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Square-bounded partitions and Catalan numbers

Matthew Bennett · Vyjayanthi Chari · R.J. Dolbin · Nathan Manning

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Abstract For each integer $k \geq 1$, we define an algorithm which associates to a partition whose maximal value is at most k a certain subset of all partitions. In the case when we begin with a partition λ which is square-bounded, i.e. $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ with $\lambda_1 = k$ and $\lambda_k = 1$, applying the algorithm ℓ times gives rise to a set whose cardinality is either the Catalan number $c_{\ell-k+1}$ (the self dual case) or twice that Catalan number. The algorithm defines a tree and we study the propagation of the tree, which is not in the isomorphism class of the usual Catalan tree. The algorithm can also be modified to produce a two-parameter family of sets and the resulting cardinalities of the sets are the ballot numbers. Finally, we give a conjecture on the rank of a particular module for the ring of symmetric functions in $2\ell + m$ variables.

Keywords Partitions · Young diagrams · Catalan numbers · Current algebras

1 Introduction

The Catalan numbers c_ℓ, where ℓ is a non-negative integer, appear in a large number combinatorial settings and in [10] one can find 66 interpretations of the Catalan

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M. Bennett · V. Chari · R.J. Dolbin (✉) · N. Manning
Department of Mathematics, University of California, Riverside, CA 92521, USA
e-mail: rjdolbin@math.ucr.edu

M. Bennett
e-mail: mbenn002@math.ucr.edu

V. Chari
e-mail: chari@math.ucr.edu

N. Manning
e-mail: nmanning@math.ucr.edu

numbers. Many of these generalize to the ballot numbers $b_{\ell,m}$ where ℓ and m are both non-negative integers and $c_\ell = b_{\ell,0}$. These numbers also appear in the representation theory of the Lie algebra $\mathfrak{sl}_2$ in the following way. Consider the $(2\ell + m)$-fold tensor product of the natural representation of $\mathfrak{sl}_2$. As a representation of $\mathfrak{sl}_2$ this tensor product is completely reducible and the multiplicity of the $(m + 1)$-dimensional irreducible representation of $\mathfrak{sl}_2$ is $b_{\ell,m}$.

The current paper was motivated by the study of the category of finite-dimensional representations of the affine Lie algebra associated to $\mathfrak{sl}_2$ and an attempt begun in [4] and [5] to develop a theory of highest weight categories after [7]. In the course of their work, Chari and Greenstein realized that one of the results required for this would be to prove that a certain naturally defined module for the ring of symmetric functions in 2ℓ-variables is free of rank equal to the Catalan number c_ℓ. In fact, it has turned out that finer results are needed, namely one would need the basis of the free module and also an extension to more general modules for the ring of symmetric functions. The conjecture is made precise in Sect. 5 of this paper.

In Sect. 2 of this paper, we define an algorithm which, when applied ℓ times to a partition $\lambda = (k \geq \lambda_2 \cdots \geq \lambda_{k-1} \geq 1)$, gives a subset of partitions with cardinality equal to the Catalan number $c_{\ell-k+1}$. In fact we prove that this algorithm defines an equivalence relation on the set $\mathcal{P}^\ell$ of all partitions $\mu = (\mu_1 \geq \cdots \geq \mu_\ell)$ which satisfy $\mu_1 \leq \ell$. The algorithm defines an ordered rooted tree which is labeled either by pairs of positive integers or by single positive integers, and thus is very different from the usual Catalan tree. In addition, our algorithm uses a certain involution τ_ℓ which defines a duality on $\mathcal{P}^\ell$. Our proofs in Sect. 3 are algebraic rather than combinatorial. A reason for this is that we were unable to find any natural bijection between the sets we describe and the usual sets giving rise to the Catalan numbers which keeps track of the duality.

In Sect. 4 we describe a generalization of the algorithm which gives rise to ballot numbers $b_{\ell,m}$. This time, the algorithm describes a set of m rooted ordered trees. To do this, we prove an alternating identity for the ballot numbers, which generalizes the well-known one [1] for Catalan numbers. In Sect. 5, we also discuss further directions in which these algorithms could be generalized.

2 The main results

2.1

Throughout the paper $\mathbf{N}$ denotes the set of natural numbers and $\mathbf{Z}_+$ the set of non-negative integers. By a partition λ with n parts, we mean a decreasing sequence

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$$

of positive integers. We denote the set of all partitions by $\mathcal{P}$. Given $\lambda \in \mathcal{P}$ set

$$\lambda \setminus \{\lambda_n\} = (\lambda_1 \geq \cdots \geq \lambda_{n-1}),$$

and for $0 < \lambda_{n+1} \leq \lambda_n$ set

$$(\lambda : \lambda_{n+1}) = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_{n+1}).$$

For $\ell, m \geq 0$, let $b_{\ell,m}$ be the ballot number given by

$$b_{\ell,m} = \binom{2\ell+m}{\ell} - \binom{2\ell+m}{\ell-1},$$

and set $c_\ell = b_{\ell,0}$.

2.2

For $k, n \in \mathbf{N}$, let $\mathcal{P}^{n,k}$ be the set of partitions with exactly n parts where no part is bigger than k, i.e.

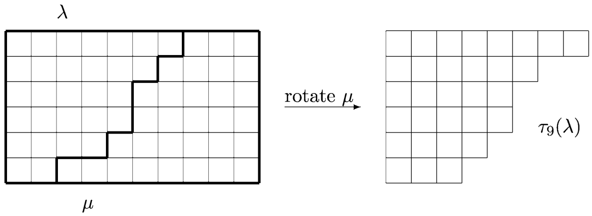
$$\mathcal{P}^{n,k} = \bigl\{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) : \lambda_1 \leq k, \lambda_n > 0\bigr\}.$$

Let $\tau_k : \mathcal{P}^{n,k} \to \mathcal{P}^{n,k}$ be defined by

$$\tau_k(\lambda_1 \geq \cdots \geq \lambda_n) = (k+1-\lambda_n \geq \cdots \geq k+1-\lambda_1).$$

Clearly τ_k is a bijection of order two. To understand the map τ_k in terms of Young diagrams, it is convenient to think of $\mathcal{P}^{n,k}$ as the set of partitions whose Young diagrams lie in an $n \times k$ rectangle and have exactly n rows. The Young diagram of $\tau_k(\lambda)$ is obtained by taking the skew diagram $(k+1)^n \setminus \lambda$ and rotating it by 180 degrees.

