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Linear extension sums as valuations on cones

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Abstract The geometric and algebraic theory of valuations on cones is applied to understand identities involving summing certain rational functions over the set of linear extensions of a poset.

Keywords Poset · Rational function identities · Valuation of cones · Lattice points · Affine semigroup ring · Hilbert series · Total residue · Root system · Weight lattice

1 Introduction

This paper presents a different viewpoint on the following two classes of rational function summations, which are both summations over the set $\mathcal{L}(P)$ of all linear

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extensions of a partial order P on the set $\{1, 2, \ldots, n\}$:

$$\Psi_P(\mathbf{x}) := \sum_{w\in\mathcal{L}(P)} w\left(\frac{1}{(x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n)}\right);$$

$$\Phi_P(\mathbf{x}) := \sum_{w\in\mathcal{L}(P)} w\left(\frac{1}{x_1(x_1+x_2)(x_1+x_2+x_3)\cdots(x_1+\cdots+x_n)}\right).$$

Recall that a *linear extension* is a permutation $w = (w(1), \ldots, w(n))$ in the symmetric group $\mathfrak{S}_n$ for which the *linear order* P_w defined by $w(1) <_{P_w} \cdots <_{P_w} w(n)$ satisfies $i <_{P_w} j$ whenever $i <_P j$.

Several known results express these sums explicitly for particular posets P as rational functions in lowest terms. In the past, these results have most often been proven by induction, sometimes in combination with techniques such as *divided differences* and more general operators on multivariate polynomials. We first explain three of these results that motivated us.

1.1 Strongly planar posets

The rational function $\Psi_P(\mathbf{x})$ was introduced by Greene [15] in his work on the Murnaghan–Nakayama formula. There he evaluated $\Psi_P(\mathbf{x})$ when P is a *strongly planar* poset in the sense that the poset $P \sqcup \{\hat{0}, \hat{1}\}$ with an extra bottom and top element has a planar embedding for its Hasse diagram, with all edges directed upward in the plane. To state his evaluation, note that in this situation, the edges of the Hasse diagram for P dissect the plane into bounded regions ρ, and the set of vertices lying on the boundary of ρ will consist of two chains having a common minimum element $\min(\rho)$ and maximum element $\max(\rho)$ in the partial order P.

Theorem A (Greene [15, Theorem 3.3]) *For any strongly planar poset P,*

$$\Psi_P(\mathbf{x}) = \frac{\prod_\rho (x_{\min(\rho)} - x_{\max(\rho)})}{\prod_{i\lessdot_P j}(x_i - x_j)},$$

where the product in the denominator runs over all covering relations $i \lessdot_P j$ or over the edges of the Hasse diagram for P, while the product in the numerator runs over all bounded regions ρ for the Hasse diagram for ρ.

1.2 Skew diagram posets

Further work on $\Psi_P(\mathbf{x})$ appeared in [7–9, 16]. For example, we will prove in Sect. 4 the following generalization of a result of the first author. Consider a *skew (Ferrers) diagrams* $D = \lambda/\mu$, in English notation as a collection of points (i, j) in the plane, where rows are numbered $1, 2, \ldots, r$ from top to bottom (the usual English convention), and the columns numbered $1, 2, \ldots, c$ from right to left (not the usual English convention). Thus the northeasternmost and southwesternmost points of D are labelled $(1, 1)$ and (r, c), respectively; see Example 4.3. Define the *bipartite poset* P_D on the set $\{x_1, \ldots, x_r, y_1, \ldots, y_c\}$ having an order relation $x_i <_{P_D} y_j$ whenever (i, j) is a point of D.

Theorem B *For any skew diagram D,*

$$\Psi_{P_D}(\mathbf{x}) = \frac{\sum_\pi \prod_{(i,j)\in D\setminus\pi}(x_i - y_j)}{\prod_{(i,j)\in D}(x_i - y_j)},$$

where the product in the numerator runs over all lattice paths π from $(1,1)$ to (r,c) inside D that take steps either one unit south or west.

In particular (Boussicault [8, Prop. 4.7.2]), when $\mu = \emptyset$, so that D is the Ferrers diagram for a partition[1] *λ, this can be rewritten*

$$\Psi_{P_D}(\mathbf{x}) = \frac{\mathfrak{S}_{\hat{w}}(\mathbf{x},\mathbf{y})}{\mathfrak{S}_w(\mathbf{x},\mathbf{y})},$$

where $\mathfrak{S}_w(\mathbf{x},\mathbf{y})$, $\mathfrak{S}_{\hat{w}}(\mathbf{x},\mathbf{y})$ are the double Schubert polynomials for the dominant permutation w having Lehmer code $\lambda = (\lambda_1,\dots,\lambda_r)$, and the vexillary permutation $\hat{w}$ having Lehmer code $\hat{\lambda} := (0, \lambda_2 - 1, \dots, \lambda_r - 1)$.

1.3 Forests

In his treatment of the character table for the symmetric group $\mathfrak{S}_n$, Littlewood [20, p. 85] used the fact that the *antichain* poset $P = \emptyset$, having no order relations on $\{1,2,\dots,n\}$ and whose set of linear extensions $\mathcal{L}(\emptyset)$ is equal to all of $\mathfrak{S}_n$, satisfies

$$\Phi_\emptyset(\mathbf{x}) = \frac{1}{x_1 x_2 \cdots x_n}. \tag{1.1}$$

The following generalization appeared more recently in [11]. Say that a poset P is a *forest* if every element is covered by at most one other element.

Theorem C (Chapoton, Hivert, Novelli, and Thibon [11, Lemma 5.3]) *For any forest poset P,*

$$\Phi_P(\mathbf{x}) = \frac{1}{\prod_{i=1}^n (\sum_{j \le_P i} x_j)}.$$

1.4 The geometric perspective of cones

Our first new perspective on these results views $\Psi_P(\mathbf{x})$, $\Phi_P(\mathbf{x})$ as instances of a well-known valuation on convex polyhedral cones K in a Euclidean space V with inner product $\langle\cdot,\cdot\rangle$:

$$s(K;\mathbf{x}) := \int_K e^{-\langle \mathbf{x}, v\rangle}\, dv.$$

[1] Such bipartite graphs were called *λ-complete* in [8] and sometimes appear in the literature under the name *Ferrers graphs*.

One can think of $s(K;\mathbf{x})$ as the multivariable *Laplace transform* applied to the $\{0,1\}$-valued characteristic function of the cone K. After reviewing the properties of this valuation in Sect. 2, we use these to establish that

$$\Psi_P(\mathbf{x}) = s\big(K_P^{\text{root}};\mathbf{x}\big),$$
$$\Phi_P(\mathbf{x}) = s\big(K_P^{\text{wt}};\mathbf{x}\big),$$

where $K_P^{\text{root}}, K_P^{\text{wt}}$ are two cones naturally associated to the poset P as follows:

$$K_P^{\text{root}} = \mathbb{R}_+\{e_i - e_j : i <_P j\},$$
$$K_P^{\text{wt}} = \big\{x \in \mathbb{R}_+^n : x_i \geq x_j \text{ for } i <_P j\big\},$$

$\mathbb{R}_+$ denotes the nonnegative real numbers. In Sects. 4 and 5, this identification is used, together with the properties of $s(K;\mathbf{x})$ from Sect. 2, to give simple geometric proofs underlying Theorems B and C above.

1.5 The algebraic perspective of Hilbert series

One gains another useful perspective when the cone K is *rational* with respect to some lattice L inside V, which holds for both $K_P^{\text{root}}, K_P^{\text{wt}}$. This allows one to compute a more refined valuation, the multigraded *Hilbert series*

$$\text{Hilb}(K\cap L;\mathbf{x}) := \sum_{v\in K\cap L} e^{\langle \mathbf{x}, v\rangle}$$

for the affine semigroup ring $k[K \cap L]$ with coefficients in any field k. As discussed in Sect. 2.4 below, it turns out that $\text{Hilb}(K\cap L;\mathbf{x})$ is a meromorphic function of $x_1,\dots,x_n$, whose Laurent expansion begins in total degree $-d$, where d is the dimension of the cone K, with this lowest term of total degree $-d$ equal to $s(K;\mathbf{x})$, up to a predictable sign. This allows one to algebraically analyze the ring $k[K\cap L]$, compute its Hilbert series, and thereby recover $s(K;\mathbf{x})$.

For example, in Sect. 8.3, it will be shown that Theorem A by Greene is the reflection of a *complete intersection* presentation for the affine semigroup ring of K_P^{root} when P is a strongly planar poset, having generators indexed by the edges in the Hasse diagram of P, and relations among the generators indexed by the bounded regions ρ.

As another example, in Sect. 6, it will be shown that Theorem C, along with the "maj" hook formula for forests due to Björner and Wachs [5, Theorem 1.2] are both consequences of an easy Hilbert series formula (Proposition 6.2 below) related to K_P^{wt} when P is a forest.

2 Cones and valuations

2.1 A review of cones

We review some facts and terminology about polyhedral cones; see, e.g., [21, Chap. 7], [23, §4.6] for background.

Let V be an n-dimensional vector space over $\mathbb{R}$. A linear function ℓ in V^* has as zero set a hyperplane H containing the origin and defines a closed halfspace H^+ consisting of the points v in V with $\ell(v) \geq 0$. A *polyhedral cone* K (containing the origin 0) in V is the intersection $K = \bigcap_i H_i^+$ of finitely many linear halfspaces H_i^+, or alternatively the *nonnegative span* $K = \mathbb{R}_+\{u_1, \ldots, u_N\}$ of finitely many generating vectors u_i in V. Its *dimension*, denoted $\dim_{\mathbb{R}} K$, is the dimension of the smallest linear subspace that contains it. One says that K is *full-dimensional* if $\dim_{\mathbb{R}} K = n = \dim_{\mathbb{R}} V$.

Say that K is *pointed* if it contains no lines. In this case, if $\{u_1, \ldots, u_N\}$ is a minimal set of vectors for which $K = \mathbb{R}_+\{u_1, \ldots, u_N\}$, then the u_i are said to span the *extreme rays* $\mathbb{R}_+ u_i$ of K; these rays are unique, although the choice of vectors u_i is unique only up to positive scalings.

Say that K is *simplicial* if its extreme rays are spanned by a linearly independent set of vectors $\{u_1, \ldots, u_N\}$, so that $N = \dim_{\mathbb{R}} K \leq n$.

In the dual space V^* one has the *dual* or *polar cone*

$$K^* := \left\{x \in V^* : \langle x, v\rangle \geq 0 \text{ for all } v \in K\right\}.$$

The following facts about duality of cones are well known:

- Under the identification $(V^*)^* = V$, one has $(K^*)^* = K$.
- A cone K is pointed (resp. full-dimensional) if and only if its dual cone K^* is full-dimensional (resp. pointed).
- A cone K is simplicial if and only if its dual cone K^* is simplicial.

2.2 The Laplace transform valuation

Choose a basis $v_1, \ldots, v_n$ for V and dual basis $x_1, \ldots, x_n$ for V^*. Then the polynomial functions $\mathbb{Q}[V]$ on V are identified with the symmetric/polynomial algebras $\mathrm{Sym}(V^*) \cong \mathbb{R}[x_1, \ldots, x_n]$ and the rational functions $\mathbb{Q}(V)$ on V with the field of fractions $\mathbb{Q}(x_1, \ldots, x_n)$.

In order to consider integrals on V, let $dv = dv_1 \cdots dv_n$ denote Lebesgue measure on $\mathbb{R}^n \cong V$ using the basis $v_1, \ldots, v_n$ for this identification.

The following proposition defining our first valuation is well known; see, e.g., [1, Proposition 2.4], [3, Proposition 5].

Proposition 2.1 *There exists a unique assignment of a rational function* $\mathrm{s}(K; \mathbf{x})$ *lying in* $\mathbb{Q}(V) = \mathbb{Q}(x_1, \ldots, x_n)$ *to each polyhedral cone* K, *having the following properties*:

(i) $\mathrm{s}(K; \mathbf{x}) = 0$ *when* K *is not pointed.*
(ii) $\mathrm{s}(K; \mathbf{x}) = 0$ *when* K *is not full-dimensional.*
(iii) *When* K *is pointed and full-dimensional, for each* $\mathbf{x}$ *in the dual cone* K^*, *the improper integral* $\int_K e^{-\langle \mathbf{x}, v\rangle} dv$ *converges to the value given by the rational function* $\mathrm{s}(K; \mathbf{x})$.
(iv) *When* K *is pointed and full-dimensional, with extreme rays spanned by* $\{u_1, \ldots, u_N\}$, *the rational function* $\mathrm{s}(K; \mathbf{x})$ *can be written with smallest denominator* $\prod_{i=1}^N \langle \mathbf{x}, u_i\rangle$.

(v) *In particular, when K is full-dimensional and simplicial, with extreme rays spanned by* $\{u_1, \dots, u_n\}$, *then*

$$\mathrm{s}(K;\mathbf{x}) = \frac{|\det[u_1,\dots,u_n]|}{\prod_{i=1}^n \langle \mathbf{x}, u_i\rangle}.$$

(vi) *The map* $\mathrm{s}(-;\mathbf{x})$ *is a solid valuation, that is, if there is a linear relation* $\sum_{i=1}^t c_i \chi_{K_i} = 0$ *among the characteristic functions* χ_{K_i} *of the cones* K_i, *there will be a linear relation*

$$\sum_{i:\dim_{\mathbb{R}} K_i = n} c_i \mathrm{s}(K_i;\mathbf{x}) = 0.$$

2.3 The semigroup ring and its Hilbert series

Now endow the n-dimensional real vector space V with a distinguished lattice L of rank n and assume that the chosen basis $v_1, \dots, v_n$ for V is also a $\mathbb{Z}$-basis for L.

