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On the cyclically fully commutative elements of Coxeter groups

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Abstract Let W be an arbitrary Coxeter group. If two elements have expressions that are cyclic shifts of each other (as words), then they are conjugate (as group elements) in W. We say that w is *cyclically fully commutative* (CFC) if every cyclic shift of any reduced expression for w is fully commutative (i.e., avoids long braid relations). These generalize Coxeter elements in that their reduced expressions can be described combinatorially by acyclic directed graphs, and cyclically shifting corresponds to source-to-sink conversions. In this paper, we explore the combinatorics of the CFC elements and enumerate them in all Coxeter groups. Additionally, we characterize precisely which CFC elements have the property that powers of them remain fully commutative, via the presence of a simple combinatorial feature called

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a *band*. This allows us to give necessary and sufficient conditions for a CFC element w to be *logarithmic*, that is, $\ell(w^k) = k \cdot \ell(w)$ for all $k \geq 1$, for a large class of Coxeter groups that includes all affine Weyl groups and simply laced Coxeter groups. Finally, we give a simple non-CFC element that fails to be logarithmic under these conditions.

Keywords Coxeter groups · Cyclic words · Fully commutative elements · Root automaton

1 Introduction

A classic result of Coxeter groups, known as Matsumoto's theorem [12], states that any two reduced expressions of the same element differ by a sequence of braid relations. If two elements have expressions that are cyclic shifts of each other (as words), then they are conjugate (as group elements). We say that an expression is cyclically reduced if every cyclic shift of it is reduced, and ask the following question, where an affirmative answer would be a "cyclic version" of Matsumoto's theorem.

Do two cyclically reduced expressions of conjugate elements differ by a sequence of braid relations and cyclic shifts?

While the answer to this question is, in general, "no," it seems to "often be true," and understanding when the answer is "yes" is a central focus of a broad ongoing research project of the last three authors. It was recently shown to hold for all Coxeter elements [6, 14], though the result was not stated in this manner. Key to this was establishing necessary and sufficient conditions for a Coxeter element $w \in W$ to be *logarithmic*, that is, for $\ell(w^k) = k \cdot \ell(w)$ to hold for all $k \geq 1$. Trying to understand which elements in a Coxeter group are logarithmic motivated this work. Here, we introduce and study a class of elements that generalize the Coxeter elements, in that they share certain key combinatorial properties.

A Coxeter element is a special case of a *fully commutative* (FC) element [16], which is any element with the property that any two reduced expressions are equivalent by only short braid relations (i.e., iterated commutations of commuting generators). In this paper, we introduce the *cyclically fully commutative* (CFC) elements. These are the elements for which every cyclic shift of any reduced expression is a reduced expression of an FC element. If we write a reduced expression for a cyclically reduced element in a circle, thereby allowing braid relations to "wrap around the end of the word," the CFC elements are those where only short braid relations can be applied. In this light, the CFC elements are the "cyclic version" of the FC elements. In particular, the cyclic version of Matsumoto's theorem for the CFC elements asks when two reduced expressions for conjugate elements w and w' are equivalent via only short braid relations and cyclic shifts. As with Coxeter elements, the first step in attacking this problem is to find necessary and sufficient conditions for a CFC element to be logarithmic.

This paper is organized as follows. After necessary background material on Coxeter groups is presented in Sect. 2, we introduce the CFC elements in Sect. 3. We motivate them as a natural generalization of Coxeter elements, in the sense that like Coxeter elements, they can be associated with canonical acyclic directed graphs, and a cyclic shift (i.e., conjugation by a generator) of a reduced expression corresponds on the graph level to converting a source into a sink. In Sect. 4, we prove a number of combinatorial properties of CFC elements and introduce the concept of a *band*, which tells us precisely when powers of a CFC element remain fully commutative (Theorem 4.9). In Sect. 5, we enumerate the CFC elements in all Coxeter groups, and we give a complete characterization of the CFC elements in groups that contain only finitely many. In Sect. 6, we formalize the root automaton of a Coxeter group in a new way. We then use it to prove a new result on reducibility, which we utilize in Sect. 7 to establish necessary and sufficient conditions for CFC elements to be logarithmic, as long as they have no "large bands" (Theorem 7.1). We conclude that in any Coxeter group without "large odd endpoints" (a class of groups includes all affine Weyl groups and simply laced Coxeter groups) a CFC element is logarithmic if and only if it is *torsion-free* (Corollary 7.2). The CFC assumption is indeed crucial for being logarithmic, as we conclude with a simple counterexample in $\widetilde{C}_2$ by dropping only the CFC condition.