As an example, we can regard the partition $\lambda = (7 \geq 6 \geq 5 \geq 5 \geq 4 \geq 2)$ as an element of $\mathcal{P}^{6,9}$, in which case we have $\tau_9(\lambda) = (8 \geq 6 \geq 5 \geq 5 \geq 4 \geq 3)$, and pictorially we get



2.3

Set $\mathcal{P}^k = \mathcal{P}^{k,k}$. Given $\lambda \in \mathcal{P}^k$ and $\ell, k \in \mathbf{N}$ with $\ell \geq k$, define subsets $\mathcal{P}^\ell(\lambda)$ of $\mathcal{P}^\ell$ inductively, by

$$\mathcal{P}^k(\lambda) = \{\lambda\} \cup \{\tau_k\lambda\}, \qquad \mathcal{P}^\ell(\lambda) = \mathcal{P}^\ell_{\mathrm{d}}(\lambda) \cup \mathcal{P}^\ell_\tau(\lambda),$$

where

$$\mathcal{P}^\ell_{\mathrm{d}}(\lambda) = \bigl\{\mu \in \mathcal{P}^\ell : \mu \setminus \{\mu_\ell\} \in \mathcal{P}^{\ell-1}(\lambda)\bigr\},$$
$$\mathcal{P}^\ell_\tau(\lambda) = \bigl\{\mu \in \mathcal{P}^\ell : \tau_\ell\mu \setminus \{\ell+1-\mu_1\} \in \mathcal{P}^{\ell-1}(\lambda)\bigr\} = \tau_\ell\mathcal{P}^\ell_{\mathrm{d}}(\lambda).$$

Clearly

$$\mathcal{P}^\ell(\tau_\ell\lambda) = \tau_\ell\mathcal{P}^\ell(\lambda) = \mathcal{P}^\ell(\lambda).$$

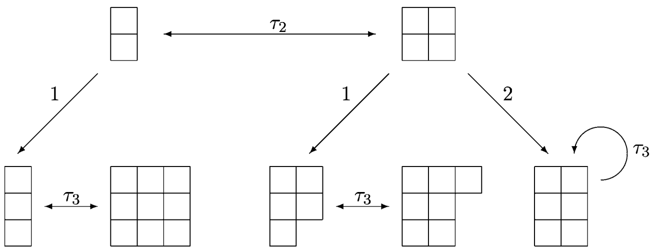
Lemma *Let $\mu \in \mathcal{P}^\ell$. Then*

$$\mu \in \mathcal{P}^\ell_{\mathrm{d}}(\lambda) \implies \mu_1 \le \ell - 1,$$
$$\mu \in \mathcal{P}^\ell_{\tau}(\lambda) \implies \mu_\ell > 1.$$

Proof The first statement is clear from the definition of $\mathcal{P}^\ell_{\mathrm{d}}(\lambda)$, while for the second, note that if $\mu = \tau_\ell \nu$ for some $\nu \in \mathcal{P}^\ell_{\mathrm{d}}(\lambda)$, then $\mu_\ell = \ell + 1 - \nu_1 \ge 2$. □

2.4

We illustrate the recursive definition of $\mathcal{P}^\ell(\lambda)$ in a simple case using Young diagrams. Consider the case when $\lambda = (1 \ge 1) \in \mathcal{P}^2$. Then, the elements of $\mathcal{P}^2(\lambda)$ and $\mathcal{P}^3(\lambda)$ are obtained as follows:



2.5

For $k \in \mathbf{Z}_+$, set

$$\mathcal{P}^k_{\mathrm{sqb}} = \{\lambda \in \mathcal{P}^k : \lambda_1 = k,\ \lambda_k = 1\}.$$

The following is the main result of this section.

Theorem 1

(i) *Let $\ell, k \in \mathbf{N}$ be such that $\ell \ge k$ and let $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$. Then,*

$$\#\mathcal{P}^\ell(\lambda) = \begin{cases} c_{\ell-k+1}, & \lambda = \tau_k\lambda, \\ 2c_{\ell-k+1}, & \lambda \ne \tau_k\lambda. \end{cases}$$

(ii) *Let $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$, $\nu \in \mathcal{P}^s_{\mathrm{sqb}}$. For all $\ell \in \mathbf{Z}_+$ with $\ell \ge \max(k, s)$, we have*

$$\mathcal{P}^\ell(\lambda) \cap \mathcal{P}^\ell(\nu) = \emptyset, \quad \textit{if } \nu \notin \{\lambda, \tau_k(\lambda)\}.$$

(iii) *We have*

$$\mathcal{P}^\ell = \bigsqcup_{\{\lambda \in \mathcal{P}^k_{\mathrm{sqb}}:\ \ell \ge k \ge 1\}} \mathcal{P}^\ell(\lambda).$$

Remark The first part of the theorem in particular proves that applying the algorithm m times to any element of $\mathcal{P}^k_{\mathrm{sqb}}$ for any $k \geq 1$ produces a set whose cardinality depends only on m and the size of the orbit of the initial element. The remaining parts prove that the algorithm gives a partition of the set $\bigcup_{\ell \geq 1} \mathcal{P}^\ell$.

2.6

We note the following corollary of the theorem, which is also a consequence of a well-known combinatorial identity.

Corollary *For $\ell \geq 1$, we have*

$$\ell \cdot c_{\ell+1} = \sum_{i=1}^{\ell} (\ell - i + 2) c_i c_{\ell-i+1}.$$

2.7

We shall prove the theorem in Sect. 3. For the rest of this section we show that our algorithm defines a tree if $\lambda = \tau_k \lambda$ and a forest with two trees if $\lambda \neq \tau_k \lambda$. We study the propagation of the tree and forest, respectively. The first step in this is to observe that the sets $\mathcal{P}^\ell_d(\lambda)$ and $\mathcal{P}^\ell_\tau(\lambda)$ need not be disjoint and to identify the intersection of the two sets.

Proposition *Let $\ell \geq k \geq 1$, $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$. We have*

$$\begin{aligned} \mathcal{P}^\ell_{\mathrm{d}}(\lambda) \cap \mathcal{P}^\ell_\tau(\lambda) &= \big\{\mu \in \mathcal{P}^\ell_{\mathrm{d}}(\lambda) : \ell - 1 \geq \mu_1 \geq \mu_\ell \geq 2\big\} \\ &= \big\{\mu \in \mathcal{P}^\ell_\tau(\lambda) : \ell - 1 \geq \mu_1 \geq \mu_\ell \geq 2\big\} \\ &= \big\{\mu \in \mathcal{P}^\ell(\lambda) : \ell - 1 \geq \mu_1 \geq \mu_\ell \geq 2\big\}. \end{aligned}$$

Proof Notice that

$$\mu \in \mathcal{P}^\ell_{\mathrm{d}}(\lambda) \cap \mathcal{P}^\ell_\tau(\lambda) \implies \mu \setminus \{\mu_\ell\}, \qquad \tau_\ell \mu \setminus \{\ell + 1 - \mu_1\} \in \mathcal{P}^{\ell-1}(\lambda),$$

and hence we get

$$\mu_1 \leq \ell - 1, \qquad \ell + 1 - \mu_\ell \leq \ell - 1, \quad \text{i.e.} \quad \ell - 1 \geq \mu_1 \geq \mu_\ell \geq 2.$$