Say that the polyhedral cone K is *rational* with respect to L if one can express $K = \mathbb{R}_+\{u_1, \dots, u_N\}$ for some elements u_i in L. The subset $K \cap L$ together with its additive structure inherited from addition of vectors in V is then called an *affine semigroup*. Our goal here is to describe how one can approach the computation of the previous valuation $\mathrm{s}(K;\mathbf{x})$ for pointed cones K through the calculation of the finely graded *Hilbert series* for this affine semigroup:

$$\mathrm{Hilb}(K\cap L;\mathbf{x}) := \sum_{v\in K\cap L} e^{\langle \mathbf{x}, v\rangle}.$$

One should clarify how to interpret this infinite series, as it lives in several ambient algebraic objects. Firstly, it lies in the abelian group $\mathbb{Z}\{\{L\}\}$ of all formal combinations

$$\sum_{v\in L} c_v\, e^{\langle \mathbf{x}, v\rangle}$$

with c_v in $\mathbb{Z}$, in which there are no restrictions on vanishing of the coefficients c_v. This set $\mathbb{Z}\{\{L\}\}$ forms an abelian group under addition but is not a ring. However, it contains the *Laurent polynomial ring*

$$\mathbb{Z}[L] \cong \mathbb{Z}\big[X_1^{\pm 1}, \dots, X_n^{\pm n}\big]$$

as the subgroup where only finitely many of the c_v are allowed to be nonzero, using the identification via the exponential change of variables

$$X_i = e^{\langle \mathbf{x}, v_i\rangle}, \quad \text{so that } X_1^{c_1}\cdots X_n^{c_n} = X^v = e^{\langle \mathbf{x}, v\rangle} \text{ if } v := \sum_{i=1}^n c_i v_i. \tag{2.1}$$

Furthermore, $\mathbb{Z}\{\{L\}\}$ forms a module over this subring $\mathbb{Z}[L]$. One can also define the $\mathbb{Z}[L]$-submodule of *summable* elements (see [21, Definition 8.3.9]), namely those

f in $\mathbb{Z}\{\{L\}\}$ for which there exists p, q in $\mathbb{Z}[L]$ with $q \neq 0$ and $q \cdot f = p$. In this situation, say that f *sums to* $\frac{p}{q}$ as an element of the fraction field

$$\mathbb{Q}(L) \cong \mathbb{Q}(X_1, \ldots, X_n).$$

General theory of affine semigroups (see, e.g., [21, Chap. 8]) says that for a rational polyhedral cone K and the semigroup $K \cap L$, the Hilbert series $\mathrm{Hilb}(K \cap L; \mathbf{x})$ is always summable. More precisely,

- when K is not pointed, $\mathrm{Hilb}(K \cap L; \mathbf{x})$ sums to zero. This is because K will not only contain a line, but also an L-rational line, and then any nonzero vector v of L lying on this line will have $(1 - e^{\langle \mathbf{x}, v\rangle}) \cdot \mathrm{Hilb}(K \cap L; \mathbf{x}) = 0$.
- when K is pointed and $\{u_1, \ldots, u_N\}$ are vectors in L that span its extreme rays, then one can show that

$$\left(\prod_{i=1}^{N} \left(1 - e^{\langle \mathbf{x}, u_i\rangle}\right)\right) \cdot \mathrm{Hilb}(K \cap L; \mathbf{x})$$

  always lies in $\mathbb{Z}[L]$.

In fact, one has the following analogue of Proposition 2.1; see, e.g., [1, Proposition 4.4], [2, Theorem 3.1], [3, Proposition 7].

Proposition 2.2 *Let V be an n-dimensional vector space V. Let L be the sublattice in V with $\mathbb{Z}$-basis $v_1, \ldots, v_n$, and V^* the dual space, with dual basis $x_1, \ldots, x_n$.*

Then there exists a well-defined and unique assignment of a rational function $\mathrm{H}(K; \mathbf{X})$ *lying in $\mathbb{Q}(X_1, \ldots, X_n)$ to each L-rational polyhedral cone K, having the following properties*:

(i) $\mathrm{H}(K; \mathbf{X}) = 0$ *when K is not pointed.*
(ii) *When K is pointed, the Hilbert series* $\mathrm{Hilb}(K \cap L; \mathbf{x})$ *sums to the element* $\frac{p}{q} = \mathrm{H}(K; \mathbf{X})$, *considered as a rational function lying in $\mathbb{Q}(L)$.*
(iii) *When K is pointed and full-dimensional, for each $\mathbf{x}$ in the dual cone K^*, the infinite sum $\sum_{v \in K \cap L} e^{\langle \mathbf{x}, v\rangle}$ converges, to the value given by the exponential substitution (2.1) into the rational function* $\mathrm{H}(K; \mathbf{X})$.
(iv) *When K is pointed and full-dimensional, with $\mathbf{u} = \{u_1, \ldots, u_N\}$ the unique primitive vectors (that is, those lying in L nearest the origin) that span its extreme rays, the rational function* $\mathrm{H}(K; \mathbf{X})$ *can be written with smallest denominator $\prod_{i=1}^{N}(1 - X^{u_i})$.*
(v) *In particular, if K is simplicial and $\mathbf{u} := \{u_1, \ldots, u_d\}$ its set of primitive vectors that span its extreme rays, define the semiopen parallelepiped*

$$\Pi_{\mathbf{u}} := \left\{ \sum_{i=1}^{n} c_i u_i : 0 \leq c_i < 1 \right\} \subset V.$$

Then one has

$$\mathrm{H}(K; \mathbf{X}) = \frac{\sum_{u \in \Pi_{\mathbf{u}} \cap L} X^u}{\prod_{i=1}^{d}(1 - X^{u_i})}. \tag{2.2}$$

(vi) *The map* $\mathrm{H}(-;\mathbf{X})$ *is a valuation*: *if there is a linear relation* $\sum_{i=1}^{t} c_i \chi_{K_i} = 0$ *among the characteristic functions* χ_{K_i} *of a collection of* (L*-rational*) *cones* K_i, *there will be a linear relation*

$$\sum_{i=1}^{t} c_i \mathrm{H}(K_i;\mathbf{X}) = 0.$$

2.4 Why $\mathrm{H}(K;\mathbf{X})$ is finer than $\mathrm{s}(K;\mathbf{x})$

When K is an L-rational cone, there is a well-known way (see, e.g., [10]) to compute the Laplace transform valuation $\mathrm{s}(K;\mathbf{x})$ from the Hilbert series valuation $\mathrm{H}(K;\mathbf{X})$ by a certain linear *residue* operation, which we now explain.

Proposition 2.3 *Let* K *be an* L*-rational pointed cone*, *with* $\{u_1,\dots,u_N\}$ *vectors in* L *that span its extreme rays. Regard* $\mathrm{H}(K;\mathbf{X})$ *as a function of the variables* $\mathbf{x} = (x_1,\dots,x_n)$ *via the exponential substitution* (2.1).

Then $\mathrm{H}(K;\mathbf{X})$ *is meromorphic in* $\mathbf{x}$, *of the form*

$$\mathrm{H}(K;\mathbf{X}) = \frac{h(K;\mathbf{x})}{\prod_{i=1}^{N} \langle \mathbf{x}, u_i\rangle}$$

where $h(K;\mathbf{x})$ *is analytic in* $\mathbf{x}$.

Furthermore, *if* $d := \dim_{\mathbb{R}} K$, *then the multivariate Taylor expansion for* $h(K;\mathbf{x})$ *starts in degree* $N-d$, *that is*,

$$h(K;\mathbf{x}) = h_{N-d}(K;\mathbf{x}) + h_{N-d+1}(K;\mathbf{x}) + \cdots,$$

where $h_i(K;\mathbf{x})$ *are homogeneous polynomials of degree* i, *and the multivariate Laurent expansion for* $\mathrm{H}(K;\mathbf{X})$ *starts in degree* $-d$, *that is*,

$$\mathrm{H}(K;\mathbf{X}) = \mathrm{H}_{-d}(\mathbf{x}) + \mathrm{H}_{-d+1}(\mathbf{x}) + \mathrm{H}_{-d+2}(\mathbf{x}) + \cdots.$$

Lastly, *when* K *is full-dimensional* (*so* $d = n$), *then*

$$\mathrm{s}(K;\mathbf{x}) = (-1)^n \frac{h_{N-n}(K;\mathbf{x})}{\prod_{i=1}^{N} \langle \mathbf{x}, u_i\rangle}) = (-1)^n \mathrm{H}_{-n}(\mathbf{x}),$$

so that $h_{N-n}(K;\mathbf{x})$ *is* $(-1)^n$ *times the numerator for* $\mathrm{s}(K;\mathbf{x})$ *accompanying the smallest denominator described in Proposition* 2.1(iv).

Proof We first check all of the assertions when K is simplicial, say with extreme rays spanned by the vectors $u_1,\dots,u_d$ in L. In this case, $N = d$, and the exponential substitution of variables (2.1) into (2.2) gives

$$\mathrm{H}(K;\mathbf{X}) = \frac{\sum_{u\in\Pi_{\mathbf{u}}} e^{\langle \mathbf{x},u\rangle}}{\prod_{i=1}^{d}(1-e^{\langle \mathbf{x},u_i\rangle})} = (-1)^d \frac{\sum_{u\in\Pi_{\mathbf{u}}} e^{\langle \mathbf{x},u\rangle}}{\prod_{i=1}^{d}\langle \mathbf{x},u_i\rangle} \prod_{i=1}^{d} \frac{\langle \mathbf{x},u_i\rangle}{e^{\langle \mathbf{x},u_i\rangle}-1}. \tag{2.3}$$

We wish to be somewhat explicit about the Taylor expansion of each factor in the last product within (2.3). To this end, recall that the function

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} B_n \frac{x^n}{n!} = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots$$

is analytic in the variable x, having power series coefficients described by the *Bernoulli numbers* B_n. Consequently, for each $i = 1, 2, \dots, d$, the factor $\frac{\langle \mathbf{x}, u_i \rangle}{e^{\langle \mathbf{x}, u_i \rangle} - 1}$ appearing in (2.3) is analytic in the variables $\mathbf{x} = (x_1, \dots, x_n)$ and has power series expansion that begins with constant term $+1$. Note that the sum

$$\sum_{u \in \Pi_{\mathbf{u}}} e^{\langle \mathbf{x}, u \rangle} \sum_{u \in \Pi_{\mathbf{u}}} \left(1 + \langle \mathbf{x}, u \rangle + \frac{1}{2} \langle \mathbf{x}, u \rangle^2 + \cdots \right)$$

is also analytic in $\mathbf{x}$, having power series expansion that begins with the constant term $|\Pi_{\mathbf{u}}|$. Thus the expansion in (2.3) begins in degree $-d$ with

$$(-1)^d \frac{|\Pi_{\mathbf{u}}|}{\prod_{i=1}^d \langle \mathbf{x}, u_i \rangle}.$$

Whenever K is full-dimensional, so that $d = n$, expressing the u_i in coordinates with respect to a $\mathbb{Z}$-basis $e_1, \dots, e_n$ for L, one has $|\Pi_{\mathbf{u}}| = |\det(u_1, \dots, u_n)|$. Comparison with Proposition 2.1(v) then shows that the proposition is correct when K is simplicial.

When K is pointed but not simplicial, it is well known (see, e.g., [23, Lemma 4.6.1]) that one can triangulate K as a complex of simplicial subcones $K_1, \dots, K_t$ whose extreme rays are all among the extreme rays $u_1, \dots, u_N$ for K. This triangulation lets one express the characteristic function χ_K in the form (cf. [23, Lemma 4.6.4]) $\chi_K = \sum_j c_j \chi_{K_j}$ where the c_j are integers, and $c_j = +1$ whenever the cone K_j has the same dimension as K. Thus by Proposition 2.1(vi), one has

$$\mathrm{H}(K; \mathbf{X}) = \sum_j c_i \mathrm{H}(K_j; \mathbf{X}),$$

which shows that $h(K; \mathbf{x}) := (\prod_{i=1}^N \langle \mathbf{x}, u_i \rangle) \mathrm{H}(K; \mathbf{X})$ is analytic in $\mathbf{x}$. Furthermore, after clearing denominators, it gives the expansion

$$h(K; \mathbf{x}) = \sum_j c_j \left(\prod_{\substack{i : u_i \text{ a ray of } K, \\ \text{but not of } K_j}} \langle \mathbf{x}, u_i \rangle \right) h(K_j; \mathbf{x}).$$

Since the simplicial cones K_j have at most n extreme rays, this shows that $h_i(K; \mathbf{x}) = 0$ for $i < N - n$ and that

$$h_{N-n}(K; \mathbf{x}) = \sum_{j : \dim_{\mathbb{R}} K_j = n} \left(\prod_{\substack{i : u_i \text{ a ray of } K, \\ \text{but not of } K_j}} \langle \mathbf{x}, u_i \rangle \right) h_0(K_j; \mathbf{x}),$$

using the fact that $c_j = +1$ whenever $\dim_{\mathbb{R}} K_j = \dim_{\mathbb{R}} K$. Dividing through by $\prod_{i=1}^{N} \langle \mathbf{x}, u_i \rangle$ and multiplying by $(-1)^n$ give

$$(-1)^n \frac{h_{N-n}(K;\mathbf{x})}{\prod_{i=1}^{N} \langle \mathbf{x}, u_i \rangle} = \sum_{j:\dim_{\mathbb{R}} K_j = n} \mathrm{s}(K_j;\mathbf{x}) = \mathrm{s}(K;\mathbf{x}),$$

where the first equality uses the simplicial case already proven, and the last equality uses Proposition 2.1(v). □

The linear operator passing from the meromorphic function $\mathrm{H}(K;\mathbf{X})$ of $\mathbf{x}$ to the rational function $\mathrm{H}_{-n}(K;\mathbf{x}) = (-1)^n \mathrm{s}(K;\mathbf{x})$ has been called taking the *total residue* in [10], where other methods for computing it are also developed.

2.5 Complete intersections

For a pointed L-rational polyhedral cone K, one approach to computing $\mathrm{H}(K;\mathbf{x})$ (and hence $\mathrm{s}(K;\mathbf{x})$) is through an algebraic analysis of the affine semigroup $K \cap L$ and its affine semigroup ring

$$R := k[K \cap L] = k\{e^u\}_{u \in (K \cap L)}$$

over some coefficient field k. We discuss this here, with the case where R is a complete intersection being particularly simple.

