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# $f$-vectors of simplicial posets that are balls

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**Abstract** Results of R. Stanley and M. Masuda completely characterize the $h$-vectors of simplicial posets whose order complexes are spheres. In this paper we examine the corresponding question in the case where the order complex is a ball. Using the face rings of these posets, we develop a series of new conditions on their $h$-vectors. We also present new methods for constructing poset balls with specific $h$-vectors. Combining this work with a new result of S. Murai we are able to give a complete characterization of the $h$-vectors of simplicial poset balls in all even dimensions, as well as odd dimensions less than or equal to five.

**Keywords** Simplicial poset · $f$-vector · Face ring · $h$-vector

## 1 Introduction

A *simplicial poset* $P$ is a finite poset containing a minimal element $\hat{0}$ such that for every $p \in P$ the closed interval $[\hat{0}, p]$ is a Boolean algebra. For any simplicial poset $P$ there is a regular CW-complex $\Gamma(P)$ such that $P$ is the face poset of $\Gamma(P)$ (see [1]). The closed faces of $\Gamma(P)$ are simplexes, but two faces can intersect in a subcomplex of their boundaries instead of just a single face. In particular, $\Gamma(P)$ can have multiple faces on the same vertex set. Throughout this paper we will identify each closed face of $\Gamma(P)$ with the corresponding element of the poset $P$.

Let $\overline{P} = P - \{\hat{0}\}$. The *order complex* of $\overline{P}$, denoted $\Delta(\overline{P})$, is the simplicial complex whose vertices are the elements of $\overline{P}$ and whose faces are the chains of $\overline{P}$. For a simplicial poset $P$, when we refer to the order complex of $P$ we mean $\Delta(\overline{P})$. The spaces $\Gamma(P)$ and $|\Delta(\overline{P})|$ are homeomorphic (in fact, $\Delta(\overline{P})$ is isomorphic to the barycentric subdivision of $\Gamma(P)$). Therefore, in the following we will often refer to

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topological properties of $\Gamma(P)$ or $|\Delta(\overline{P})|$ as being properties of the poset $P$. In this paper we study simplicial posets that are balls.

The *ith face number* of $\Gamma(P)$, denoted $f_i(\Gamma(P))$, is the number of $i$-dimensional faces of $\Gamma(P)$. Equivalently, $f_i(\Gamma(P))$ is the number of elements $p \in P$ such that $[\hat{0}, p]$ is a Boolean algebra of rank $i+1$. In particular, $f_{-1}(\Gamma(P)) = 1$ corresponding to the empty face in $\Gamma(P)$ or the element $\hat{0}$ in $P$. The *dimension* of $\Gamma(P)$ is the largest $i$ such that $f_i(\Gamma(P))$ is non-zero. We define $f_i(P)$, the $i$th face number of the poset $P$, by $f_i(P) = f_i(\Gamma(P))$. If the poset $P$ is clear from the context we often write $f_i$ instead of $f_i(P)$ or $f_i(\Gamma(P))$.

Let $d-1$ be the dimension of $P$. We record all of the face numbers of $P$ in a single vector $f(P) = (f_{-1}, f_0, f_1, \dots, f_{d-1})$ called the *f-vector* of $P$. It will often be easier to work with an equivalent encoding of the face numbers called the *h-vector*. The entries of the $h$-vector $(h_0, h_1, \dots, h_d)$ are obtained from the face numbers by the relation

$$\sum_{i=0}^{d} h_i x^i = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i}.$$

In the case where $\Gamma(P)$ is a simplicial complex, this definition of the $h$-vector of $\Gamma(P)$ agrees with the standard definition of the $h$-vector of a simplicial complex. Note that if $P$ is a simplicial poset of dimension $d-1$ then $h_d = \sum_{i=0}^{d}(-1)^{d-i} f_{i-1} = (-1)^{d-1}\tilde{\chi}(\Gamma(P))$. In particular, if $P$ is a $(d-1)$-ball then $h_d = 0$. In many cases we will study the differences between consecutive entries of the $h$-vector. We therefore define $g_i := h_i - h_{i-1}$, with $g_0 := h_0 = 1$.

A significant area of study is characterizing the possible $f$-vectors of various types of simplicial posets. Complete characterizations are already known for Cohen–Macaulay posets (see Sect. 2), spheres, products of spheres, and real projective spaces. For simplicial posets that are spheres, sufficiency was proved by Stanley [9, Theorem 4.3, Remark 4] and necessity by Masuda [4, Corollary 1.2].

**Theorem 1** *Let* $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{N}^{d+1}$. *Then there exists a simplicial poset* $P$ *with* $h(P) = \mathbf{h}$ *and* $\Gamma(P)$ *homeomorphic to a* $(d-1)$*-sphere if and only if* $h_0 = 1$, $h_i = h_{d-i}$ *for* $0 \le i \le d$, *and either* $h_i > 0$ *for* $0 \le i \le d$ *or* $\sum_{i=0}^{d} h_i$ *is even.*

The equations $h_i = h_{d-i}$ are a generalized version of the Dehn–Sommerville equations. Because of this symmetry in the $h$-vector, whenever $d$ is odd $\sum_{i=0}^{d} h_i$ is always even and the final condition in Theorem 1 is automatically satisfied. In the case where $d$ is even the parity of $\sum_{i=0}^{d} h_i$ is equal to the parity of $h_{d/2}$. Extending these ideas and using the concept of crystallizations to obtain new constructions, Murai recently characterized the $f$-vectors of $\mathbb{R}P^n$ and products of spheres [8].

In this paper we investigate the question of characterizing the $h$-vectors of posets $P$ such that $\Gamma(P)$ is a ball. Section 3 relates the $h$-vector of a poset that is a ball to the $h$-vector of its boundary poset, which is a sphere. This allows us to translate known conditions on the $h$-vectors of spheres to conditions on our ball. However, as in the case of simplicial *complexes* that are balls, these conditions are not sufficient to characterize the $h$-vectors of balls [2].

In Sects. 4 and 5 we develop additional necessary conditions on the $h$-vectors of balls. In Sect. 4 we show that when $h_1$ of the boundary sphere is zero there is a surjective map from the face ring of the ball modulo a linear system of parameters to the face ring of the boundary sphere modulo a related linear system of parameters. This result gives a series of new inequalities relating the entries of the $h$-vectors of the ball and the boundary sphere. We also state a recent generalization of this result due to Murai for the case when a general $h_i$ of the boundary sphere is zero.

Section 5 gives a number of conditions on the $h$-vector of a ball that force the sum of the entries of the $h$-vector to be even. All of these conditions require that some entry of $h$-vector of the ball is zero (besides $h_d$, which is always zero) and that some entry of the $h$-vector of the boundary sphere is also zero. The first two results in this section follow from counting arguments involving the incidences of facets and codimension-one faces of the ball. The other two conditions are derived by adding to our ball the cone over the boundary of the ball, resulting in a sphere of the same dimension as the original ball. We then look at the restriction map from the face ring of this new sphere to the face ring our original ball and use this map to transfer known conditions on the sphere to conditions on the ball.

In Sect. 6 we present constructions for obtaining simplicial posets that are balls and have specific $h$-vectors. These constructions all use the idea of shellings [1, Definition 4.1] to ensure that the resulting complex is a ball and that it has the desired $h$-vector. We present two general constructions as well as a third result that yields additional $h$-vectors in dimension five. We conclude the paper in Sect. 7 by using the previous results to give a complete characterization of the $h$-vectors of simplicial posets that are balls in all even dimensions as well as in dimensions three and five. We also discuss what types of problems remain in the higher odd dimension cases.

## 2 Notation and background

In this section we provide background on many of the ideas mentioned in the introduction, as well as some additional useful results of Masuda about simplicial poset spheres.

### 2.1 Shellings of $\Gamma(P)$

The *facets* of any CW-complex are the maximal faces with respect to inclusion. A CW-complex is *pure* if all of its facets have the same dimension. Let $P$ be a simplicial poset of dimension $d-1$. Then the facets of $\Gamma(P)$ correspond to the maximal elements of $P$.

Consider the case where $\Gamma(P)$ is a pure complex. A *shelling* of $\Gamma(P)$ is an ordering $F_1, F_2, \ldots, F_t$ of the (closed) facets of $\Gamma(P)$ such that $F_j \cap (\bigcup_{i=1}^{j-1} F_i)$ is a union of (closed) *facets* of $\partial F_j$. This is equivalent to the definition of a shelling of a CW-complex used by Björner [1, Definition 4.1] specialized to the case of $\Gamma(P)$ for a simplicial poset $P$. Define the *restriction face* of $F_k$, denoted $\sigma(F_k)$, to be the set of vertices $v$ of $F_k$ such that the facet of $\partial F_k$ not containing $v$ is in $\bigcup_{i=1}^{k-1} F_i$. Then the entries of the $h$-vector of $P$ are given by $h_j = |\{F_k : |\sigma(F_k)| = j\}|$. We also use $\sigma(F_k)$ to refer to the face of $F_k$ containing exactly the vertices in the set $\sigma(F_k)$.