To prove the reverse inclusion we proceed by induction on ℓ. If $\ell \in \{k, k+1\}$ then there does not exist $\mu \in \mathcal{P}^\ell$ with $2 \leq \mu_\ell \leq \mu_1 \leq \ell - 1$ and hence induction begins. For the inductive step, we must prove that if $\mu \in \mathcal{P}^\ell_{\mathrm{d}}(\lambda)$ is such that $\ell - 1 \geq \mu_1 \geq \mu_\ell \geq 2$, then $\tau_\ell \mu \in \mathcal{P}^\ell_{\mathrm{d}}(\lambda)$, i.e. that $\tau_\ell \mu \setminus \{\ell + 1 - \mu_1\} \in \mathcal{P}^{\ell-1}(\lambda)$. This is equivalent to proving that $\tau_{\ell-1}(\tau_\ell \mu \setminus \{\ell + 1 - \mu_1\}) \in \mathcal{P}^{\ell-1}(\lambda)$, i.e. that

$$\mu' = (\mu_2 - 1 \geq \cdots \geq \mu_{\ell-1} - 1 \geq \mu_\ell - 1) \in \mathcal{P}^{\ell-1}(\lambda) \tag{2.1}$$

and in fact we claim that $\mu' \in \mathcal{P}_{\mathrm{d}}^{\ell-2}(\lambda)$. Consider the case when $\mu \setminus \{\mu_\ell\} \in \mathcal{P}_{\mathrm{d}}^{\ell-1}(\lambda)$; then the induction hypothesis applies and we get $\tau_{\ell-1}(\mu \setminus \{\mu_\ell\}) \in \mathcal{P}_{\mathrm{d}}^{\ell-1}(\lambda)$, i.e. that

$$\nu = (\ell - \mu_{\ell-1} \geq \cdots \geq \ell - \mu_2) \in \mathcal{P}^{\ell-2}(\lambda),$$

and hence

$$\mu' = \tau_{\ell-2}\nu = (\mu_2 - 1 \geq \cdots \geq \mu_{\ell-1} - 1) \in \mathcal{P}^{\ell-2}(\lambda),$$

and we are done. Now suppose that $\mu \setminus \{\mu_\ell\} \in \mathcal{P}_{\tau}^{\ell-1}(\lambda)$. This means precisely that $\tau_{\ell-1}(\mu \setminus \{\mu_\ell\}) \in \mathcal{P}_{\mathrm{d}}^{\ell-1}(\lambda)$ and the preceding argument repeats and proves this case. The other statements of the proposition are now clear. □

Corollary *For all $\ell \geq k$, we have*

$$\begin{aligned}\mathcal{P}^{\ell}(\lambda) &= \mathcal{P}_{\mathrm{d}}^{\ell}(\lambda) \sqcup \left\{\mu \in \mathcal{P}_{\tau}^{\ell}(\lambda) : \mu_1 = \ell\right\} \\ &= \mathcal{P}_{\tau}^{\ell}(\lambda) \sqcup \left\{\mu \in \mathcal{P}_{\mathrm{d}}^{\ell}(\lambda) : \mu_\ell = 1\right\}.\end{aligned}$$

Proof Suppose that $\mu \in \mathcal{P}^{\ell}(\lambda)$ and $\mu \notin \mathcal{P}_{\mathrm{d}}^{\ell}(\lambda)$. The proposition implies that we must then have either $\mu_1 = \ell$ or $\mu_\ell = 1$. Since $\mu \in \mathcal{P}_{\tau}^{\ell}(\lambda)$, this means by (2.3) that $\mu_\ell \neq 1$ and hence we have $\mu_1 = \ell$ which proves the first equality of the corollary. The second follows by applying τ_ℓ. □

2.8

Given any $\mu = (\mu_1 \geq \cdots \geq \mu_\ell) \in \mathcal{P}^{\ell}$, set

$$\mathbf{d}(\mu) = \left\{(\mu : j) : 1 \leq j \leq \mu_\ell\right\} \cup \left\{\tau_{\ell+1}(\mu : 1)\right\} \subset \mathcal{P}^{\ell+1}.$$

Notice that if $\mu \neq \mu'$ then $\mathbf{d}(\mu) \cap \mathbf{d}(\mu') = \emptyset$: it is clear that $(\mu : j) = (\mu' : j')$ for some j, j' implies that $\mu = \mu'$, and if $(\mu, j) = \tau_{\ell+1}(\mu' : 1)$ for some j, then we would have $\mu_1 = \ell + 1$, which is impossible since $\mu \in \mathcal{P}^{\ell}$. By Corollary 2.7 we see that

$$\mathcal{P}^{\ell+1}(\lambda) = \bigsqcup_{\nu \in \mathcal{P}^{\ell}(\lambda)} \mathbf{d}(\nu). \tag{2.2}$$

Let $\lambda \in \mathcal{P}_{\mathrm{sqb}}^{k}$ be such that $\lambda = \tau_k(\lambda)$ and define a tree $\mathbf{T}_\lambda$ as follows. The set of vertices of the tree is

$$\mathcal{P}(\lambda) = \bigsqcup_{\ell \geq k} \mathcal{P}^{\ell}(\lambda),$$

and two vertices $\mu, \nu \in \mathcal{P}(\lambda)$ are connected by an edge precisely if $\nu \in \mathbf{d}(\mu)$ or vice-versa. The tree $\mathbf{T}_\lambda$ is naturally rooted at λ and the elements of $\mathcal{P}^{\ell}(\lambda)$ are those vertices with a path of length $\ell - k$ to the root. The vertices of the tree at any given level come with a natural total order defined as follows. Suppose that we have fixed an ordering of the vertices $\mathcal{P}^{\ell}(\lambda)$; then the order on $\mathcal{P}^{\ell+1}(\lambda)$ is as follows:

$$\nu \prec \nu' \implies \mu \prec \mu' \quad \forall \mu \in \mathbf{d}(\nu)\ \mu' \in \mathbf{d}(\nu')$$

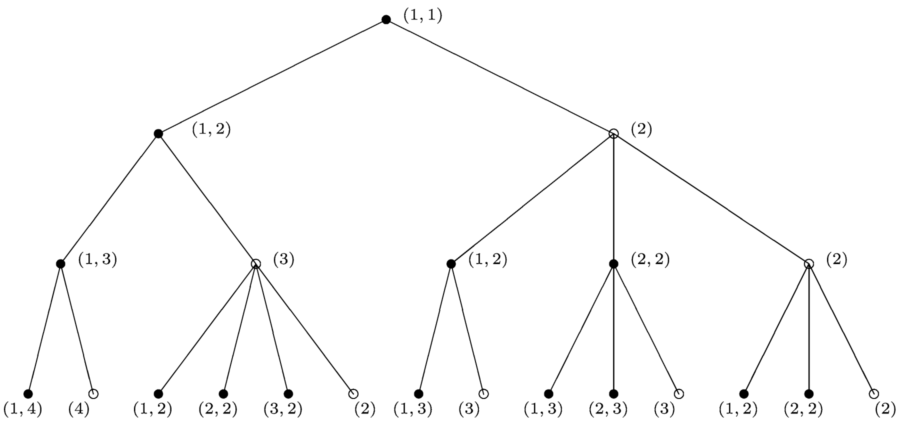
and the ordering on $\mathbf{d}(\nu)$ is given by

$$(\nu:1) \prec (\nu:2) \prec \cdots \prec (\nu:\nu_\ell) \prec \tau(\nu:1).$$

In the case when $\lambda \neq \tau_k\lambda$ the preceding construction gives a forest of two trees $\mathbf{F}_\lambda$ rooted at λ and $\tau_k\lambda$ respectively.

2.9

We shall now see that the propagation of the tree $\mathbf{T}_\lambda$ is independent of λ. For this, we define another rooted tree $\mathbf{T}$ as follows. The vertices of the tree will be labeled either by a pair of integers or a single integer. The root v_0 of the tree will be labeled $(1,1)$. We then define the labels of descendants of a node based on the label of the original node: $\mathbf{d}((m,n)) = \{(i,n+1) : 1 \le i \le m\} \cup \{(n+1)\}$, and $\mathbf{d}((p)) = \{(i,2) : 1 \le i \le p\} \cup \{(2)\}$. Note that the label $(1,1)$ never appears again and is uniquely associated to the root. The following picture shows the labeling at the first few levels.