For any semigroup elements $u_1, \dots, u_m$ in $K \cap L$, one can introduce a polynomial ring $S := k[U_1, \dots, U_m]$ and a ring homomorphism $S \to R$ sending $U_i \mapsto e^{u_i}$. This map makes R into an S-module. One also has a fine L-multigrading on R and S for which $\deg(U_i) = \deg(e^{u_i}) = u_i$. This makes R an L-graded module over the L-graded ring S. It is not hard to see that R is a *finitely-generated* S-module if and only if $\{u_1, \dots, u_m\}$ contain at least one vector spanning each extreme ray of K.

When $u_1, \dots, u_m$ generate (not necessarily minimally) the semigroup $K \cap L$, the map $S \to R$ is surjective, and its kernel I is often called the *toric ideal* for $u_1, \dots, u_m$.

Proposition 2.4 ([21, *Theorem* 7.3], [25, *Lemma* 4.1]) *One can generate the toric ideal $I = \ker(S \to R)$ by finitely many L-homogeneous elements chosen among the binomials $U^\alpha - U^\beta$ for which $\alpha, \beta \in \mathbb{N}^m$ and $\sum_{i=1}^m \alpha_i u_i = \sum_{j=1}^m \beta_j u_j$.*

As $R = S/I$, and because S has Krull dimension m while R has Krull dimension $d := \dim_{\mathbb{R}} K$, the number of generators for the ideal I is at least $m - d$. The theory of Cohen–Macaulay rings says that, since the polynomial algebra S is Cohen–Macaulay, whenever the ideal I in S can be generated by *exactly* $m - d$ elements $f_1, \dots, f_{m-d}$, then these elements must form an *S-regular sequence*: for each $i \geq 1$, the image of f_i forms a nonzero divisor in the quotient $S/(f_1, \dots, f_{i-1})$. In this case, the presentation $R = S/I = S/(f_1, \dots, f_{m-d})$ is said to present R as a *complete intersection*. A simple particular case of this occurs when the toric ideal I is principal, as in Example 2.6 and in Corollary 8.2. By a standard calculation using the nonzero divisor condition (see, e.g., [21, §13.4, p. 264]) one concludes the following factorization for $\mathrm{H}(K;\mathbf{X})$ and $\mathrm{s}(K;\mathbf{x})$.

Proposition 2.5 *Let K be a pointed L-rational cone for which the associated affine semigroup ring $R = k[K \cap L]$ can be presented as a complete intersection*

$$R = S/I = k[U_1, \dots, U_m]/(f_1, \dots, f_{m-d})$$

where $U_i = e^{u_i}$ for some generators $u_1, \dots, u_m$ of $K \cap L$, and where $f_1, \dots, f_{m-d}$ are L-homogeneous elements of S with degrees $\delta_1, \dots, \delta_{m-d}$. Then

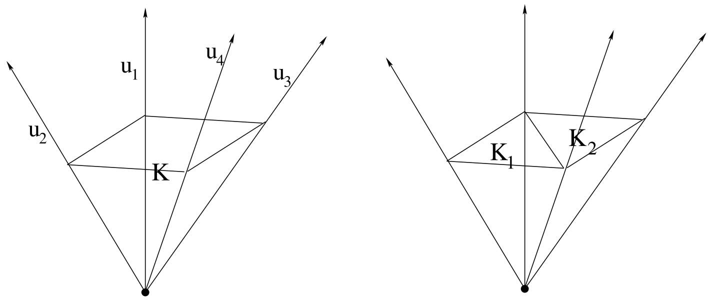
$$\mathrm{H}(K; \mathbf{X}) = \frac{\prod_{i=1}^{m-d}(1 - \mathbf{X}^{\delta_i})}{\prod_{j=1}^{m}(1 - \mathbf{X}^{u_j})},$$

and if $d = n$, then

$$\mathrm{s}(K; \mathbf{x}) = \frac{\prod_{i=1}^{m-n}\langle \mathbf{x}, \delta_i \rangle}{\prod_{j=1}^{m}\langle \mathbf{x}, u_j \rangle}.$$

Example 2.6 Let $V = \mathbb{R}^3$ with standard basis e_1, e_2, e_3, and let K be the full-dimensional, pointed cone whose extreme rays are generated by the four vectors

$$\begin{aligned} u_1 &= e_1, \\ u_2 &= e_1 + e_2, \\ u_3 &= e_1 + e_3, \\ u_4 &= e_1 + e_2 + e_3. \end{aligned}$$



Note that K is not simplicial, but it can be expressed as $K = K_1 \cup K_2$ where K_1, K_2 are the full-dimensional unimodular simplicial cones generated by the two bases for the lattice $L = \mathbb{Z}^3$ given by $\{u_1, u_2, u_4\}, \{u_1, u_3, u_4\}$, respectively. Their intersection $K_1 \cap K_2$ is the two-dimensional simplicial cone generated by $\{u_1, u_4\}$.

Therefore, applying properties (vi) and then (v) from Proposition 2.1, one can compute

$$\begin{aligned} \mathrm{s}(K; \mathbf{x}) &\overset{\text{(vi)}}{=} \mathrm{s}(K_1; \mathbf{x}) + \mathrm{s}(K_2; \mathbf{x}) \\ &\overset{\text{(v)}}{=} \frac{1}{x_1(x_1 + x_2)(x_1 + x_2 + x_3)} + \frac{1}{x_1(x_1 + x_3)(x_1 + x_2 + x_3)} \\ &= \frac{2x_1 + x_2 + x_3}{x_1(x_1 + x_2)(x_1 + x_3)(x_1 + x_2 + x_3)}. \end{aligned}$$

Alternatively, one could first compute H$(K,\mathbf{X})$ via Proposition 2.2(vi) and (v):

$$\begin{aligned}
\mathrm{H}(K;\mathbf{X}) &\overset{\text{(vi)}}{=} \mathrm{H}(K_1;\mathbf{X})+\mathrm{H}(K_2;\mathbf{X})-\mathrm{H}(K_1\cap K_2;\mathbf{X})\\
&\overset{\text{(v)}}{=} \frac{1}{(1-X_1)(1-X_1X_2)(1-X_1X_2X_3)}\\
&\quad+\frac{1}{(1-X_1)(1-X_1X_3)(1-X_1X_2X_3)}-\frac{1}{(1-X_1)(1-X_1X_2X_3)}\\
&= \frac{1-X_1^2X_2X_3}{(1-X_1)(1-X_1X_2)(1-X_1X_3)(1-X_1X_2X_3)}.
\end{aligned} \tag{2.4}$$

Then one could recover s$(K;\mathbf{x})$ by first making the exponential substitution (2.1), then expanding the analytic part H$(K;\mathbf{X})$ as a power series in $\mathbf{x}$, and using this to extract the homogeneous component $\mathrm{H}_{-3}(\mathbf{x})$ of degree $-3=-n$:

$$\begin{aligned}
&\mathrm{H}(K;\mathbf{X})\\
&= \frac{1-e^{2x_1+x_2+x_3}}{(1-e^{x_1})(1-e^{x_1+x_2})(1-e^{x_1+x_3})(1-e^{x_1+x_2+x_3})}\\
&= \frac{1}{x_1(x_1+x_2)(x_1+x_3)(x_1+x_2+x_3)}\\
&\quad\cdot\big(1-e^{2x_1+x_2+x_3}\big)\left(\frac{x_1}{1-e^{x_1}}\right)\left(\frac{x_1+x_2}{1-e^{x_1+x_2}}\right)\left(\frac{x_1+x_3}{1-e^{x_1+x_3}}\right)\left(\frac{x_1+x_2+x_3}{1-e^{x_1+x_2+x_3}}\right)\\
&= \frac{-(2x_1+x_2+x_3)+(\text{terms of degree at least 2})}{x_1(x_1+x_2)(x_1+x_3)(x_1+x_2+x_3)}\\
&\quad\cdot\big(1+o(x_1)\big)\big(1+o(x_1+x_2)\big)\big(1+o(x_1+x_3)\big)\big(1+o(x_1+x_2+x_3)\big)\\
&= (-1)^3\underbrace{\left(\frac{2x_1+x_2+x_3}{x_1(x_1+x_2)(x_1+x_3)(x_1+x_2+x_3)}\right)}_{\mathrm{s}(K;\mathbf{x})}+(\text{terms of degree at least }-2)
\end{aligned}$$

in agreement with our previous computation.

Alternatively, one can obtain H$(K;\mathbf{X})$ and s$(K;\mathbf{x})$ from Proposition 2.5, since we claim that $R=k[K\cap L]$ has this complete intersection presentation:

$$R\cong S/I=k[U_1,U_2,U_3,U_4]/(U_1U_4-U_2U_3).$$

To see this, start by observing that the map

$$\begin{aligned}
S=k[U_1,U_2,U_3,U_4] &\xrightarrow{\varphi} R\\
U_i &\longmapsto e^{u_i}
\end{aligned}$$

is surjective, since K was covered by the two unimodular cones K_1 and K_2. Note that there is a unique (up to scaling) linear dependence

$$u_1+u_4=u_2+u_3 \quad (=2e_1+e_2+e_3) \tag{2.5}$$

among $\{u_1, u_2, u_3, u_4\}$. Hence, $I = \ker\varphi$ contains the principal ideal $(U_1U_4 - U_2U_3)$. Furthermore, Proposition 2.4 implies that I is generated by binomials of the form $U^\alpha - U^\beta$ where $\sum_{i=1}^4 \alpha_i u_i = \sum_{j=1}^4 \beta_j u_j$. Due to the uniqueness of the dependence (2.5), one must have

$$\alpha_1 = \alpha_4 = \beta_2 = \beta_3 > 0 \quad \text{and} \quad \alpha_2 = \alpha_3 = \beta_1 = \beta_4 = 0.$$

Thus, $U^\alpha - U^\beta = (U_1U_4)^{\alpha_1} - (U_2U_3)^{\alpha_1}$, which lies in the ideal $(U_1U_4 - U_2U_3)$. Thus, $I = \ker\varphi = (U_1U_4 - U_2U_3)$.

3 Identifying Ψ_P and Φ_P

Recall from the introduction that for a poset P on $\{1, 2, \ldots, n\}$, we wish to associate two polyhedral cones. The first is

$$K_P^{\mathrm{wt}} := \left\{x \in \mathbb{R}_+^n : x_i \geq x_j \text{ for } i <_P j\right\}$$

inside the vector space $\mathbb{R}^n$ with standard basis $e_1, \ldots, e_n$ spanning the appropriate lattice $L^{\mathrm{wt}} = \mathbb{Z}^n$. The second is

$$K_P^{\mathrm{root}} = \mathbb{R}_+\{e_i - e_j : i <_P j\}$$

inside the codimension one subspace $V^{\mathrm{root}} \cong \mathbb{R}^{n-1}$ of $\mathbb{R}^n$ where the sum of coordinates $x_1 + \cdots + x_n = 0$. We consider this subspace to have Lebesgue measure normalized to make the basis $\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n\}$ for the appropriate lattice $L^{\mathrm{root}} \cong \mathbb{Z}^{n-1}$ span a parallelepiped of volume 1.

Proposition 3.1 *For any poset P on $\{1, 2, \ldots, n\}$, one has*

$$\Psi_P(\mathbf{x}) := \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)}\right) = s\left(K_P^{\mathrm{root}}; \mathbf{x}\right),$$

$$\Phi_P(\mathbf{x}) := \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{x_1(x_1 + x_2)(x_1 + x_2 + x_3)\cdots(x_1 + \cdots + x_n)}\right) = s\left(K_P^{\mathrm{wt}}; \mathbf{x}\right).$$

Proof (Cf. Gessel [14, proof of Theorem 1]) Proceed by induction on the number of pairs $\{i, j\}$ in $[n]$ that are *incomparable* in P. In the base case where there are no such pairs, P is a linear order, of the form P_w for some w in $\mathfrak{S}_n$, with $\mathcal{L}(P_w) = \{w\}$, and the cones $K_{P_w}^{\mathrm{wt}}, K_{P_w}^{\mathrm{root}}$ are simplicial and unimodular, having extreme rays spanned by, respectively,

$$(e_{w(1)} - e_{w(2)}, e_{w(2)} - e_{w(3)}, \ldots, e_{w(n-1)} - e_{w(n)}) \quad \text{and}$$
$$(e_{w(1)}, e_{w(1)} + e_{w(2)}, \ldots, e_{w(1)} + e_{w(2)} + \cdots + e_{w(n)}).$$

Thus, Proposition 2.1(v) gives the desired equalities in this case.