### 2.2 Cones of posets

Given a simplicial poset $P$, the *cone* over $P$ is the simplicial poset $P \times [1,2]$ where $[1,2]$ is the poset of two elements with $2 > 1$. More specifically, the elements of $P \times [1,2]$ are the ordered pairs $(p,i)$ where $p \in P$ and $i \in \{1,2\}$ and the covering relations are:

– If $p$ covers $q$ in $P$, then $(p,i)$ covers $(q,i)$ in $P \times [1,2]$ for $i \in \{1,2\}$.
– For all $p \in P$, $(p,2)$ covers $(p,1)$ in $P \times [1,2]$.

Topologically, $\Gamma(P \times [1,2])$ is the cone over $\Gamma(P)$. The dimension of $P \times [1,2]$ is one greater than that of $P$. A straightforward calculation shows that the $h$-vector of $P \times [1,2]$ is the same as that of $P$ except being augmented by $h_{d+1} = 0$.

### 2.3 The face ring of a simplicial poset

We now describe Stanley's idea of the face ring of a simplicial poset [9]. Let $k$ be an infinite field. Define $S := k[x_p : p \in \overline{P}]$ to be the polynomial ring over $k$ with variables indexed by the elements of $\overline{P}$. We define a grading on $S$ by letting the degree of $x_p$ be one more than the dimension of the face in $\Gamma(P)$ corresponding to $p$ (so $[\hat{0}, p]$ is a Boolean algebra of rank equal to the degree of $x_p$). For elements $p$ and $q$ in $P$ the *meet* of $p$ and $q$, denoted $p \wedge q$, is the largest element that is less than both $p$ and $q$. Since $P$ is a simplicial poset we know that the element $p \wedge q$ is well defined whenever $p$ and $q$ have a common upper bound. Define $I_P$ to be the ideal of $S$ generated by all elements of the form $x_p x_q - x_{p \wedge q} \sum_r x_r$ where the sum is over all minimal upper bounds $r$ of $p$ and $q$. In the case where $p$ and $q$ have no common upper bound in $P$, this reduces to the element $x_p x_q$. If $p \wedge q = \hat{0}$ then let $x_{p \wedge q} = 1$. Define the *face ring* of $P$ to be $A_P := S/I_P$.

### 2.4 Cohen–Macaulay simplicial posets

A simplicial poset $P$ is *Cohen–Macaulay* if its order complex $\Delta(\overline{P})$ is a Cohen–Macaulay simplicial complex. For a simplicial complex $\Delta$, Munkres [6] proved that $\Delta$ is Cohen–Macaulay if and only if for all points $p \in |\Delta|$ and all $i < \dim \Delta$, $\tilde{H}_i(|\Delta|; k) = H_i(|\Delta|, |\Delta| - p; k) = 0$. In particular, whenever $|\Delta(\overline{P})| \cong \Gamma(P)$ is a ball or sphere $P$ is Cohen–Macaulay. Stanley proved that a simplicial poset $P$ is Cohen–Macaulay if and only if its face ring $A_P$ is a Cohen–Macaulay ring [9, Corollary 3.7]. Using this result, Stanley [9, Theorem 3.10] showed that if $Q$ is a Cohen–Macaulay simplicial poset then $h_0(Q) = 1$ and $h_i(Q) \geq 0$ for $i \geq 1$ (he also proved that these are sufficient conditions to characterize the $h$-vectors of Cohen–Macaulay simplicial posets). Stanley's book [11] is a good reference for more information on Cohen–Macaulay rings and complexes.

Let $T = T_0 \oplus T_1 \oplus \cdots$ be a finitely generated graded algebra over the (infinite) field $k = T_0$. The *Hilbert function* of $T$ is $F(T,i) := \dim_k T_i$ where $i \geq 0$. Let $d$ be the Krull dimension of $T$. Then a *linear system of parameters (l.s.o.p)* for $T$ is a collection of elements $\theta_1, \ldots, \theta_d \in T_1$ such that $T$ is finitely generated as a $k[\theta_1, \ldots, \theta_d]$-module. From [9, Theorem 3.10] we know that when $P$ is a Cohen–Macaulay simplicial poset, $\dim A_P = \dim(\Gamma(P)) + 1$ and an l.s.o.p. for $A_P$ exists. Further, when

$P$ is a Cohen–Macaulay simplicial poset and $\theta_1, \ldots, \theta_d$ is an l.s.o.p. for $A_P$ we have $F(A_P/(\theta_1, \ldots, \theta_d), i) = h_i(P)$ [9, Theorem 3.8].

### 2.5 Additional results about simplicial poset spheres

As mentioned in the introduction, Theorem 1 gives a complete characterization of the possible $h$-vectors of simplicial posets that are spheres. In addition to the numerical result we will often need some of the stronger statements that were used to prove the necessity of the claim. The following theorem is discussed on pages 343–344 of the original proof of necessity due to Masuda [4] and is Theorem 2 in the paper [5] published two years later by Miller and Reiner giving a simplified proof of Masuda's result.

**Theorem 2** *Let $P$ be a simplicial poset such that $\Gamma(P)$ is a $(d-1)$-sphere. If $h_i(P) = 0$ for some $i$ strictly between zero and $d$, then for every subset $V = \{v_1, \ldots, v_d\}$ of the vertices of $\Gamma(P)$ the number of facets of $\Gamma(P)$ with vertex set $V$ is even.*

One consequence of the proof of Theorem 2 is the following result that relates the parity of the number of facets on a vertex set to the product of the variables in the algebra $A_P$ corresponding to those vertices.

**Proposition 3** *Let $P$ be a simplicial poset such that $\Gamma(P)$ is a $(d-1)$-sphere and let $V = \{v_1, \ldots, v_d\}$ be a subset of the vertices of $\Gamma(P)$. Let $\Theta = \theta_1, \ldots, \theta_d$ be an l.s.o.p. for $A_P$. If $x_{v_1} \cdots x_{v_d}$ is zero in $A_P/\Theta$ then there are an even number of facets of $\Gamma(P)$ with vertex set $V$.*

## 3 The *h*-vector of the boundary of a simplicial poset

The goal of this section is to relate the $h$-vector of a simplicial poset $P$ such that $\Gamma(P)$ is a manifold with boundary to the $h$-vector of the boundary complex. In the case of balls, this will allow us to use Theorem 1 about the $h$-vectors of spheres to restrict the possible $h$-vectors of balls.

Our starting point is a paper by I.G. Macdonald [3, Theorem 2.1] that gives the desired relationship for the case of simplicial complexes whose geometric realizations are manifolds with boundary. The entire first section of Macdonald's paper is done in the generality of cell complexes and applies in our case. When Macdonald proves Theorem 2.1, the only property of simplicial complexes that he uses is the fact that for each simplex $y$ in the complex the interval $[\hat{0}, y]$ in the face poset is a Boolean algebra. Since this fact is true for simplicial posets, Macdonald's result holds in this more general setting as well. Expressing his result in terms of $h$- and $g$-vectors we have the following theorem.

**Theorem 4** *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a manifold with boundary. Then*

$$h_{d-i}(P) - h_i(P) = \binom{d}{i}(-1)^{d-1-i}\tilde{\chi}\big(\Gamma(P)\big) - g_i\big(\partial\Gamma(P)\big)$$

*for all $0 \le i \le d$.*

In the special case where $\Gamma(P)$ is a $(d-1)$-ball this reduces to the equation $h_i(P) - h_{d-i}(P) = g_i(\partial\Gamma(P))$. In particular, for $0 \le j \le d$ we have

$$h_j\big(\partial\Gamma(P)\big) = \sum_{i=0}^{j}\big(h_i(P) - h_{d-i}(P)\big). \tag{1}$$

By this result and the Dehn–Sommerville equations, in the case where $d$ is odd we have

$$\sum_{i=0}^{d} h_i(P) \equiv h_{(d-1)/2}\big(\partial\Gamma(P)\big) \equiv \sum_{i=0}^{d-1} h_i\big(\partial\Gamma(P)\big) \mod 2. \tag{2}$$

Using this relationship we now give a first set of necessary conditions on the $h$-vectors of simplicial poset balls. As discussed in Sect. 2, any simplicial poset $P$ such that $\Gamma(P)$ is a ball is a Cohen–Macaulay poset. Combining Stanley's characterization of the $h$-vectors of Cohen–Macaulay posets with Theorem 1 and (1) and (2) we have the following.