Proposition *For $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$, we have an isomorphism of trees $\psi : \mathbf{T}_\lambda \cong \mathbf{T}$ such that $\psi(\lambda) = v_0$ and if $\nu \in \mathcal{P}^\ell(\lambda)$ then the vertex $\psi(\nu)$ has label $(\nu_\ell, \ell+1-\nu_1)$, if $\nu \in \mathcal{P}^\ell_{\mathrm{d}}(\lambda)$, and label (ν_ℓ) if $\nu \in \mathcal{P}^\ell_\tau(\lambda)$.*

Proof We define the isomorphism inductively and note that mapping the root λ of $\mathbf{T}_\lambda$ to the root v_0 of $\mathbf{T}$ gives the desired labels. Let $\mathbf{T}^s$, $\mathbf{T}^s_\lambda$ be the subtree of $\mathbf{T}$ and $\mathbf{T}_\lambda$ respectively, consisting of the first s propagations of the root. Assume that we have defined the isomorphism $\psi : (\mathbf{T}_\lambda)^{s-1} \to \mathbf{T}^{s-1}$ with the desired properties. By (2.2) it suffices to show that we can extend ψ to a map from $\mathbf{d}(\nu) \to \mathbf{T}$ for all $\nu \in \mathcal{P}^{s+k}(\lambda)$. Suppose first that $\nu \in \mathcal{P}^{s+k}_{\mathrm{d}}(\lambda)$, in which case $\psi(\nu)$ has label $(\nu_{s+k}, s+k+1-\nu_1)$. This means that the vertex $\psi(\nu)$ has ν_{s+k} descendants with labels $(j, s+k+2-\nu_1)$, $1 \le j \le \nu_{s+k}$ and one descendant with label $(s+k+2-\nu_1)$. Thus if $(\nu : j) \in \mathbf{d}(\nu)$, we let $\psi((\nu : j))$ be the vertex which is the descendant of $\psi(\nu)$ with label $(j, s+k+2-\nu_1)$ and ψ maps $\tau_{s+k+1}(\nu:1)$ to the vertex with label $(s+k+2-\nu_1)$.

Similarly, if $\nu \in \mathcal{P}^{s+k}_\tau(\lambda)$ then $\psi(\nu)$ has label (ν_ℓ) and hence the vertex $\psi(\nu)$ has ν_ℓ descendants with labels $\{(k,2) : 1 \le k \le \nu_\ell\}$ and one descendant with label $\{(2)\}$.

This time, we assign to a descendant $(\nu : m)$ the vertex labeled $(m, 2)$ and to the descendant $\tau_{s+k+1}(\nu : 1)$ the vertex labeled (2). This establishes a bijection between the vertices of $\mathbf{T}_\lambda$ and the vertices of $\mathbf{T}$, which by construction is now an isomorphism of trees. □

3 Proof of Theorem 1

3.1

For $r, k \in \mathbf{Z}_+$, $\lambda \in \mathcal{P}^k$, set

$$\mathcal{P}^\ell(\lambda, r) = \{\mu \in \mathcal{P}^\ell(\lambda) : \mu_\ell \geq r\},$$
$$e_{\ell,r}(\lambda) = \#\mathcal{P}^\ell(\lambda, r).$$

The subsets $\mathcal{P}^\ell_{\mathrm{d}}(\lambda, r)$, $\mathcal{P}^\ell_\tau(\lambda, r)$ are defined in the obvious way. Note that by applying τ_ℓ, we get

$$e_{\ell,r}(\lambda) = \#\{\mu \in \mathcal{P}^\ell(\lambda) : \mu_1 \leq \ell + 1 - r\}. \tag{3.1}$$

As a consequence, we see that $e_{\ell,0}(\lambda) = e_{\ell,1}(\lambda) = \#\mathcal{P}^\ell(\lambda)$. To prove part (i) of the main theorem we must prove that $e_{\ell,1}(\lambda) = c_{\ell-k+1}$. We do this by showing that the $e_{\ell,r}(\lambda)$ satisfy a suitable recurrence relation and by determining the initial conditions; this is the content of the next proposition.

Proposition *Let $r, \ell \in \mathbf{N}$ and assume $\ell \geq k$. We have*

$$e_{\ell,r}(\lambda) = \sum_{s \geq r-1} e_{\ell-1,s}(\lambda) = e_{\ell-1,r-1}(\lambda) + e_{\ell,r+1}(\lambda). \tag{3.2}$$

Moreover,

$$e_{\ell,r}(\lambda) = 0, \quad r > \ell - k + 1, \tag{3.3}$$
$$e_{\ell,\ell-k+1}(\lambda) = \#\mathcal{P}^k(\lambda). \tag{3.4}$$

3.2

Before proving the proposition, we deduce part (i) of the theorem. It is clear that the system of recurrence relations with the initial conditions given in the proposition has a unique solution. It is also well known [1] and is a simple matter to check that if we set

$$e_{\ell,r}(\lambda) = \begin{cases} b_{\ell-k-r+1,r}, & \text{if } \lambda = \tau_k\lambda, \\ 2b_{\ell-k-r+1,r}, & \text{if } \lambda \neq \tau_k\lambda, \end{cases}$$

then the recurrence relation and the initial conditions are satisfied. Since

$$e_{\ell,1}(\lambda) = \#\mathcal{P}^\ell(\lambda) = \begin{cases} c_{\ell-k+1}, & \text{if } \lambda = \tau(\lambda), \\ 2c_{\ell-k+1}, & \text{else}, \end{cases}$$

part (i) is proved.

3.3

To prove (3.2) it is clear that

$$\mathcal{P}^{\ell}_{\mathrm{d}}(\lambda, r) = \bigsqcup_{s \geq r} \{(\mu : s) : \mu \in \mathcal{P}^{\ell-1}(\lambda, s)\},$$

and hence

$$\#\mathcal{P}^{\ell}_{\mathrm{d}}(\lambda, r) = \sum_{s \geq r} e_{\ell-1,s}(\lambda).$$

By Corollary 2.7 we may write

$$\mathcal{P}^{\ell}(\lambda, r) = \mathcal{P}^{\ell}_{\mathrm{d}}(\lambda, r) \sqcup \{\mu \in \mathcal{P}^{\ell}_{\tau}(\lambda, r) : \mu_1 = \ell\}.$$

We have a bijection of sets

$$\{\mu \in \mathcal{P}^{\ell}_{\tau}(\lambda, r) : \mu_1 = \ell\} \to \{\nu \in \mathcal{P}^{\ell-1}(\lambda) : \nu_1 \leq \ell + 1 - r\},$$

given by

$$\mu \to \mu \setminus \{1\} \quad \nu \to \tau_\ell(\nu : 1),$$

hence by using (3.1) we see that

$$\#\{\mu \in \mathcal{P}^{\ell}_{\tau}(\lambda, r) : \mu_1 = \ell\} = \#\{\nu \in \mathcal{P}^{\ell-1}(\lambda) : \nu_1 \leq \ell + 1 - r\} = e_{\ell-1,r-1}(\lambda),$$

which proves (3.2).

3.4

The initial conditions (3.3) and (3.4) are clearly immediate consequences of the following.

