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Gröbner bases of contraction ideals

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Abstract We investigate Gröbner bases of contraction ideals under monomial homomorphisms. As an application, we generalize the result of Aoki–Hibi–Ohsugi–Takemura and Ohsugi–Hibi for toric ideals of nested configurations.

Keywords Gröbner bases · Toric ideal · Nested configuration

1 Introduction

In algebraic combinatorics, the theory of toric ideal is used for investigating the structure of a combinatorial model. Conti–Traverso [2] have given an algorithm for solving the integer programming problem using the toric ideal. In recent years, applications of the toric ideal in statistics have been successfully developed since the pioneering work of Diaconis–Sturmfels [6]. They have given algebraic algorithms for sampling from a finite sample space using Markov chain Monte Carlo methods. In this paper, we investigate the structure of a combinatorial model constructed from several small combinatorial models.

We denote by $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ the set of non-negative integers. For a multi-index $\boldsymbol{a} = {}^t(a_1, \dots, a_r) \in \mathbb{Z}^r$ and variables $\boldsymbol{x} = (x_1, \dots, x_r)$, we write $\boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} \cdots x_r^{a_r}$. We set $|\boldsymbol{a}| = a_1 + \cdots + a_r$. Rings appearing in this paper may equip two or more graded ring structures. We call $K[x_1, \dots, x_n]$, $\deg(x_i) = 1$, a *standard graded* polynomial ring. To avoid confusion, for a ring with a graded ring structure given by an object $*$ (e.g. a weight vector $\boldsymbol{w}$, or an abelian group $\mathbb{Z}^d$), we say that elements or ideals are $*$-homogeneous or $*$-graded if they are homogeneous with respect to the graded ring structure given by $*$. For a $*$-homogeneous polynomial f, we denote by $\deg_*(f)$ the degree of f with respect to $*$, and call it the $*$-degree of f.

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We omit $*$ when we consider the standard grading on polynomial rings. In this paper, "quadratic" means "of degree at most two". We say that a monomial ideal J satisfies a property P (e.g. quadratic, square-free, or of degree at most m) if the minimal system of monomial generators of J satisfies P.

Let $\mathcal{A} = (a_{ij})_{1\le i\le m, 1\le j\le n}$, be an $m \times n$ integer matrix, and $\boldsymbol{b} \in \mathbb{Z}^m$. We identify the matrix $\mathcal{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$, $\boldsymbol{a}_j = {}^t(a_{1j}, \dots, a_{mj})$, with the configuration $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\} \subset \mathbb{Z}^m$. We denote by $\mathrm{Fiber}_{\mathcal{A}}(\boldsymbol{b}) = \{\boldsymbol{a} \in \mathbb{N}^n \mid \mathcal{A} \cdot \boldsymbol{a} = \boldsymbol{b}\}$ the *$\mathcal{A}$-fiber space* of $\boldsymbol{b}$. Integer programming is the problem of finding a vector $\boldsymbol{a}_0$ that maximizes (or minimizes) $\boldsymbol{w} \cdot \boldsymbol{a}$ over $\mathrm{Fiber}_{\mathcal{A}}(\boldsymbol{b})$. In some statistical models, sample spaces are described as an $\mathcal{A}$-fiber space. Let K be a field. We define the K-algebra homomorphism $\phi_{\mathcal{A}} : K[x_1, \dots, x_n] \to K[y_1^{\pm 1}, \dots, y_m^{\pm 1}]$, $x_j \mapsto \boldsymbol{y}^{\boldsymbol{a}_j}$. We set $K[\mathcal{A}] = K[\boldsymbol{y}^{\boldsymbol{a}_1}, \dots, \boldsymbol{y}^{\boldsymbol{a}_n}]$. We call the binomial prime ideal $P_{\mathcal{A}} := \mathrm{Ker}\, \phi_{\mathcal{A}} = \langle \boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}} \mid \boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^n, \mathcal{A} \cdot \boldsymbol{a} = \mathcal{A} \cdot \boldsymbol{b}\rangle \subset K[\boldsymbol{x}] = K[x_1, \dots, x_n]$ the *toric ideal* of $\mathcal{A}$. We are mainly interested in the problem when $P_{\mathcal{A}}$ admits a quadratic initial ideal or a square-free initial ideal. We call $\mathcal{A}$ a *standard graded configuration* if there exists a vector $0 \ne \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Q}^m$ such that $\lambda \cdot \boldsymbol{a}_j = 1$ for all j. If $\mathcal{A}$ is standard graded, then $P_{\mathcal{A}}$ is homogeneous in the usual sense, and some algebraic properties of $K[\mathcal{A}] \cong K[\boldsymbol{x}]/P_{\mathcal{A}}$ can be derived from Gröbner bases of $P_{\mathcal{A}}$. If $P_{\mathcal{A}}$ admits square-free initial ideal, then $K[\mathcal{A}]$ is normal, and if $P_{\mathcal{A}}$ admits quadratic initial idea, then $K[\mathcal{A}]$ is a Koszul algebra, that is, the residue field K has a linear minimal graded free resolution.

We will investigate a toric ideal $P_{\mathcal{C}}$ such that $\mathcal{C}$ is a product of two matrices $\mathcal{B}$ and $\mathcal{A}$. Defining ideal of Veronese subrings of toric algebras, Segre products of toric ideals, and toric fiber products of toric ideals are examples of toric ideals of form $P_{\mathcal{C}}$ with $\mathcal{C} = \mathcal{B} \cdot \mathcal{A}$. This type of matrix appears when one consider nested selection: Suppose that there exist m types, $C_1, \dots, C_m$, of items. We make r types of group, $B_1, \dots, B_r$, combining these items, and express them by column vectors $\boldsymbol{b}_j = {}^t(b_{1j}, \dots, b_{mj})$, $1 \le j \le r$, where b_{ij} is the number of items of type C_i contained in B_j. Then we construct n types of family, $A_1, \dots, A_n$, combining the groups $B_1, \dots, B_r$, and express them by column vectors $\boldsymbol{a}_j = (a_{1j}, \dots, a_{rj})$, $1 \le j \le n$. Let $\mathcal{B} = (b_{ij})_{1\le i\le m, 1\le j\le r}$, $\mathcal{A} = (a_{ij})_{1\le i\le r, 1\le j\le n}$ and $\mathcal{C} = \mathcal{B} \cdot \mathcal{A}$. For $\boldsymbol{c} = {}^t(c_1, \dots, c_n) \in \mathbb{N}^n$, the set of combinations of families which contain c_i items of type C_i is expressed by the $\mathcal{C}$-fiber space of $\boldsymbol{c}$. Since $\phi_{\mathcal{B}} \circ \phi_{\mathcal{A}} = \phi_{\mathcal{B}\cdot\mathcal{A}}$, $P_{\mathcal{B}\cdot\mathcal{A}} = \phi_{\mathcal{A}}^{-1}(P_{\mathcal{B}})$. For a ring homomorphism $\phi : S \to R$ and an ideal $I \subset R$, we call $\phi^{-1}(I) \subset R$ the *contraction ideal* of I under ϕ. Thus the problem is reduced to the study of Gröbner bases of contraction ideals under monomial homomorphisms.

In Sect. 2, we will show that under a suitable condition, $P_{\mathcal{A}}^{-1}(I)$ admits a quadratic (resp. square-free) initial ideal if both of $P_{\mathcal{A}}$ and I admit quadratic (resp. square-free) initial ideals. We first prove this in the case where I is a monomial ideal (Theorem 2.7). For a general ideal I, by extending a method developed by Sullivant [10], we give a sufficient condition under which the initial ideal of $\phi_{\mathcal{A}}^{-1}(I)$ coincides with the initial ideal of the contraction ideal $\phi_{\mathcal{A}}^{-1}(\mathrm{in}_{\prec}(I))$ of the initial ideal of I. The main theorem of this paper is the following.

Theorem 1 (Theorem 2.20) *Let $K[\boldsymbol{y}] = K[y_1, \dots, y_s]$ be a $\mathbb{Z}^d$-graded polynomial ring with $\deg_{\mathbb{Z}^d}(y_i) = \boldsymbol{v}_i \in \mathbb{Z}^d$. Let $\mathfrak{H} \subset \mathbb{Z}^d$ be a finitely generated subsemigroup, and*

$\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\} \subset \mathbb{N}^s$ *a system of generators of the semigroup*

$$\left\{\boldsymbol{a} \in \mathbb{N}^s \mid \deg_{\mathbb{Z}^d}\left(\boldsymbol{y}^{\boldsymbol{a}}\right) \in \mathfrak{H}\right\} = \left\{\boldsymbol{a} = {}^t(a_1, \ldots, a_s) \in \mathbb{N}^s \mid a_1\boldsymbol{v}_1 + \cdots + a_s\boldsymbol{v}_s \in \mathfrak{H}\right\}.$$

We set $\phi_{\mathcal{A}} : K[x_1, \ldots, x_r] \to K[\boldsymbol{y}]$, $x_j \mapsto \boldsymbol{y}^{\boldsymbol{a}_j}$. *Let* $I \subset K[\boldsymbol{y}]$ *be a* $\mathbb{Z}^d$*-graded ideal. Then the following hold.*

(1) *If both of* I *and* $P_{\mathcal{A}}$ *admit initial ideals of degree at most* m, *then so does* $\phi_{\mathcal{A}}^{-1}(I)$.
(2) *If both of* I *and* $P_{\mathcal{A}}$ *admit square-free initial ideals*, *then so does* $\phi_{\mathcal{A}}^{-1}(I)$.

The configuration $\mathcal{A}$ in Theorem 1 corresponds to a special type of selection. For example, suppose that there exist three items A_1, A_2, A_3 whose (weight, volume) are $(1, 30)$, $(1, 10)$, and $(3, 10)$, respectively. The set of combinations of these items such that the ratio of the total weight to the total volume is $1/20$ can be expressed by the semigroup

$$\left\{{}^t(a_1, a_2, a_3) \in \mathbb{N}^3 \mid a_1 \cdot (1, 30) + a_2 \cdot (1, 10) + a_3 \cdot (3, 10) \in \mathbb{N} \cdot (1, 20)\right\},$$

and its system of generators is $\mathcal{A} = \{{}^t(1, 1, 0), {}^t(5, 0, 1)\}$.

Using Theorem 1, we generalize the results of Aoki–Hibi–Ohsugi–Takemura [1] and Ohsugi–Hibi [8].

Theorem 2 (Theorem 3.5) *Let* $0 < d \in \mathbb{N}$, *and let* $K[\boldsymbol{z}^{\pm 1}] = K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ *be a* $\mathbb{Q}^d$*-graded Laurent polynomial ring with* $\deg_{\mathbb{Q}^d}(z_i) = \boldsymbol{v}_i \in \mathbb{Q}^d$. *Let* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_s \in \mathbb{Q}^d$ *be rational vectors that are linearly independent over* $\mathbb{Q}$. *For* $1 \le i \le s$, *take* $\mathcal{B}_i = \{\boldsymbol{b}_j^{(i)} \mid 1 \le j \le \lambda_i\} \subset \{\boldsymbol{b} \in \mathbb{Z}^n \mid \deg_{\mathbb{Q}^d}(\boldsymbol{z}^{\boldsymbol{b}}) = \boldsymbol{u}_i\}$, *and set* $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_s$. *Let* $\mathcal{A} \subset \mathbb{N}^s$ *be a standard graded configuration. We set*

$$\mathcal{A}[\mathcal{B}_1, \ldots, \mathcal{B}_s] := \left\{\sum_{j=1}^{\lambda_1} a_j^{(1)}\boldsymbol{b}_j^{(1)} + \cdots + \sum_{j=1}^{\lambda_s} a_j^{(s)}\boldsymbol{b}_j^{(s)} \;\middle|\; a_j^{(i)} \in \mathbb{N}, \left(\sum_{j=1}^{\lambda_1} a_j^{(1)}, \ldots, \sum_{j=1}^{\lambda_s} a_j^{(s)}\right) \in \mathcal{A}\right\}.$$

Then the following hold.