In the inductive step, if i, j are incomparable in P, then either order relation $i < j$ or the reverse $j < i$ may be added to P (followed by taking the transitive closure), to obtain two posets $P_{i<j}, P_{j<i}$. Note that

$$\mathcal{L}(P) = \mathcal{L}(P_{i<j}) \sqcup \mathcal{L}(P_{j<i}),$$

and hence,

$$\begin{aligned} \Psi_P(\mathbf{x}) &= \Psi_{P_{i<j}}(\mathbf{x}) + \Psi_{P_{j<i}}(\mathbf{x}), \\ \Phi_P(\mathbf{x}) &= \Phi_{P_{i<j}}(\mathbf{x}) + \Phi_{P_{j<i}}(\mathbf{x}). \end{aligned} \tag{3.1}$$

It only remains to show that $\mathrm{s}(K_P^{\mathrm{root}}; \mathbf{x})$ and $\mathrm{s}(K_P^{\mathrm{wt}}; \mathbf{x})$ satisfy this same recurrence. If one introduces into the binary relation P *both* relations $i \leq j$ and $j \leq i$ before taking the transitive closure, then one obtains a *quasiorder* or *preorder* that we denote $P_{i=j}$. It is natural to also introduce the (non-full-dimensional) cone $K_{P_{i=j}}^{\mathrm{wt}}$ lying inside the hyperplane where $x_i = x_j$, and the (non-pointed) cone $K_{P_{i=j}}^{\mathrm{root}}$ containing the line $\mathbb{R}(e_i - e_j)$. One then has these decompositions

$$\begin{aligned} K_P^{\mathrm{wt}} &= K_{P_{i<j}}^{\mathrm{wt}} \cup K_{P_{j<i}}^{\mathrm{wt}} \quad \text{with } K_{P_{i<j}}^{\mathrm{wt}} \cap K_{P_{j<i}}^{\mathrm{wt}} = K_{P_{i=j}}^{\mathrm{wt}}, \\ K_{P_{i=j}}^{\mathrm{root}} &= K_{P_{i<j}}^{\mathrm{root}} \cup K_{P_{j<i}}^{\mathrm{root}} \quad \text{with } K_{P_{i<j}}^{\mathrm{root}} \cap K_{P_{j<i}}^{\mathrm{root}} = K_P^{\mathrm{root}} \end{aligned}$$

leading to these relations among characteristic functions of cones:

$$\begin{aligned} \chi_{K_P^{\mathrm{wt}}} + \chi_{K_{P_{i=j}}^{\mathrm{wt}}} &= \chi_{K_{P_{i<j}}^{\mathrm{wt}}} + \chi_{K_{P_{j<i}}^{\mathrm{wt}}}, \\ \chi_{K_P^{\mathrm{root}}} + \chi_{K_{P_{i=j}}^{\mathrm{root}}} &= \chi_{K_{P_{i<j}}^{\mathrm{root}}} + \chi_{K_{P_{j<i}}^{\mathrm{root}}}. \end{aligned} \tag{3.2}$$

From this one concludes using Proposition 2.1(vi) that

$$\begin{aligned} \mathrm{s}\big(K_P^{\mathrm{wt}}; \mathbf{x}\big) &= \mathrm{s}\big(K_{P_{i<j}}^{\mathrm{wt}}; \mathbf{x}\big) + \mathrm{s}\big(K_{P_{j<i}}^{\mathrm{wt}}; \mathbf{x}\big), \\ \mathrm{s}\big(K_P^{\mathrm{root}}; \mathbf{x}\big) &= \mathrm{s}\big(K_{P_{i<j}}^{\mathrm{root}}; \mathbf{x}\big) + \mathrm{s}\big(K_{P_{j<i}}^{\mathrm{root}}; \mathbf{x}\big) \end{aligned}$$

since Proposition 2.1(i) implies $\mathrm{s}(K_{P_{i=j}}^{\mathrm{wt}}; \mathbf{x}) = \mathrm{s}(K_{P_{i=j}}^{\mathrm{root}}; \mathbf{x}) = 0$. Comparing with (3.1), the result follows by induction. □

Remark 3.2 The parallel between the relations in (3.2) is *not* a coincidence. It reflects a general duality [2, Corollary 2.8] relating identities among characteristic functions of cones K_i and their polar dual cones K_i^*:

$$\sum_i c_i \chi_{K_i} = 0 \quad \text{if and only if} \quad \sum_i c_i \chi_{K_i^*} = 0. \tag{3.3}$$

While it is not true that the cones K_P^{wt} and K_P^{root} are polar dual to each other, this is *almost* true, as we now explain.

The dual space to the hyperplane $x_1 + \cdots + x_n = 0$, which is the ambient space for K_P^{root} is the quotient space $\mathbb{R}^n/\ell$ where ℓ is the line $\mathbb{R}(e_1 + \cdots + e_n)$. Thus, identities among characteristic functions of cones K_P^{root} give rise via (3.3) to identities among the characteristic functions of their dual cones $(K_P^{\mathrm{root}})^*$ inside this quotient space. The cone K_P^{wt} maps via the quotient mapping $\mathbb{R}^n \to \mathbb{R}^n/\ell$ to the dual cone $(K_P^{\mathrm{root}})^*$. Moreover, one can check that the intersection $K_P^{\mathrm{wt}} \cap \ell$ is exactly the half-line/ray

$$\ell^+ := \mathbb{R}_+(e_1 + \cdots + e_n).$$

Therefore, identities among characteristic functions of the cones $(K_P^{\mathrm{root}})^*$ "lift" to the same identity among characteristic functions of the cones K_P^{wt}.

We are still lying slightly here, since just as in (3.2), one must not only consider the cones $K_P^{\mathrm{wt}}, K_P^{\mathrm{root}}$ for posets on $\{1, 2, \ldots, n\}$, but also for preposets. See [22, §3.3] for more on this preposet-cone dictionary for the cones K_P^{wt}.

We remark also that this duality is the source of our terminology $K^{\mathrm{root}}, K^{\mathrm{wt}}$ for these cones, as the hyperplane $x_1 + \cdots + x_n = 0$ is the ambient space for the *root lattice* of type A_{n-1}, while the dual space $\mathbb{R}^n/\ell$ is the ambient space for its dual lattice, the *weight lattice* of type A_{n-1}.

4 Application: skew diagram posets and Theorem B

Recall from the introduction that to a *skew (Ferrers) diagrams* $D = \lambda/\mu$, thought of as a collection of points (i, j) in the plane occupying rows $1, 2, \ldots, r$ numbered top to bottom, and columns $1, 2, \ldots, c$ numbered right to left, we associate a bipartite poset P_D on the set $\{x_1, \ldots, x_r, y_1, \ldots, y_c\}$ having an order relation $x_i <_{P_D} y_j$ whenever (i, j) is a point of D.

We wish to prove Theorem B from the introduction, evaluating $\Psi_{P_D}(\mathbf{x})$ for every skew diagram D. Without loss of generality, we will assume for the remainder of this section that the skew diagram D is connected in the sense that its poset P_D is connected; otherwise both sides of Theorem B vanish (for the left side, via Corollary 5.2, and for the right side because the sum is empty).

We exhibit a known triangulation for the cone $K_{P_D}^{\mathrm{root}}$. The cone $K_{P_D}^{\mathrm{root}}$ lives in the codimension one subspace V^{root} of the product space $\mathbb{R}^{r+c} = \mathbb{R}^r \times \mathbb{R}^c$ with standard basis vectors $e_1, \ldots, e_r$ and $f_1, \ldots, f_c$, and dual coordinates $x_1, \ldots, x_r$ and $y_1, \ldots, y_c$. Here $K_{P_D}^{\mathrm{root}}$ is the nonnegative span of the vectors $\{e_i - f_j : (i, j) \in D\}$. Note that each of these vectors lies in the following affine hyperplane H of V^{root}:

$$H := \bigl\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^r \times \mathbb{R}^c : x_1 + \cdots + x_r = 1 \text{ and } y_1 + \cdots + y_c = -1\bigr\}. \tag{4.1}$$

Thus, it suffices to triangulate the polytope $\mathcal{P}_D$, which is the convex hull of these vectors inside this affine hyperplane H.

Consider the skew diagram D as the componentwise partial order on its elements (i, j). One finds that D is a *distributive lattice*, in which the meet $\wedge$ and join $\vee$ of two elements $(i, j), (i', j')$ are their componentwise minimums and maximums:

$$(i, j) \wedge (i', j') = (\min(i, i'), \min(j, j')),$$
$$(i, j) \vee (i', j') = (\max(i, i'), \max(j, j')).$$

Consequently, by Birkhoff's theorem on the structure of finite distributive lattices [23, Theorem 3.4.1], the lattice D is isomorphic to the lattice of order ideals for the subposet $\mathrm{Irr}(D)$ of *join-irreducible* elements of D.

For any finite poset Q, Stanley [24] and before that, Geissinger [13] considered a convex polytope called the *order polytope* of $\mathcal{O}(Q)$, which one can think of as the convex hull within $\mathbb{R}^Q$ of the characteristic vectors of order ideals of Q; see [24, Corollary 1.3].

Proposition 4.1 *The convex hull $\mathcal{P}_D$ of the vectors $\{e_i - f_j : (i,j) \in D\}$ is affinely isomorphic to the order polytope $\mathcal{O}(\mathrm{Irr}(D))$ for the poset $\mathrm{Irr}(D)$.*

Proof Identify the join-irreducibles (i,j) in $\mathrm{Irr}(D)$ with basis vectors

$$\epsilon_1, \dots, \epsilon_{r-1}, \phi_1, \dots, \phi_{c-1}$$

in $\mathbb{R}^{r-1} \times \mathbb{R}^{c-1}$ as follows:

- if (i,j) covers $(i-1,j)$, identify (i,j) with ϵ_{i-1},
- if (i,j) covers $(i,j-1)$, identify (i,j) with ϕ_{j-1}.

One can then check that a general element (i,j) of D corresponds to an order ideal in $\mathrm{Irr}(D)$ whose elements are identified with $\{\epsilon_1, \dots, \epsilon_{i-1}, \phi_1, \dots, \phi_{j-1}\}$. Thus, the order polytope $\mathcal{O}(\mathrm{Irr}(D))$ is simply the convex hull of vectors

$$\big\{\epsilon_1 + \cdots + \epsilon_{i-1} + \phi_1 + \cdots + \phi_{j-1} : (i,j) \in D\big\}.$$

The linear morphism

$$\psi: \quad \begin{aligned} \mathbb{R}^r \times \mathbb{R}^c &\longrightarrow \mathbb{R}^{r-1} \times \mathbb{R}^{c-1}, \\ e_i &\longmapsto \epsilon_1 + \cdots + \epsilon_{i-1}, \\ f_j &\longmapsto \phi_1 + \cdots + \phi_{j-1}, \end{aligned}$$

restricts to an affine isomorphism $H \to \mathbb{R}^{r-1} \times \mathbb{R}^{c-1}$ sending $e_i - f_j$ to

$$\epsilon_1 + \cdots + \epsilon_{i-1} + \phi_1 + \cdots + \phi_{j-1}.$$

Therefore, ψ restricts further to an isomorphism between $\mathcal{P}_D$ and $\mathcal{O}(\mathrm{Irr}(D))$. □

Corollary 4.2 *For any skew diagram D, the cone $K_{P_D}^{\mathrm{root}}$ has a triangulation into unimodular cones K_π indexed by lattice paths π from $(1,1)$ to (r,c). Furthermore, the extreme rays of K_π are spanned by the vectors $\{e_i - f_j\}_{(i,j)\in\pi}$.*

Consequently, as asserted in Theorem B, one has

$$\Psi_{P_D}(\mathbf{x}) = \sum_\pi \frac{1}{\prod_{(i,j)\in\pi}(x_i - y_j)} = \frac{\sum_\pi \prod_{(i,j)\in D\setminus\pi}(x_i - y_j)}{\prod_{(i,j)\in D}(x_i - y_j)}.$$

In particular, when D is the Ferrers diagram D of a partition λ, one has

$$\Psi_{P_D}(\mathbf{x}) = \frac{\mathfrak{S}_{\hat{w}}(\mathbf{x},\mathbf{y})}{\mathfrak{S}_w(\mathbf{x},\mathbf{y})}$$

where $\mathfrak{S}_w(\mathbf{x},\mathbf{y})$, $\mathfrak{S}_{\hat{w}}(\mathbf{x},\mathbf{y})$ *are the double Schubert polynomials for the dominant permutation* w *having Lehmer code* $\lambda = (\lambda_1, \ldots, \lambda_r)$, *and the vexillary permutation* $\hat{w}$ *having Lehmer code* $\hat{\lambda} := (0, \lambda_2 - 1, \ldots, \lambda_r - 1)$.

Proof Stanley [24, §5] describes a triangulation of the order polytope $\mathcal{O}(Q)$ whose maximal simplices correspond to linear extensions π of Q or to maximal chains π in the distributive lattice of order ideals $J(Q)$. For $Q = \mathrm{Irr}(D)$, so that $J(Q) = D$, these linear extensions π correspond to lattice paths from $(1,1)$ to (r,c) in the diagram D. Here the vertices spanning the maximal simplex in the triangulation corresponding to π are the characteristic vectors of the order ideals on the chain π.

Thus, one obtains a corresponding triangulation for the polytope, which is the intersection of $K_{P_D}^{\mathrm{root}}$ with the affine hyperplane in (4.1), in which the vertices of the maximal simplex corresponding to π are $\{e_i - f_j : (i,j) \in \pi\}$. Looking instead at the positive cone $K_\pi := \{e_i - f_j : (i,j) \in \pi\}$ spanned by these vectors therefore gives a triangulation of the cone $K_{P_D}^{\mathrm{root}}$.

The cones K_π are unimodular: one can easily check, via induction on $r+c$, that for any lattice path π from $(1,1)$ to (r,c), the $\mathbb{Z}$-linear span of the vectors $\{e_i - f_j\}_{(i,j)\in\pi}$ contains *all* vectors of the form

$$\begin{aligned} &e_i - e_j \quad \text{for } 1 \le i \ne j \le r,\\ &f_i - f_j \quad \text{for } 1 \le i \ne j \le c,\\ &e_i - f_j \quad \text{for } 1 \le i \le r \text{ and } 1 \le j \le c. \end{aligned}$$

Therefore by Proposition 3.1 and Proposition 2.1(vi), one has

$$\begin{aligned} \Psi_{P_D} &= \mathrm{s}\bigl(K_{P_D}^{\mathrm{root}}; \mathbf{x}\bigr) = \sum_\pi \mathrm{s}(K_\pi; \mathbf{x}) \\ &= \sum_\pi \frac{1}{\prod_{(i,j)\in\pi}(x_i - y_j)} = \frac{\sum_\pi \prod_{(i,j)\in D\setminus\pi}(x_i - y_j)}{\prod_{(i,j)\in D}(x_i - y_j)}. \end{aligned}$$

When D is the Ferrers diagram of a partition λ, this denominator product $\prod_{(i,j)\in D}(x_i - y_j)$ is the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x},\mathbf{y})$ for the dominant permutation w that has Lehmer code λ; see, e.g., [18, §9.4], [19, (6.14)], or one can argue similarly to the argument for the numerator sum given in the next paragraph.