**Theorem 5** *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball. Then $h_0(P) = 1$, $h_d(P) = 0$, $h_i(P) \ge 0$ for all $i$, and $\sum_{i=0}^{j}(h_i(P) - h_{d-i}(P)) \ge 0$ for $0 \le j \le \lfloor (d-1)/2 \rfloor$. Further, if $d$ is odd and $\sum_{i=0}^{j}(h_i(P) - h_{d-i}(P)) = 0$ for some $0 \le j \le \lfloor (d-1)/2 \rfloor$, then $\sum_{i=0}^{d} h_i(P)$ is even.*

## 4 Inequalities relating a ball and its boundary

Consider a simplicial poset ball such that $h_1$ of the boundary sphere is zero. In the following we derive inequalities relating the $h$-vectors of the boundary sphere and the ball in this case. The idea of the argument follows that of a similar result by Stanley [10, Theorem 2.1] for the $h$-vectors of simplicial complexes.

One of the main tools in this proof is a useful characterization of linear systems of parameters for the ring $A_P$. Fix an ordering $\{v_1, \ldots, v_n\}$ of the vertices of $\Gamma(P)$. Let $\theta_1, \ldots, \theta_d$ be a collection of homogeneous degree-one elements of $A_P$. We can write each element of our collection as a linear combination of the $x_{v_j}$, $\theta_i = \sum_{j=1}^{n} \Theta_{i,j} x_{v_j}$. This gives a $d \times n$ matrix $\Theta_{i,j}$ whose rows correspond to the $\theta_i$.

Let $F$ be a facet of $\Gamma(P)$. Define $\Theta_F$ to be the $d \times d$ submatrix of $\Theta_{i,j}$ obtained by restricting to the columns corresponding to the vertices of $F$. Then we have the following characterization of the collections of degree one elements that are linear systems of parameters.

**Lemma 6** *Let $P$ be a simplicial poset and let $\theta_1, \ldots, \theta_d$ be a collection of homogeneous degree one elements of $A_P$. Then $\theta_1, \ldots, \theta_d$ is an l.s.o.p. for $A_P$ if and only if* $\det(\Theta_F) \ne 0$ *for all facets $F$ of $\Gamma(P)$.*

The only if part of the lemma was proved by Masuda [4, Lemma 3.1] and Miller and Reiner [5, p. 1051]. The if direction follows from Proposition 5 of Miller and Reiner's paper. In this proposition Miller and Reiner show that for an l.s.o.p. $\theta_1,\dots,\theta_d$, $A_P/(\theta_1,\dots,\theta_d)$ is spanned $k$-linearly by the images of the $x_G$ for all elements $G\in P$. The only property of the l.s.o.p. used in the proof is the non-zero determinant assumption in the above lemma.

Let $P$ be a $(d-1)$-ball. Then $\partial(\Gamma(P))$ is a $(d-2)$-sphere. If $h_1(\partial(\Gamma(P)))=0$ we know that $\partial(\Gamma(P))$ has only $d-1$ vertices. Therefore every facet of $\partial(\Gamma(P))$ has the same vertex set. Let $F$ be a facet of $\Gamma(P)$ such that a codimension-one face of $F$ is in $\partial(\Gamma(P))$. Let $v$ be the vertex of $F$ that is not $\partial(\Gamma(P))$. Note that all of the vertices of $\Gamma(P)$ not in $F$ are interior vertices.

Let $\theta_1,\dots,\theta_d$ be an l.s.o.p. for $A_P$. By Lemma 6 we know that $\Theta_F$ has non-zero determinant. Thus the span of $\{\theta_1,\dots,\theta_d\}$ contains some element $\theta'=x_v+\sum_{w\notin F}c_wx_w$ where the sum is over the vertices of $\Gamma(P)$ not in $F$ and the $c_w$ are constants in $k$. This allows us to choose a new l.s.o.p. $\theta'_1,\dots,\theta'_d$ for $A_P$ such that $\theta'_d$ is a linear combination of interior vertices of $\Gamma(P)$. By Lemma 6 we know that $\det(\Theta'_G)\neq 0$ for all facets $G$ of $\Gamma(P)$.

Let $Q$ be the face poset of $\partial\Gamma(P)$. Let $f: A_P\to A_Q$ be given by setting all variables corresponding to faces in $\Gamma(P)\backslash\partial\Gamma(P)$ equal to zero. Identify the l.s.o.p. $\theta'_1,\dots,\theta'_d$ with its image under $f$ in $A_Q$. Let $H$ be any facet of $\partial\Gamma(P)$. The last row of $\Theta'_H$ is all zeros, so the $(d-1)\times(d-1)$ minor given by the first $(d-1)$ rows must have a non-zero determinant. Again using Lemma 6 we see that $\theta'_1,\dots,\theta'_{d-1}$ is an l.s.o.p. for $\partial\Gamma(P)$.

Therefore, $f$ induces a degree preserving surjection

$$f: A_P/\big(\theta'_1,\dots,\theta'_d\big)\to A_Q/\big(\theta'_1,\dots,\theta'_{d-1}\big).$$

Hence $h_i(P)=F(A_P/(\theta'_1,\dots,\theta'_d),i)\geq F(A_Q/(\theta'_1,\dots,\theta'_{d-1}),i)=h_i(Q)$. We have therefore proved the following theorem.

**Theorem 7** *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball and $h_1(\partial(\Gamma(P)))=0$. Then $h_i(P)\geq h_i(\partial(\Gamma(P)))$ for all $i\geq 0$.*

Now consider the case of a ball $P$ such that $h_n(\partial(\Gamma(P)))=0$ for some $n>0$. If we let $\{\theta_1,\dots,\theta_d\}$ be a generically chosen set of linear forms, then Murai [7] noted that there is a surjection

$$g: A_P/\big(\theta_1,\dots,\theta_{d-1},\theta_d^n\big)\to A_Q/(\theta_1,\dots,\theta_{d-1}).$$

Using this surjection along with the fact

$$\dim_k\big(A_P/(\theta_1,\dots,\theta_{d-1})\big)_l=h_0(P)+h_1(P)+\cdots+h_l(P)$$

Murai was able to prove the following generalization of the previous theorem.

**Theorem 8** (Murai) *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball and $h_n(\partial(\Gamma(P)))=0$. Then*

$$h_l(\partial P)\leq h_l(P)+h_{l-1}(P)+\cdots+h_{l-(n-1)}(P)\quad \textit{for } l\geq n.$$

## 5 Parity conditions on the sum of the $h_i(P)$

If $P$ is a simplicial poset ball of even dimension (so $d$ is odd) we already know from Theorem 5 that if any $h_k(\partial\Gamma(P))$ is zero then $\sum_{i=0}^{d} h_i(P)$ is even. For odd-dimensional balls the situation is not as simple. In this section we derive a series of different conditions under which the sum of the $h_i(P)$ must be even. All of these conditions involve some $h_k(\partial\Gamma(P))$ and some $h_j(P)$ being zero. However, we will see in Section 6 that for odd-dimensional balls there are cases where the $h$-vector of the boundary sphere has a zero entry and $\sum_{i=0}^{d} h_i(P)$ is odd.

### 5.1 Conditions from counting arguments

Our first two examples of this new type of condition follow from counting arguments involving the faces of our complexes. The main idea in both proofs is the following connection between a zero in the $h$-vector of the boundary sphere and a parity condition on the incidences between facets and codimension-one faces.

**Lemma 9** *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball and $h_k(\partial\Gamma(P)) = 0$ for some $k$ strictly between zero and $d-1$. Then every set of $d-1$ vertices of $P$ is contained in an even number of facets (possibly zero).*

*Proof* Let $S$ be a set of $d-1$ vertices of $\Gamma(P)$. If $S$ is not contained in any facet of $\Gamma(P)$ we are done. Otherwise, let $F$ be a face of $\Gamma(P)$ with vertex set $S$. If $F$ is an interior face of $\Gamma(P)$ then since $\Gamma(P)$ is a manifold there are exactly two facets of $\Gamma(P)$ that have $F$ as a codimension-one face. If $F$ is a boundary face of $\Gamma(P)$ then there is exactly one facet of $\Gamma(P)$ that has $F$ as a codimension-one face. Further, since some $h_k(\partial\Gamma(P)) = 0$, by Theorem 2 the number of boundary faces of $\Gamma(P)$ with vertex set $S$ is even. Since no single facet of $\Gamma(P)$ can have multiple faces with the same vertex set, the total number of facets of $\Gamma(P)$ that contain $S$ is even. □

Now consider the case where $\Gamma(P)$ is a ball, $h_1(P) = 0$, and $h_k(\partial\Gamma(P)) = 0$ for some $k$. By Lemma 9, any set of $d-1$ vertices of $P$ is contained in an even number of facets. In terms of the face numbers, $h_1(P) = 0$ implies $f_0(P) = d$, meaning that all of the vertices of $\Gamma(P)$ are in every facet. Therefore, every set of $d-1$ vertices of $P$ is in every facet and hence the total number of facets of $\Gamma(P)$ is even. Recalling that $\sum_{i=0}^{d} h_i(P)$ is the number of facets of $\Gamma(P)$ we have the following proposition.