(1) *If both of* $P_{\mathcal{B}}$ *and* $P_{\mathcal{A}}$ *admit initial ideals of degree at most* m, *then so does* $P_{\mathcal{A}[\mathcal{B}_1, \ldots, \mathcal{B}_d]}$.
(2) *If both of* $P_{\mathcal{B}}$ *and* $P_{\mathcal{A}}$ *admit square-free initial ideals*, *then so does* $P_{\mathcal{A}[\mathcal{B}_1, \ldots, \mathcal{B}_d]}$.

The configuration $\mathcal{A}[\mathcal{B}_1, \ldots, \mathcal{B}_d]$ in Theorem 2 is a generalization of nested configuration defined in [1], and it appears when one considers a special type of nested selection. Theorem 2 also contains the result of Sullivant [10] (toric fiber products).

2 Gröbner bases of contraction ideals

Let $R = K[x_1, \ldots, x_r]$ and $S = K[y_1, \ldots, y_s]$ be polynomial rings over K, and I an ideal of S. Let $\mathcal{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r)$, $\boldsymbol{a}_i \in \mathbb{N}^s$, and $\phi_{\mathcal{A}} : R \to S$, $x_j \mapsto \boldsymbol{y}^{\boldsymbol{a}_j}$. We investigate Gröbner bases of the contraction ideal $\phi_{\mathcal{A}}^{-1}(I)$ of I.

Assume that $\boldsymbol{a}_i = \boldsymbol{a}_j$, and let $\prec$ be a term order on R such that $x_i \prec x_j$. Let $\mathcal{A}' = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_r)$. Then the union of $\{x_j - x_i\}$ and a Gröbner basis of $P_{\mathcal{A}'} \subset K[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r]$ with respect to the term order induced by $\prec$ is a Gröbner basis of $P_{\mathcal{A}}$. Thus we may identify the matrix $\mathcal{A}$ with the configuration $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$ when we investigate square-freeness or degree bound of initial ideals.

2.1 Preliminaries on Gröbner bases

We recall the theory of Gröbner bases. See [3, 4, 9] for details.

We write $\mathrm{in}_\prec(f)$ (resp. $\mathrm{in}_{\boldsymbol{w}}(f)$) for the initial term (resp. initial form) of a polynomial f with respect to a term order $\prec$ (resp. a weight vector $\boldsymbol{w}$) following [9]. We call $\mathrm{in}_{\boldsymbol{w}}(I) = \langle \mathrm{in}_{\boldsymbol{w}}(f) \mid f \in I \rangle$ (resp. $\mathrm{in}_\prec(I) = \langle \mathrm{in}_\prec(f) \mid f \in I \rangle$) the *initial ideal* of I with respect to $\boldsymbol{w}$ (resp. $\prec$). A monomial not in $\mathrm{in}_\prec(I)$ is called a *standard monomial* of I with respect to $\prec$. We say that a finite collection of polynomials $G \subset I$ is a *pseudo-Gröbner basis* of I with respect to $\boldsymbol{w}$ if $\langle \mathrm{in}_{\boldsymbol{w}}(g) \mid g \in G \rangle = \mathrm{in}_{\boldsymbol{w}}(I)$. If G is a pseudo-Gröbner basis and $\mathrm{in}_{\boldsymbol{w}}(g)$ is a monomial for all $g \in G$, we call G a Gröbner basis of I with respect to $\boldsymbol{w}$.

Proposition 2.1 ([9] Proposition 1.11) *For any term order $\prec$ and any ideal $I \subset R$, there exists a vector $\boldsymbol{w} \in \mathbb{N}^r$ such that* $\mathrm{in}_\prec(I) = \mathrm{in}_{\boldsymbol{w}}(I)$.

We use a term order given by a weight vector with a term order as a tie-breaker.

Definition 2.2 For a weight vector $\boldsymbol{w}$ and a term order $\prec$, we define a term order $\prec_{\boldsymbol{w}}$ as follows: $\boldsymbol{x}^{\boldsymbol{a}} \prec_{\boldsymbol{w}} \boldsymbol{x}^{\boldsymbol{b}}$ if $\boldsymbol{w} \cdot \boldsymbol{a} < \boldsymbol{w} \cdot \boldsymbol{b}$, or $\boldsymbol{w} \cdot \boldsymbol{a} = \boldsymbol{w} \cdot \boldsymbol{b}$ and $\boldsymbol{x}^{\boldsymbol{a}} \prec \boldsymbol{x}^{\boldsymbol{b}}$.

Proposition 2.3 ([9] Proposition 1.8) $\mathrm{in}_\prec(\mathrm{in}_{\boldsymbol{w}}(I)) = \mathrm{in}_{\prec_{\boldsymbol{w}}}(I)$.

A Gröbner basis of I with respect to $\prec_{\boldsymbol{w}}$ is a pseudo-Gröbner basis of I with respect to $\boldsymbol{w}$, but the converse is not true in general.

2.2 In the case of monomial ideals

In this subsection, we consider contractions of monomial ideals.

Definition 2.4 For a monomial ideal J, we denote by $\delta(J)$ the maximum of the degrees of a system of minimal generators of J.

We define the monomial ideal generated by standard monomials in the contraction ideal of a monomial ideal.

Definition 2.5 Let $I \subset S$ be a monomial ideal. Let $L_{\prec}^{(\mathcal{A})}(I)$ be a monomial ideal generated by all monomials in $\phi_{\mathcal{A}}^{-1}(I) \setminus \mathrm{in}_{\prec}(P_{\mathcal{A}})$. We denote by $M_{\prec}^{(\mathcal{A})}(I)$ the minimal system of monomial generators $L_{\prec}^{(\mathcal{A})}(I)$.

For monomial ideals $I_1, I_2 \subset S$, $L_{\prec}^{(\mathcal{A})}(I_1 + I_2) = L_{\prec}^{(\mathcal{A})}(I_1) + L_{\prec}^{(\mathcal{A})}(I_2)$ as $\phi_{\mathcal{A}}(u) \in I_1 + I_2$ if and only if $\phi_{\mathcal{A}}(u) \in I_1$ or $\phi_{\mathcal{A}}(u) \in I_2$ for a monomial u. In particular, if $I = \langle \boldsymbol{y}^{\boldsymbol{b}_1}, \dots, \boldsymbol{y}^{\boldsymbol{b}_n} \rangle$, then

$$L_{\prec}^{(\mathcal{A})}(I) = L_{\prec}^{(\mathcal{A})}\bigl(\boldsymbol{y}^{\boldsymbol{b}_1}\bigr) + \cdots + L_{\prec}^{(\mathcal{A})}\bigl(\boldsymbol{y}^{\boldsymbol{b}_n}\bigr).$$

Lemma 2.6 *Let $I \subset S$ be a monomial ideal. Then the following hold*:

(1) $\delta(L_{\prec}^{(\mathcal{A})}(I)) \le \delta(I)$.
(2) *If I is generated by square-free monomials, then $L_{\prec}^{(\mathcal{A})}(I)$ is generated by square-free monomials.*

Proof It is enough to treat in the case where I is a principal monomial ideal. Assume that I is generated by a monomial v. Let $u \in \phi_{\mathcal{A}}^{-1}(I) \setminus \mathrm{in}_{\prec}(P_{\mathcal{A}})$ be a monomial.

(1) Let $\delta := \delta(I) = \deg(v)$ and $m = \deg(u)$. Assume that $m > \delta$. It is enough to show that there exists a monomial $u' \in \phi_{\mathcal{A}}^{-1}(I)$ of degree strictly less than m such that u' divides u. We prove this by induction on δ. It is trivial in the case where $\delta = 0$. Assume that $\delta \ge 1$. We may assume, without loss of generality, that x_1 divides u. Let $\tilde{v} = \gcd(v, \phi_{\mathcal{A}}(x_1))$. If $\tilde{v} = 1$, we can take u/x_1 as u'. If $\tilde{v} \ne 1$, then $v/\tilde{v}$ is a monomial of degree at most $\delta - 1$, and $u/x_1 \in \phi_{\mathcal{A}}^{-1}(\langle v/\tilde{v} \rangle)$. By the hypothesis of induction, there exists a monomial $u'' \in \phi_{\mathcal{A}}^{-1}(v/\tilde{v})$ such that u'' divides u/x_1 and $\deg(u'') < m - 1$. Then $u' = x_1 \cdot u''$ is a monomial with desired conditions.
(2) We may assume, without loss of generality, that $v = \prod_{j=1}^{t} y_j$ for some $t \le s$. Let $\boldsymbol{x}^{\boldsymbol{a}} = \prod_{i=1}^{r} x_i^{a_i} \in \phi_{\mathcal{A}}^{-1}(I)$ be a monomial. It is enough to show that there exists a square-free monomial in $\phi_{\mathcal{A}}^{-1}(I)$ that divides $\boldsymbol{x}^{\boldsymbol{a}}$. For $1 \le k \le t$, there exists $1 \le i(k) \le r$ such that $a_{i(k)} \ne 0$ and y_k divides $\phi_{\mathcal{A}}(x_{i(k)})$. Let $\Lambda = \{i(1), \dots, i(t)\}$. Then $\prod_{i \in \Lambda} x_i$ is a square-free monomial in $\phi_{\mathcal{A}}^{-1}(I)$ which divides $\boldsymbol{x}^{\boldsymbol{a}}$.

□

Theorem 2.7 *Let $I \subset S$ be a monomial ideal. Let $G_{\mathcal{A}}$ be a Gröbner basis of $P_{\mathcal{A}}$ with respect to $\prec$. Then the following hold*:

(1) *$G_{\mathcal{A}} \cup M_{\prec}^{(\mathcal{A})}(I)$ is a Gröbner basis of $\phi_{\mathcal{A}}^{-1}(I)$ with respect to $\prec$.*
(2) $\delta(\mathrm{in}_{\prec}(\phi_{\mathcal{A}}^{-1}(I))) \le \max\{\delta(I), \delta(\mathrm{in}_{\prec}(P_{\mathcal{A}}))\}$.
(3) *If I and $\mathrm{in}_{\prec}(P_{\mathcal{A}})$ are generated by square-free monomials, then $\mathrm{in}_{\prec}(\phi_{\mathcal{A}}^{-1}(I))$ is also generated by square-free monomials.*

Proof It is clear that $G_{\mathcal{A}} \cup M_{\prec}^{(\mathcal{A})}(I) \subset \phi_{\mathcal{A}}^{-1}(I)$. Let $f \in \phi_{\mathcal{A}}^{-1}(I)$, and g be the remainder of f when divided by $G_{\mathcal{A}}$. Then any term of g is not in $\mathrm{in}_{\prec}(P_{\mathcal{A}})$. Hence different monomials appearing in g map to different monomials under $\phi_{\mathcal{A}}$. Since I is a monomial ideal, it follows that all terms of g are in $L_{\prec}^{(\mathcal{A})}(I)$. Thus the remainder of g when divided by $M_{\prec}^{(\mathcal{A})}(I)$ is zero. Therefore a remainder of f on division by $G \cup M_{\prec}^{(\mathcal{A})}(I)$ is zero. This implies (1).

We conclude (2) and (3) immediately from (1) and Lemma 2.6. $\square$

2.3 Reduction to the case of monomial ideals

Let I be an ideal of S. We fix a weight vector $\boldsymbol{w} = (w_1, \dots, w_s) \in \mathbb{N}^s$ on $S = K[y_1, \dots, y_s]$ such that $\mathrm{in}_{\boldsymbol{w}}(I)$ is a monomial ideal. We take

$$\phi_{\mathcal{A}}^* \boldsymbol{w} := \boldsymbol{w} \cdot \mathcal{A} = \big(\deg_{\boldsymbol{w}} \phi_{\mathcal{A}}(x_1), \dots, \deg_{\boldsymbol{w}} \phi_{\mathcal{A}}(x_r)\big)$$

as a weight vector on $R = K[x_1, \dots, x_r]$. We define $\mathbb{N}$-graded structures on R and S by $\boldsymbol{w}$ and $\phi_{\mathcal{A}}^* \boldsymbol{w}$, respectively; $R = \bigoplus_{i \in \mathbb{N}} R_i$ and $S = \bigoplus_{i \in \mathbb{N}} S_i$ where R_i and S_i are the K-vector spaces spanned by all monomials of weight i with respect to $\phi_{\mathcal{A}}^* \boldsymbol{w}$ and $\boldsymbol{w}$, respectively. Then $\phi_{\mathcal{A}}$ is a homogeneous homomorphism of graded rings of degree 0, that is, $\phi_{\mathcal{A}}(R_i) \subset S_i$. Hence $P_{\mathcal{A}}$ is a $\phi_{\mathcal{A}}^* \boldsymbol{w}$-homogeneous ideal.