There are various ways to identify the numerator sum $\sum_\pi \prod_{(i,j)\in D\setminus\pi}(x_i - y_j)$ as $\mathfrak{S}_{\hat{w}}(\mathbf{x},\mathbf{y})$. One way is to check that each lattice path π in D gives rise as follows to a *reduced pipe dream* for $\hat{w}$ in the terminology of Knutson and Miller [17] (see also [21, §16.1]): the $+$'s occur with the (row, column) indices (i,j) given by the lattice points *not* visited by π. Thus, the numerator sum is the expansion of $\mathfrak{S}_{\hat{w}}(\mathbf{x},\mathbf{y})$ as a sum over reduced pipe dreams for $\hat{w}$; see Fomin and Kirillov [12, Proposition 6.2], or Miller and Sturmfels [21, Corollary 16.30]. □

Example 4.3 Consider the skew diagram

$$D = (4,4,2)/(1,1,0) = \begin{matrix} \cdot & \bullet & \bullet & \bullet \\ \cdot & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \end{matrix}$$

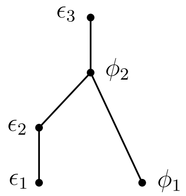
whose rows and columns we index as follows:

$$\begin{array}{ccccc} & y_4 & y_3 & y_2 & y_1 \\ x_1 & \cdot & (1,3) & (1,2) & (1,1) \\ x_2 & \cdot & (2,3) & (2,2) & (2,1) \\ x_3 & (3,4) & (3,3) & & \end{array}$$

Thinking of D as a distributive lattice via the componentwise order on the labels (i,j), one can label its five join-irreducibles $\mathrm{Irr}(D)$ by the basis vectors $\epsilon_1, \epsilon_2, \epsilon_3, \phi_1, \phi_2$ as in the above proof:

$$\begin{array}{cccc} \cdot & \epsilon_2 & \epsilon_1 & \bullet \\ \cdot & \bullet & \bullet & \phi_1 \\ \epsilon_3 & \phi_2 & & \end{array}$$

The poset Q of join-irreducible elements of D has the following Hasse diagram:



In this way, the elements of D correspond to the order ideals of Q and to the vertices of the order polytope $\mathcal{O}(\mathrm{Irr}(D))$ as follows:

$$\begin{array}{cccc} \cdot & \epsilon_1+\epsilon_2 & \epsilon_1 & 0 \\ \cdot & \begin{array}{c}\epsilon_1+\epsilon_2\\+\phi_1\end{array} & \begin{array}{c}\epsilon_1\\+\phi_1\end{array} & \phi_1 \\ \begin{array}{c}\epsilon_1+\epsilon_2+\epsilon_3\\+\phi_1+\phi_2\end{array} & \begin{array}{c}\epsilon_1+\epsilon_2\\+\phi_1+\phi_2\end{array} & & \end{array}$$

There are three paths π from $(1,1)$ to $(r,c)=(4,3)$, giving rise to the three terms in $\Psi_{P_D}(\mathbf{x})$:

$$\begin{array}{cccc} \cdot & (1,3) & (1,2) & (1,1) \\ \cdot & (2,3) & & \\ (3,4) & (3,3) & & \end{array} \qquad \frac{1}{(x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_3)(x_3-y_3)(x_3-y_4)}$$

$$\begin{array}{cccc} \cdot & \cdot & (1,2) & (1,1) \\ \cdot & (2,3) & (2,2) & \\ (3,4) & (3,3) & & \end{array} \qquad + \frac{1}{(x_1-y_1)(x_1-y_2)(x_2-y_2)(x_2-y_3)(x_3-y_3)(x_3-y_4)}$$

$$\begin{array}{cccc} \cdot & \cdot & \cdot & (1,1) \\ \cdot & (2,3) & (2,2) & (2,1) \\ (3,4) & (3,3) & & \end{array} + \frac{1}{(x_1-y_1)(x_2-y_1)(x_2-y_2)(x_2-y_3)(x_3-y_3)(x_3-y_4)}$$

5 Extreme rays and Theorem C

Our goal here is to identify the extreme rays of the cones $K_P^{\mathrm{wt}}, K_P^{\mathrm{root}}$. Once achieved, this gives the denominators of $\Psi_P(\mathbf{x}), \Phi_P(\mathbf{x})$ and allows one to decide when the cones are simplicial, leading to Theorem C.

Recall that an *order ideal* of a poset P is a subset J of its elements such that, for any pair i, j of comparable elements ($i \leq_P j$), if $j \in J$, then $i \in J$.

Proposition 5.1 *Let P be a poset on $\{1, 2, \ldots, n\}$.*

(i) *The cone K_P^{root} has extreme rays spanned by $\{e_i - e_j\}_{i \lessdot_P j}$.*
(ii) *The cone K_P^{wt} has extreme rays spanned by the characteristic vectors*

$$e_J := \chi_J = \sum_{j \in J} e_j$$

for the connected nonempty order ideals J in P.

Proof For (i), note that K_P^{root} is the cone nonnegatively spanned by $\{e_i - e_j : i <_P j\}$, and since $i <_P j <_P k$ implies

$$e_i - e_k = (e_i - e_j) + (e_j - e_k) \in \mathbb{R}_+\{e_i - e_j, e_j - k\},$$

its extreme rays must be spanned by some subset of $\{e_i - e_j : i \lessdot_P j\}$. On the other hand, for each covering relation $i \lessdot_P j$, one can exhibit a linear functional f that vanishes on $e_i - e_j$ and is strictly negative on the rest of the vectors spanning K_P^{root} as follows. Choose a linear extension $w = (w(1), \ldots, w(n))$ in $\mathcal{L}(P)$ such that i, j appear adjacent in the linear order, say $w(k) = i$ and $w(k+1) = j$, and define the functional $f : \mathbb{R}^n \to \mathbb{R}$ by the values

$$\begin{aligned} f(e_{w(m)}) &= m && \text{for } m = 1, 2, \ldots, k-1; \\ f(e_{w(k)}) = f(e_i) = k &= f(e_j) = f(e_{w(k+1)}); && \\ f(e_{w(m)}) &= m - 1 && \text{for } m = k+2, k+3, \ldots, n. \end{aligned}$$

For (ii), note that K_P^{wt} is described by the system of inequalities

$$\begin{cases} x_i \geq 0 & \text{for all } i; \\ x_i \geq x_j & \text{for } i <_P j. \end{cases}$$

We first claim that K_P^{wt} is the nonnegative span of characteristic vectors e_J for order ideals J of P: if $x = (x_1, \ldots, x_n)$ lies in K_P^{wt}, and its coordinates x_i take on the distinct positive values $c_1 < c_2 < \cdots < c_t$, then (setting $c_0 := 0$), one has

$$x = \sum_{r=1}^{t} (c_r - c_{r-1}) e_{J_r}$$

where J_r is the order ideal of P defined by

$$J_r := \{ j \in \{1, 2, \ldots, n\} : x_j \geq c_r \}.$$

Furthermore, if an order ideal J of P decomposes into connected components as $J = \sqcup_i J^{(i)}$, then each $J^{(i)}$ is itself a (connected) order ideal, and $e_J = \sum_i e_{J^{(i)}}$.

Therefore, the extreme rays of the cone must be spanned by some subset of the vectors e_J for connected order ideals J. On the other hand, for any connected order ideal J, one can exhibit the line $\mathbb{R}e_J$ spanned by e_J as the intersection of $n-1$ linearly independent hyperplanes that come from inequalities valid on K_P^{wt} as follows. Consider the Hasse diagram for J as a connected graph and pick a spanning tree T among its edges. Then the line $\mathbb{R}e_J$ is the set of solutions to the system

$$\begin{cases} x_i = 0 & \text{for } i \notin J; \\ x_i = x_j & \text{for } i \lessdot_P j \text{ or } i \gtrdot_P j \text{ with } \{i, j\} \in T. \end{cases}$$ □

Proposition 2.1 then immediately implies the following:

Corollary 5.2 *Let P be a poset on $\{1, 2, \ldots, n\}$.*

(i) *If P is disconnected, then the cone K_P^{root} is not full-dimensional, and $\Psi_P(\mathbf{x}) = 0$. If P is connected, the cone K_P^{root} is full-dimensional, and the smallest denominator for $\Psi_P(\mathbf{x})$ is $\prod_{i \lessdot_P j}(x_i - x_j)$.*

(ii) *The cone K_P^{wt} is always full-dimensional, and the smallest denominator for $\Phi_P(\mathbf{x})$ is $\prod_J(\sum_{j \in J} x_j)$ where the product runs over all connected order ideals J in P.*

Theorem C is now simply a consequence of the analysis of the simplicial cases.

Corollary 5.3 *The cone K_P^{root} is simplicial if and only if the Hasse diagram for P contains no cycles. In this case it is also unimodular. Hence, the Hasse diagram for P is a spanning tree on $\{1, 2, \ldots, n\}$ if and only if*

$$\Psi_P(\mathbf{x}) = \frac{1}{\prod_{i \lessdot_P j}(x_i - x_j)}.$$

The cone K_P^{wt} is simplicial if and only if P is a forest in the sense that every element is covered by at most one other element. In this case it is also unimodular. Hence, P is a forest if and only if

$$\Phi_P(\mathbf{x}) = \frac{1}{\prod_{i=1}^{n}(\sum_{j \le_P i} x_j)}.$$

Proof According to Proposition 5.1, the extreme rays of the cone K_P^{root} are the vectors $\{e_i - e_j : i \lessdot_P j\}$, which are linearly independent if and only if there are no cycles in the Hasse diagram for P. Furthermore, when there are no such cycles, an easy leaf induction shows that the cone is unimodular. The rest of the assertions follow.

To analyze K_P^{wt}, first note that when P is a forest, the connected order ideals of P are exactly the principal order ideals $P_{\le i} := \{j : j \le_P i\}$ for $i = 1, 2, \ldots, n$. Not only are their characteristic vectors $e_{P_{\le i}}$ linearly independent, but if one orders the labels i

according to any linear extension of P, one finds that these vectors $e_{P_{\leq i}}$ form the columns of a unitriangular matrix, which is therefore unimodular.

When P is not a forest, it remains to show that the cone K_P^{wt} cannot be simplicial. There must exist two elements i, j incomparable in P whose principal order ideals have nonempty intersection $P_{\leq i} \cap P_{\leq j}$. Decompose $P_{\leq i} \cap P_{\leq j} = \sqcup_{\ell=1}^{t} J^{(\ell)}$ into its connected components $J^{(\ell)}$. Then each of these components $J^{(\ell)}$ will be a nonempty connected ideal, as will be $P_{\leq i}, P_{\leq j}$ and their union $P_{\leq i} \cup P_{\leq j}$. This leads to the following linear relation:

$$e_{P_{\leq i}} + e_{P_{\leq j}} = e_{P_{\leq i} \cup P_{\leq j}} + \sum_{i=1}^{t} e_{J^{(\ell)}}.$$

Since Proposition 5.1 implies the vectors involved in this relation all span extreme rays of the cone K_P^{wt}, the cone is not simplicial in this case. □

An interesting special case of the preceding result leads to a special role played by *dominant* or 132-*avoiding* permutations when considering posets of order dimension two, that is, the subposets of the componentwise order on $\mathbb{R}^2$. Björner and Wachs [5, Theorems 6.8, 6.9] showed that P has order dimension two if and only if one can relabel the elements i in $[n]$ so that $\mathcal{L}(P)$ forms a principal order ideal $[e, w]$ in the *weak Bruhat order* on $\mathfrak{S}_n$.

Corollary 5.4 *When $\mathcal{L}(P) = [e, w]$ for some permutation w, the cone K_P^{wt} is simplicial if and only if w is* 132*-avoiding.*

Proof When $\mathcal{L}(P) = [e, w]$, one can check that P has the following order relations: $i <_P j$ exactly when $i <_{\mathbb{Z}} j$ and (i, j) are *noninversion* values for w, that is, $w^{-1}(i) < w^{-1}(j)$, or i appears earlier than j in the list notation $(w(1), \ldots, w(n))$.

By Corollary 5.3, the cone P is not simplicial if and only if P is not a forest, that is, if and only if there exist i, j which are incomparable in P and have a common lower bound $h <_P i, j$. Hence, by the previous paragraph, one must have $h <_{\mathbb{Z}} i$ and $h <_{\mathbb{Z}} j$, with h appearing earlier than both i, j in the list notation for w. Without loss of generality, $i <_{\mathbb{Z}} j$ by reindexing, and then the incomparability of i, j in P forces j to appear earlier than i in the list notation. That is, $h <_{\mathbb{Z}} i <_{\mathbb{Z}} j$ occur in the order (h, j, i) within w, forming an occurrence of the pattern $(1, 3, 2)$. □

Example 5.5 Among the permutations w in $\mathfrak{S}_3$, five out of the six are dominant or 132-avoiding; only $w = (1, 3, 2)$ is not. It has $[e, w] = \mathcal{L}(P) = \{(1, 2, 3), (1, 3, 2)\}$, and K_P^{wt} is the nonsimplicial cone considered in Example 2.6, having extreme rays spanned by $\{e_1, e_1 + e_2, e_1 + e_3, e_1 + e_2 + e_3\}$, and

$$\mathrm{s}(K; \mathbf{x}) = \frac{2x_1 + x_2 + x_3}{x_1(x_1 + x_2)(x_1 + x_3)(x_1 + x_2 + x_3)}.$$

6 P-partitions, forests, and the Hilbert series for K_P^{wt}

We digress here to discuss the Hilbert series for the affine semigroup $K \cap L$ for the cone $K = K_P^{\mathrm{wt}}$ inside the lattice $L = L^{\mathrm{wt}}$. Analyzing this when P is a forest leads to a common generalization of both Theorem C and the "maj" hook formula for forests of Björner and Wachs.

One can think of $K \cap L$ as the semigroup of *weak P-partitions* in the sense of Stanley [23, §4.5], namely functions $f : P \to \mathbb{N}$ which are order-reversing: $f(i) \geq f(j)$ for $i <_P j$. Within this semigroup $K \cap L$, Stanley also considers the semigroup ideal $\mathcal{A}(P)$ of *P-partitions* (in the strong sense), that is, those order-reversing functions $f : P \to \mathbb{N}$ which in addition satisfy the strict inequality $f(i) > f(j)$ whenever (i, j) is in the *descent set*

$$\mathrm{Des}(P) := \bigl\{(i, j) : i \lessdot_P j \text{ and } i >_{\mathbb{Z}} j\bigr\}.$$

The main lemma of P-partition theory [23, Theorem 7.19.4] asserts the disjoint decomposition[2]

$$\mathcal{A}(P) = \bigsqcup_{w \in \mathcal{L}(P)} \mathcal{A}(w).$$

Equivalently, in terms of the Hilbert series of the semigroup ideal $\mathcal{A}(P)$ defined by

$$\mathrm{H}\bigl(\mathcal{A}(P); \mathbf{X}\bigr) := \sum_{f \in \mathcal{A}(P)} \mathbf{X}^f$$

where $\mathbf{X}^f := \prod_{i=1}^n X_i^{f(i)}$, this says that

$$\mathrm{H}\bigl(\mathcal{A}(P); \mathbf{X}\bigr) = \sum_{w \in \mathcal{L}(P)} \mathrm{H}\bigl(\mathcal{A}(P_w), \mathbf{X}\bigr). \tag{6.1}$$

This simple equation is more powerful than it looks at first glance. Define the notation $\mathbf{X}^A := \prod_{j \in A} X_j$ for subsets $A \subset \{1, 2, \dots, n\}$.