**Proposition 10** *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball, $h_1(P) = 0$, and $h_k(\partial\Gamma(P)) = 0$ for some $k$ strictly between zero and $d-1$. Then $\sum_{i=0}^{d} h_i(P)$ is even.*

We can extend the result of Proposition 10 to the case $h_2(P) = 0$ (instead of $h_1(P) = 0$) using a somewhat more involved argument based on the same ideas.

**Proposition 11** *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball, $h_2(P) = 0$, and $h_k(\partial\Gamma(P)) = 0$ for some $k$ strictly between zero and $d-1$. Then $\sum_{i=0}^{d} h_i(P)$ is even.*

*Proof* Pick a facet $F_0$ of $\Gamma(P)$ with vertex set $V=\{v_1,\dots,v_d\}$. Let $\Delta_0$ be the induced subcomplex on the vertex set $V$; $\Delta_0$ consists of all of the faces of $\Gamma(P)$ whose vertices are contained in the set $V$. Note that $\Delta_0$ contains at least $\binom{d}{2}$ edges.

Let $F_1$ be a facet of $\Gamma(P)-\Delta_0$ that intersects $\Delta_0$ in a face of dimension $d-2$. Since $\Gamma(P)$ is a manifold, unless $\Gamma(P)=\Delta_0$ such a facet $F_1$ must exist. If $\Gamma(P)=\Delta_0$ then $\Gamma(P)$ has $d$ vertices, so $h_1(P)=0$ and we are in the case of Proposition 10. Otherwise, let $w_1$ be the vertex of $F_1$ not in $V$ and let $\Delta_1$ be the induced subcomplex of $\Gamma(P)$ on the vertex set $V\cup\{w_1\}$. There must be at least $d-1$ edges in $\Delta_1-\Delta_0$ in order for the facet $F_1$ to exist.

We can continue to build our complex in this manner until we reach $\Delta_{h_1}=\Gamma(P)$. This results in a minimum of $\binom{d}{2}+h_1(P)\cdot(d-1)$ edges in our complex. However, since $h_2(P)=0$ this is exactly the number of edges in $\Gamma(P)$. So we must have added the minimum number of edges at each step in our construction. In particular, for $1\le i\le h_1$, all of the facets of $\Delta_i$ that contain $w_i$ must have the same vertex set as $F_i$.

By Lemma 9 we know that every set of $d-1$ vertices of $\Gamma(P)$ is contained in an even number of facets. In particular, let $S$ be a set of $d-1$ vertices of $F_{h_1}$ that includes the vertex $w_{h_1}$. The facets that contain the vertices of $S$ are exactly those facets whose vertex set equals the vertex set of $F_{h_1}$. Therefore, there must be an even number of facets on the vertex set of $F_{h_1}$.

Since we are only interested in the parity of the number of facets on each vertex set we can now ignore the contribution of the facets on the vertex set of $F_{h_1}$ and repeat the above argument on the complex $\Delta_{h_1-1}$ and the facet $F_{h_1-1}$. Continuing in this manner we see that there are an even number of facets on all of the sets of $d$ vertices of $\Gamma(P)$. Therefore $\Gamma(P)$ has an even number of facets, as desired. □

### 5.2 The cone over the boundary of $\Gamma(P)$

Let $P$ be a simplicial poset such that $\Gamma(P)$ is a manifold with boundary and let $Q$ be the face poset of $\partial(\Gamma(P))$. Define the cone over the boundary of $\Gamma(P)$ to be $SP:=P\cup(Q\times[1,2])$ with each element $(q,1)\in(Q\times[1,2])$ identified with the element in $P$ corresponding to $q$. The covering relations in $SP$ are all of the covering relations in $P$ along with all of the covering relations in $Q\times[1,2]$. In the case where $\Gamma(P)$ is a $(d-1)$-ball, $\Gamma(SP)$ is a $(d-1)$-sphere.

We now consider the relationship between the algebras $A_P$ and $A_{SP}$. Let $v$ be the cone point of $SP$; $v$ is the vertex corresponding to $(\hat{0},2)$ in $Q\times[1,2]$. There is a natural surjective map $f:A_{SP}\to A_P$ given by setting all of the variables corresponding to faces containing $v$ equal to zero. If $\Theta=\theta_1,\dots,\theta_d$ is an l.s.o.p. for $A_{SP}$, then by Lemma 6 the image of $\Theta$ under $f$ (which we also write as $\Theta$) is an l.s.o.p. for $A_P$. Therefore, there is an induced map $f:A_{SP}/\Theta\to A_P/\Theta$ with kernel generated (modulo $\Theta$) by monomials containing a variable corresponding to a face containing $v$. We use this map $f$ to prove the following lemma.

**Lemma 12** *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball and $h_k(P)=0$ for some $k$ strictly between zero and $d$. Let $V=\{v_1,\dots,v_{d-1},v_d\}$ be a set of vertices of $\Gamma(P)$ such that $v_d$ is an interior vertex. Then $\Gamma(P)$ has an even number of facets with vertex set $V$.*

*Proof* Let $\Theta$ be an l.s.o.p. for $A_P$ and let $v$ be the cone point of $SP$ as above. Let $m := x_{v_1} \cdots x_{v_k}$ be a monomial in $(A_{SP})_k$. Since the dimension of $(A_P/\Theta)_k$ is $h_k(P) = 0$ we know that $m$ is in the kernel of the map $f : A_{SP}/\Theta \to A_P/\Theta$ defined above. Therefore, in $A_{SP}/\Theta$ we can write $m$ as a linear combination of monomials each containing a variable corresponding to a face containing $v$. Since $v_d$ is an interior vertex of $\Gamma(P)$, $x_{v_d}m$ is zero in $A_{SP}/\Theta$. Thus by Proposition 3 we know that there must be an even number of facets of $\Gamma(SP)$ with vertex set $V$. Since the cone point $v$ is not in $V$, the facets of $\Gamma(SP)$ with vertex set $V$ are exactly the same as the facets of $\Gamma(P)$ with vertex set $V$, proving the desired result. □

### 5.3 The case $h_1(\partial\Gamma(P)) = 0$

Using Lemma 12 we now prove the following necessary condition.

**Proposition 13** *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball, $h_1(\partial\Gamma(P)) = 0$, and $h_k(P) = 0$ for some $0 < k < d$. Then $\sum_{i=0}^{d} h_i(P)$ is even.*

*Proof* Since $h_1(\partial\Gamma(P)) = 0$ we know that $\partial\Gamma(P)$ has only $d-1$ vertices. Therefore every facet of $\Gamma(P)$ contains an interior vertex. By Lemma 12 there are an even number of facets (possibly zero) on every set of $d$ vertices of $\Gamma(P)$. Hence $\sum_{i=0}^{d} h_i(P)$, which is the total number of facets of $\Gamma(P)$, is even. □

### 5.4 The case $h_1(\partial\Gamma(P)) = 1$

A slightly more complicated argument allows us to extend the result of Proposition 13 to the case where $h_1(\partial\Gamma(P)) = 1$ and some higher $h_j(\partial(\Gamma(P))$ is zero.

**Proposition 14** *Let $P$ be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball, $h_1(\partial\Gamma(P)) = 1$, $h_j(\partial\Gamma(P)) = 0$ for some $1 < j < d-1$, and $h_k(P) = 0$ for some $0 < k < d$. Then $\sum_{i=0}^{d} h_i(P)$ is even.*

*Proof* The assumption $h_1(\partial\Gamma(P)) = 1$ implies that $\partial\Gamma(P)$ has $d$ vertices. Let $W = \{w_1, \dots, w_d\}$ be the set of exterior vertices of $\Gamma(P)$. Let $V$ be a set of $d$ vertices of $\Gamma(P)$. If $V \neq W$ then $V$ contains some interior vertex, so by Lemma 12 we know that there are an even number of facets of $\Gamma(P)$ with vertex set $V$. In particular, given any set $S$ of $(d-1)$ vertices of $\Gamma(P)$ there are an even number (possibly zero) of facets with vertex set $V$ that contain $S$.