In the case where the equality $\mathrm{in}_{\phi_{\mathcal{A}}^* \boldsymbol{w}}(\phi_{\mathcal{A}}^{-1}(I)) = \phi_{\mathcal{A}}^{-1}(\mathrm{in}_{\boldsymbol{w}}(I))$ holds, we can reduce to the case of monomial ideal. It is easy to show that $\mathrm{in}_{\phi_{\mathcal{A}}^* \boldsymbol{w}}(\phi_{\mathcal{A}}^{-1}(I)) \subset \phi_{\mathcal{A}}^{-1}(\mathrm{in}_{\boldsymbol{w}}(I))$ (see Lemma 2.11 (1)). In the case of toric fiber product, the converse inclusion holds true [10]. However, the equality $\mathrm{in}_{\phi_{\mathcal{A}}^* \boldsymbol{w}}(\phi_{\mathcal{A}}^{-1}(I)) = \phi_{\mathcal{A}}^{-1}(\mathrm{in}_{\boldsymbol{w}}(I))$ does not hold in general.

Example 2.8 Let $R = K[x_1, x_2]$ and $S = K[y_1, y_2]$ be polynomial rings, $\boldsymbol{w} = (2, 1)$ a weight vector on S, and $\mathcal{A} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\phi_{\mathcal{A}}(x_1) = y_1$, $\phi_{\mathcal{A}}(x_2) = y_1 y_2$, and $\phi_{\mathcal{A}}^* \boldsymbol{w} = (2, 3)$. Let I be an ideal generated by $f = y_1 + y_2 \in S$. Then

$$\phi_{\mathcal{A}}^{-1}(I) = \langle x_1 - y_1, x_2 - y_1 y_2, f \rangle \cap R = \big\langle x_1^2 + x_2 \big\rangle$$

and thus $\mathrm{in}_{\phi_{\mathcal{A}}^* \boldsymbol{w}}(\phi_{\mathcal{A}}^{-1}(I)) = \langle x_1^2 \rangle$. On the other hand, $\mathrm{in}_{\boldsymbol{w}}(f) = y_1$ and thus

$$\phi_{\mathcal{A}}^{-1}\big(\mathrm{in}_{\boldsymbol{w}}(I)\big) = \langle x_1, x_2 \rangle.$$

Therefore $\mathrm{in}_{\phi_{\mathcal{A}}^* \boldsymbol{w}}(\phi_{\mathcal{A}}^{-1}(I)) \neq \phi_{\mathcal{A}}^{-1}(\mathrm{in}_{\boldsymbol{w}}(I))$.

Theorem 2.9 *Let the notation be as above. Suppose, in addition, that the equality*

$$\mathrm{in}_{\phi_{\mathcal{A}}^* \boldsymbol{w}}\big(\phi_{\mathcal{A}}^{-1}(I)\big) = \phi_{\mathcal{A}}^{-1}\big(\mathrm{in}_{\boldsymbol{w}}(I)\big)$$

holds. Then the following hold:

(1) $\mathrm{in}_{\prec_{\phi_{\mathcal{A}}^* \boldsymbol{w}}}(\phi_{\mathcal{A}}^{-1}(I)) = \mathrm{in}_{\prec}(P_{\mathcal{A}}) + L_{\prec}^{(\mathcal{A})}(\mathrm{in}_{\boldsymbol{w}}(I))$.

(2) $\delta(\mathrm{in}_{\prec_{\phi_{\mathcal{A}}^* \boldsymbol{w}}}(\phi_{\mathcal{A}}^{-1}(I))) \leq \max\{\delta(\mathrm{in}_{\boldsymbol{w}}(I)), \delta(\mathrm{in}_{\prec}(P_{\mathcal{A}}))\}$.

(3) *If both of* $\mathrm{in}_{\boldsymbol{w}}(I)$ *and* $\mathrm{in}_{\prec}(P_{\mathcal{A}})$ *are generated by square-free monomials, then* $\mathrm{in}_{\prec_{\phi_{\mathcal{A}}^* \boldsymbol{w}}}(\phi_{\mathcal{A}}^{-1}(I))$ *is generated by square-free monomials.*

Proof Since

$$\operatorname{in}_{\prec_{\phi^*_{\mathcal{A}}\boldsymbol{w}}}\bigl(\phi_{\mathcal{A}}^{-1}(I)\bigr) = \operatorname{in}_{\prec}\bigl(\operatorname{in}_{\phi^*_{\mathcal{A}}\boldsymbol{w}}\bigl(\phi_{\mathcal{A}}^{-1}(I)\bigr)\bigr) = \operatorname{in}_{\prec}\bigl(\phi_{\mathcal{A}}^{-1}\bigl(\operatorname{in}_{\boldsymbol{w}}(I)\bigr)\bigr),$$

and $\operatorname{in}_{\boldsymbol{w}}(I)$ is a monomial ideal, we conclude the assertion by applying Theorem 2.7 to the monomial ideal $\operatorname{in}_{\boldsymbol{w}}(I)$. □

In the rest of this paper, we investigate when the equality $\operatorname{in}_{\phi^*_{\mathcal{A}}\boldsymbol{w}}(\phi_{\mathcal{A}}^{-1}(I)) = \phi_{\mathcal{A}}^{-1}(\operatorname{in}_{\boldsymbol{w}}(I))$ holds.

2.4 Pseudo-Gröbner bases

We naturally extend the definition of pseudo-Gröbner bases to ideals of $\mathbb{N}$-graded rings.

Definition 2.10 Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be an $\mathbb{N}$-graded ring and $f = \sum_i f_i \in A$ $(f_i \in A_i)$. We define $\operatorname{in}_A(f) = f_d$ where $d = \deg(f) = \max\{i \mid f_i \neq 0\}$. For an ideal $I \subset A$, we define

$$\operatorname{in}_A(I) = \bigl\langle \operatorname{in}_A(f) \mid f \in I \bigr\rangle \subset A.$$

We say that a finite collection of polynomials $G \subset I$ is a pseudo-Gröbner basis of I if $\langle \operatorname{in}_A(g) \mid g \in G\rangle = \operatorname{in}_A(I)$.

It is easy to show that a pseudo-Gröbner basis of I generates I. Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ and $B = \bigoplus_{i\in\mathbb{N}} B_i$ be graded rings, and $\phi : A \to B$ a graded ring homomorphism of degree 0, that is, $\phi(A_i) \subset B_i$ for all i.

Lemma 2.11 *Let I be an ideal of B. Then the following hold*:

(1) $\operatorname{in}_A(\phi^{-1}(I)) \subset \phi^{-1}(\operatorname{in}_B(I))$.
(2) *If ϕ is surjective*, *then* $\operatorname{in}_A(\phi^{-1}(I)) = \phi^{-1}(\operatorname{in}_B(I))$.

Proof (1) Let $f = \sum_{i=1}^d f_i \in \phi^{-1}(I)$ where $f_i \in A_i$ and $f_d \neq 0$. Then $\operatorname{in}_A(f) = f_d$, $\phi(f) = \sum_{i=1}^d \phi(f_i) \in I$, and $\phi(f_i) \in B_i$. Hence $\phi(f_d) = 0$ or $\phi(f_d) = \operatorname{in}_B(\phi(f)) \in \operatorname{in}_B(I)$, and thus $\phi(\operatorname{in}_A(f)) \in \operatorname{in}_B(I)$.

(2) Since $A/\operatorname{Ker}\phi \cong B$ as $\mathbb{N}$-graded rings, and $\phi^{-1}(I)/\operatorname{Ker}\phi \cong I$ as $\mathbb{N}$-graded ideals, $\operatorname{in}_A(\phi^{-1}(I))$ coincides with $\phi^{-1}(\operatorname{in}_B(I))$ module $\operatorname{Ker}\phi$. Since $\operatorname{Ker}\phi$ is a homogeneous ideal of A, $\operatorname{Ker}\phi \subset \operatorname{in}_A(\phi^{-1}(I))$, and it is clear that $\operatorname{Ker}\phi \subset \phi^{-1}(\operatorname{in}_B(I))$. Hence we conclude the assertion. □

2.5 Sufficient condition so that initial commutes with contraction

Now, we return to the problem when the equality $\operatorname{in}_{\phi^*_{\mathcal{A}}\boldsymbol{w}}(\phi_{\mathcal{A}}^{-1}(I)) = \phi_{\mathcal{A}}^{-1}(\operatorname{in}_{\boldsymbol{w}}(I))$ holds. The homomorphism $\phi_{\mathcal{A}} : R \to S$ can be decomposed into the surjection $R \to K[\mathcal{A}]$ and the inclusion $K[\mathcal{A}] \hookrightarrow S$. Since $K[\mathcal{A}]$ has an $\mathbb{N}$-graded ring structure induced by $\boldsymbol{w}$, we can consider pseudo-Gröbner bases of ideals of $K[\mathcal{A}]$ in the sense of Definition 2.10. Note that $\mathrm{in}_{K[\mathcal{A}]}(f)=\mathrm{in}_{\boldsymbol{w}}(f)$ for $f\in K[\mathcal{A}]$. By Lemma 2.11, the equality

$$\mathrm{in}_{\phi^*_{\mathcal{A}}\boldsymbol{w}}\big(\phi_{\mathcal{A}}^{-1}(I)\big)=\phi_{\mathcal{A}}^{-1}\big(\mathrm{in}_{\boldsymbol{w}}(I)\big)$$

holds if and only if the equality

$$\mathrm{in}_{K[\mathcal{A}]}\big(I\cap K[\mathcal{A}]\big)=\mathrm{in}_{\boldsymbol{w}}(I)\cap K[\mathcal{A}]$$

holds. To obtain a sufficient condition for this equality to hold, we define a class of subrings of a graded ring.

Definition 2.12 Let $\mathfrak{G}$ be a semigroup, and $S=\bigoplus_{\boldsymbol{v}\in\mathfrak{G}}S_{\boldsymbol{v}}$ a $\mathfrak{G}$-graded ring. For a subsemigroup $\mathfrak{H}\subset\mathfrak{G}$, we define

$$S^{(\mathfrak{H})}=\bigoplus_{\boldsymbol{v}\in\mathfrak{H}}S_{\boldsymbol{v}},$$

a graded subring of S.

We consider the case where S is a multi-graded polynomial ring; let $S=K[\boldsymbol{y}]=K[y_1,\dots,y_s]$ be a $\mathbb{Z}^d$-graded polynomial ring with $\deg_{\mathbb{Z}^d}(y_i)=\boldsymbol{v}_i$, $\boldsymbol{v}_i\in\mathbb{Z}^d$. Set $\mathcal{V}=(\boldsymbol{v}_1,\dots,\boldsymbol{v}_s)$. Let $\mathfrak{H}$ be a finitely generated subsemigroup of $\mathbb{Z}^d$. Then

$$\begin{aligned}S^{(\mathfrak{H})}&=K\big[\boldsymbol{y}^{\boldsymbol{a}}\mid \boldsymbol{a}\in\mathbb{N}^s,\deg_{\mathbb{Z}^d}\big(\boldsymbol{y}^{\boldsymbol{a}}\big)\in\mathfrak{H}\big]\\&=K\big[\boldsymbol{y}^{\boldsymbol{a}}\mid \boldsymbol{a}={}^t(a_1,\dots,a_s)\in\mathbb{N}^s,\mathcal{V}\cdot\boldsymbol{a}=a_1\boldsymbol{v}_1+\cdots+a_s\boldsymbol{v}_s\in\mathfrak{H}\big].\end{aligned}$$

We will prove that $S^{(\mathfrak{H})}$ is Noetherian, equivalently, $\{\boldsymbol{a}\in\mathbb{N}^s\mid\mathcal{V}\cdot\boldsymbol{a}\in\mathfrak{H}\}$ is finitely generated as a semigroup.