Proposition 6.1 *For any forest poset P on $\{1, 2, \dots, n\}$, one has*

$$\mathrm{H}\bigl(\mathcal{A}(P); \mathbf{X}\bigr) = \frac{\prod_{(i,j) \in \mathrm{Des}(P)} \mathbf{X}^{P_{\leq i}}}{\prod_{i=1}^n (1 - \mathbf{X}^{P_{\leq i}})}. \tag{6.2}$$

In particular, (6.1) *becomes*

$$\frac{\prod_{i \in \mathrm{Des}(P)} \mathbf{X}^{P_{\leq i}}}{\prod_{i=1}^n (1 - \mathbf{X}^{P_{\leq i}})} = \sum_{w \in \mathcal{L}(P)} \frac{\prod_{i : w_i > w_{i+1}} \mathbf{X}^{\{w_1, w_2, \dots, w_i\}}}{\prod_{i=1}^n (1 - \mathbf{X}^{\{w_1, w_2, \dots, w_i\}})}. \tag{6.3}$$

[2] This disjoint decomposition is closely related to the triangulation of K_P^{wt} that appeared implicitly in the proof of Proposition 3.1, modeled on Gessel's proof of the main P-partition lemma in [14, Theorem 1]).

Proof When P is a forest, we claim that $\mathcal{A}(P)$ is actually a *principal ideal* within $K \cap L$, generated by the P-partition f_0 for which $f_0(i)$ is the number of descent edges encountered along the unique path in the Hasse diagram from i to a maximal element of P. Alternatively f_0 is the sum of characteristic functions of the subtrees $P_{\leq i}$ for which one has (i, j) in $\mathrm{Des}(P)$ (here j is the unique element covering i in P). In other words, $\mathcal{A}(P) = f_0 + K \cap L$, and consequently,

$$\mathrm{H}\big(\mathcal{A}(P); \mathbf{X}\big) = \mathbf{X}^{f_0} \cdot \mathrm{H}(K; \mathbf{X}) = \left(\prod_{(i,j)\in \mathrm{Des}(P)} \mathbf{X}^{P_{\leq i}}\right) \cdot \mathrm{H}(K; \mathbf{X}).$$

But then Corollary 5.3 implies that $K \cap L$ is a unimodular cone having extreme rays spanned by the characteristic vectors of the subtrees $P_{\leq i}$, and hence,

$$\mathrm{H}(K; \mathbf{X}) = \prod_{i=1}^{n} \big(1 - \mathbf{X}^{P_{\leq i}}\big). \tag{6.4}$$

The rest follows from the observation that when one considers a permutation w as a linearly ordered poset P_w having $w(1) <_{P_w} \cdots <_{P_w} w(n)$, it is an example of a forest, in which $P_{\leq i} = \{w(1), w(2), \ldots, w(i)\}$. □

This has two interesting corollaries. The first is that by applying the total residue operator discussed in Sect. 2.4 to (6.4), one obtains a second derivation of Theorem C.

The second is that by setting $X_j = q$ for all j in (6.3), one immediately deduces the major index q-hook formula for forests of Björner and Wachs [5, Theorem 1.2]:

Corollary 6.2 *When P is a forest*,

$$\sum_{w\in \mathcal{L}(P)} q^{\mathrm{maj}(w)} = q^{\mathrm{maj}(P)} \frac{[n]!_q}{\prod_{i=1}^{n} [h(i)]_q}$$

where

$$\begin{aligned}
\mathrm{maj}(P) &:= \sum_{(i,j)\in \mathrm{Des}(P)} |P_{\leq i}|,\\
h(i) &:= |P_{\leq i}|,\\
[n]_q &:= \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1},\\
[n]!_q &:= [n]_q [n-1]_q \cdots [2]_q [1]_q.
\end{aligned}$$

7 Generators for the affine semigroups

The two families of cones $K_P^{\mathrm{root}}, K_P^{\mathrm{wt}}$ share a pleasant property: the generating sets for their affine semigroups are as small as possible. This will be used in Sect. 8.

Proposition 7.1 *For P any poset on $\{1,2,\dots,n\}$, both cones $K = K_P^{\mathrm{root}}, K_P^{\mathrm{wt}}$, and the appropriate lattices $L = L^{\mathrm{root}}, L^{\mathrm{wt}}$ have the affine semigroup $K \cap L$ generated by the primitive lattice vectors* (*the vectors nearest the origin*) *lying on the extreme rays of K.*

Proof It suffices to produce a triangulation of K into unimodular cones, each of whose extreme rays is a subset of these extreme rays of K.

For K_P^{root}, this essentially follows from the fact that the root system of type A_{n-1} is totally unimodular—every simplicial cone generated by a subset of roots $e_i - e_j$ is a unimodular cone. Thus, one can pick such a triangulation of K_P^{root} into simplicial subcones K introducing no new extreme rays arbitrarily, as in [23, Lemma 4.6.1].

For K_P^{wt}, one must be more careful in producing a triangulation of K_P^{wt} into unimodular cones introducing no new extreme rays.[3]

Proceed as in the proof of Proposition 3.1 via induction on the number $|\mathcal{L}(P)|$ of linear extensions, but using as base cases the situation where P is a forest, so that K_P^{wt} is a unimodular cone by Corollary 5.3.

In this inductive step, assuming that P is not a forest, there exist two elements i, j which are incomparable in P with a common lower bound $h <_P i, j$. As in the proof of Proposition 3.1, one has

$$\mathcal{L}(P) = \mathcal{L}(P_{i<j}) \sqcup \mathcal{L}(P_{j<i})$$

and hence a decomposition

$$K_{\mathcal{L}(P)} = K_{\mathcal{L}(P_{i<j})} \cup K_{\mathcal{L}(P_{j<i})}. \tag{7.1}$$

Note that induction applies to both $P_{i<j}$ and $P_{j>i}$ since they have fewer linear extensions. By the symmetry between i and j, it only remains to show that the extreme rays of $K_{\mathcal{L}(P_{i<j})}$ are a subset of those for $K_{\mathcal{L}(P)}$, or equivalently, that any subset $J \subseteq [n]$ which induces a connected order ideal of $P_{i<j}$ will also induce a connected order ideal of P.

First note that J will also be an order ideal in P, since P has fewer order relations than $P_{i<j}$. Given any two elements a, b in J, there will be a path

$$a = a_0, a_1, \dots, a_m = b \tag{7.2}$$

in J where each pair $a_\ell, a_{\ell+1}$ are comparable in $P_{i<j}$. If any pair $a_\ell, a_{\ell+1}$ are incomparable in P, this means either $a_\ell \le i$ and $j \le a_{\ell+1}$, or the same holds swapping the indices $\ell, \ell+1$. In either case, j must also lie in the ideal J of $P_{i<j}$, and hence h and i lie in J too. Thus, one can replace the single step $(a_\ell, a_{\ell+1})$ in the path (7.2) with the longer sequence $(a_\ell, i, h, j, a_{\ell+1})$ of steps, or the same swapping the indices ℓ, $\ell+1$. □

[3] It is not clear, a priori, that every simplicial cone spanned by a subset of the extreme rays of K_P^{wt} is unimodular, e.g. consider the cone spanned by these three rays: $e_1+e_2,\ e_1+e_3,\ e_2+e_3$.

8 Analysis of the semigroup for K_P^{root}

In the following subsections, we focus on the cone $K = K_P^{\text{root}}$ with lattice $L = L^{\text{root}}$ and attempt to analyze the structure of the affine semigroup $K \cap L$ and its semigroup ring $R = k[K \cap L]$ over a field k. Ultimately this leads to Corollary 8.10, giving a complete intersection presentation for R when the poset P is strongly planar, lifting Greene's Theorem A from the introduction to a statement about affine semigroup structure.

8.1 Generating the toric ideal

The affine semigroup $R = k[K \cap L]$ is naturally a subalgebra of a Laurent polynomial algebra

$$R = k\big[t_i t_j^{-1}\big]_{i <_P j} \subset k\big[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}\big].$$

On the other hand, recall from Proposition 7.1 that the affine semigroup $K \cap L$ is generated by the primitive vectors $\{e_i - e_j : i \lessdot_P j\}$ on its extreme rays. Therefore, one can present R as a quotient via the surjection

$$\begin{aligned} S := k[U_{ij}]_{i \lessdot_P j} &\longrightarrow R \\ U_{ij} &\longmapsto t_i t_j^{-1}. \end{aligned}$$

Defining as in Sect. 2.5 the toric ideal $I := \ker(S \to R)$, one has $R \cong S/I$.

It therefore helps to know generators for I in analyzing R and trying to compute its Hilbert series. As in Proposition 2.4, I is always generated by certain binomials. However, there is a smaller generating set of binomials available in this situation.

Say that a set of edges C in the (undirected) Hasse diagram for P forms a *circuit*[4] if they can be directed to form a cycle, and they are minimal with respect to inclusion having this property. Having fixed a circuit C and having fixed one of the two ways to orient C as a directed cycle, say that an edge $\{i, j\}$ of C having $i \lessdot_P j$ goes *with* C if $\{i, j\}$ is directed toward j in C and goes *against* C otherwise. Define two monomials

$$W(C) := \prod_{i \lessdot_P j \text{ with } C} U_{ij},$$

$$A(C) := \prod_{i \lessdot_P j \text{ against } C} U_{ij}$$

and define the *circuit binomial*

$$U(C) := W(C) - A(C).$$

Proposition 8.1 *For any poset P on $[n]$, the toric ideal $I = \ker(S \to R)$ where $S = k[U_{ij}]_{i \lessdot_P j}$ is generated by the circuit binomials $\{U(C)\}$ as C runs through all circuits of the undirected Hasse diagram of P.*

[4]Sometimes these are called *simple cycles*.

Proof Proposition 2.4 says that I is generated by binomials of the form

$$\prod_{i \lessdot_P j} U_{ij}^{a_{ij}} - \prod_{i \lessdot_P j} U_{ij}^{b_{ij}} \tag{8.1}$$

where a_{ij}, b_{ij} are nonnegative integers such that

$$\sum_{i \lessdot_P j} a_{ij}(e_i - e_j) = \sum_{i \lessdot P j} b_{ij}(e_j - e_i)$$

or equivalently

$$\sum_{i \lessdot_P j} a_{ij}(e_i - e_j) - b_{ij}(e_j - e_i) = 0.$$

In looking for a smaller set of generators for I, note that one may assume that if $a_{ij} \neq 0$, then $b_{ij} = 0$, else one could cancel factors of U_{ij} from the binomial in (8.1). This means that the nonnegative integers $a_{ij}, b_{i,j}$ can be thought of as the multiplicities on a collection $\mathcal{C}$ of directed arcs that either go up or down along edges in P, with the $\mathcal{C}$-indegree equalling the $\mathcal{C}$-outdegree at every vertex. Thus, $\mathcal{C}$ can be decomposed into collections supported on various circuits $C_1, \dots, C_t$ of edges (allowing multiplicity among the C_i). One then finds that the binomial (8.1) lies in the ideal generated by the circuit binomials $\{U(C_i)\}_{i=1}^t$ using the following calculation and induction on t:

$$\begin{aligned}\prod_{i \lessdot_P j} U_{ij}^{a_{ij}} - \prod_{i \lessdot_P j} U_{ij}^{b_{ij}} &= \prod_{i=1}^t W(C_i) - \prod_{i=1}^t A(C_i)\\ &= \underbrace{\big(W(C_1) - A(C_1)\big)}_{U(C_1)} \prod_{i=2}^t W(C_i)\\ &\quad + A(C_1)\left(\prod_{i=2}^t W(C_i) - \prod_{i=2}^t A(C_i)\right).\end{aligned}$$

□

For example, using this (together with Proposition 2.5) allows one to immediately compute $\mathrm{H}(K_P^{\mathrm{root}}; \mathbf{X})$ and $\Psi_P(\mathbf{x}) = \mathrm{s}(K_P^{\mathrm{root}}; \mathbf{x})$ in the case where the Hasse diagram of P has only one circuit, as done for Ψ_P by other means in [7, 9].

Corollary 8.2 *Let P be a poset whose Hasse diagram has only one circuit C. Considering the elements on C as a subposet, let* $\max(C)$ *and* $\min(C)$ *denote its maximal and minimal elements.*

Then the complete intersection presentation $R = k[K \cap L] \cong S/(U(C))$ implies

$$\mathrm{H}(K_P; \mathbf{X}) = \left(1 - \prod_{i \in \min(C)} X_i \cdot \prod_{j \in \max(C)} X_j^{-1}\right) \frac{1}{\prod_{i \lessdot_P j}(1 - X_i X_j^{-1})},$$

$$\Psi_P(\mathbf{x}) = \left(\sum_{i \in \min(C)} x_i - \sum_{j \in \max(C)} x_j \right) \frac{1}{\prod_{i \lessdot_P j} (x_i - x_j)},$$

assuming that P is connected for the latter formula.

8.2 The biconnected component reduction

Since the ideal $I = \ker(S \to R)$ is generated by the circuits within the undirected Hasse diagram for P, decomposing the Hasse diagram into its biconnected components provides a reduction in understanding the structure of R, which we explain next.

First we recall the notion of biconnected components in an undirected graph $G = (V, E)$. Say that two edges are *circuit-equivalent* if there is a circuit C of edges that passes through both. Consider the equivalence classes E_i of the transitive closure of this relation.[5] If V_i is the set of vertices which are at least the extremity of one edge in E_i, let the *biconnected components* of G be the subgraphs $G_i = (V_i, E_i)$.