2 Coxeter groups

A *Coxeter group* is a group W with a distinguished set of generating involutions S with presentation

$$\left\langle s_1, \dots, s_n \;\middle|\; (s_i s_j)^{m_{i,j}} = 1 \right\rangle,$$

where $m_{i,j} := m(s_i, s_j) = 1$ if and only if $s_i = s_j$. The exponents $m(s,t)$ are called *bond strengths*, and it is well known that $m(s,t) = |st|$. We define $m(s,t)$ to be ∞ if there is no exponent $k > 0$ such that $(st)^k = 1$. A Coxeter group is *simply laced* if each $m(s,t) \le 3$. If $S = \{s_1, \dots, s_n\}$, the pair (W, S) is called a *Coxeter system* of *rank* n. A Coxeter system can be encoded by a unique *Coxeter graph* Γ having vertex set S and edges $\{s,t\}$ for each $m(s,t) \ge 3$. Moreover, each edge is labeled with its corresponding bond strength, although typically the labels of 3 are omitted because they are the most common. If Γ is connected, then W is called *irreducible*.

Let S^* denote the free monoid over S. If a word $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is equal to w when considered as an element of W, we say that w is an *expression* for w. (Expressions will be written in sans serif font for clarity.) If furthermore, m is minimal, we say that w is a *reduced expression* for w, and we call m the *length* of w, denoted $\ell(w)$. If every cyclic shift of w is a reduced expression for some element in W, then we say that w is *cyclically reduced*. A group element $w \in W$ is cyclically reduced if every reduced expression for w is cyclically reduced.

The *left descent set* of $w \in W$ is the set $D_L(w) = \{s \in S \mid \ell(sw) < \ell(w)\}$, and the *right descent set* is defined analogously as $D_R(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$. If $s \in D_L(w)$ (respectively, $D_R(w)$), then s is said to be *initial* (respectively, *terminal*). It is well known that if $s \in S$, then $\ell(sw) = \ell(w) \pm 1$, and so $\ell(w^k) \le k \cdot \ell(w)$. If equality holds for all $k \in \mathbb{N}$, we say that w is *logarithmic*.

For each integer $m \geq 0$ and distinct generators $s, t \in S$, define

$$\langle s, t\rangle_m = \underbrace{stst\cdots}_{m} \in S^*.$$

The relation $\langle s, t\rangle_{m(s,t)} = \langle t, s\rangle_{m(s,t)}$ is called a *braid relation*, and is additionally called a *short braid relation* if $m(s,t) = 2$. (Some authors call $\langle s, t\rangle_{m(s,t)} = \langle t, s\rangle_{m(s,t)}$ a short braid relation if $m(s,t) = 3$, and a commutation relation if $m(s,t) = 2$.) The short braid relations generate an equivalence relation on S^*, and the resulting equivalence classes are called *commutation classes*. If two reduced expressions are in the same commutation class, we say that they are *commutation equivalent*. An element $w \in W$ is *fully commutative* (FC) if all of its reduced expressions are commutation equivalent, and we denote the set of FC elements by $\mathrm{FC}(W)$. For consistency, we say that an expression $\mathsf{w} \in S^*$ is FC if it is a reduced expression for some $w \in \mathrm{FC}(W)$. If w is not FC, then it is commutation equivalent to a word w' for which either ss or $\langle s, t\rangle_{m(s,t)}$ appears as a consecutive subword, with $m(s,t) \geq 3$ (this is not immediately obvious; see Proposition 4.2).