Definition 2.13 We say that a semigroup $\mathfrak{H}\subset\mathbb{Z}^d$ is *normal* if $\mathfrak{H}=L\cap C$ for some sublattice $L\subset\mathbb{Z}^d$ and finitely generated rational cone $C\subset\mathbb{R}^d$.

It is well-known that normal semigroups are finitely generated (Gordan's Lemma).

Lemma 2.14 *Let $\mathfrak{H}_1,\mathfrak{H}_2\subset\mathbb{Z}^d$ be finitely generated semigroups. Then $\mathfrak{H}_1\cap\mathfrak{H}_2$ is also a finitely generated semigroup.*

Proof It is enough to show that $K[\mathfrak{H}_1\cap\mathfrak{H}_2]=K[\boldsymbol{x}^a\mid\boldsymbol{a}\in\mathfrak{H}_1\cap\mathfrak{H}_2]$ is a Noetherian ring. Let $\overline{\mathfrak{H}_i}=\mathbb{Z}\mathfrak{H}_i\cap\mathbb{R}_{\geq 0}\mathfrak{H}_i$ for $i=1,2$. Then $\overline{\mathfrak{H}_1}$, $\overline{\mathfrak{H}_2}$, and $\overline{\mathfrak{H}_1}\cap\overline{\mathfrak{H}_2}$ are finitely generated semigroups by Gordan's Lemma. Thus there exists $0\neq d_i\in\mathbb{N}$ such that $d_i\overline{\mathfrak{H}_i}\subset\mathfrak{H}_i$. Let $d=d_1d_2$. Then $K[d(\overline{\mathfrak{H}_1}\cap\overline{\mathfrak{H}_2})]\subset K[\mathfrak{H}_1\cap\mathfrak{H}_2]\subset K[\overline{\mathfrak{H}_1}\cap\overline{\mathfrak{H}_2}]$. Since $K[d(\overline{\mathfrak{H}_1}\cap\overline{\mathfrak{H}_2})]$ is Noetherian, and $K[\overline{\mathfrak{H}_1}\cap\overline{\mathfrak{H}_2}]$ is a finitely generated $K[d(\overline{\mathfrak{H}_1}\cap\overline{\mathfrak{H}_2})]$-module, $K[\mathfrak{H}_1\cap\mathfrak{H}_2]$ is also a finitely generated $K[d(\overline{\mathfrak{H}_1}\cap\overline{\mathfrak{H}_2})]$-module. Thus $K[\mathfrak{H}_1\cap\mathfrak{H}_2]$ is a Noetherian ring. Therefore $\mathfrak{H}_1\cap\mathfrak{H}_2$ is a finitely generated semigroup. □

Lemma 2.15 *Let $\mathcal{V}=(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)$, $\boldsymbol{v}_i\in\mathbb{Z}^m$, be an $m\times n$ integer matrix, and $\mathfrak{H}\subset\mathbb{Z}^m$ a finitely generated semigroup. Then $\{\boldsymbol{a}\in\mathbb{N}^n\mid\mathcal{V}\cdot\boldsymbol{a}\in\mathfrak{H}\}$ is a finitely generated semigroup.*

Proof By Lemma 2.14, $(\sum_{i=1}^{n}\mathbb{Z}\boldsymbol{v}_i)\cap\mathfrak{H}$ is a finitely generated semigroup. Thus there exist $\boldsymbol{a}_1,\dots,\boldsymbol{a}_\ell\in\mathbb{Z}^m$ such that $\{\mathcal{V}\cdot\boldsymbol{a}_1,\dots,\mathcal{V}\cdot\boldsymbol{a}_\ell\}$ is a system of generators of the semigroup $(\sum_{i=1}^{n}\mathbb{Z}\boldsymbol{v}_i)\cap\mathfrak{H}$. Let $L=\{\boldsymbol{a}\in\mathbb{Z}^m\mid\mathcal{V}\cdot\boldsymbol{a}=0\}$. Then $\{\boldsymbol{a}\in\mathbb{Z}^n\mid\mathcal{V}\cdot\boldsymbol{a}\in\mathfrak{H}\}=L+\sum_{i=1}^{\ell}\mathbb{N}\cdot\boldsymbol{a}_i$, and it is a finitely generated semigroup. Thus $\{\boldsymbol{a}\in\mathbb{N}^n\mid\mathcal{V}\cdot\boldsymbol{a}\in\mathfrak{H}\}=\{\boldsymbol{a}\in\mathbb{Z}^n\mid\mathcal{V}\cdot\boldsymbol{a}\in\mathfrak{H}\}\cap\mathbb{N}^n$ is also a finitely generated semigroup by Lemma 2.14. □

By Lemma 2.15, $S^{(\mathfrak{H})}$ is Noetherian for a $\mathbb{Z}^d$-graded polynomial ring S and a finitely generated semigroup $\mathfrak{H}\subset\mathbb{Z}^d$.

Notation 2.16 *Let $S=K[y_1,\dots,y_s]$, and $d>0$ be a positive integer. We fix a $d\times s$ integer matrix $\mathcal{V}=(\boldsymbol{v}_1,\dots,\boldsymbol{v}_s)$ with the column vectors $\boldsymbol{v}_1,\dots,\boldsymbol{v}_s\in\mathbb{Z}^d$. We define a $\mathbb{Z}^d$-graded structure on S by setting $\deg_{\mathbb{Z}^d}(y_i)=\boldsymbol{v}_i$. Then $\deg_{\mathbb{Z}^d}\boldsymbol{y}^{\boldsymbol{a}}=\mathcal{V}\cdot\boldsymbol{a}$ for $\boldsymbol{a}\in\mathbb{N}^s$, and $S=\bigoplus_{\boldsymbol{v}\in\mathbb{Z}^d}S_{\boldsymbol{v}}$ where $S_{\boldsymbol{v}}$ is the K-vector space spanned by all monomials in S of multi-degree $\boldsymbol{v}$. Let $\mathfrak{H}$ be a finitely generated subsemigroup of $\mathbb{Z}^d$. Let $\mathcal{A}_{\mathfrak{H}}=\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_r\}\subset\mathbb{N}^s$ be a system of generators of $\{\boldsymbol{a}={}^t(a_1,\dots,a_s)\mid\mathcal{V}\cdot\boldsymbol{a}\in\mathfrak{H}\}$ as a semigroup. Then $S^{(\mathfrak{H})}=K[\mathcal{A}_{\mathfrak{H}}]$. Let $R^{[\mathfrak{H}]}=K[x_1,\dots,x_r]$ a polynomial ring over K, and $\phi_{\mathcal{A}_{\mathfrak{H}}}:R^{[\mathfrak{H}]}\to S$, $x_i\mapsto\boldsymbol{y}^{\boldsymbol{a}_i}$, the monomial homomorphism corresponding to $\mathcal{A}_{\mathfrak{H}}$.*

We remark that $\mathcal{A}_{\mathfrak{H}}$ is not always a standard graded configuration.

Definition 2.17 For $\boldsymbol{v}\in\mathbb{Z}^d$, we define

$$C_{\mathfrak{H}}(\boldsymbol{v})=\bigoplus_{\boldsymbol{u}\in(-\boldsymbol{v}+\mathfrak{H})\cap\mathbb{Z}^d}S_{\boldsymbol{u}},$$

a $\mathbb{Z}^d$-graded $K[\mathcal{A}_{\mathfrak{H}}]$-submodule of S. Let $\Gamma_{\mathfrak{H}}(\boldsymbol{v})$ be the minimal system of generators of $C_{\mathfrak{H}}(\boldsymbol{v})$ as an $K[\mathcal{A}_{\mathfrak{H}}]$-module consisting of monomials in S.

If $S_{\boldsymbol{v}}\neq 0$, then $C_{\mathfrak{H}}(\boldsymbol{v})\cong f\cdot C_{\mathfrak{H}}(\boldsymbol{v})\subset K[\mathcal{A}_{\mathfrak{H}}]$ for any $0\neq f\in S_{\boldsymbol{v}}$. Hence $C_{\mathfrak{H}}(\boldsymbol{v})$ is isomorphic to an ideal of $K[\mathcal{A}_{\mathfrak{H}}]$ up to shift of grading. In particular, $C_{\mathfrak{H}}(\boldsymbol{v})$ is finitely generated over $K[\mathcal{A}_{\mathfrak{H}}]$.

Lemma 2.18 *Let the notation be as in Notation 2.16. Fix a weight vector $\boldsymbol{w}\in\mathbb{N}^s$ on S, and regard S and $K[\mathcal{A}_{\mathfrak{H}}]$ as $\mathbb{N}$-graded rings. Let I be a $\mathbb{Z}^d$-graded ideal with $\mathbb{Z}^d$-homogeneous system of generators $F=\{f_1,\dots,f_\ell\}$ with $\deg_{\mathbb{Z}^d}(f_i)=\boldsymbol{v}_i\in\mathbb{Z}^d$. Then the following hold*:

(1) *$I\cap K[\mathcal{A}_{\mathfrak{H}}]$ is generated by $\{\boldsymbol{y}^{\boldsymbol{a}}\cdot f_i\mid 1\le i\le\ell,\ \ \boldsymbol{y}^{\boldsymbol{a}}\in\Gamma_{\mathfrak{H}}(\boldsymbol{v}_i)\}$.*
(2) $\mathrm{in}_{K[\mathcal{A}_{\mathfrak{H}}]}(I\cap K[\mathcal{A}_{\mathfrak{H}}])=\mathrm{in}_{\boldsymbol{w}}(I)\cap K[\mathcal{A}_{\mathfrak{H}}]$.
(3) *If F is a pseudo-Gröbner basis of I with respect to $\boldsymbol{w}$, then*

$$\left\{\boldsymbol{y}^{\boldsymbol{a}}\cdot f_i\mid 1\le i\le\ell,\ \ \boldsymbol{y}^{\boldsymbol{a}}\in\Gamma_{\mathfrak{H}}(\boldsymbol{v}_i)\right\}$$

is a pseudo-Gröbner basis of $I\cap K[\mathcal{A}_{\mathfrak{H}}]$ in sense of Definition 2.10.

Proof (1) For $1 \le i \le \ell$ and $\boldsymbol{y}^{\boldsymbol{a}} \in \Gamma_{\mathfrak{H}}(\boldsymbol{v}_i)$, $\boldsymbol{y}^{\boldsymbol{a}} \cdot f_i$ is a $\mathbb{Z}^d$-homogeneous element whose degree is in $\mathfrak{H}$ by the definition of $\Gamma_{\mathfrak{H}}(\boldsymbol{v}_i)$, thus $\boldsymbol{y}^{\boldsymbol{a}} \cdot f_i \in K[\mathcal{A}]$. Let J be the ideal of $K[\mathcal{A}_{\mathfrak{H}}]$ generated by $\{\boldsymbol{y}^{\boldsymbol{a}} \cdot f_i \mid 1 \le i \le \ell, \quad \boldsymbol{y}^{\boldsymbol{a}} \in \Gamma_{\mathfrak{H}}(\boldsymbol{v}_i)\}$. Then $J \subset I \cap K[\mathcal{A}_{\mathfrak{H}}]$.