If there are no facets with vertex set $W$ then we are done, so assume $F$ is a facet with vertex set $W$. Let $W'$ be a set of $d-1$ distinct elements of $W$. Since $h_j(\partial\Gamma(P)) = 0$ we know by Theorem 2 that the number of boundary faces of $\Gamma(P)$ with vertex set $W'$ is even. Because $\Gamma(P)$ is a manifold each interior face with vertex set $W'$ is contained in two facets of $\Gamma(P)$ and each boundary face with vertex set $W'$ is contained in one facet of $\Gamma(P)$. Therefore there are an even number of facets of $\Gamma(P)$ that contain the vertices $W'$. As argued above there are an even number of facets on vertex sets other than $W$ that contain $W'$. Thus in total there are an even number of facets on vertex set $W$. Hence we have an even total number of facets, which gives the desired result. □

## 6 Constructions

We now turn our attention to constructing posets $P$ with prescribed $h$-vectors such that $\Gamma(P)$ is a ball. The balls that we construct are all shellable. We use the following result of Björner [1, Proposition 4.3] to prove that the complexes that we build are actually balls.

**Proposition 15** *Let $\Gamma(P)$ be a shellable CW-complex of dimension $d-1$. If every $(d-2)$-cell is a face of at most two $(d-1)$-cells and some $(d-2)$-cell is a face of only one $(d-1)$-cell then $\Gamma(P)$ is homeomorphic to a $(d-1)$-ball.*

The first theorem of this section presents our basic construction method. The remainder of the section gives some extensions of this construction that allow us to obtain additional $h$-vectors.

**Theorem 16** *Let $\mathbf{h}=(h_0,h_1,\dots,h_{d-1},h_d)\in\mathbb{N}^{d+1}$ with $h_0=1$ and $h_d=0$. Let $\partial h_j=\sum_{i=0}^{j}(h_i-h_{d-i})$.*

1. *If $\partial h_j>0$ for $0\le j\le\lfloor(d-1)/2\rfloor$ then there exists a poset $P$ such that $\Gamma(P)$ is a $(d-1)$-ball and $h(P)=\mathbf{h}$.*
2. *Alternatively, let $0<n<\lfloor(d-1)/2\rfloor$ be the smallest number such that $\partial h_n=0$. If $\sum_{i=0}^{d}h_i$ is even and $\partial h_l\le\sum_{i=0}^{n-1}h_{l-i}$ for $n+1\le l\le d-(n+1)$ then there exists a poset $P$ such that $\Gamma(P)$ is a $(d-1)$-ball and $h(P)=\mathbf{h}$.*

Many of the conditions in Theorem 16 are related to the restrictions on the $h$-vectors of balls in Theorem 5. Also note that the inequalities of Theorem 16 are known to be necessary by Murai's result, Theorem 8.

*Proof* The structure of our proof is as follows. We first present the notation that we will we use to describe the facets of our ball. We then recursively construct our ball by explaining how each facet of the ball is attached to the union of the previous facets. In Claim 1 we prove that this process results in a well-defined CW-complex. In Claim 2 we show that the complex is shellable with the desired restriction faces. Finally, in Claim 3 we show we have actually constructed a ball by proving that the conditions of Proposition 15 are satisfied.

We begin with the notation for the facets of our ball. The facets are denoted by $F_i$ for $1\le i\le\sum_{i=0}^{d}h_i$. Each facet $F_i$ contains $d$ vertices. We label these vertices $\{1\}_i,\{2\}_i,\dots,\{d\}_i$. For any set $S\subseteq[d]$ we denote by $\{S\}_i$ the face of $F_i$ containing exactly the vertices $\{\{l\}_i\}_{l\in S}$. For example $\{1,2,3,4\}_3$ is the face of $F_3$ containing the vertices $\{1\}_3$, $\{2\}_3$, $\{3\}_3$, and $\{4\}_3$. We use the notation $\{a:b\}$ and $\{a:b\}_c$ to refer to $\{a,a+1,\dots,b\}$ and $\{a,a+1,\dots,b\}_c$, respectively.

Next we describe recursively how to construct the ball using the facets $F_i$. Let $\Delta_j$ be the complex $\bigcup_{i=1}^{j}F_i$. For each facet $F_i$ we describe below the identifications of faces of $F_i$ with faces in $\Delta_{i-1}$. Most of the vertices of $F_i$ will be identified with vertices of $\Delta_{i-1}$, but in some cases $F_i$ may contain a new vertex. For example, we may state that $\{1\}_2$ is identified with $\{1\}_1$ or that $\{1\}_2$ is a new face. In general, we choose the vertex labels such that two identified faces contain vertices labeled by the same numbers.

First we consider the case where all of the $\partial h_i$ are strictly positive. Let $a = \sum_{i=0}^{d} h_i$, which is the total number of facets in our shelling. For $1 \le k \le a-1$ let $c_k$ be the integer such that $\sum_{i=0}^{c_k-1} h_i < k+1 \le \sum_{i=0}^{c_k} h_i$. Thus $c_k$ measures the location where the sum of the entries of the vector $\mathbf{h}$ reaches $k+1$. Set $c_0 = 0$. Then $|\{k : c_k = j\}| = h_j$. As an example, if $\mathbf{h} = (1,2,0,0,1,0)$ then $a = 4$, $c_0 = 0$, $c_1 = 1$, $c_2 = 1$, and $c_3 = 4$.

We begin the shelling with the facet $F_1$ which cannot have any identifications with any previous facets, hence $|\sigma(F_1)| = 0$. The remaining facets will be added in pairs $F_i, F_{i+1}$ where $i$ is even. The restriction faces of $F_i$ and $F_{i+1}$ will be $\{1 : c_{i/2}\}_i$ and $\{c_{i/2}+1 : c_{i/2} + c_{a-i/2}\}_{i+1}$, respectively. We are pairing a facet contributing to the start of the $h$-vector with a facet contributing to the end of the $h$-vector and then working our way inward to the center of the $h$-vector with the subsequent pairs of facets. We stop after adding the facet $F_a$.

We now describe how the facets $F_i$ and $F_{i+1}$ are attached to our complex. For $i$ even, we introduce a new face $\{1 : c_{i/2}\}_i$. Let $S \subseteq [d]$. If $S \supseteq \{1 : c_{i/2}\}$ then $\{S\}_i$ cannot be identified with any face in a previous facet. If $S \supseteq \{c_{i/2-1}+1 : c_{i/2}\}$ but $S \not\supseteq \{1 : c_{i/2-1}\}$ then identify $\{S\}_i$ with $\{S\}_{i-1}$. For all other sets $S \subseteq [d]$ identify $\{S\}_i$ with $\{S\}_1$. The fact that these identifications are well defined follows from the case $k = i$ of Claim 1 below.

Continuing the shelling, $F_{i+1}$ is identified with $F_i$ except we replace the face $\{c_{i/2}+1 : c_{i/2}+c_{a-i/2}\}_i$ by a new face $\{c_{i/2}+1 : c_{i/2}+c_{a-i/2}\}_{i+1}$ with the same boundary. The fact that all of the $\partial h_i$ are positive ensures that $c_{i/2}+c_{a-i/2}$ never exceeds $d$, so the construction can proceed as described.

**Claim 1** *Fix $i$ even with $2 \le i \le a$ and $k$ even with $2 \le k \le i$. Let $S \subseteq [d]$ be such that $S \not\supseteq \{c_{i/2-1}+1 : c_{i/2}\}$ and $S \not\supseteq \{1 : c_{k/2-1}\}$. Then $\{S\}_{k-1} = \{S\}_1$.*

*Proof of Claim 1* Our proof is by induction on $i$. The base case $i = 2$ is trivial. Assuming the result for $i = i' - 2$ we prove it for $i = i'$. The inductive hypothesis allows us to assume that the construction of $\Delta_{i'-1}$ is well defined.

We prove the case $i = i'$ by induction on $k$. Again, the base case $k = 2$ is trivial. We assume the claim for $k = k' - 2$ and prove it for $k = k'$.

We first show that $\{S\}_{k'-1} = \{S\}_{k'-2}$. By the construction of the odd index facets, this follows from showing $S \not\supseteq \{c_{(k'-2)/2}+1 : c_{(k'-2)/2}+c_{a-(k'-2)/2}\}$. By assumption $S \not\supseteq \{c_{i'/2-1}+1 : c_{i'/2}\}$. Thus it is enough to show

$$c_{(k'-2)/2}+1 \le c_{i'/2-1}+1 \quad \text{and} \quad c_{i'/2} \le c_{(k'-2)/2}+c_{a-(k'-2)/2}. \tag{3}$$

The first inequality in (3) follows from the monotonicity of the $c_l$. Again using the monotonicity of the $c_l$ and the fact that $i' \le a$ we have

$$c_{i'/2} \le c_{a-i'/2} \le c_{a-(k'-2)/2} \le c_{(k'-2)/2}+c_{a-(k'-2)/2},$$

proving the second inequality.