The braid relations generate a coarser equivalence relation on S^*. Matsumoto's theorem [7, Theorem 1.2.2] says that an equivalence class containing a reduced expression must consist entirely of reduced expressions and that the set of all such equivalence classes under this coarser relation is in 1–1 correspondence with the elements of W.

Theorem 2.1 (Matsumoto's theorem) *In a Coxeter group W, any two reduced expressions for the same group element differ by braid relations.*

Now, consider an additional equivalence relation $\sim_\kappa$, generated by cyclic shifts of words, i.e.,

$$s_{x_1}s_{x_2}\cdots s_{x_m} \longmapsto s_{x_2}s_{x_3}\cdots s_{x_m}s_{x_1}. \tag{2.1}$$

The resulting equivalence classes were studied in [11] and are in general, finer than conjugacy classes, but they often coincide. Determining conditions for when κ-equivalence and conjugacy agree would lead to a "cyclic version" of Matsumoto's theorem for some class of elements, and is one of the long-term research goals of the last three authors.

Definition 2.2 Let W be a Coxeter group. We say that a conjugacy class C satisfies the *cyclic version of Matsumoto's theorem* if any two cyclically reduced expressions of elements in C differ by braid relations and cyclic shifts.

One only needs to look at type A_n (the symmetric group Sym_{n+1}) to find an example of where the cyclic version of Matsumoto's theorem fails. Any two simple generators in A_n are conjugate, e.g., $s_1s_2(s_1)s_2s_1 = s_2$. However, for longer words, such examples appear to be less common, and we would like to characterize them.

The *support* of an expression $\mathsf{w} \in S^*$ is simply the set of generators that appear in it. As a consequence of Matsumoto's theorem, it is also well defined to speak of the support of a group element $w \in W$, as the set of generators appearing in any reduced expression for w. We denote this set by $\mathrm{supp}(w)$ and let $W_{\mathrm{supp}(w)}$ be the (standard parabolic) subgroup of W that it generates. If $W_{\mathrm{supp}(w)} = W$ (i.e., $\mathrm{supp}(w) = S$), we say that w has *full support*. If $W_{\mathrm{supp}(w)}$ has no finite factors, or equivalently, if every connected component of $\Gamma_{\mathrm{supp}(w)}$ (i.e., the subgraph of Γ induced by the support of w) describes an infinite Coxeter group, then we say that w is *torsion-free*. The following result is straightforward.

Proposition 2.3 *Let W be a Coxeter group. If $w \in W$ is logarithmic, then w is cyclically reduced and torsion-free.*

Proof If w is not cyclically reduced, then there exists a sequence of cyclic shifts of some reduced expression of w that results in a nonreduced expression. In this case, there exists $w_1, w_2 \in W$ such that $w = w_1 w_2$ (reduced) while $\ell(w_2 w_1) < \ell(w)$. This implies that

$$\ell\bigl(w^2\bigr) = \ell(w_1 w_2 w_1 w_2) \le \ell(w_1) + \ell(w_2 w_1) + \ell(w_2) < 2\ell(w),$$

and hence w is not logarithmic. If w is not torsion-free, then we can write $w = w_1 w_2$ with every generator in w_1 commuting with every generator in w_2, and $0 < |w_1| = k < \infty$. Now,

$$\ell\bigl(w^k\bigr) = \ell\bigl(w_1^k w_2^k\bigr) = \ell\bigl(w_2^k\bigr) < k \cdot \ell(w),$$

and so w is not logarithmic. □

We ask when the converse of Proposition 2.3 holds. In 2009, it was shown to hold for Coxeter elements [14], and in this paper, we show that it holds for all CFC elements that lack a certain combinatorial feature called a "large band." As a corollary, we can conclude that in any group without "large odd endpoints," a CFC element is logarithmic if and only if it is torsion free. This class of groups includes all affine Weyl groups and simply laced Coxeter groups. Additionally, we give a simple counterexample when the CFC condition is dropped.

3 Coxeter and cyclically fully commutative elements

A common example of an FC element is a *Coxeter element*, which is an element for which every generator appears exactly once in each reduced expression. The set of Coxeter elements of W is denoted by $\mathrm{C}(W)$. As mentioned at the end of the previous section, the converse of Proposition 2.3 holds for Coxeter elements, and this follows easily from a recent result in [14] together with the simple fact that Coxeter elements are trivially cyclically reduced.