For the converse inclusion, take a $\mathbb{Z}^d$-homogeneous element $g \in I \cap K[\mathcal{A}_{\mathfrak{H}}]$, $\deg(g) = \boldsymbol{v}$, and write $g = \sum h_i f_i$ where h_i's are $\mathbb{Z}^d$-homogeneous elements with $\deg(h_i f_i) = \boldsymbol{v}$. Since $\boldsymbol{v} \in \mathfrak{H}$, we have $h_i f_i \in I \cap K[\mathcal{A}_{\mathfrak{H}}]$, and thus $\deg_{\mathbb{Z}^d}(h_i) + \boldsymbol{v}_i \in \mathfrak{H}$. Hence $h_i \in C_{\mathfrak{H}}(\boldsymbol{v}_i)$. Therefore it follows that $g \in J$.

(2), (3) Assume that $\{f_1, \dots, f_\ell\}$ is a pseudo-Gröbner basis with respect to $\boldsymbol{w}$. Since $\mathrm{in}_{\boldsymbol{w}}(I)$ is also $\mathbb{Z}^d$-graded ideal and $\mathrm{in}_{\boldsymbol{w}}(f_i) \in S_{\boldsymbol{v}_i}$, the contraction ideal $\mathrm{in}_{\boldsymbol{w}}(I) \cap K[\mathcal{A}_{\mathfrak{H}}]$ is generated by $\{\boldsymbol{y}^{\boldsymbol{a}} \cdot \mathrm{in}_{\boldsymbol{w}}(f_i) \mid 1 \le i \le \ell, \quad \boldsymbol{y}^{\boldsymbol{a}} \in \Gamma_{\mathfrak{H}}(\boldsymbol{v}_i)\}$. As $\boldsymbol{y}^{\boldsymbol{a}} \cdot \mathrm{in}_{\boldsymbol{w}}(f_i) = \mathrm{in}_{\boldsymbol{w}}(\boldsymbol{y}^{\boldsymbol{a}} \cdot f_i)$, we conclude $\mathrm{in}_{K[\mathcal{A}_{\mathfrak{H}}]}(I \cap K[\mathcal{A}_{\mathfrak{H}}]) = \mathrm{in}_{\boldsymbol{w}}(I) \cap K[\mathcal{A}_{\mathfrak{H}}]$. □

Proposition 2.19 *Let the notation be as in Notation 2.16. Let $\boldsymbol{w}' := \phi^*_{\mathcal{A}_{\mathfrak{H}}} \boldsymbol{w}$. Then*

$$\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}\bigl(\mathrm{in}_{\boldsymbol{w}}(I)\bigr) = \mathrm{in}_{\boldsymbol{w}'}\bigl(\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}(I)\bigr).$$

Proof We also denote by $\phi_{\mathcal{A}_{\mathfrak{H}}}$ the surjection $R \to K[\mathcal{A}_{\mathfrak{H}}]$. By Lemma 2.18,

$$\mathrm{in}_{K[\mathcal{A}_{\mathfrak{H}}]}\bigl(I \cap K[\mathcal{A}_{\mathfrak{H}}]\bigr) = \mathrm{in}_{\boldsymbol{w}}(I) \cap K[\mathcal{A}_{\mathfrak{H}}],$$

and by Lemma 2.11 (2),

$$\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}\bigl(\mathrm{in}_{K[\mathcal{A}_{\mathfrak{H}}]}\bigl(I \cap K[\mathcal{A}_{\mathfrak{H}}]\bigr)\bigr) = \mathrm{in}_{\boldsymbol{w}'}\bigl(\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}\bigl(I \cap K[\mathcal{A}_{\mathfrak{H}}]\bigr)\bigr).$$

Thus $\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}(\mathrm{in}_{\boldsymbol{w}}(I) \cap K[\mathcal{A}_{\mathfrak{H}}]) = \mathrm{in}_{\boldsymbol{w}'}(\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}(I \cap K[\mathcal{A}_{\mathfrak{H}}]))$. Since $\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}(J \cap K[\mathcal{A}_{\mathfrak{H}}]) = \phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}(J)$ for any ideal $J \subset S$, we conclude that $\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}(\mathrm{in}_{\boldsymbol{w}}(I)) = \mathrm{in}_{\boldsymbol{w}'}(\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}(I))$. □

Theorem 2.20 *Let $S = K[y_1, \dots, y_s]$ be a $\mathbb{Z}^d$-graded polynomial ring with $\deg_{\mathbb{Z}^d}(y_i) = \boldsymbol{v}_i \in \mathbb{Z}^d$. Let $\mathfrak{H} \subset \mathbb{Z}^d$ be a finitely generated subsemigroup, and $\mathcal{A}_{\mathfrak{H}} = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_r\} \subset \mathbb{N}^s$ a system of generators of the semigroup $\{\boldsymbol{a} \in \mathbb{N}^s \mid \deg_{\mathbb{Z}^d}(\boldsymbol{y}^{\boldsymbol{a}}) \in \mathfrak{H}\}$. We set $R^{[\mathfrak{H}]} = K[x_1, \dots, x_r]$, and $\phi_{\mathcal{A}_{\mathfrak{H}}} : R^{[\mathfrak{H}]} \to S$, $x_j \mapsto \boldsymbol{y}^{\boldsymbol{a}_j}$. Let $I \subset S$ a $\mathbb{Z}^d$-graded ideal. Take a term order $\prec$ on $R^{[\mathfrak{H}]}$, and a weight vector $\boldsymbol{w} \in \mathbb{N}^s$ on S such that $\mathrm{in}_{\boldsymbol{w}}(I)$ is a monomial ideal. Let $\boldsymbol{w}' := \phi^*_{\mathcal{A}_{\mathfrak{H}}} \boldsymbol{w}$. Then the following hold:*

(1) $\delta(\mathrm{in}_{\prec_{\boldsymbol{w}'}}(\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}(I))) \le \max\{\delta(\mathrm{in}_{\boldsymbol{w}}(I)), \delta(\mathrm{in}_{\prec}(P_{\mathcal{A}_{\mathfrak{H}}}))\}$.
(2) *If both of $\mathrm{in}_{\boldsymbol{w}}(I)$ and $\mathrm{in}_{\prec}(P_{\mathcal{A}_{\mathfrak{H}}})$ are generated by square-free monomials, then $\mathrm{in}_{\prec_{\boldsymbol{w}'}}(\phi^{-1}_{\mathcal{A}_{\mathfrak{H}}}(I))$ is also generated by square-free monomials.*

Proof The assertions follow from Theorem 2.9 and Proposition 2.19. □

Note that Theorem 2.20 holds true even if $\mathcal{A}_{\mathfrak{H}}$ is not a standard graded configuration.

2.6 Pseudo-Gröbner bases and Gröbner bases of contraction ideals

We will give a method to construct a pseudo Gröbner basis of $\phi_{\mathcal{A}_{\mathfrak{H}}}^{-1}(I)$, and investigate when it become a Gröbner basis. First, we fix the notation in this subsection.

Notation 2.21 *Let* $S = K[y_1, \dots, y_s]$, $\mathcal{A}_{\mathfrak{H}}$, $R^{[\mathfrak{H}]} = K[x_1, \dots, x_r]$, *and* $\phi_{\mathcal{A}_{\mathfrak{H}}} : R^{[\mathfrak{H}]} \to S$ *be as in Notation 2.16. Let* $\prec$ *be a term order on* $R^{[\mathfrak{H}]}$, *and* $G_{\mathcal{A}_{\mathfrak{H}}}$ *the reduced Gröbner basis of* $P_{\mathcal{A}_{\mathfrak{H}}}$ *with respect to* $\prec$. *Let* $I \subset S$ *be a* $\mathbb{Z}^d$*-graded ideal, and fix a weight vector* $\boldsymbol{w} \in \mathbb{N}^s$ *on* S *such that* $\mathrm{in}_{\boldsymbol{w}}(I)$ *is a monomial ideal. Let* $\boldsymbol{w}' := \phi_{\mathcal{A}_{\mathfrak{H}}}^* \boldsymbol{w} = \boldsymbol{w} \cdot \mathcal{A}$.

Definition 2.22 For $0 \neq q \in K[\mathcal{A}_{\mathfrak{H}}]$, there is the unique polynomial $\tilde{q} \in R^{[\mathfrak{H}]}$ such that $\phi_{\mathcal{A}_{\mathfrak{H}}}(\tilde{q}) = q$ and any term of $\tilde{q}$ is not in $\mathrm{in}_{\prec}(P_{\mathcal{A}_{\mathfrak{H}}})$. We define $\mathrm{lift}_{\prec}(q) = \tilde{q}$. For a subset $Q \subset K[\mathcal{A}_{\mathfrak{H}}]$, we define $\mathrm{lift}_{\prec}(Q) = \{\mathrm{lift}_{\prec}(q) \mid q \in Q\}$.

Remark 2.23

(1) For $q \in K[\mathcal{A}_{\mathfrak{H}}]$, take a polynomial $p \in R^{[\mathfrak{H}]}$ such that $\phi_{\mathcal{A}_{\mathfrak{H}}}(p) = q$. Then $\mathrm{lift}_{\prec}(q)$ is the remainder of p on division by $G_{\mathcal{A}_{\mathfrak{H}}}$ with respect to $\prec$.
(2) If $u \in K[\mathcal{A}_{\mathfrak{H}}]$ is a monomial, then $\mathrm{lift}_{\prec}(u)$ is a monomial such that $\deg_{\boldsymbol{w}}(u) = \deg_{\boldsymbol{w}'}(\mathrm{lift}_{\prec}(u))$ since the remainder of a monomial on division by a $\boldsymbol{w}'$-homogeneous binomial ideal is a monomial with the same degree. Therefore, if Q is a set of monomials, then so is $\mathrm{lift}_{\prec}(Q)$.
(3) Let $q \in K[\mathcal{A}_{\mathfrak{H}}] \subset S$. If $\mathrm{in}_{\boldsymbol{w}}(q)$ is a monomial, then $\mathrm{in}_{\boldsymbol{w}'}(\mathrm{lift}_{\prec}(q))$ is also a monomial and $\deg_{\boldsymbol{w}'}(\mathrm{lift}_{\prec}(q)) = \deg_{\boldsymbol{w}}(q)$ by (2). Furthermore, $\mathrm{in}_{\boldsymbol{w}'}(\mathrm{lift}_{\prec}(q)) = \mathrm{lift}_{\prec}(\mathrm{in}_{\boldsymbol{w}}(q))$, and $\phi_{\mathcal{A}_{\mathfrak{H}}}(\mathrm{in}_{\boldsymbol{w}'}(\mathrm{lift}_{\prec}(q))) = \mathrm{in}_{\boldsymbol{w}}(q)$.
(4) Since $R^{[\mathfrak{H}]} / \mathrm{Ker}\, \phi_{\mathcal{A}_{\mathfrak{H}}} \cong K[\mathcal{A}_{\mathfrak{H}}]$ as $\mathbb{N}$-graded rings, for an ideal J of $K[\mathcal{A}_{\mathfrak{H}}]$ with a system of generators Q, we have $\phi_{\mathcal{A}_{\mathfrak{H}}}^{-1}(J) = \langle \mathrm{lift}_{\prec}(Q) \rangle + \mathrm{Ker}\, \phi_{\mathcal{A}_{\mathfrak{H}}}$.

Proposition 2.24 *Let* J *be an ideal in* $K[\mathcal{A}_{\mathfrak{H}}]$ *with a pseudo-Gröbner basis* $Q = \{q_1, \dots, q_\ell\}$ (*in the sense of Definition 2.10 with a graded ring structure given by* $\boldsymbol{w}$). *Then* $\mathrm{lift}_{\prec}(F) \cup G_{\mathcal{A}_{\mathfrak{H}}}$ *is a pseudo-Gröbner basis of* $\phi_{\mathcal{A}_{\mathfrak{H}}}^{-1}(J)$ *with respect to* $\boldsymbol{w}'$.

Proof This easily follows from the above remarks. □

Combining Proposition 2.24 and Lemma 2.18, we can obtain a pseudo-Gröbner basis of $\phi_{\mathcal{A}_{\mathfrak{H}}}^{-1}(I)$.