Corollary 8.3 *If the Hasse diagram for P has biconnected components $P_1, \ldots, P_t$ (regarding each as the Hasse diagram for a poset P_ℓ), then one can express the semigroup ring R_P for P as a tensor product of graded k-algebras*:

$$R_P \cong R_{P_1} \otimes_k \cdots \otimes_k R_{P_t},$$

and therefore

$$\mathrm{H}\big(K_P^{\mathrm{root}}; \mathbf{X}\big) = \prod_{\ell=1}^{t} \mathrm{H}(K_{P_\ell}; \mathbf{X});$$

$$\Psi_P(\mathbf{x}) = \prod_{\ell=1}^{t} \Psi_{P_\ell}(\mathbf{x}).$$

Proof Express R_P as S/I. Since every edge of the Hasse diagram lies in a unique biconnected component P_i $(1 \le i \le t)$, one has $S \cong \otimes_{\ell=1}^{t} S_{P_\ell}$ with $S_{P_\ell} := k[U_{ij}]_{i \lessdot_{P_\ell} j}$. Since each circuit C is supported on a set of edges that lies within a single biconnected component P_ℓ, Proposition 8.1 implies $I = \bigoplus_{\ell=1}^{t} I_{P_\ell}$ where I_{P_ℓ} is the toric ideal $\ker(S_{P_\ell} \to R_{P_\ell})$. The first assertion follows, and the remaining assertions follow from the first. ☐

Remark 8.4 The argument above works in a more general context. Namely, if the ambient vector space V, lattice L, and cone K have compatible direct sum decompositions

$$V = V_1 \oplus \cdots \oplus V_\ell,$$

[5] Actually, this relation is already transitive, although we will not need this here.

$$L = L_1 \oplus \cdots \oplus L_\ell,$$
$$K = K_1 \oplus \cdots \oplus K_\ell,$$

then the semigroup ring $R := k[K \cap L]$ has a tensor product decomposition

$$R \cong R_1 \otimes_k \cdots \otimes_k R_\ell,$$

where $R_i = k[K_i \cap L_i]$ for $i = 1, \ldots, \ell$.

8.3 Notches and disconnecting chains

Note that Corollary 8.3 provides a somewhat trivial sufficient condition for $\Psi_P(\mathbf{x})$ to factor. Our goal here is a less trivial such condition on P, including a ring-theoretic explanation of the factorization due to disconnecting chains from [9, Theorem 7.1]. This is provided by the following operation, which sometimes applies to the Hasse diagram for P.

Definition 8.5 In a finite poset P, say that a triple of elements (a, b, c) forms a *notch of* $\vee$*-shape* (dually, a *notch of* $\wedge$*-shape*) if $a \lessdot_P b, c$ (dually, $a \gtrdot_P b, c$) and, in addition, b, c lie in different connected components of the poset $P \setminus P_{\leq a}$ (dually, $P \setminus P_{\geq a}$).

When (a, b, c) forms a notch of either shape in a poset P, say that the quotient poset $\bar{P} := P/\{b \equiv c\}$, having one fewer element and one fewer Hasse diagram edge, is obtained from P by *closing the notch*, and that P is obtained from $\bar{P}$ by *opening a notch*.

It should be noted that when (a, b, c) forms a $\vee$-shaped notch, the two elements b, c have no common upper bounds in P. This eliminates several pathologies which could occur in the formation of the quotient poset $\bar{P} = P/\{b \equiv c\}$, e.g., double edges other than the edge $\{a, b\}$, $\{a, c\}$, oriented cycles, and creation of a new edge in the quotient that is the transitive closure of other edges.

For example, in Fig. 1, the poset P_2 contains a notch of $\vee$-shape $(3, 5, 5')$, and the poset P_1 is obtained from P_2 by closing this notch.

We state the following result relating $K_{\bar{P}}^{\mathrm{root}}$, K_P^{root} in the case where the notch is $\vee$-shaped; the result for a $\wedge$-shaped notch is analogous.

Theorem 8.6 *When $\bar{P}$ is obtained from P closing a $\vee$-shaped notch (a, b, c), the affine semigroup ring $R_{\bar{P}}$ is obtained from the ring R_P by modding out the nonzero divisor $t_a t_b^{-1} - t_a t_c^{-1}$:*

$$R_{\bar{P}} \cong R_P / (t_a t_b^{-1} - t_a t_c^{-1}). \tag{8.2}$$

In particular,

$$\mathrm{H}\big(K_{\bar{P}}^{\mathrm{root}}; \mathbf{X}\big) = \big(1 - X_a X_b^{-1}\big)\big[\mathrm{H}\big(K_P^{\mathrm{root}}; \mathbf{X}\big)\big]_{X_b = X_c},$$
$$\Psi_{\bar{P}}(\mathbf{x}) = (x_a - x_b)\big[\Psi_P(\mathbf{x})\big]_{x_b = x_c},$$

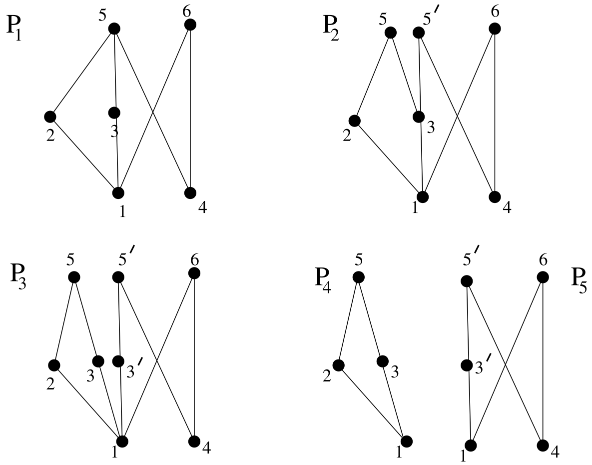


Fig. 1 Examples of opening and closing notches

so that $\Psi_{\bar{P}}(\mathbf{x})$ *and* $[\Psi_P(\mathbf{x})]_{x_b=x_c}$ *have exactly the same numerator polynomials when written over the denominator* $\prod_{i \lessdot_{\bar{P}} j}(x_i - x_j)$, *and a complete intersection presentation for* R_P *leads to such a presentation for* $R_{\bar{P}}$.

Example 8.7 Before delving into the proof, we illustrate how Theorem 8.6, together with some of the foregoing results, helps to analyze the ring R_P, as well as the Hilbert series $\mathrm{H}(K_P^{\mathrm{root}}; \mathbf{X})$ and hence $\Psi_P(\mathbf{x})$.

Consider the posets shown in Fig. 1. As mentioned earlier, P_1 is obtained from P_2 by closing the $\vee$-shaped notch $3 < 5, 5'$. In addition, P_2 is obtained from P_3 by closing the $\vee$-shaped notch $1 < 3, 3'$. Lastly, note that P_4, P_5 are the two biconnected components of P_3.

In analyzing R_{P_1}, therefore, one can start with P_4, P_5, which each have a unique circuit, and apply Corollary 8.2 to write down these simple (complete intersection) presentations:

$$R_{P_4} \cong k[U_{12}, U_{25}, U_{13}, U_{35}]/(U_{12}U_{25} - U_{13}U_{35}),$$

$$R_{P_5} \cong k[U_{13'}, U_{16}, U_{3'5'}, U_{45'}, U_{46}]/(U_{13'}U_{3'5'}U_{46} - U_{16}U_{45'}).$$

Applying Corollary 8.3 yields the following tensor product (complete intersection) presentation for R_{P_3}:

$$\begin{aligned} R_{P_3} &\cong R_{P_4} \otimes R_{P_5} \\ &\cong k[U_{12}, U_{25}, U_{13}, U_{35}, U_{13'}, U_{16}, U_{3'5'}, U_{45'}, U_{46}] \\ &\quad /(U_{12}U_{25} - U_{13}U_{35},\ U_{13'}U_{3'5'}U_{46} - U_{16}U_{45'}). \end{aligned}$$

Applying Theorem 8.6 to close the notch at $1 < 3, 3'$ yields the following complete intersection presentation for R_{P_2}:

$$\begin{aligned} R_{P_2} \cong\ & k[U_{12}, U_{25}, U_{13}, U_{35}, U_{16}, U_{35'}, U_{45'}, U_{46}] \\ & /(U_{12}U_{25} - U_{13}U_{35},\ U_{13}U_{35'}U_{46} - U_{16}U_{45'}). \end{aligned}$$

Applying Theorem 8.6 once more to close the notch at $3 < 5, 5'$ yields the following complete intersection presentation for R_{P_1}:

$$R_{P_1} \cong k[U_{12}, U_{25}, U_{13}, U_{35}, U_{16}, U_{45}, U_{46}] \\ /(U_{12}U_{25} - U_{13}U_{35},\ U_{13}U_{35}U_{46} - U_{16}U_{45}).$$

Consequently, from Theorem 2.5 one has

$$\mathrm{H}\big(K_{P_1}^{\mathrm{root}}; \mathbf{X}\big) = \frac{(1 - X_1X_5^{-1})(1 - X_1X_4X_5^{-1}X_6^{-1})}{\prod_{i \lessdot_{P_1} j}(1 - X_iX_j^{-1})},$$

$$\Psi_{P_1}(\mathbf{x}) = \frac{(x_1 - x_5)(x_1 + x_4 - x_5 - x_6)}{\prod_{i \lessdot_{P_1} j}(x_i - x_j^{-1})}.$$

Proof of Theorem 8.6 Define $S_P := k[U_{ij}]_{i \lessdot_P j}$, so that

$$R_P := k\big[K_P^{\mathrm{root}} \cap L^{\mathrm{root}}\big] \cong S_P/I_P$$

where I_P is the kernel of the map $S_P \to R_P$ sending U_{ij} to $t_it_j^{-1}$.

Define a map $S_P \xrightarrow{\phi} R_{\bar{P}}$ sending most variables U_{ij} to $t_it_j^{-1}$, except that both U_{ab}, U_{ac} get sent to $t_at_b^{-1}$. We wish to describe the ideal $J := \ker(S_P \to R_{\bar{P}})$ and in particular to show that

$$J = I_P + (U_{ab} - U_{ac}). \tag{8.3}$$

This would imply (8.2): the map ϕ is surjective since it hits a set of generators for $R_{\bar{P}}$, and hence

$$\begin{aligned} R_{\bar{P}} &\cong S_P/J \\ &= S_P/\big(I_P + (U_{ab} - U_{ac})\big) \\ &\cong (S_P/I_P)/\big(\bar{U}_{ab} - \bar{U}_{bc}\big) \\ &\cong R_P/\big(t_at_b^{-1} - t_at_c^{-1}\big). \end{aligned}$$

To prove the equality of ideals asserted in (8.3), one checks that the two ideals are included in each other. The inclusion $I_P + (U_{ab} - U_{ac}) \subseteq J$ is not hard: both U_{ab}, U_{ac} are sent by ϕ to $t_at_b^{-1}$, so the binomial $U_{ab} - U_{ac}$ is in the kernel J, and since circuits C in the directed graph P remain circuits in the quotient directed graph $\bar{P}$, Proposition 8.1 implies the inclusion $I_P \subseteq J$.

For the reverse inclusion $J \subseteq I_P + (U_{ab} - U_{ac})$, first note that one can reinterpret the ideal J: it is the toric ideal for the presentation of the semigroup $R_{\bar{P}}$ in which the Hasse diagram edge $a \lessdot_{\bar{P}} bc$ has been "doubled" into two parallel directed edges associated with the same monomial $t_at_b^{-1}$, but hit by two variables U_{ab}, U_{ac} from S_P. Denote by $\bar{P}^+$ this directed graph obtained from the Hasse diagram for $\bar{P}$ by doubling this edge. The definition of $\bar{P}^+$ is illustrated on Fig. 2.

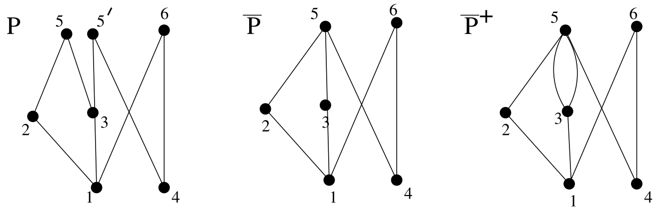


Fig. 2 An example of P, $\bar{P}$, $\bar{P}^+$

The analysis from Proposition 8.1 then shows that J is generated by the circuit binomials $U(C)$ as C runs through the circuits of $\bar{P}^+$.

It remains to show that for every circuit C in the directed graph $\bar{P}^+$, the circuit binomial $U(C)$ lies in $I_P + (U_{ab} - U_{ac})$.

If this circuit C in $\bar{P}^+$ does not pass through the collapsed vertex bc in $\bar{P}^+$, then C is also a circuit in P, and hence $U(C)$ already lies in I_P.

If this circuit C does pass through vertex bc, we distinguish two cases. Consider the partition of the set $E_{bc} = E_b \sqcup E_c$ of edges incident to bc in $\bar{P}^+$, where E_b (resp. E_c) is the subset of edges whose preimage in P is incident to b (resp. c). If the two edges of C incident to bc lie in the same set of this partition, then, as before, C is also a circuit in P, and hence $U(C)$ already lies in I_P.

Consider now the last case where C does pass through vertex bc, but the two edges of C incident to bc lie respectively in E_b and E_c. Since b, c lie in different connected components of $P \setminus P_{\leq a}$, the circuit C must pass through at least one vertex $d \leq_P a$. Use this to create two directed cycles C_b, C_c in P:

- C_b follows b to d along the same path π_{bd} chosen by C, then follows d to a along any saturated chain π_{da} in P between them, and finally from a to b.
- C_c follows a to d reversing the same saturated chain π_{da}, then follows d to c along the same path π_{dc} chosen by C, and finally goes from c to a.

One then has the following relation in S_P:

$$\begin{aligned} U(C) = {} & U(C_b) \cdot W(\pi_{dc}) \\ & + U(C_c) \cdot A(\pi_{bd}) \\ & + (U_{ab} - U_{ac}) \cdot W(\pi_{dc}) \cdot A(\pi_{bd}) \cdot W(\pi_{da}) \end{aligned} \tag{8.4}$$

where for a path π of edges in the Hasse diagram, one defines monomials

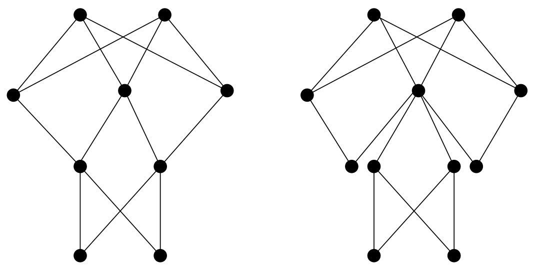
$$\begin{aligned} W(\pi) &:= \prod_{\substack{i \lessdot_P j: \\ i \to j \text{ appears in } \pi}} U_{ij}, \quad A(\pi) \\ &:= \prod_{\substack{i \lessdot_P j: \\ i \leftarrow j \text{ appears in } \pi}} U_{ij}. \end{aligned}$$

Relation (8.4) shows that $U(C)$ lies in $I_P + (U_{ab} - U_{ac})$, as desired.