Theorem 3.1 *In any Coxeter group, a Coxeter element is logarithmic if and only if it is torsion-free.*

Proof The forward direction is immediate from Proposition 2.3. For the converse, if $c \in \mathrm{C}(W)$ is torsion-free, then $c = c_1 c_2 \cdots c_m$, where each c_i is a Coxeter element of an infinite irreducible parabolic subgroup $W_{\mathrm{supp}(c_i)}$. Theorem 1 of [14] says that in an infinite irreducible Coxeter group, Coxeter elements are logarithmic, and it follows that for any $k \in \mathbb{N}$,

$$\ell(c^k) = \ell(c_1^k \cdots c_m^k) = \ell(c_1^k) + \cdots + \ell(c_m^k) = k \cdot \ell(c_1) + \cdots + k \cdot \ell(c_m) = k \cdot \ell(c),$$

and hence c is logarithmic. □

The proof of Theorem 1 of [14] is combinatorial and relies on a natural bijection between the set $\mathrm{C}(W)$ of Coxeter elements and the set $\mathrm{Acyc}(\Gamma)$ of acyclic orientations of the Coxeter graph. Specifically, if $c \in \mathrm{C}(W)$, let (Γ, c) denote the graph where the edge $\{s_i, s_j\}$ is oriented as (s_i, s_j) if s_i appears before s_j in c. (Some authors reverse this convention, orienting $\{s_i, s_j\}$ as (s_i, s_j) if s_i appears after s_j in c.) The vertex s_{x_i} is a source (respectively, sink) of (Γ, c) if and only if s_{x_i} is initial (respectively, terminal) in c. Conjugating a Coxeter element $c = s_{x_1} \cdots s_{x_n}$ by s_{x_1} cyclically shifts the word to $s_{x_2} \cdots s_{x_n} s_{x_1}$, and on the level of acyclic orientations, this corresponds to converting the source vertex s_{x_1} of (Γ, c) into a sink, which takes the orientation (Γ, c) to $(\Gamma, s_{x_1} c s_{x_1})$. This generates an equivalence relation $\sim_\kappa$ on $\mathrm{Acyc}(\Gamma)$ and on $\mathrm{C}(W)$, which has been studied recently in [11]. Two acyclic orientations (Γ, c) and (Γ, c') are κ-equivalent if and only if there is a sequence $x_1, \ldots, x_k$ such that $c' = s_{x_k} \cdots s_{x_1} c s_{x_1} \cdots s_{x_k}$ and $s_{x_{i+1}}$ is a source vertex of $(\Gamma, s_{x_i} \cdots s_{x_1} c s_{x_1} \cdots s_{x_i})$ for each $i = 1, \ldots, k-1$. Thus, two Coxeter elements $c, c' \in \mathrm{C}(W)$ are κ-equivalent if they differ by a sequence of length-preserving conjugations, i.e., if they are conjugate by a word $w = s_{x_1} \cdots s_{x_k}$ such that

$$\ell(c) = \ell(s_{x_i} \cdots s_{x_1} c s_{x_1} \cdots s_{x_i})$$

for each $i = 1, \ldots, k$. Though this is in general a stronger condition than just conjugacy, the following recent result by H. Eriksson and K. Eriksson shows that they are equivalent for Coxeter elements, thus establishing the cyclic version of Matsumoto's theorem for Coxeter elements.

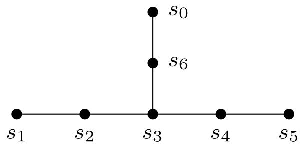
Theorem 3.2 (Eriksson–Eriksson [6]) *Let W be a Coxeter group, and $c, c' \in \mathrm{C}(W)$. Then c and c' are conjugate if and only if $c \sim_\kappa c'$.*

It is well known (see [15]) that $|\mathrm{Acyc}(\Gamma)| = T_\Gamma(2, 0)$, where T_Γ is the Tutte polynomial [19] of Γ. In [10], it was shown that for any undirected graph Γ, there are exactly $T_\Gamma(1, 0)$ κ-equivalence classes in $\mathrm{Acyc}(\Gamma)$. Applying this to Theorem 3.2, we get the following result.