Definition 2.25 For a finite set $F = \{f_1, \dots, f_\ell\} \subset S$, $\deg_{\mathbb{Z}^d} f_i = \boldsymbol{v}_i$, we define

$$\mathrm{Lift}_{\prec}(F) := \mathrm{lift}_{\prec}\bigl(\bigl\{\boldsymbol{y}^{\boldsymbol{a}} \cdot f_i \mid 1 \le i \le \ell,\ \ \boldsymbol{y}^{\boldsymbol{a}} \in \Gamma_{\mathfrak{H}}(\boldsymbol{v}_i)\bigr\}\bigr).$$

The notation $\mathrm{Lift}_{\prec}(F)$ was introduced in the case of toric fiber product by Sullivant in [10].

Proposition 2.26 *Let the notation be as in Notation 2.21. Let $F = \{f_1, \ldots, f_\ell\}$ a pseudo-Gröbner basis of I with respect to $\boldsymbol{w}$ consisting of $\mathbb{Z}^d$-homogeneous polynomials. Then the union $\mathrm{Lift}_\prec(F) \cup G_{\mathcal{A}_\mathfrak{H}}$ is a pseudo-Gröbner basis of $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)$ with respect to $\boldsymbol{w}'$.*

We remark that $G_{\mathcal{A}_\mathfrak{H}} \cup \mathrm{Lift}_\prec(F)$ is not always a Gröbner basis even if I is a principal monomial ideal.

Example 2.27 Let $S = K[y_1, y_2, y_3]$ be an $\mathbb{N}$-graded ring with $\deg(y_1) = \deg(y_2) = \deg(y_3) = 1$. Let $\mathfrak{H} = \{2n \mid n \in \mathbb{N}\} \subset \mathbb{N}$. Then $\mathcal{A}_\mathfrak{H} = \{{}^t(2,0), {}^t(1,1), {}^t(0,2)\}$, $R^{[\mathfrak{H}]} = K[x_1, x_2, x_3]$, and $\phi_{\mathcal{A}_\mathfrak{H}} : R^{[\mathfrak{H}]} \to S$, $x_1 \mapsto y_1^2$, $x_2 \mapsto y_1 y_2$, $x_3 \mapsto y_2^2$. Let $\prec$ be the lexicographic order on $R^{[\mathfrak{H}]}$ such that $x_1 \prec x_2 \prec x_3$. Then the reduced Gröbner basis $G_{\mathcal{A}_\mathfrak{H}}$ of $P_{\mathcal{A}_\mathfrak{H}}$ is $\{\underline{x_1 x_3} - x_2^2\}$. Let $I = \langle y_2 y_3^3 \rangle$ and $F = \{y_2 y_3^3\}$. Then $\mathrm{Lift}_\prec(F) = \{x_2 x_3\}$ and $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I) = \langle x_2 x_3, x_1 x_3 - x_2^2 \rangle$. Let $\boldsymbol{w}$ be any weight vector on S. Since $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)$ is a $\boldsymbol{w}'$-homogeneous ideal, $G_{\mathcal{A}_\mathfrak{H}} \cup \mathrm{Lift}_\prec(F) = \{x_2 x_3, x_1 x_3 - x_2^2\}$ is pseudo-Gröbner basis of $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)$ with respect to $\boldsymbol{w}'$. However, $G_{\mathcal{A}_\mathfrak{H}} \cup \mathrm{Lift}_\prec(F)$ is not a Gröbner basis of $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)$. Recall that $G_{\mathcal{A}_\mathfrak{H}} \cup M_\prec^{(\mathcal{A})}(I)$ is a Gröbner basis of $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)$ with respect to $\prec_{\boldsymbol{w}'}$ by Theorem 2.7. We have $M_\prec^{(\mathcal{A})}(I) = \{x_2 x_3, x_2^3\}$, and $G_{\mathcal{A}_\mathfrak{H}} \cup M_\prec^{(\mathcal{A})}(I) = \{x_2 x_3, x_1 x_3 - x_2^2, x_2^3\}$ is a Gröbner basis of $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)$ with respect to $\prec_{\boldsymbol{w}'}$.

Note that x_2^3 is obtained from the S-polynomial $S(x_2 x_3, x_1 x_3 - x_2^2) = x_2^3$, and it has degree 3 which is strictly greater than $\deg(x_2 x_3) = \deg(x_1 x_3 - x_2^2) = 2$.

We will give a sufficient condition for the pseudo-Gröbner basis constructed in Proposition 2.26 to be a Gröbner basis.

Proposition 2.28 *Let the notation be as in Notation 2.21. Assume that $\mathcal{A}_\mathfrak{H}$ is a standard graded configuration. Suppose that $F = \{f_1, \ldots, f_\ell\}$ is a Gröbner basis of I with respect to $\boldsymbol{w}$. Let $L_i = L_\prec^{(\mathcal{A})}(\mathrm{in}_{\boldsymbol{w}}(f_i))$ and $M_i = M_\prec^{(\mathcal{A})}(\mathrm{in}_{\boldsymbol{w}}(f_i))$. Assume that for each i, there exists $\delta_i \in \mathbb{N}$ such that $\deg(u) = \delta_i$ for all $u \in M_i$. Then $G_{\mathcal{A}_\mathfrak{H}} \cup \mathrm{Lift}_\prec(F)$ is a Gröbner basis of $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)$ with respect to $\prec_{\boldsymbol{w}'}$.*

Proof Note that $P_{\mathcal{A}_\mathfrak{H}}$ is a homogeneous ideal as $\mathcal{A}_\mathfrak{H}$ is a standard graded configuration. As $G_{\mathcal{A}_\mathfrak{H}} \cup \mathrm{Lift}_\prec(F)$ is a pseudo-Gröbner basis of $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)$ and $G_{\mathcal{A}_\mathfrak{H}}$ consists of $\boldsymbol{w}'$-homogeneous polynomials, the initial ideal $\mathrm{in}_{\boldsymbol{w}'}(\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I))$ is generated by $G_{\mathcal{A}_\mathfrak{H}} \cup \{\mathrm{in}_{\boldsymbol{w}'}(g) \mid g \in \mathrm{Lift}_\prec(F)\}$. Since

$$\mathrm{in}_{\prec_{\boldsymbol{w}'}}\bigl(\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)\bigr) = \mathrm{in}_\prec\bigl(\mathrm{in}_{\boldsymbol{w}'}\bigl(\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)\bigr)\bigr),$$

$G_{\mathcal{A}_\mathfrak{H}} \cup \mathrm{Lift}_\prec(F)$ is a Gröbner basis of $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I)$ if and only if $G_{\mathcal{A}_\mathfrak{H}} \cup \{\mathrm{in}_{\boldsymbol{w}'}(g) \mid g \in \mathrm{Lift}_\prec(F)\}$ is a Gröbner basis of $\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(\mathrm{in}_{\boldsymbol{w}}(I)) = \mathrm{in}_{\boldsymbol{w}'}(\phi_{\mathcal{A}_\mathfrak{H}}^{-1}(I))$. By Remark 2.23 (3), it follows that

$$\bigl\{\mathrm{in}_{\boldsymbol{w}'}(g) \mid g \in \mathrm{Lift}_\prec(F)\bigr\} = \mathrm{Lift}_\prec\bigl(\bigl\{\mathrm{in}_{\boldsymbol{w}}(f) \mid f \in F\bigr\}\bigr).$$

Thus it is enough to show that

$$G_{\mathcal{A}_{\mathfrak{H}}} \cup \mathrm{Lift}_{\prec}\big(\big\{\mathrm{in}_{\boldsymbol{w}}(f) \mid f \in F\big\}\big)$$

is a Gröbner basis of $\phi_{\mathcal{A}_{\mathfrak{H}}}^{-1}(\mathrm{in}_{\boldsymbol{w}}(I))$ with respect to $\prec_{\boldsymbol{w}'}$. Since $\{\mathrm{in}_{\boldsymbol{w}}(f) \mid f \in F\}$ is a system of generators of the monomial ideal $\mathrm{in}_{\boldsymbol{w}}(I)$, it is enough to prove this theorem for $\mathrm{in}_{\boldsymbol{w}}(I)$. Thus we may, and do assume that I is a monomial ideal and $F = \{f_1, \dots, f_\ell\}$ is the minimal system of monomial generators of I. Then $G_{\mathcal{A}_{\mathfrak{H}}} \cup (\bigcup_{i=1}^{\ell} M_i)$ is a Gröbner basis of $\phi_{\mathcal{A}_{\mathfrak{H}}}^{-1}(I)$ by Theorem 2.7.

Note that $\mathrm{Lift}_{\prec}(F)$ is a set of monomials, and $G_{\mathcal{A}_{\mathfrak{H}}} \cup \mathrm{Lift}_{\prec}(F)$ is a system of generators of $\phi_{\mathcal{A}_{\mathfrak{H}}}^{-1}(I)$. We will prove that $G_{\mathcal{A}_{\mathfrak{H}}} \cup \mathrm{Lift}_{\prec}(F)$ is a Gröbner basis of $\phi_{\mathcal{A}_{\mathfrak{H}}}^{-1}(I)$ with respect to $\prec$ using Buchberger's criterion. It is enough to show that the remainder of the S-polynomial $S(u, g)$ when divided by $G_{\mathcal{A}_{\mathfrak{H}}} \cup \mathrm{Lift}_{\prec}(F)$ is zero for all $u \in \mathrm{Lift}_{\prec}(F)$ and $g \in G_{\mathcal{A}_{\mathfrak{H}}}$. Let $u \in \mathrm{Lift}_{\prec}(F)$. Then $u \in L_i$ for some i, and thus $\deg(u) \geq \delta_i$. For any $g \in G_{\mathcal{A}_{\mathfrak{H}}}$, as $u \notin \mathrm{in}_{\prec}(P_{\mathcal{A}_{\mathfrak{H}}})$, it follows that $u \neq \mathrm{in}_{\prec}(g)$ and thus the degree of the S-polynomial $S(u, g)$ is strictly greater than δ_i. Let u' be a remainder of $S(u, g)$ when divided by $G_{\mathcal{A}_{\mathfrak{H}}} \cup \mathrm{Lift}_{\prec}(F)$. Since $G_{\mathcal{A}_{\mathfrak{H}}} \cup \mathrm{Lift}_{\prec}(F)$ is a set of homogeneous binomials and monomials, u' is zero or a monomial in L_i of degree $\deg(S(u, g)) > \delta_i$. Hence $\mathrm{in}_{\prec}(u') = u'$ is zero or not a member of the minimal system of monomial generators of $\mathrm{in}_{\boldsymbol{w}'}(\phi_{\mathcal{A}_{\mathfrak{H}}}^{-1}(I))$. If $u' \neq 0$ for some $u \in \mathrm{Lift}_{\prec}(F)$, this contradicts to the next lemma. □

Lemma 2.29 *Let $I \subset K[x_1, \dots, x_r]$ be a homogeneous ideal with a homogeneous system of generators $G = \{g_1, \dots, g_\ell\}$. Assume that G is not a Gröbner basis of I. Then there exist $1 \leq i < j \leq \ell$ such that the initial of $\overline{S(g_i, g_j)}^G$ is a member of the minimal system of monomial generators of $\mathrm{in}_{\prec}(I)$ where $\overline{S(g_i, g_j)}^G$ denotes the remainder of the S-polynomial $S(g_i, g_j)$ when divided by G.*