Lemma

$$\mu \in \mathcal{P}^{\ell}(\lambda) \implies \mu_\ell \leq \ell - k + 1, \quad \mu_1 \geq k, \tag{3.5}$$

$$\{\mu \in \mathcal{P}^{\ell}(\lambda) : \mu_\ell = \ell - k + 1\} \subset \{\mu \in \mathcal{P}^{\ell}(\lambda) : \mu_1 = \ell\}, \tag{3.6}$$

$$\#\{\mu \in \mathcal{P}^{\ell}(\lambda) : \mu_\ell = \ell - k + 1\} = \#\mathcal{P}^{k}(\lambda). \tag{3.7}$$

Proof To prove (3.5) we proceed by induction on $\ell - k$. If $\ell = k$, then the result holds since $\lambda \in \mathcal{P}^{k}_{\mathrm{sqb}}$ and by the definition of $\mathcal{P}^{k}(\lambda)$. Assume we have proved (3.5) for $\ell - k < s$ and let $\mu \in \mathcal{P}^{k+s}(\lambda)$. If $\mu \in \mathcal{P}^{k+s}_{\mathrm{d}}(\lambda)$ (resp. $\mu \in \mathcal{P}^{k+s}_{\tau}(\lambda)$), then we have

$$\mu_{k+s} \leq \mu_{k+s-1} \leq s \quad (\text{resp. } k + s + 1 - \mu_{k+s} \geq k, \text{ i.e., } \mu_{k+s} \leq s + 1),$$

and the inductive step is proved.

To prove (3.6) notice that it is obviously true if $\ell = k$. If $\ell > k$ and

$$\mu_\ell = \ell - k + 1 \geq 2, \quad \mu_1 < \ell,$$

then Proposition 2.7 applies and we get $\mu \in \mathcal{P}_{\mathrm{d}}^{\ell}(\lambda)$. Applying (3.5) to $\mu \setminus \{\mu_\ell\}$ gives $\mu_\ell \leq \mu_{\ell-1} \leq \ell - k$ which contradicts our assumption.

Suppose that $\mu \in \mathcal{P}^{\ell}(\lambda)$ is such that $\mu_\ell = \ell - k + 1$. Using (3.6) we see that $\mu_1 = \ell$. In particular, $\mu \notin \mathcal{P}_{\mathrm{d}}^{\ell}(\lambda)$, forcing

$$\tau_\ell \mu \setminus \{1\} = (k \geq \cdots \geq \ell + 1 - \mu_2) \in \mathcal{P}^{\ell-1}(\lambda),$$

and hence we get

$$\tau_{\ell-1}\big(\tau_\ell \mu \setminus \{1\}\big) = (\mu_2 - 1 \geq \cdots \geq \mu_\ell - 1 = \ell - k) \in \mathcal{P}^{\ell-1}(\lambda).$$

In other words the assignment $\mu \to \tau_{\ell-1}(\tau_\ell \mu \setminus \{1\})$ defines a bijection $\{\mu \in \mathcal{P}^{\ell}(\lambda) : \mu_\ell = \ell - k + 1\} \to \{\mu \in \mathcal{P}^{\ell-1}(\lambda) : \mu_\ell = \ell - k\}$ and hence (3.7) follows. □

3.5

To prove part (ii) of the theorem, assume that $\nu \notin \{\lambda, \tau_k \lambda\}$ and without loss of generality that $k \geq s$. To see that $\lambda \notin \mathcal{P}^{k}(\nu)$, notice that $\lambda \notin \mathcal{P}_{\mathrm{d}}^{k}(\nu)$ since $\lambda_1 = k$ and by Lemma 2.3 we also have $\lambda \notin \mathcal{P}_{\tau}^{k}(\nu)$ since $\lambda_k = 1$.

Suppose that $\mu \in \mathcal{P}^{\ell}(\lambda) \cap \mathcal{P}^{\ell}(\nu)$ for some $\ell > \max(s, k)$. If $2 \leq \mu_\ell \leq \mu_1 \leq \ell - 1$, then it follows from Proposition 2.7 that $\mu \in \mathcal{P}_{\mathrm{d}}^{\ell}(\lambda) \cap \mathcal{P}_{\mathrm{d}}^{\ell}(\nu)$, i.e., that

$$\mu \setminus \{\mu_\ell\} \in \mathcal{P}^{\ell-1}(\lambda) \cap \mathcal{P}^{\ell-1}(\nu),$$

which contradicts the induction hypothesis. If $\mu_1 = \ell$, then $\mu \notin \mathcal{P}_{\mathrm{d}}^{\ell}(\lambda) \cup \mathcal{P}_{\mathrm{d}}^{\ell}(\nu)$ and so we must have

$$\tau_\ell \mu \in \mathcal{P}_{\mathrm{d}}^{\ell}(\lambda) \cap \mathcal{P}_{\mathrm{d}}^{\ell}(\nu).$$

But this implies that

$$\tau_\ell \mu \setminus \{1\} \in \mathcal{P}^{\ell-1}(\lambda) \cap \mathcal{P}^{\ell-1}(\nu)$$

which is again impossible. The final case to consider is when $\mu_\ell = 1$ and this is now immediate by applying τ_ℓ to the previous case.

3.6

The following proposition proves part (iii) of the theorem.

Proposition (i) *Let* $\lambda \in \mathcal{P}^{k}$, $\nu \in \mathcal{P}^{s}$ *and* $\mu \in \mathcal{P}^{\ell}$ *with* $k \leq s \leq \ell$. *Then*

$$\mu \in \mathcal{P}^{\ell}(\nu), \ \nu \in \mathcal{P}^{s}(\lambda) \implies \mu \in \mathcal{P}^{\ell}(\lambda).$$

(ii) *Let* $\mu \in \mathcal{P}^{\ell}$ *for some* $\ell \in \mathbf{N}$. *Then* $\mu \in \mathcal{P}^{\ell}(\lambda)$ *for some* $\lambda \in \mathcal{P}_{\mathrm{sqb}}^{k}$, $1 \leq k \leq \ell$.

Proof We proceed by induction on $\ell - s$. If $\ell = s$, then we have $\mu = \nu$ or $\mu = \tau_s \nu$ and the statement follows since $\mathcal{P}^s(\lambda)$ is τ_s-stable. If $\ell > s$ and $\mu \in \mathcal{P}^\ell_{\mathrm{d}}(\nu)$, then by the induction hypothesis, we have $\mu \setminus \{\mu_\ell\} \in \mathcal{P}^{\ell-1}(\lambda)$ and hence by definition, $\mu \in \mathcal{P}^\ell(\lambda)$. Otherwise we have $\tau_\ell \mu \setminus \{\ell - \mu_1 + 1\} \in \mathcal{P}^{\ell-1}(\lambda)$ and hence $\tau_\ell \mu \in \mathcal{P}^\ell(\lambda)$. Part (i) follows by using the fact that $\mathcal{P}^\ell(\lambda)$ is τ_ℓ-stable.