For the remaining assertions, note that since R_P is a subalgebra of the Laurent polynomial ring, it is an integral domain, and therefore $t_a t_b^{-1} - t_a t_c^{-1}$ is a nonzero divisor of R_P. After identifying the grading variables $x_b = x_c$, this element $t_a t_b^{-1} - t_a t_c^{-1}$ becomes homogeneous of degree $e_a - e_b$. □

Opening notches in a poset P provides a flexible way to understand some previously observed factorizations of the numerator of $\Psi_P(\mathbf{x})$, while at the same time giving information about the semigroup ring $k[K_P^{\text{root}} \cap L_P^{\text{root}}]$ and its Hilbert series.

Example 8.8 One way to explain the factorization of the numerator of $\Psi_P(\mathbf{x})$ for the example from [9, Fig. 2] is to successively "open two notches," as shown here



and then apply Corollary 8.3 to the poset on the right, which has two biconnected components.

Example 8.9 In [9, §7] it was explained how a disconnecting chain

$$\sigma = (p_1 \lessdot_P p_2 \lessdot_P \cdots \lessdot_P p_{t-1} \lessdot p_t)$$

in P, that is, one for which $P \setminus \sigma$ has several connected components, leads to a factorization of the numerator of $\Psi_P(\mathbf{x})$ into factors indexed by each such component. After fixing one of the connected components Q of $P \setminus \sigma$, one can use several operations of opening notches, beginning with one that creates two elements p_t, p_t' covering p_{t-1}, and continuing down the chain σ, to "peel off" a copy of $Q \sqcup \sigma$ until it is attached to $P \setminus Q$ only at the vertex p_1. At this stage use Corollary 8.3, to recover the factorization of [9, Theorem 7.1].

We omit a detailed discussion to avoid the use of heavy notation. However, Example 8.7 illustrates the principle.

Lastly, one can use this to deduce a stronger form of Theorem A from the introduction. For a strongly planar poset P and a bounded region ρ of the plane enclosed by its Hasse diagram, recall that $\min(\rho)$, $\max(\rho)$ denote the P-minimum, P-maximum elements among the elements of P lying on ρ. Name the elements on the unique two maximal chains from $\min(\rho)$ to $\max(\rho)$ that bound ρ as follows:

$$\begin{aligned} \min(\rho) &=: i_0 \lessdot_P i_1 \lessdot_P \cdots \lessdot_P i_{r-1} \lessdot_P i_r := \max(\rho), \\ \min(\rho) &=: j_0 \lessdot_P j_1 \lessdot_P \cdots \lessdot_P j_{s-1} \lessdot_P j_s := \max(\rho). \end{aligned} \tag{8.5}$$

Lastly, let f_ρ be the following binomial in the polynomial algebra $S := k[U_{ij}]_{i \lessdot_P j}$:

$$f_\rho := \prod_{p=1}^{r} U_{i_{p-1} i_p} - \prod_{q=1}^{s} U_{j_{q-1} j_q}.$$

In other words, f_ρ is the circuit binomial $U(C)$ for the directed circuit C that goes up and down the two maximal chains in (8.5) bounding ρ.

Corollary 8.10 *For any strongly planar poset P on $\{1, 2, \ldots, n\}$, one has a complete intersection presentation for its semigroup ring $k[K_P^{\text{root}} \cap L^{\text{root}}]$ as the quotient S/I where $S := k[U_{ij}]_{i \lessdot_P j}$, and I is the ideal generated by the $\{f_\rho\}$ as ρ runs through all bounded regions for the Hasse diagram of P.*

Consequently,

$$\mathrm{H}\big(K_P^{\text{root}}; \mathbf{X}\big) = \frac{\prod_\rho (1 - X_{\min(\rho)} X_{\max(\rho)}^{-1})}{\prod_{i \lessdot_P j} (1 - X_i X_j^{-1})},$$

$$\Psi_P(\mathbf{x}) = \frac{\frac{\prod_\rho (x_{\min(\rho)} - x_{\max(\rho)})}{\prod_{i \lessdot_P j} (x_i - x_j)}}{},$$

where the last equality assumes that P is connected.

Proof Use induction on the number of bounded regions ρ. In the base cases where there are no such regions or one such region, apply Corollary 5.3 or 8.2, respectively.

In the inductive step, find a disconnecting chain for P that separates at least two bounded regions, as in [9, Proposition 7.4]. Use Proposition 8.6 repeatedly to open notches down this chain, until the resulting poset has two biconnected components attached at one vertex of the chain, and apply Corollary 8.3, as in Example 8.9. □

9 Reinterpreting the main transformation

Our goal in this final section is to reinterpret geometrically a very flexible identity that was used to deduce most of the results on $\Psi_P(\mathbf{x})$ in [9] and called there the *main transformation*:

Theorem [9, Theorem 4.1] *Let C be one of the two possible orientations of a circuit in the Hasse diagram for a poset P. Let $W \subset C$ be the edges of C which are directed upward in P. Then*

$$\sum_{E \subset W} (-1)^{|E|} \Psi_{P \setminus E}(\mathbf{x}) = 0 \tag{9.1}$$

where $P \setminus E$ is the poset whose Hasse diagram is obtained from that of P by removing the edges in E.

Remark 9.1 In fact, (9.1) was deduced in [9, Theorem 4.1] from a geometric identity equivalent to the following:

$$\sum_{E \subset W} (-1)^{|E|} \chi_{K^{\mathrm{wt}}_{P \setminus E}} = 0. \tag{9.2}$$

Using the duality discussed in Remark 3.2, identity (9.2) implies the following geometric identity underlying (9.1):

$$\sum_{E \subset W} (-1)^{|E|} \chi_{K^{\mathrm{root}}_{P \setminus E}} = 0. \tag{9.3}$$

Remark 9.2 In [9], identity (9.1) was used to prove some statements on Ψ by induction on the number of independent cycles (the *cyclomatic number*) in the Hasse diagram for P: terms indexed by nonempty subsets E correspond to posets $P \setminus E$ with fewer independent cycles. In the base case for such inductive proofs, the Hasse diagram is acyclic and possibly disconnected, so that either $\Psi_P(\mathbf{x}) = 0$, or Corollary 5.3 applies.

Furthermore, in [9, Sect. 6], it was shown how the choice of an embedding of the Hasse diagram of P onto a surface, together with a rooting at one of its half-edges, leads to a good choice of circuits C in the induction. This expresses $\Psi_P(\mathbf{x}) = \sum_i \Psi_{P_i}(\mathbf{x})$ for various posets P_i with tree Hasse diagrams that can be described explicitly in terms of the embedding and rooting. Using (9.3), one can show that this corresponds to an explicit triangulation for the cone K_P^{root} into subcones $K_{P_i}^{\mathrm{root}}$, in which each subcone uses no new extreme rays.

Unfortunately, iterating (9.2) does not in general lead to proofs for results on $\Phi_P(\mathbf{x})$ via induction on cyclomatic number, as the base cases with no cycles correspond to cones K_P^{wt} which are not necessarily simplicial; see Corollary 5.3.

Remark 9.3 Unlike (3.1), this identity (9.3) involves only pointed cones.

Our goal here is to point out how the geometric statement (9.3) generalizes to other families of cones and vectors. We begin with a geometric generalization of the notion of a circuit C in the Hasse diagram for P and its subset of upward edges $W \subset C$.

Definition 9.4 Given two subsets of W, V of vectors in $\mathbb{R}^d$, say that W *is cyclic*[6] *with respect to* V if there exists a positive linear combination of W lying in $\mathbb{R}_+ V$, that is, $\sum_{w \in W} a_w w = \sum_{v \in V} b_v v$ for some real numbers $a_w > 0, b_v \geq 0$.

Example 9.5 Let C be one of the two possible orientations of a directed circuit in the Hasse diagram for a poset P. Let $W \subset C$ be the edges of C which are directed upward in P. Then $\{e_i - e_j : (i, j) \in W\}$ is cyclic with respect to the set $V := \{e_i - e_j :$

[6] In the special case where V is empty, this is the notion of W being a *totally cyclic* collection of vectors from oriented matroid theory; see [6, Definition 3.4.7].

$i \lessdot_P j, (i,j) \notin W\}$, due to the relation

$$\sum_{\substack{i \lessdot_P j:\\(i,j)\in W}} e_i - e_j = \sum_{\substack{i \lessdot_P j:\\(j,i)\in C\setminus W}} e_i - e_j.$$

Bearing this example in mind, the following proposition gives the desired generalization of (9.1) and (9.3).

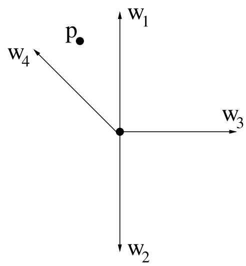
Proposition 9.6 *For subsets W, V of vectors in $\mathbb{R}^d$ where W is cyclic with respect to V, one has the identity among characteristic vectors of cones*

$$\sum_{B\subset W} (-1)^{|B|} \chi_{\mathbb{R}_+(V\cup B)} = 0,$$

and therefore

$$\sum_{B\subset W} (-1)^{|B|} s\big(\mathbb{R}_+(V\cup B); \mathbf{x}\big) = 0.$$

Example 9.7 Consider the set of vectors $W = \{w_1, w_2, w_3, w_4\}$ in $\mathbb{R}^2$ shown below, and let V be the empty set. The set W is easily seen to be cyclic with respect to V.



Consider the point p depicted. The subsets $B \subset W$ for which p lies in the cone $\mathbb{R}_+(V\cup B)$, so that $\chi_{\mathbb{R}_+(V\cup B)}(p) = 1$, are

$$\{w_1, w_4\},\ \{w_3, w_4\},\ \{w_1, w_2, w_4\},\ \{w_1, w_3, w_4\},$$
$$\{w_2, w_3, w_4\},\ \{w_1, w_2, w_3, w_4\}.$$

The sum of $(-1)^{|B|}$ over these sets B vanishes, as predicted by the proposition. However, note that this does not hold for trivial reasons, e.g., these sets B do not form an interval in the boolean lattice.

Proof of Proposition 9.6. Up to a rescaling of the vectors in W, one can assume that $u := \sum_{w\in W} w$ lies in $\mathbb{R}_+ V$.

One must show that for every point $p \in \mathbb{R}^d$, one has

$$\sum_{\substack{B\subset W:\\ p\in\mathbb{R}_+(V\cup B)}} (-1)^{|B|} = 0. \tag{9.4}$$

If p does not lie in the cone $\mathbb{R}_+(V \cup W)$, this holds because the left side is an empty sum. So without loss of generality p lies in $\mathbb{R}_+(V \cup W)$, meaning that the set

$$X_p := \left\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{R}_+^W \times \mathbb{R}_+^V : p = \sum_{w \in W} a_w w + \sum_{v \in V} b_v v \right\}$$

is a nonempty convex polyhedral cone inside $\mathbb{R}^W \times \mathbb{R}^V$. Cover X_p by the family of subsets $\{X_p(w_0)\}_{w_0 \in W}$ defined by

$$X_p(w_0) := \left\{ (\mathbf{a}, \mathbf{b}) \in X_p : a_{w_0} = \min(\mathbf{a}) \right\}.$$

These sets $X_p(w_0)$ are also convex polyhedral subsets, although possibly empty. The nerve of this covering of X_p is the abstract simplicial complex consisting of all subsets $A \subset W$ for which $\bigcap_{w_0 \in A} X_p(w_0)$ is nonempty. A standard nerve lemma (e.g., [4, Theorem 10.7]) implies that the geometric realization of this nerve is homotopy equivalent to the contractible space X_p, and hence its *(reduced) Euler characteristic* $\sum_A (-1)^{|A|-1}$ vanishes, where the sum runs over subsets A with $\bigcap_{w_0 \in A} X_p(w_0)$ nonempty. Thus, (9.4) will follow from this claim:

Claim *The set $\bigcap_{w_0 \in A} X_p(w_0)$ is nonempty if and only if p lies in $\mathbb{R}_+(V \cup (W \setminus A))$.*

For the "if" assertion of the claim, note that if p lies in $\mathbb{R}_+(V \cup (W \setminus A))$, then any expression

$$p = \sum_{w \in W \setminus A} a_w w + \sum_{v \in V} b_v v$$

leads to a similar expression

$$p = \sum_{w \in W} a_w w + \sum_{v \in V} b_v v$$

by defining $a_{w_0} := 0$ for all w_0 in A. Furthermore, the coefficients in the latter expression give an element $(\mathbf{a}, \mathbf{b})$ lying in $\bigcap_{w_0 \in A} X_p(w_0)$.

For the "only if" assertion, assuming that $\bigcap_{w_0 \in A} X_p(w_0)$ is nonempty, pick $(\mathbf{a}, \mathbf{b})$ lying in this set. Thus, $p = \sum_{w \in W} a_w w + \sum_{v \in V} b_v v$, and one has $\mu := \min(\mathbf{a}) = a_{w_0}$ for all w_0 in A. Rewriting this as

$$p = \sum_{w_0 \in A} \mu \cdot w_0 + \sum_{w \in W \setminus A} a_w w + \sum_{v \in V} b_v v$$

and using the fact that $u = \sum_{w \in W} w$ lies in $\mathbb{R}_+ V$, one can rewrite

$$p = \underbrace{\sum_{w \in W \setminus A} (a_w - \mu) w}_{\in \mathbb{R}_+(W \setminus A)} \quad + \quad \underbrace{\mu \cdot u + \sum_{v \in V} b_v v}_{\in \mathbb{R}_+ V}.$$

Therefore, p lies in $\mathbb{R}_+(V \cup (W \setminus A))$. □

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