Proof First, note that if $\overline{S(g_i, g_j)}^G \neq 0$, then $\deg \overline{S(g_i, g_j)}^G \geq \max\{\deg(g_i), \deg(g_j)\}$. Assume, to the contrary, the initial of $\overline{S(g_i, g_j)}^G$ is zero or not a member of the minimal system of monomial generators of $\mathrm{in}_{\prec}(I)$ for all $1 \leq i < (j \leq \ell)$. Let $F = \{\boldsymbol{x}^{\boldsymbol{b}^{(1)}}, \dots, \boldsymbol{x}^{\boldsymbol{b}^{(m)}}\}$ be the minimal system of monomial generators of $\mathrm{in}_{\prec}(I)$. We may assume that $\boldsymbol{x}^{\boldsymbol{b}^{(1)}}$ is the monomial of minimal degree among monomials in F which are not in $\langle \mathrm{in}_{\prec}(g) \mid g \in G\rangle$. Let $G' \subset I$ be a finite subset of I such that $G \cup G'$ is a minimal Gröbner basis of I computed from G by Buchberger's algorithm. Then there exists $h \in G' \backslash G$ such that $\boldsymbol{x}^{\boldsymbol{b}^{(1)}} = \mathrm{in}_{\prec}(h)$. By the procedure of Buchberger's algorithm, $\deg(\boldsymbol{x}^{\boldsymbol{b}^{(1)}}) = \deg h \geq \deg \overline{S(g_i, g_j)}^G$ for some $1 \leq i < j \leq \ell$ with $\overline{S(g_i, g_j)}^G \neq 0$. Let $\boldsymbol{x}^{\boldsymbol{b}^{(k)}} \in F$ such that $\boldsymbol{x}^{\boldsymbol{b}^{(k)}}$ divides the initial of $\overline{S(g_i, g_j)}^G$. By the assumption, the initial of $\overline{S(g_i, g_j)}^G$ does not coincide with $\boldsymbol{x}^{\boldsymbol{b}^{(k)}}$, and thus the degree of $\boldsymbol{x}^{\boldsymbol{b}^{(k)}}$ is strictly less than $\overline{S(g_i, g_j)}^G$. Therefore $\deg(\boldsymbol{x}^{\boldsymbol{b}^{(1)}}) > \deg(\boldsymbol{x}^{\boldsymbol{b}^{(k)}})$. Since any term of $\overline{S(g_i, g_j)}^G$ is not in $\langle \mathrm{in}_{\prec}(g) \mid g \in G\rangle$, we have $\boldsymbol{x}^{\boldsymbol{b}^{(k)}} \notin \langle \mathrm{in}_{\prec}(g) \mid g \in G\rangle$. This is a contradiction. □

3 Applications

We will present some applications of Theorem 2.20. For a given positive integer n, $[n]=\{1,2,\dots,n\}$ denotes the set of the first n positive integers.

3.1 Veronese configurations

Let $S=K[y_1,\dots,y_s]=\bigoplus_{i\in\mathbb{N}} S_i$ be a standard graded polynomial ring, that is, $\deg(y_i)=1$ for all i. Let d be a positive integer, and

$$\mathcal{A}_d=\left\{\boldsymbol{a}={}^t(a_1,\dots,a_s)\in\mathbb{N}^s \mid |\boldsymbol{a}|=d\right\}$$

the Veronese configuration. Then $S^{(\mathbb{N}\cdot d)}=K[\mathcal{A}_d]$ is the dth Veronese subring of S. Let $R=K[x_{\boldsymbol{a}} \mid \boldsymbol{a}\in\mathcal{A}_d]$ be a polynomial ring, and $\phi_{\mathcal{A}_d}: R\to S$, $x_{\boldsymbol{a}}\mapsto \boldsymbol{y}^{\boldsymbol{a}}$, the monomial homomorphism corresponding to $\mathcal{A}_d$. It is known that there exist a lexicographic order on R such that $\mathrm{in}_\prec(P_{\mathcal{A}_d})$ is generated by square-free monomial of degree two [5].

Theorem 3.1 *Let $I\subset S$ be a homogeneous ideal, $\boldsymbol{w}$ a weight vector on S such that $\mathrm{in}_{\boldsymbol{w}}(I)$ is a monomial ideal, and $\prec$ a term order on R such that $\mathrm{in}_\prec(P_{\mathcal{A}_d})$ is generated by square-free monomial of degree two. We denote the weight vector $\phi^*_{\mathcal{A}_d}\boldsymbol{w}$ by $\boldsymbol{w}'$. Then the following hold*:

(1) $\delta(\mathrm{in}_{\prec_{\boldsymbol{w}'}}(\phi^{-1}_{\mathcal{A}_d}(I))\le\max\{2,\delta(\mathrm{in}_{\boldsymbol{w}}(I))\}$.

(2) *If $\mathrm{in}_{\boldsymbol{w}}(I)$ is generated by square-free monomials, then $\mathrm{in}_{\prec_{\boldsymbol{w}'}}(\phi^{-1}_{\mathcal{A}_d}(I))$ is generated by square-free monomials.*

Proof The assertion immediately follows from Theorem 2.20. □

Eisenbud–Reeves–Totaro proved in [7] that if K is an infinite field, the coordinates $y_1,\dots,y_s$ of S are generic, and $\prec$ is a certain reversed lexicographic order, then it holds that $\delta(\mathrm{in}_{\prec_{\boldsymbol{w}'}}(\phi^{-1}_{\mathcal{A}_d}(I))\le\max\{2,\delta(\mathrm{in}_{\boldsymbol{w}}(I))/d\}$.

3.2 Toric fiber products

We recall toric fiber products defined in [10]. Let d and $s_1,\dots,s_d,t_1,\dots,t_d$ be positive integers, and $\mathcal{A}=\{\boldsymbol{u}_1,\dots,\boldsymbol{u}_d\}\subset\mathbb{Z}^d$. Let

$$S_1=K[\boldsymbol{y}]=K\left[y_j^{(i)} \mid i\in[d], j\in[s_i]\right],$$

$$S_2=K[\boldsymbol{z}]=K\left[z_k^{(i)} \mid i\in[d], k\in[t_i]\right],$$

be $\mathbb{Z}^d$-graded polynomial rings with

$$\deg\left(y_j^{(i)}\right)=\deg\left(z_k^{(i)}\right)=\boldsymbol{u}_i$$

for $i\in[d]$, $j\in[s_i]$, $k\in[t_i]$. Then

$$S:=S_1\otimes_K S_2\cong K\left[y_j^{(i)}, z_k^{(i)} \mid i\in[d], j\in[s_i], k\in[t_i]\right]$$

carries a $\mathbb{Z}^d \times \mathbb{Z}^d$-graded ring structure by setting

$$\deg_S\big(y_j^{(i)}\big) = (\boldsymbol{u}_i, \boldsymbol{0}), \qquad \deg_S\big(z_k^{(i)}\big) = (\boldsymbol{0}, \boldsymbol{u}_i)$$

for $i \in [d]$, $j \in [s_i]$, $k \in [t_i]$ in S. Let $R = K[x_{jk}^{(i)} \mid i \in [d], j \in [s_i], k \in [t_i]]$ be a polynomial ring, and $\phi : R \to S$ the monomial homomorphism $\phi(x_{jk}^{(i)}) = y_j^{(i)} z_k^{(i)}$. Let $I_1 \subset S_1$ and $I_2 \subset S_2$ be $\mathbb{Z}^d$-graded ideals. We denote $(I_1 \otimes S_2) + (S_1 \otimes I_2) \subset S$ simply by $I_1 + I_2$. The ideal

$$I_1 \times_{\mathcal{A}} I_2 := \phi^{-1}(I_1 + I_2)$$

is called the *toric fiber product* of I_1 and I_2.

Assume that $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d$ are linearly independent over $\mathbb{Q}$. Let

$$\Delta = \big\{(\boldsymbol{v}, \boldsymbol{v}) \mid \boldsymbol{v} \in \mathbb{Z}^d\big\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$$

be the diagonal subsemigroup of $\mathbb{Z}^d \times \mathbb{Z}^d$. Since $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d$ are linearly independent, we have

$$S^{(\Delta)} \cong K\big[y_j^{(i)} z_k^{(i)} \mid i \in [d], j \in [s_i], k \in [t_i]\big].$$

We denote by ϕ_Δ the above homomorphism ϕ in this case.

Let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ be weight vectors of S_1 and S_2 such that $\mathrm{in}_{\boldsymbol{w}_1}(I_1)$ and $\mathrm{in}_{\boldsymbol{w}_2}(I_2)$ are monomial ideals, and set $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2)$, the weight order of S. Let G_1 and G_2 be Gröbner bases of I_1 and I_2 with respect to $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, respectively.

Theorem 3.2 *Let the notation be as above. Assume that $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d$ are linearly independent over $\mathbb{Q}$. Let $\prec$ be the lexicographic term order on R such that $x_{j_1k_1}^{(i_1)} \prec x_{j_2k_2}^{(i_2)}$ if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$ or $i_1 = i_2$ and $j_1 = j_2$ and $k_1 > k_2$. Then the following hold*:

(1) $\delta(\mathrm{in}_{\prec_{\phi_\Delta^* \boldsymbol{w}}}(I_1 \times_{\mathcal{A}} I_2)) \le \max\{2, \delta(\mathrm{in}_{\boldsymbol{w}_1}(I_1)), \delta(\mathrm{in}_{\boldsymbol{w}_1}(I_2))\}$.
(2) *If both of* $\mathrm{in}_{\boldsymbol{w}_1}(I_1)$ *and* $\mathrm{in}_{\boldsymbol{w}_2}(I_2)$ *are generated by square-free monomials, then* $\mathrm{in}_{\prec_{\phi_\Delta^* \boldsymbol{w}}}(I_1 \times_{\mathcal{A}} I_2)$ *is generated by square-free monomials.*

Proof By [10] Proposition 2.6, the Gröbner basis of $\mathrm{Ker}\,\phi_\Delta$ with respect to $\prec$ is

$$\big\{\underline{x_{j_1k_2}^{(i)} x_{j_2k_1}^{(i)}} - x_{j_1k_1}^{(i)} x_{j_2k_2}^{(i)} \mid i \in [d],\ 1 \le j_1 < j_2 \le s_i,\ 1 \le k_1 < k_2 \le t_i\big\}$$

where underlined terms are initial. Since $G = G_1 \cup G_2$ is a Gröbner basis of $I_1 + I_2$ with respect to $\boldsymbol{w}$, we have

$$\delta\big(\mathrm{in}_{\boldsymbol{w}}(I_1 + I_2)\big) = \max\big\{\delta\big(\mathrm{in}_{\boldsymbol{w}_1}(I_1)\big), \delta\big(\mathrm{in}_{\boldsymbol{w}_1}(I_2)\big)\big\}.$$

As $I_1 + I_2$ is a $\mathbb{Z}^d \times \mathbb{Z}^d$-graded ideal, the assertions follow from Theorem 2.20. □

In case of toric fiber product, the pseudo-Gröbner basis constructed as in Proposition 2.26 from a Gröbner basis of $I_1 + I_2$ is a Gröbner basis of $I_1 \times_{\mathcal{A}} I_2$. This is mentioned in [10] Corollary 2.10, but the proof contains a minor gap (the author claims that the pseudo-Gröbner basis is a Gröbner basis without proof). One can fill this gap using Proposition 2.28.

Theorem 3.3 *Let G_1 and G_2 be Gröbner bases of I_1 and I_2 with respect to weight vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, respectively, and set $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2)$. Then the pseudo-Gröbner basis of $I_1 \times_{\mathcal{A}} I_2$ constructed from $G := G_1 \cup G_2$ as in Proposition 2.26 is a Gröbner basis of $I_1 \times_{\mathcal{A}} I_2$ with respect to $\prec_{\phi_\Delta^* \boldsymbol{w}}$.*

Proof Let $g \in G$. By Proposition 2.28, it is enough to show that $\deg(\boldsymbol{x}^{\boldsymbol{b}}) = \deg(g)$ for all $\boldsymbol{x}^{\boldsymbol{b}} \in M_\prec^{(\Delta)}(\mathrm{in}_{\boldsymbol{w}}(g))$ to prove this theorem. Since $g \in S_1$ or $g \in S_2$, we may assume, without loss of generality, that $g \in S_1 = K[\boldsymbol{y}]$. Set $\mathrm{in}_{\boldsymbol{w}}(g) = \boldsymbol{y}^{\boldsymbol{a}}$.