To prove (ii), we proceed by induction on ℓ. If $\ell = 1$, then

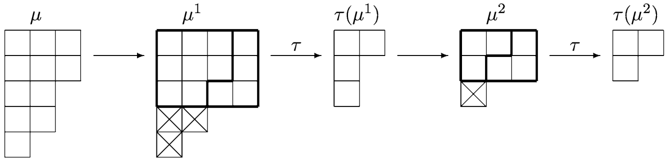
$$\mathcal{P}^1 = \mathcal{P}^1_{\mathrm{sqb}} = \{1\},$$

and we are done. Assume now that we have proved the result for all integers less than ℓ and let $\mu \in \mathcal{P}^\ell$. If $\mu_1 = \ell$ and $\mu_\ell = 1$, there is nothing to prove. If $\mu_1 < \ell$, then set $\nu = \mu \backslash \{\mu_\ell\}$. Clearly $\mu \in \mathcal{P}^\ell(\nu)$ and since $\nu \in \mathcal{P}^{\ell-1}$ we see by the induction hypothesis that $\nu \in \mathcal{P}^s(\lambda)$ for some $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$. Applying part (i) of the proposition shows that $\mu \in \mathcal{P}^\ell(\lambda)$. Finally, consider the case $\mu_1 = \ell$ and $\mu_\ell > 1$. Then we have $\nu = \tau_\ell \mu \in \mathcal{P}^{\ell,\ell-1}$ and we are now in the previous case and so

$$\nu = \tau_\ell \mu \in \mathcal{P}^\ell(\lambda),$$

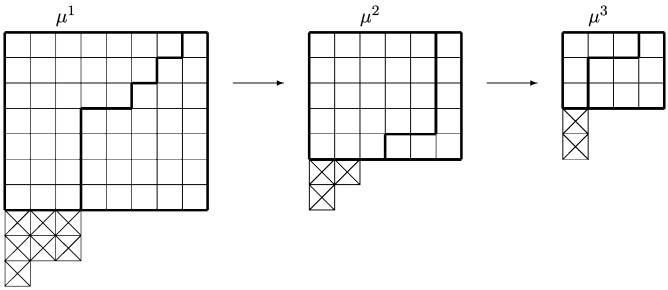
for some $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$. The result again follows since $\mathcal{P}^\ell(\lambda)$ is τ_ℓ-stable. □

The proof of the preceding proposition suggests an algorithm to find, for $\mu \in \mathcal{P}^\ell$, the unique $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$ such that $\mu \in P^\ell(\lambda)$. Suppose $\mu = (\mu_1 \geq \cdots \geq \mu_\ell) \notin \mathcal{P}^k_{\mathrm{sqb}}$. Then set $\mu^1 = (\mu_1 \geq \cdots \geq \mu_{\mu_1})$. Clearly, μ is descended from μ^1 and hence also from $\tau_{\mu_1}(\mu^1)$. Now repeat with $\tau_{\mu_1}(\mu^1)$ in place of μ, setting μ^i to be the partition obtained after iterating this process i times. We illustrate this in the following two simple examples. Take $\mu = (3 \geq 3 \geq 2 \geq 2 \geq 1)$. Then



So we have $\mu \in \mathcal{P}^5(2 \geq 1)$.

If $\mu = (7 \geq 6 \geq 5 \geq 3 \geq 3 \geq 3 \geq 3 \geq 3 \geq 3 \geq 1)$, then



So we have $\mu \in \mathcal{P}^{10}(3 \geq 1 \geq 1)$.

4 From the Catalan numbers to the Ballot numbers

In this section, we generalize the first part of Theorem 1. Namely, given $m \in \mathbf{N}$, we modify the algorithm defined in Sect. 2 so that if we start with a suitable set of m elements, then applying the algorithm ℓ times gives a set of cardinality equal to the ballot number $b_{\ell,m}$. We use the binomial identity

$$\binom{r}{s} = \binom{r-1}{s} + \binom{r-1}{s-1},$$

freely and without comment throughout the rest of the section. Note that in particular, this gives

$$b_{\ell,m} = b_{\ell,m-1} + b_{\ell-1,m+1}. \tag{4.1}$$

4.1

Fix $m \in \mathbf{N}$, and let

$$\Omega_m = \{\{j\} : 1 \le j \le m\} \subset \mathcal{P}^{1,m}.$$

We generalize the definition of the sets $\mathcal{P}^\ell(\lambda)$ given in Sect. 2 as follows. Define subsets $\mathcal{P}^\ell(\Omega_m)$ by

$$\begin{aligned}
&\mathcal{P}^1_{\mathrm{d}}(\Omega_m) = \Omega_m = \mathcal{P}^1_\tau(\Omega_m) = \tau_m \Omega_m,\\
&\mathcal{P}^\ell_{\mathrm{d}}(\Omega_m) = \{(\mu : j) \in \mathcal{P}^{\ell,\ell+m-1} : 1 \le j \le \mu_{\ell-1},\ \mu \in \mathcal{P}^{\ell-1}(\Omega_m)\},\\
&\mathcal{P}^\ell_\tau(\Omega_m) = \tau_{\ell+m-1}\mathcal{P}^\ell_{\mathrm{d}}(\Omega_m),\\
&\mathcal{P}^\ell(\Omega_m) = \mathcal{P}^\ell_{\mathrm{d}}(\Omega_m) \cup \mathcal{P}^\ell_\tau(\Omega_m).
\end{aligned}$$

The main result of this section is

Theorem *For $\ell, m \in \mathbf{N}$, we have*

$$\#\mathcal{P}^\ell(\Omega_m) = b_{\ell,m-1}.$$

4.2

The proof of the theorem is very similar to the corresponding result in Sect. 2. An inspection of Proposition 2.7 and its corollary shows that the proof works in our more general situation and we have

Proposition *For all $\ell \ge 1$, the set $\mathcal{P}^\ell(\Omega_m)$ is the disjoint union of the following sets: for all $\ell \ge 1$, we have*

$$\begin{aligned}
\mathcal{P}^\ell(\Omega_m) &= \mathcal{P}^\ell_{\mathrm{d}}(\Omega_m) \sqcup \{\mu \in \mathcal{P}^\ell_\tau(\Omega_m) : \mu_1 = \ell\}\\
&= \mathcal{P}^\ell_\tau(\Omega_m) \sqcup \{\mu \in \mathcal{P}^\ell_{\mathrm{d}}(\Omega_m) : \mu_\ell = 1\}.
\end{aligned}$$

4.3

For $s, \ell \in \mathbf{N}$, $\ell \geq 1$, set

$$\mathcal{P}^{\ell}(\Omega_m, s) = \{\mu \in \mathcal{P}^{\ell}(\Omega_m) : \mu_\ell \geq s\},$$
$$e_{\ell,s}(\Omega_m) = \#\mathcal{P}^{\ell}(\Omega_m, s)$$

and observe that $e_{\ell,0}(\Omega_m) = e_{\ell,1}(\Omega_m) = \#\mathcal{P}^{\ell}(\Omega_m)$. Note that by applying $\tau_{\ell+m-1}$, we get

$$e_{\ell,s}(\Omega_m) = \#\{\mu \in \mathcal{P}^{\ell}(\Omega_m) : \mu_1 \leq \ell + m - s\}. \tag{4.2}$$

We now determine the recurrence relation and the initial conditions satisfied by the $e_{\ell,s}(\Omega_m)$.