We complete the proof of Claim 1 by showing $\{S\}_{k'-2} = \{S\}_1$. By assumption, $S \not\supseteq \{1 : c_{k'/2-1}\} = \{1 : c_{(k'-2)/2}\}$. Thus, if $S \supseteq \{c_{(k'-2)/2-1} + 1 : c_{(k'-2)/2}\}$ then $S \not\supseteq \{1 : c_{(k'-2)/2-1}\}$. In this case, by our construction of the even index facets we have $\{S\}_{k'-2} = \{S\}_{k'-3}$ and by the inductive hypothesis $\{S\}_{k'-3} = \{S\}_1$, giving the desired result. If $S \not\supseteq \{c_{(k'-2)/2-1} + 1 : c_{(k'-2)/2}\}$ then our construction identifies $\{S\}_{k'-2}$ and $\{S\}_1$, completing the proof. □

Let $\{\hat{j}\}_k$ be the codimension-one face of $F_k$ that does not contain the vertex $\{j\}_k$.

**Claim 2** *Let $2 \le k \le a$. Then*

$$\text{for } k = i \text{ even} \qquad F_i \cap \Delta_{i-1} = \bigcup_{j=1}^{c_{i/2}} \{\hat{j}\}_i$$

$$\text{and for } k = i+1 \text{ odd} \quad F_{i+1} \cap \Delta_i = \bigcup_{j=c_{i/2}+1}^{c_{i/2}+c_{a-i/2}} \{\hat{j}\}_{i+1}.$$

*Hence the $F_i$ form a shelling order with $|\sigma(F_i)| = c_{i/2}$ and $|\sigma(F_{i+1})| = c_{a-i/2}$.*

*Proof of Claim 2* For $1 \le j \le c_{i/2-1}$ each of the faces $\{\hat{j}\}_i$ is also in $F_{i-1}$ while for $c_{i/2-1} < j \le c_{i/2}$ the face $\{\hat{j}\}_i$ is also a face of $F_1$. Therefore $\bigcup_{j=1}^{c_{i/2}} \{\hat{j}\}_i \subseteq (F_i \cap \Delta_{i-1})$. To see the reverse inclusion note that any face in $F_i \backslash (\bigcup_{j=1}^{c_{i/2}} \{\hat{j}\}_i)$ contains the face $\{1 : c_{i/2}\}_i$, which is a new face and therefore not in $\Delta_{i-1}$.

The proof for $F_{i+1}$ is handled in a similar manner with $\{\hat{j}\}_{i+1} \subseteq F_i$ for $c_{i/2} + 1 \le j \le c_{i/2} + c_{a-i/2}$ and $\{c_{i/2} + 1 : c_{i/2} + c_{a-i/2}\}_{i+1} \notin \Delta_i$. The last part of Claim 2 now follows from the definition of a shelling. □

**Claim 3** *For $1 \le p \le a$ each codimension-one face of $\Delta_p$ is contained in at most two facets and there exists a codimension-one face of $\Delta_p$ that is contained in only one facet.*

*Proof of Claim 3* We first show that each codimension-one face is contained in at most two facets. We do this by showing that any codimension-one face in $F_l \cap \Delta_{l-1}$ for $1 < l \le a$ is either a face of $F_{l-1} \backslash \Delta_{l-2}$ or a face of $F_1 \backslash (\bigcup_{i=2}^{l-1} F_i)$.

Let $l > 0$ be even and consider $F_l \cap \Delta_{l-1}$. By Claim 2 we need to consider the faces $\{\hat{j}\}_l$ for $1 \le j \le c_{l/2}$. Using the attachment rules given before Claim 1, for $1 \le j \le c_{l/2-1}$ we see that $\{\hat{j}\}_l$ is a face in $F_{l-1} \backslash \Delta_{l-2}$, while for $c_{l/2-1} + 1 \le j \le c_{l/2}$ the face $\{\hat{j}\}_l$ is in $F_1 \backslash (\bigcup_{i=2}^{l-1} F_i)$.

Now consider the case $F_{l+1} \cap \Delta_l$. By Claim 2 we are interested in the faces $\{\hat{j}\}_{l+1}$ for $c_{l/2} + 1 \le j \le c_{l/2} + c_{a-l/2}$. Again using the attachment rules given before Claim 1, all of these are faces in $F_l \backslash \Delta_{l-1}$, as desired.

Using Claim 2, the faces $\{\hat{j}\}_1$ for $j > c_{\lfloor a/2 \rfloor}$ never appear in any $F_l$ with $l > 1$. Since $h_d = 0$ we know $c_{\lfloor a/2 \rfloor} < d$, so the codimension-one face $\{\hat{d}\}_1$ is only contained in the facet $F_1$, completing the proof of Claim 3. □

By Claim 3 and Proposition 15 we know that $\Delta_a$ is a ball. Using the $|\sigma(F_i)|$ from Claim 2 to count the contribution of each facet to the $h$-vector along with the fact $|\{i : c_i = j\}| = h_j$ we know that $\Delta_a$ has the desired $h$-vector.

We now consider the case where $0 < n < \lfloor (d-1)/2 \rfloor$ is the smallest integer such that $\partial h_n = 0$. Define a new vector $\mathbf{h}' = (h'_0, h'_1, \dots, h'_{d-1}, h'_d)$ by $h'_i = h_i$ for $i \neq d-n$ and $h'_{d-n} = h_{d-n} - 1$. From the definition of the $\partial h_i$, in order for $\partial h_{n-1} > 0$ and $\partial h_n = 0$ we must have $h_{d-n} > 0$, so the vector $\mathbf{h}'$ has non-negative entries. Additionally, for $n \le i \le \lfloor (d-1)/2 \rfloor$ we have $\partial h'_i = \partial h_i + 1$ ensuring that all of the $\partial h'_i$ are strictly positive. We can therefore apply the construction of the previous case to create a ball with $h$-vector $\mathbf{h}'$. In what follows we take $a = \sum_{i=0}^d h'_i = (\sum_{i=0}^d h_i) - 1$ to match the definition of $a$ in the previous case. Similarly, $c_k$ for $0 \le k \le a-1$ measures the location where the sum of the entries of $\mathbf{h}'$ reaches $k+1$.

By assumption $\sum_{i=0}^d h_i$ is even, hence the construction of the ball with $h$-vector $\mathbf{h}'$ ends with the facet $F_a$ with $a$ odd. We complete the construction of a ball with $h$-vector $\mathbf{h}$ by adding a facet $F_{a+1}$. We attach $F_{a+1}$ to the ball $\Delta_a$ using the rules given before Claim 1 for attaching an even index facet but acting as if $\tilde{c}_{(a+1)/2} = d-n$, so $\{1 : d-n\}_{a+1}$ is a new face (when referring to the facet $F_{a+1}$ we use the notation $\tilde{c}_{(a+1)/2} = d-n$ to avoid confusion with $c_{(a+1)/2}$ as defined from the vector $\mathbf{h}'$). To complete the proof we must extend the results of Claims 1, 2, and 3 to include the additional facet $F_{a+1}$.

First we prove Claim 1 for the case $i = a+1$. The proof is the same as for smaller $i$ values except that the second inequality in (3) requires a different justification. Rewriting this inequality, for $1 \le j \le (a-1)/2$ we must show

$$d - n \le c_j + c_{a-j}.$$

First consider the case $c_j \ge d - 2n$. Since $n < \lfloor (d-1)/2 \rfloor$, adding the equations $\sum_{m=0}^d h_m = a+1$ and $\sum_{m=0}^n (h_m - h_{d-m}) = 0$ and then removing some non-negative terms from the left-hand side yields $\sum_{m=0}^n h_m \le (a+1)/2$. Hence $\sum_{m=0}^n h'_m \le (a+1)/2$ and $c_{(a+1)/2} \ge n$. Since $a - j \ge (a+1)/2$ we have $c_{a-j} \ge c_{(a+1)/2} \ge n$, completing the proof of this case.

For the case $1 \le c_j \le d - 2n - 1$ note that we can rewrite the assumption $\partial h_l \le \sum_{m=0}^{n-1} h_{l-m}$ as

$$\sum_{m=0}^{l-n} h_m \le \sum_{m=0}^{l} h_{d-m} \quad \text{or} \quad \sum_{m=1}^{l-n} h'_m \le \sum_{m=0}^{l} h'_{d-m}$$

where $n+1 \le l \le d-(n+1)$. By the second inequality, choosing $l$ such that $c_j = l - n$ we have $c_{a-j} \ge d - l$. Therefore $c_j + c_{a-j} \ge d - n$, as desired.

We extend Claim 2 by showing

$$F_{a+1} \cap \Delta_a = \bigcup_{j=1}^{d-n} \{\hat{j}\}_{a+1}. \tag{4}$$

This follows from the proof of the even case of Claim 2 by treating $\tilde{c}_{(a+1)/2} = d-n$.