Let $\boldsymbol{x}^{\boldsymbol{b}} \in M_\prec^{(\Delta)}(\mathrm{in}_{\boldsymbol{w}}(g))$. Then $\boldsymbol{y}^{\boldsymbol{a}}$ divide $\phi_\Delta(\boldsymbol{x}^{\boldsymbol{b}})$. Since the degree of $\phi_\Delta(\boldsymbol{x}^{\boldsymbol{b}})$ in $\boldsymbol{y}$ is the same as $\deg(\boldsymbol{x}^{\boldsymbol{b}})$ by the definition of ϕ_Δ, we have $\deg(\boldsymbol{x}^{\boldsymbol{b}}) \geq \deg(\boldsymbol{y}^{\boldsymbol{a}})$. By Lemma 2.6 (1), it holds that $\deg(\boldsymbol{x}^{\boldsymbol{b}}) \leq \deg(\boldsymbol{y}^{\boldsymbol{a}})$. Hence we conclude $\deg(\boldsymbol{x}^{\boldsymbol{b}}) = \deg(\boldsymbol{y}^{\boldsymbol{a}}) = \deg(g)$. $\square$

3.3 Generalized nested configurations

Let n, s and $\lambda_1, \dots, \lambda_s$ be positive integers. Let $\mathcal{B}_i = \{\boldsymbol{b}_1^{(i)}, \dots, \boldsymbol{b}_{\lambda_i}^{(i)}\} \subset \mathbb{Z}^n$, $1 \leq i \leq s$, and $\mathcal{A} \subset \mathbb{N}^s$ be standard graded configurations. The *(generalized) nested configuration* arising from $\mathcal{A}$ and $\mathcal{B}_1, \dots, \mathcal{B}_s$ is the configuration

$$\mathcal{A}[\mathcal{B}_1, \dots, \mathcal{B}_s] := \left\{ \sum_{j=1}^{\lambda_1} a_j^{(1)} \boldsymbol{b}_j^{(1)} + \cdots + \sum_{j=1}^{\lambda_s} a_j^{(s)} \boldsymbol{b}_j^{(s)} \;\middle|\; a_j^{(i)} \in \mathbb{N}, \left(\sum_{j=1}^{\lambda_1} a_j^{(1)}, \dots, \sum_{j=1}^{\lambda_s} a_j^{(s)} \right) \in \mathcal{A} \right\}.$$

The original definition of nested configurations by Aoki–Hibi–Ohsugi–Takemura [1] is the case where there exist $0 < n_1, \dots, n_s \in \mathbb{N}$ such that $\mathbb{N}^n = \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_s}$ and $\mathcal{B}_i \subset \mathbb{N}^{n_i}$.

Let $F = \bigoplus_{i=1}^s \bigoplus_{j=1}^{\lambda_i} \mathbb{Z}\boldsymbol{e}_j^{(i)}$ be a free $\mathbb{Z}$-module of rank $\lambda_1 + \cdots + \lambda_s$. Let $\mathcal{E}_i = \{\boldsymbol{e}_1^{(i)}, \dots, \boldsymbol{e}_{\lambda_i}^{(i)}\} \subset F$ for $1 \leq i \leq s$, and $\mathcal{A} \subset \mathbb{N}^s$ a configuration. We set

$$S = K[\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_s] \cong K\big[y_j^{(i)} \mid i \in [s],\ j \in [\lambda_i]\big],$$

the $\mathbb{N}^s$-graded polynomial ring with $\deg_{\mathbb{N}^s} y_j^{(i)} = \boldsymbol{e}_i$. Then

$$S^{(\mathbb{N}\mathcal{A})} = K\big[\mathcal{A}[\mathcal{E}_1, \dots, \mathcal{E}_s]\big].$$

A Gröbner basis of $P_{\mathcal{A}[\mathcal{E}_1, \dots, \mathcal{E}_s]}$ is given in [8].

Theorem 3.4 ([8] Theorem 2.5) *Let the notation be as above. Then the following holds*:

(1) *If $P_{\mathcal{A}}$ admits an initial ideal of degree at most m, then so does $P_{\mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_s]}$.*
(2) *If $P_{\mathcal{A}}$ admits a square-free initial ideal, then so does $P_{\mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_s]}$.*

By Theorem 3.4 and Theorem 2.20, for the monomial homomorphism $\phi_{\mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_s]}: K[x_{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_s]] \to S$ and a $\mathbb{Z}^s$-deal $I \subset S$, one can describe the initial ideal of the contraction ideal $\phi^{-1}_{\mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_s]}(I)$ using Gröbner bases of $P_{\mathcal{A}}$ and I. In the case where I is a toric ideal, we have the following.

Theorem 3.5 *Let $0 < d \in \mathbb{N}$, and let $K[\boldsymbol{z}^{\pm 1}] = K[z_1^{\pm 1},\dots,z_n^{\pm 1}]$ be a $\mathbb{Q}^d$-graded Laurent polynomial ring with $\deg_{\mathbb{Q}^d}(z_i) = \boldsymbol{v}_i \in \mathbb{Q}^d$. Let $\boldsymbol{u}_1,\dots,\boldsymbol{u}_s \in \mathbb{Q}^d$ be rational vectors that are linearly independent over $\mathbb{Q}$. Take $\mathcal{B}_i = \{\boldsymbol{b}_j^{(i)} \mid 1 \le j \le \lambda_i\} \subset \{\boldsymbol{b} \in \mathbb{Z}^n \mid \deg_{\mathbb{Q}^d}(\boldsymbol{z}^{\boldsymbol{b}}) = \boldsymbol{u}_i\}$ for $1 \le i \le s$, and set $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_s$. Let $\mathcal{A} \subset \mathbb{N}^s$ be a standard graded configuration. Then the following hold.*

(1) *If both of $P_{\mathcal{B}}$ and $P_{\mathcal{A}}$ admit initial ideals of degree at most m, then so does $P_{\mathcal{A}[\mathcal{B}_1,\dots,\mathcal{B}_s]}$.*
(2) *If both of $P_{\mathcal{B}}$ and $P_{\mathcal{A}}$ admit square-free initial ideals, then so does $P_{\mathcal{A}[\mathcal{B}_1,\dots,\mathcal{B}_s]}$.*

Proof Let $R = K[x_{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_s]]$, $S = K[y_j^{(i)} \mid i \in [s],\ j \in [\lambda_i]]$ be polynomial rings, and set $\phi_{\mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_s]}: R \to S$, $x_{\boldsymbol{a}} \mapsto \boldsymbol{y}^{\boldsymbol{a}}$, and $\phi_{\mathcal{B}}: S \to K[\boldsymbol{z}^{\pm 1}] = K[z_1^{\pm 1},\dots,z_n^{\pm 1}]$, $y_j^{(i)} \mapsto \boldsymbol{z}^{\boldsymbol{b}_j^{(i)}}$. Then $\phi_{\mathcal{B}} \circ \phi_{\mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_s]} = \phi_{\mathcal{A}[\mathcal{B}_1,\dots,\mathcal{B}_s]}$. Since $\boldsymbol{u}_1,\dots,\boldsymbol{u}_s$ are linearly independent, $P_{\mathcal{B}} = \operatorname{Ker}\phi_{\mathcal{B}}$ is a $\mathbb{Z}^s$-graded ideal. By Theorem 3.4 and Theorem 2.20, we conclude the assertion. □

Example 3.6 Assume that there are four types of ingredient, z_1, z_3, z_3, z_4, and three manufacturers, B_1, B_2, B_3. Assume that each ingredient z_i is equipped with a property vector $\boldsymbol{v}_i = (v_{i1}, v_{i2}, v_{i3}) \in \mathbb{N}^3$ as in Table 1. Each of the manufacturers provides products combining $z_1,\dots,z_4$. A product is expressed as a monomial $z_1^{b_1} z_2^{b_2} z_3^{b_3} z_4^{b_4}$ where b_i is the number of z_i contained in the product. Assume that property vectors are additive, that is, the property vector of $z_1^{b_1} z_2^{b_2} z_3^{b_3} z_4^{b_4}$ is $b_1\boldsymbol{v}_1 + \dots + b_4\boldsymbol{v}_4$. Suppose that each manufacturer B_j sells the products with a fixed property vector $\boldsymbol{w}_j$ as in Table 2, and we set $\mathcal{B}_j$ the corresponding configuration. Suppose each customer choose two manufacturers and buys one product from each chosen manufacturer, and

Table 1 Ingredient

Ingredient	Property 1	Property 2	Property 3
z_1	600	30	20
z_2	400	30	10
z_3	700	20	30
z_4	1200	40	50

Table 2 Products

Manufacturer	Products	Property vector $\boldsymbol{w}_j$
B_1	$\bar{y}_1 := z_1^2 z_3 z_4$, $\bar{y}_2 := z_1 z_2 z_3^3$	(3100, 120, 120)
B_2	$\bar{y}_3 := z_1 z_3 z_4^2$, $\bar{y}_4 := z_2 z_3^3 z_4$	(3700, 130, 150)
B_3	$\bar{y}_5 := z_1^2 z_4^2$, $\bar{y}_6 := z_1 z_2 z_3^2 z_4$, $\bar{y}_7 := z_2^2 z_3^4$	(3600, 140, 140)

the chosen two manufacturers are expressed by columns of $\mathcal{A}$.

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \mathcal{B}_1 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 3 \\ 1 & 0 \end{pmatrix},$$

$$\mathcal{B}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \\ 2 & 1 \end{pmatrix}, \qquad \mathcal{B}_3 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}.$$

Then there are 16 patterns of customer's choice of products, which are expressed by columns of $\mathcal{A}[\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3]$;

$$\mathcal{A}[\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3] = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

where $\mathcal{E}_1 = (\boldsymbol{e}_1, \boldsymbol{e}_2)$, $\mathcal{E}_2 = (\boldsymbol{e}_3, \boldsymbol{e}_4)$, and $\mathcal{E}_3 = (\boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7)$. Then $\mathcal{A}[\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3]$ is the product of $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ and $\mathcal{A}[\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3]$;

$$\mathcal{A}[\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3] = \begin{pmatrix} 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 & 2 & 1 & 3 & 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 1 & 2 & 1 & 2 & 3 \\ 2 & 4 & 4 & 6 & 1 & 3 & 5 & 3 & 5 & 7 & 1 & 3 & 5 & 3 & 5 & 7 \\ 3 & 2 & 2 & 1 & 3 & 2 & 1 & 2 & 1 & 0 & 4 & 3 & 2 & 3 & 2 & 1 \end{pmatrix}.$$

Suppose that there are 1000 customers, and the choices of the customers is

$$\boldsymbol{a}_0 = {}^t(101, 59, 80, 21, 129, 62, 78, 83, 47, 51, 98, 70, 12, 58, 31, 20)$$

where the kth component of $\boldsymbol{a}_0$ is the number of customers whose choice corresponds to the kth column of $\mathcal{A}[\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3]$. Then the ith component of $\mathcal{A}[\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3] \cdot \boldsymbol{a}_0 = {}^t(2447, 1003, 3267, 2286)$ is the number of z_i in the whole of the sold products. We consider all the possibilities of 1000 customers choices such that the number of z_i in the whole of the sold products is the same as $\boldsymbol{a}_0$ for all $i = 1, 2, 3, 4$.

This space is expressed as the $\mathcal{A}[\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3]$-fiber space of $\mathcal{A}[\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3] \cdot \boldsymbol{a}_0 = {}^t(2447, 1003, 3267, 2286)$. The Gröbner basis of $P_{\mathcal{B}}$ with respect to the lexicographic order $\prec_{\text{lex}}$ with $y_7 \prec_{\text{lex}} \cdots \prec_{\text{lex}} y_1$ is

$$\{y_4y_1 - y_3y_2, y_6y_1 - y_5y_2, y_7y_1 - y_6y_2, y_6y_3 - y_5y_4, y_7y_3 - y_6y_4, y_7y_5 - y_6^2\},$$

thus $\text{in}_{\prec_{\text{lex}}}(P_{\mathcal{B}})$ is generated by square-free quadratic monomials. The toric ideal $P_{\mathcal{A}} = \langle x_1x_3 - x_2^2 \rangle$ admits a square-free quadratic initial ideal. Therefore $P_{\mathcal{A}[\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3]}$ also admits a square-free quadratic initial ideal by Theorem 3.5.

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