Proposition *For $\ell \geq 1$ and $s \geq 0$, we have*

$$e_{\ell,s}(\Omega_m) = \sum_{j \geq s-1} e_{\ell-1,s}(\Omega_m) = e_{\ell-1,s-1}(\Omega_m) + e_{\ell,s+1}(\Omega_m). \tag{4.3}$$

Moreover,

$$e_{\ell,s}(\Omega_m) = 0 \quad \text{if } s > \ell + m - 1, \tag{4.4}$$
$$e_{\ell,\ell+m-1}(\Omega_m) = 1, \tag{4.5}$$
$$e_{1,s}(\Omega_m) = \max(0, m - s + 1). \tag{4.6}$$

Proof It is immediate from Proposition 4.2 and (4.2) that (4.3) holds. Equation (4.4) holds since by definition $\mu \in \mathcal{P}^{\ell}(\Omega_m)$ implies $\mu_1 \leq \ell + r - 1$. Let $\mu \in \mathcal{P}^{\ell}(\Omega_m)$ be such that $\mu_\ell \geq \ell + r - 1$. Then we must have, $\mu = (\ell + r - 1 \geq \ell + r - 1 \geq \cdots \geq \ell + r - 1)$. Further, this element is $\tau_{\ell+r-1}(1 \geq 1 \cdots \geq 1)$, and we clearly have $(1 \geq 1 \geq \cdots \geq 1) \in \mathcal{P}^{\ell}_{\mathrm{d}}(\Omega_m)$. This proves (4.5). Finally, (4.6) is an immediate consequence of the definitions of $e_{\ell,s}(\Omega_m)$ and of Ω_m. □

4.4

It is clear that the integers $e_{\ell,s}(\Omega_m)$ are completely determined by Proposition 4.3. The proof of the theorem is completed by the following proposition which gives closed formulas for the $e_{\ell,s}(\Omega_m)$.

Proposition *For $\ell, m \geq 1$ and $s \geq 0$, we have*

$$e_{\ell,s}(\Omega_m) = \begin{cases} \binom{m+2\ell-s-1}{\ell}, & s \geq \ell, \\ \sum_{j \geq 0} (-1)^j \binom{s-j}{j} b_{\ell-j,m-1}, & 1 \leq s \leq \ell - 1. \end{cases}$$

In particular, $e_{\ell,1}(\Omega_m) = b_{\ell,m-1}$.

Proof Notice first that the numbers on the right hand side satisfy (4.4), (4.5) and (4.6). The proposition follows if we prove that they also satisfy (4.3). If $s \geq \ell$ (resp. $s < \ell - 1$), then we must check that

$$\binom{m+2\ell-s-1}{\ell} = \binom{m+2\ell-s-2}{\ell-1} + \binom{m+2\ell-s-2}{\ell},$$

and

$$\begin{aligned}&\sum_{j\geq 0}(-1)^j\binom{s-j}{j}b_{\ell-j,m-1}\\&\quad=\sum_{j\geq 0}(-1)^j\binom{s-j-1}{j}b_{\ell-j-1,m-1}+\sum_{j\geq 0}(-1)^j\binom{s-j+1}{j}b_{\ell-j,m-1}.\end{aligned}$$

The first one is just the usual binomial identity, while for the second, observe that

$$\begin{aligned}&\sum_{j\geq 0}(-1)^j\binom{s-j}{j}b_{\ell-j,m-1}-\sum_{j\geq 0}(-1)^j\binom{s-j+1}{j}b_{\ell-j,m-1}\\&\quad=\sum_{j\geq 1}(-1)^{j+1}\binom{s-j}{j-1}b_{\ell-j,m-1}\\&\quad=\sum_{j\geq 0}(-1)^j\binom{s-j-1}{j}b_{\ell-j-1,m-1}.\end{aligned}$$

It remains to consider the case when $s = \ell - 1$, i.e. we have to verify that

$$\begin{aligned}&\sum_{j\geq 0}(-1)^j\binom{\ell-j-1}{j}b_{\ell-j,m-1}\\&\quad=\sum_{j\geq 0}(-1)^j\binom{\ell-j-2}{j}b_{\ell-j-1,m-1}+\binom{m+\ell-1}{\ell}.\end{aligned}$$

This amounts to proving (by replacing j with $j+1$ on the right hand side and using the binomial identity again) that

$$\sum_{j\geq 0}(-1)^j\binom{\ell-j}{j}b_{\ell-j,m-1}=\binom{m+\ell-1}{\ell}.$$

This is probably well known but we isolate it as a separate lemma and give a proof, since we were unable to find a reference in general. □

Lemma *For $\ell, m \geq 0$, we have*

$$\sum_{j\geq 0}(-1)^j\binom{\ell-j}{j}b_{\ell-j,m}=\binom{m+\ell}{\ell}. \tag{4.7}$$

Proof Note that if $m=0$, then this formula is known for all ℓ, [1] since the $b_{\ell,0}$ are Catalan numbers. Assume now that we have proved it for all pairs (ℓ,m') with $m'<m$. To prove it for (ℓ,m) we proceed again by induction on ℓ. If $\ell=0$, the equation is just $b_{0,m}=1$ which follows from the definition. Assuming the result for (ℓ,m), we prove it for $(\ell+1,m)$ as follows. Consider:

$$\begin{aligned}
&\sum_{j\geq 0}(-1)^j\binom{\ell+1-j}{j}b_{\ell+1-j,m}\\
&\quad=\sum_{j\geq 0}(-1)^j\left(\binom{\ell+2-j}{j}-\binom{\ell+1-j}{j-1}\right)(b_{\ell+2-j,m-1}-b_{\ell+2-j,m-2})\\
&\quad=\sum_{j\geq 0}(-1)^j\binom{\ell+2-j}{j}(b_{\ell+2-j,m-1}-b_{\ell+2-j,m-2})\\
&\qquad+\sum_{j\geq 0}(-1)^j\binom{\ell-j}{j}(b_{\ell+1-j,m-1}-b_{\ell+1-j,m-2})\\
&\quad=\sum_{j\geq 0}(-1)^j\binom{\ell+2-j}{j}(b_{\ell+2-j,m-1}-b_{\ell+2-j,m-2})\\
&\qquad+\sum_{j\geq 0}(-1)^j\binom{\ell-j}{j}b_{\ell-j,m}.
\end{aligned}$$

For the inductive step to work, we must have the result for $m=1$ as well. For this, note that $b_{\ell+1-j,1}=b_{\ell+2-j,0}$ and $b_{\ell,-1}=0$ for all ℓ. Hence, we get

$$\begin{aligned}
&\sum_{j\geq 0}(-1)^j\binom{\ell+1-j}{j}b_{\ell+1-j,1}\\
&\quad=\sum_{j\geq 0}(-1)^j\binom{\ell+2-j}{j}b_{\ell+2-j,0}+\sum_{j\geq 0}(-1)^j\binom{\ell-j}{j}b_{\ell-j,1}\\
&\quad=2+\ell
\end{aligned}$$

as required. In the general case, the induction hypothesis applies to all the terms on the right hand side and we get

$$\begin{aligned}
\sum_{j\geq 0}(-1)^j\binom{\ell+1-j}{j}b_{\ell+1-j,m}&=\binom{r+\ell+1}{\ell+2}-\binom{m+\ell}{\ell+2}+\binom{m+\ell}{\ell}\\
&=\binom{m+\ell+1}{\ell+1},
\end{aligned}$$

and the proof is complete. □

5 Concluding Remarks and a Conjecture

5.1

As we mentioned in the introduction, our motivation for this paper came from the study of the representation theory of affine Lie algebras and the current algebra associated to a simple Lie algebra. There is a well-known relationship [2, 6, 8] between the ring of symmetric functions in infinitely many variables and the universal enveloping algebra of the affine Lie algebra, and this relationship plays an important role in the finite-dimensional representation theory of these algebras.