To allow $p = a + 1$ in Claim 3 we consider the codimension-one faces in $F_{a+1} \cap \Delta_a$. By (4) these are $\{\hat{j}\}_{a+1}$ for $1 \le j \le d - n$. Using the attachment rules for the $F_i$, for $1 \le j \le c_{(a-1)/2}$ we see that $\{\hat{j}\}_{a+1}$ is a face in $F_a \backslash \Delta_{a-1}$, while for $c_{(a-1)/2} + 1 \le j \le d - n$ the face $\{\hat{j}\}_{a+1}$ is in $F_1 \backslash (\bigcup_{i=2}^{a} F_i)$.

The faces $\{\hat{j}\}_1$ for $j > d - n$ never appear in any $F_l$ with $l > 1$. Since $n > 0$ the codimension-one face $\{\hat{d}\}_1$ is only contained in the facet $F_1$, completing the proof of the extended version of Claim 3. $\square$

We next present a slight augmentation of the previous theorem that allows us to deal with some additional cases involving $h$-vectors that have a single sequence of non-zero entries.

**Theorem 17** *Let* $\mathbf{h} = (h_0, h_1, \dots, h_{d-1}, h_d) \in \mathbb{N}^{d+1}$ *with* $h_0 = 1$. *Assume that there exists* $k \in \{1, 2, \dots, d - 1\}$ *such that* $h_j = 0$ *for* $j > k$ *and* $h_j > 0$ *for* $1 \le j \le k$. *Define* $\mathbf{h}' = (1, h_1 - 1, h_2 - 1, \dots, h_k - 1, 0, \dots, 0)$. *If* $\mathbf{h}'$ *satisfies the conditions of Theorem* 16 *then there exists a poset* $P$ *such that* $\Gamma(P)$ *is a* $(d - 1)$*-ball and* $h(P) = \mathbf{h}$.

*Proof* We once again construct a shellable CW-complex with the desired $h$-vector. We begin by building a *simplicial complex* on vertex set $[d + 1]$. We think of the faces of this simplicial complex as subsets of $[d + 1]$ as well as topological simplexes.

For $0 \le i \le k$ define $G_i = [d + 1] - \{i + 1\}$, a face of our simplicial complex. Let $\Delta_j$ be the simplicial complex whose facets are $\{G_i\}_{i=0}^{j}$. Then for $1 \le i \le k$

$$G_i \cap \Delta_{i-1} = \bigcup_{j=1}^{i} \bigl(G_i - \{j\}\bigr).$$

Hence $G_1, \dots, G_k$ is a shelling order for $\Delta_k$ with $\sigma(G_i) = \{1, 2, \dots, i\}$ and $|\sigma(G_i)| = i$.

We complete the proof by performing the construction of Theorem 16 on the vector $\mathbf{h}'$ with a few alterations. We omit the initial facet $F_1$ in Theorem 16. Instead we attach all of our additional facets to the boundary of $\Delta_k$. In our new construction we replace the vertices $\{j\}_1$, $1 \le j \le d$, from Theorem 16 with the vertices $[d + 1] \backslash \{k + 2\}$ of $\Delta_k$, identifying vertices in the order preserving way. We then replace the faces of $\partial F_1$ from Theorem 16 with the faces defined by the corresponding sets of vertices of $\Delta_k$.

We claim that performing the construction of Theorem 16 with this alteration gives a shellable ball, with a shelling order given by concatenating the order $G_1, G_2, \dots, G_k$ with the order given by Theorem 16. To prove this, we take every facet of $\partial F_1$ that is contained in a later facet in the construction of Theorem 16 and show that the corresponding $(d - 1)$-subset of $[d + 1] \backslash \{k + 2\}$ is a face of $\partial \Delta_k$. Let $H = [d + 1] \backslash \{j, k + 2\}$ be a $(d - 1)$-subset of $[d + 1] \backslash \{k + 2\}$. For $1 \le j \le k + 1$, the only facet of $\Delta_k$ that contains $H$ is $G_{j-1}$, so $H$ is in $\partial \Delta_k$. For $k + 3 \le j \le d + 1$, the facet of $\partial F_1$ corresponding to $H$ will never be used in the construction of Theorem 16 since $h_l = 0$ for $l > k$.

Totaling the contributions of all of the $|\sigma(G_i)|$ shows that the ball created in this manner has the desired $h$-vector. $\square$

We now present one final construction specific to dimension five. This construction allows us to complete the characterization of the possible $h$-vectors of five-balls in the following section.

**Proposition 18** *Let* $\mathbf{h} = (h_0, h_1, h_2, h_3, h_4, 0, 0) \in \mathbb{N}^7$ *with* $h_0 = 1$, $h_1, h_2 \neq 0$, $\sum_{i=0}^{4} h_i$ *odd, and* $\partial h_2 = 1 + h_1 + h_2 - h_4 = 0$. *Then there exists a poset* $P$ *such that* $\Gamma(P)$ *is a five-ball and* $h(P) = \mathbf{h}$.

*Proof* We begin this construction by using Theorem 16 to create a 5-ball with $h$-vector $(1, 0, 0, h_3 - 1, 0, 0, 0)$. Note that since $1 + h_1 + h_2 - h_4 = 0$, the parity of $h_3$ is the same as the parity of $\sum_{i=0}^{4} h_i$, which we assumed to be odd. Therefore $h_3 - 1$ is even and non-negative.

When we complete this construction the facets of the boundary of our ball are

$$\{1,2,3,4,5\}_1, \qquad \{1,2,3,4,6\}_1, \qquad \{1,2,3,5,6\}_1,$$
$$\{2,3,4,5,6\}_{h_3}, \qquad \{1,3,4,5,6\}_{h_3}, \quad \text{and} \quad \{1,2,4,5,6\}_{h_3}.$$

The fact that the first three faces are on the boundary follows from the discussion at the end of Claim 3 in the proof of Theorem 16. The last three faces are on the boundary because $F_{h_3}$ is the last facet added to our ball and $\{4,5,6\}_{h_3}$ is a new face when $F_{h_3}$ is added in the construction of Theorem 16.

Further, all of the faces of $F_{h_3}$ are identified with the corresponding faces of $F_1$ except for $\{1,2,3\}_{h_3} = \{1,2,3\}_{h_3-1}$, $\{4,5,6\}_{h_3}$, and all faces containing one of these two faces (there is also the easier case where $h_3 = 1$ and we have only one facet in this initial part of the shelling).

We now describe the next six facets of our shelling, altering our notation slightly to make the description easier to follow.

$F_{h_3+1} = \{1,2,3,4,5,7\}_{h_3+1}$ where $\{7\}_{h_3+1}$ is a new vertex.
 $\{1,2,3,4,5\}_{h_3+1}$ is identified with $\{1,2,3,4,5\}_1$.
 Hence $\sigma(F_{h_3+1}) = \{7\}_{h_3+1}$.
$F_{h_3+2} = \{1,2,3,4,6,7\}_{h_3+2}$.
 $\{1,2,3,4,6\}_{h_3+2}$ is identified with $\{1,2,3,4,6\}_1$.
 $\{1,2,3,4,7\}_{h_3+2}$ is identified with $\{1,2,3,4,7\}_{h_3+1}$.
 Hence $\sigma(F_{h_3+2}) = \{6,7\}_{h_3+2}$.
$F_{h_3+3} = \{1,2,3,5,6,7\}_{h_3+3}$.
 $\{1,2,3,5,6\}_{h_3+3}$ is identified with $\{1,2,3,5,6\}_1$.
 $\{1,2,3,5,7\}_{h_3+3}$ is identified with $\{1,2,3,5,7\}_{h_3+1}$.
 $\{1,2,3,6,7\}_{h_3+3}$ is identified with $\{1,2,3,6,7\}_{h_3+2}$.
 Hence $\sigma(F_{h_3+3}) = \{5,6,7\}_{h_3+3}$.
$F_{h_3+4} = \{1,2,4,5,6,7\}_{h_3+4}$ with $\{4,5,6\}_{h_3+4} = \{4,5,6\}_{h_3}$.
 $\{1,2,4,5,6\}_{h_3+4}$ is identified with $\{1,2,4,5,6\}_{h_3}$.
 $\{1,2,4,5,7\}_{h_3+4}$ is identified with $\{1,2,4,5,7\}_{h_3+1}$.
 $\{1,2,4,6,7\}_{h_3+4}$ is identified with $\{1,2,4,6,7\}_{h_3+2}$.

