Then r is an element of $\mathbb{Z}_p^*$ of order $4q$. There are exactly $2(q-1)$ elements of order $4q$ in $\mathbb{Z}_p^*$, that is r^k, where $k \in \mathbb{Z}_{4q}^*$. The graph $X(r^k; 2q, p)$ has edge set $\{\{u_i^j, u_{i+1}^{j \pm r^{ki}}\} \mid i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p\}$, and vertex set $\{u_i^j \mid i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p\}$ for each $k \in \mathbb{Z}_{4q}^*$. It is easy to see that $X(r^k; 2q, p) = X(r^{k+2q}; 2q, p)$. Note that $(r^k)^i = (r^{2q-k})^{2q-i}$ or $-(r^{2q-k})^{2q-i}$ for each $i \in \mathbb{Z}_p$. One may easily show that the permutation $u_i^j \mapsto u_{2q-i+1}^j$, $(j \in \mathbb{Z}_p$ and $i \in \mathbb{Z}_{2q})$ on $\{u_i^j \mid i \in \mathbb{Z}_{2q}, j \in \mathbb{Z}_p\}$ is a graph isomorphism from $X(r^k; 2q, p)$ to $X(r^{2q-k}; 2q, p)$. It follows that $X \cong X(r^k; 2q, p)$ for some odd integer k satisfying $1 \le k \le q-1$. Thus, $X \cong \mathcal{NC}_{2pq}^{r^k}$ for some odd integer k between 1 and $q-1$. □

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