5.2

To explain our motivation further, let $\mathfrak{sl}_2$ be the Lie algebra of 2×2 trace zero complex matrices and let $\mathbf{C}[t]$ be the polynomial algebra in one variable. The associated current algebra is $\mathfrak{sl}_2 \otimes \mathbf{C}[t]$ with the Lie bracket given by,

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg, \quad x, y \in \mathfrak{sl}_2, \ f, g \in \mathbf{C}[t].$$

Clearly, $\mathfrak{sl}_2 \otimes \mathbf{C}[t]$ has a natural $\mathbf{Z}_+$-grading given by powers of t and we let $\mathcal{G}$ be the category of $\mathbf{Z}_+$-graded modules for this Lie algebra. The category is not semi-simple and one is interested in its homological properties; some of these have been studied in [4] and [5]. There are three families of interesting modules in this category, all indexed by $\mathbf{Z}_+ \times \mathbf{Z}_+$: the irreducible modules $V(n, r)$, their projective covers $P(n, r)$ and an intermediate family of modules $W(n, r)$ called the global Weyl modules. This is very similar to the situation in the BGG category $\mathcal{O}$ for semisimple Lie algebras. The global Weyl modules do share some of the properties of the Verma modules, [3] and it is natural to ask if, for instance, a version of BGG-reciprocity holds in $\mathcal{G}$. Although the global Weyl modules are infinite-dimensional, one can define a suitable notion of the multiplicity of $V(n, r)$ in $W(m, s)$, which allows one to formulate a statement analogous to BGG-reciprocity.

The modules $P(n, r)$ have a natural decreasing filtration and we would like to prove that the successive quotients $P(n, r)^{2m}$, $m \geq 0$ are isomorphic to a finite direct sum of global Weyl modules. The module $P(n, r)^{2m}$ has a set of generators which is indexed by all partitions $\mu_1 \geq \cdots \geq \mu_m$, with exactly m parts. The first challenge is to prove that in fact a finite subset of these generators is enough and we can do this by using results of [8] which gives us an upper bound for μ_1. This proves simultaneously that $P(n, r)^{2m}$ is a quotient of a direct sum of global Weyl modules. However, the results of [8] do not immediately give the minimal set of generators. But calculations in small cases showed that elements indexed by the set $\mathcal{P}^m(\Omega_n)$ were in fact minimal. This was the first motivation for this paper: to come up with a natural algorithm which would allow us to identify a conjecturally minimal set of generators for $P(n, r)^{2m}$.

5.3

The next step in the program would be to prove that $P(n, r)^{2m}$ is actually a direct sum of global Weyl modules, and this too has been checked for small values of m.

The global Weyl module $W(n+2m)$ also admits the structure of a right module for the ring of symmetric functions in $(n+2m)$-variables and is in fact a free module for this ring of rank 2^{n+2m}. Using [3] and the structure of the global Weyl modules [6], we can show that the desired BGG-reciprocity would follow, if in addition we can establish the following conjecture, which can be formulated without any mention of representation theory.

For $r \geq 1$, let $\mathbf{C}[x_1, \ldots, x_r]$ be the polynomial ring in r-variables, S_r be the symmetric group on r letters, and let

$$\Lambda_r = \mathbf{C}[x_1, \ldots, x_r]^{S_r}$$

be the ring of invariants under the canonical action of S_r on the polynomial ring. Given elements $a, b \in \mathbf{C}[x_1, \ldots, x_r]$ and $m \geq 0$, set

$$\mathbf{p}_0(a,b) = 1, \qquad \mathbf{p}_m(a,b) = \sum_{j=0}^{m-1} a^{m-j-1} b^j.$$

Given a partition $\mu = (\mu_1 \geq \cdots \geq \mu_s)$, $\mu_s > 0$, let $\mathrm{comp}(\mu) \subset \mathbf{Z}_+^s$ be the set of all distinct elements arising from permutations of $(\mu_1, \ldots, \mu_s) \in \mathbf{Z}_+^s$.

From now on, we consider the case when $r = 2\ell + m$ for some $\ell, m \geq 1$. Given $\mu \in \mathcal{P}^\ell$, we set

$$\mathbf{p}(\mu) = \sum_{\mu' \in \mathrm{comp}(\mu)} \mathbf{p}_{\mu'_1}(x_1, x_2) \cdots \mathbf{p}_{\mu'_\ell}(x_{2\ell-1}, x_{2\ell}).$$

Let $\mathbf{M}(\ell, m)$ be the Λ_r-submodule of $\mathbf{C}[x_1, \ldots, x_r]$ spanned by the elements $\mathbf{p}(\mu)$, $\mu \in \mathcal{P}^\ell$. We can now state our conjecture.

Conjecture *The Λ_r-module $\mathbf{M}(\ell, m)$ is free with basis*

$$\{\mathbf{p}(\mu) : \mu \in \mathcal{P}^\ell(\Omega_m)\},$$

and in particular is of rank $b_{\ell,m-1}$.

We have checked that the conjecture is true for all m if $\ell = 1, 2$ and for $\ell = 3, 4$ for $m = 0, 1, 2$.

5.4

There are other natural generalizations of the algorithm. Namely, we could start with any partition $\lambda \in \mathcal{P}$ and define a subset $\mathcal{Q}(\lambda)$ by setting

$$\mathcal{Q}^1(\lambda) = \{\lambda\} \cup \{\tau_{\lambda_1+\lambda_k-1}(\lambda)\},$$

and then defining $\mathcal{Q}^\ell_{\mathrm{d}}(\lambda)$ in the obvious way and

$$\mathcal{Q}^\ell_\tau(\lambda) = \tau_{\lambda_1+\lambda_k+\ell-1}\mathcal{Q}^\ell_{\mathrm{d}}(\lambda).$$

Computations for small values of ℓ and specific λ do yield sequences of numbers found in [9] for the cardinality of the sets. The abstract result needed, however, to compute the recurrence relations in general is the analog of Corollary 2.7. The corollary is definitely false in this generality and it should be interesting to find the correct statement.

6 Index of notation

Section 2.1: $\mathbf{N}$, $\mathbf{Z}_+$, $\lambda \setminus \{\lambda_n\}$, $(\lambda : \lambda_{n+1})$, $b_{\ell,m}$, c_ℓ.
Section 2.2: $\mathcal{P}^{n,k}$, τ_k.
Section 2.3: $\mathcal{P}^k$, $\mathcal{P}^\ell(\lambda)$, $\mathcal{P}^\ell_{\rm d}(\lambda)$, $\mathcal{P}^\ell_\tau(\lambda)$.
Section 2.5: $\mathcal{P}^k_{\rm sqb}$.
Section 3.1: $\mathcal{P}^\ell(\lambda, r)$, $\mathcal{P}^\ell_{\rm d}(\lambda, r)$, $\mathcal{P}^\ell_\tau(\lambda, r)$, $e_{\ell,r}(\lambda)$.
Section 4.1: Ω_m, $\mathcal{P}^\ell(\Omega_m)$, $P^\ell_{\rm d}(\Omega_m)$, $P^\ell_\tau(\Omega_m)$.
Section 4.3: $\mathcal{P}^\ell(\Omega_m, s)$, $e_\ell(\Omega, s)$.

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