$\{1,2,5,6,7\}_{h_3+4}$ is identified with $\{1,2,5,6,7\}_{h_3+3}$.
Hence $\sigma(F_{h_3+4}) = \{4,5,6,7\}_{h_3+4}$.
$F_{h_3+5} = \{1,3,4,5,6,7\}_{h_3+5}$ with $\{4,5,6\}_{h_3+5} = \{4,5,6\}_{h_3}$ and new face $\{4,5,6,7\}_{h_3+5}$.
$\{1,3,4,5,6\}_{h_3+5}$ is identified with $\{1,3,4,5,6\}_{h_3}$.
$\{1,3,4,5,7\}_{h_3+5}$ is identified with $\{1,3,4,5,7\}_{h_3+1}$.
$\{1,3,4,6,7\}_{h_3+5}$ is identified with $\{1,3,4,6,7\}_{h_3+2}$.
$\{1,3,5,6,7\}_{h_3+5}$ is identified with $\{1,3,5,6,7\}_{h_3+3}$.
Hence $\sigma(F_{h_3+5}) = \{4,5,6,7\}_{h_3+5}$.
$F_{h_3+6} = \{2,3,4,5,6,7\}_{h_3+6}$ with $\{4,5,6\}_{h_3+6} = \{4,5,6\}_{h_3}$ and new face $\{4,5,6,7\}_{h_3+6}$.
$\{2,3,4,5,6\}_{h_3+6}$ is identified with $\{2,3,4,5,6\}_{h_3}$.
$\{2,3,4,5,7\}_{h_3+6}$ is identified with $\{2,3,4,5,7\}_{h_3+1}$.
$\{2,3,4,6,7\}_{h_3+6}$ is identified with $\{2,3,4,6,7\}_{h_3+2}$.
$\{2,3,5,6,7\}_{h_3+6}$ is identified with $\{2,3,5,6,7\}_{h_3+3}$.
Hence $\sigma(F_{h_3+6}) = \{4,5,6,7\}_{h_3+6}$.
Examining the $|\sigma(F_i)|$ shows that the ball we have constructed has $h$-vector $(1,1,1,h_3,3,0,0)$.

We finish building our ball using a slightly altered version of the construction of Theorem 16 on the vector $(1,h_1-1,h_2-1,0,h_4-3,0,0)$. In place of the initial facet from Theorem 16 we use the final facet $F_{h_3+6}$ from the above construction (with the order preserving identification of the two facets' vertices).

Consider the construction of Theorem 16 for the $h$-vector $(1,h_1-1,h_2-1,0,h_4-3,0,0)$. We have $c_{(a-1)/2} \le 2$, hence by the proof of Claim 3 the only codimension-one faces of $F_1$ that are attached to facets $F_i$ with $i>1$ during the construction are $\{\hat{1}\}_1$ and $\{\hat{2}\}_1$.

When we replace the initial facet in Theorem 16 with $F_{h_3+6}$ from the above construction, the corresponding co-dimension one faces that will be attached to later facets are $\{3,4,5,6,7\}_{h_3+6}$ and $\{2,4,5,6,7\}_{h_3+6}$. These two faces are both on the boundary of our above constructed ball. Thus we can finish the shelling in this manner, creating a ball with the desired $h$-vector. □

## 7 A summary of known conditions

Using the results of the previous two sections we now fully characterize all of the $h$-vectors of simplicial posets that are balls in all even dimensions as well as dimensions three and five.

**Theorem 19** (All Even Dimensions) *Let $d$ be an odd integer and let $\mathbf{h} = (h_0,h_1,\ldots,h_{d-1},h_d) \in \mathbb{Z}^{d+1}$ with $h_0 = 1$ and $h_d = 0$. Define $\partial h_j = \sum_{i=0}^{j}(h_i - h_{d-i})$ for $0 \le j \le d-1$. Then there exists a simplicial poset $P$ such that $\Gamma(P)$ is a $(d-1)$-ball and $\mathbf{h} = h(P)$ if and only if the following all hold.*

1. *$h_i \ge 0$ for $1 \le i \le d-1$.*
2. *$\partial h_i \ge 0$ for $0 \le i \le (d-1)/2$.*
3. *If $\partial h_i = 0$ for any $1 \le i \le (d-1)/2$ then $\sum_{j=0}^{d-1} h_j$ is even.*

4. *If* $\partial h_i = 0$ *for any* $1 \le i \le (d-1)/2$ *then* $\partial h_l \le \sum_{j=0}^{i-1} h_{l-j}$ *for* $i+1 \le l \le d-(i+1)$.

*Proof* For the first three conditions necessity follows directly from Theorem 5, while for the last condition it is a result of Theorem 8. When the second condition is satisfied with all strict inequalities sufficiency is proved using the first case of Theorem 16; otherwise we use the second case of Theorem 16. □

**Proposition 20** (Dimension 3) *Let* $\mathbf{h} = (1, h_1, h_2, h_3, 0) \in \mathbb{Z}^5$. *Then there exists a simplicial poset* $P$ *such that* $\Gamma(P)$ *is a three-ball and* $\mathbf{h} = h(P)$ *if and only if the following all hold.*

1. $h_i \ge 0$ *for* $1 \le i \le 3$.
2. $h_3 \le h_1 + 1$.
3. *If* $h_1 = 0$ *and* $h_3 = 1$ *then* $h_2$ *is even.*

*Proof* The necessity of the first two conditions follows directly from Theorem 5, while the third condition is a consequence of Proposition 10.

When $h_3 < h_1 + 1$ sufficiency follows from the first case of Theorem 16. If $h_3 = h_1 + 1$ and $h_2$ is even the second case of Theorem 16 gives the desired result. Otherwise, $h_3 = h_1 + 1 > 1$ and $h_2$ is odd which means all of the $h_i$ for $0 \le i \le 3$ are non-zero and we can apply Theorem 17 to obtain the needed construction. □

**Proposition 21** (Dimension 5) *Let* $\mathbf{h} = (h_0, h_1, h_2, h_3, h_4, h_5, h_6) \in \mathbb{Z}^7$ *with* $h_0 = 1$ *and* $h_6 = 0$. *Let* $\partial h_j = \sum_{i=0}^{j} (h_i - h_{6-i})$ *for* $0 \le j \le 5$. *Then there exists a simplicial poset* $P$ *such that* $\Gamma(P)$ *is a five-ball and* $\mathbf{h} = h(P)$ *if and only if the following all hold.*

1. $h_i \ge 0$ *for* $1 \le i \le 5$.
2. $\partial h_1 \ge 0$. *If* $\partial h_1 = 0$ *then* $h_i \ge \partial h_i$ *for* $0 \le i \le 5$. *If* $\partial h_1 = 0$ *and* $h_j = 0$ *for some* $1 \le j \le 4$ *then* $\sum_{i=0}^{5} h_i$ *is even.*
3. $\partial h_2 \ge 0$. *If* $\partial h_2 = 0$ *and* $h_1 = 0$ *or* $h_2 = 0$ *then* $\sum_{i=0}^{5} h_i$ *is even.*

*Proof* The necessity of condition one and the first inequality in the other two conditions follow from Theorem 5. The additional inequalities when $\partial h_1 = 0$ come from Theorem 7. The remainder of condition two comes from Proposition 13. For the third condition, the case $h_1 = 0$ follows from Proposition 10 while the case $h_2 = 0$ is a result of Proposition 11.

For sufficiency, if there are strict inequalities in the second and third conditions we use the first case of Theorem 16. If $\partial h_1 = 0$ and $\sum_{i=0}^{5} h_i$ is even then we apply the second case of Theorem 16. If $\partial h_1 = 0$ and $\sum_{i=0}^{5} h_i$ is odd then all of the $h_i$ for $1 \le i \le 5$ are non-zero, so we apply Theorem 17. It is not hard to check that reducing the elements of the $h$-vector by one will preserve the inequalities $h_i \ge \partial h_i$ for $0 \le i \le 5$. The case $h_3 \ge \partial h_3$ uses the fact that the sum of the $h_i$ is odd.

If $\partial h_1 > 0$ and $\partial h_2 = 0$, the inequality in the second case of Theorem 16 is always trivially satisfied. This gives the needed construction whenever $\sum_{i=0}^{5} h_i$ is even. For the case where $\sum_{i=0}^{5} h_i$ is odd, note that $h_3$ must be odd and hence non-zero. Therefore $h_i > 0$ for $1 \leq i \leq 4$. When $h_5 > 0$ we can apply Theorem 17 to obtain the desired construction; otherwise we use Proposition 18. □

For even-dimensional balls ($d$ odd), when any entry of the boundary $h$-vector is zero the sum of the $h_i$ of the ball must have even parity. This allows us to give a complete characterization of the $h$-vectors of even-dimensional balls. In the odd-dimensional case, this relationship is lost and more subtle behavior occurs. In particular, whether or not some of the $h_i$ of the ball are zero needs to be considered, resulting in some of the more complicated conditions and the extra construction in the dimension five case. To solve the characterization problem in higher odd dimensions we still need a general framework to describe what types of conditions on the $h$-vectors of the balls and their boundary spheres force an even number of facets. Additionally, as we saw in dimension five, more constructions are needed to obtain all possible $h$-vectors for odd-dimensional balls.

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