Corollary 3.3 *In any Coxeter group W, the $T_\Gamma(2, 0)$ Coxeter elements fall into exactly $T_\Gamma(1, 0)$ conjugacy classes, where T_Γ is the Tutte polynomial.*

The proof of Theorem 3.2 hinges on torsion-free Coxeter elements being logarithmic, and as mentioned, the proof of this involves combinatorial properties of the acyclic orientation construction and source-to-sink equivalence relation. Thus, we are motivated to extend these properties to a larger class of elements. Indeed, the acyclic

Fig. 1 The Coxeter graph of type $\widetilde{E}_6$



orientation construction above generalizes to the FC elements. If $w \in \mathrm{FC}(W)$, then (Γ, w) is the graph where the vertices are the disjoint union of letters in any reduced expression of w, and a directed edge is present for each pair of noncommuting letters, with the orientation denoting which comes first in w. Since $w \in \mathrm{FC}(W)$, the graph (Γ, w) is well defined. Though the acyclic orientation construction extends from $\mathrm{C}(W)$ to $\mathrm{FC}(W)$, the source-to-sink operation does not. The problem arises because a cyclic shift of a reduced expression for an FC element need not be FC. This motivates the following definition.

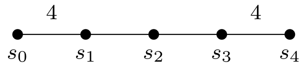
Definition 3.4 An element $w \in W$ is *cyclically fully commutative* (CFC) if every cyclic shift of every reduced expression for w is a reduced expression for an FC element.

We denote the set of CFC elements of W by $\mathrm{CFC}(W)$. They are precisely those whose reduced expressions, when written in a circle, avoid $\langle s, t\rangle_m$ subwords for $m = m(s,t) \geq 3$, and as such they are the elements for which the source-to-sink operation extends in a well-defined manner. However, acyclic directed graphs are not convenient to capture this generalization—they are much better handled as periodic heaps [8].

Example 3.5 Here are some examples and nonexamples of CFC elements. We will return to Examples (iv) and (v) at the end of Sect. 7.

(i) Any Coxeter element is an example of a CFC element, because Coxeter elements are FC, and any cyclic shift of a Coxeter element is also a Coxeter element.
(ii) Consider the Coxeter group of type A_3 with generators s_1, s_2, s_3 labeled so that s_1 and s_3 commute. The element $s_2s_1s_3s_2$ is a reduced expression for an FC element w. However, w is not cyclically reduced because the above expression has a cyclic shift $s_2s_2s_1s_3$ that reduces to s_1s_3, and so w is not CFC.
(iii) The Coxeter group of type $\widetilde{A}_2$ has generators s_1, s_2, s_3 with $m(s_i, s_j) = 3$ for $i \neq j$. The element $s_1s_3s_1s_2$ is cyclically reduced but not FC, because $s_1s_3s_1s_2 = s_3s_1s_3s_2$. If we increase the bond strength $m(s_1, s_3)$ from 3 to ∞, it becomes FC. However, it is still not CFC because conjugating it by s_1 yields the element $s_3s_1s_2s_1 = s_3s_2s_1s_2$.
(iv) Next, consider the affine Weyl group of type $\widetilde{E}_6$ (see Fig. 1). The element $w = s_1s_3s_2s_4s_3s_5s_4s_6s_0s_3s_2s_6$ is a CFC element of $W(\widetilde{E}_6)$, and it turns out that w is logarithmic.
(v) Now, consider the affine Weyl group of type $\widetilde{C}_4$ (see Fig. 2). Let $w_1 = s_0s_2s_4s_1s_3$ and $w_2 = s_0s_1s_2s_3s_4s_3s_2s_1$ be elements in $W(\widetilde{C}_4)$. It is quickly seen that both elements are CFC with full support, and as we shall be able to prove later, both w_1 and w_2 are logarithmic.

Fig. 2 The Coxeter graph of type $\widetilde{C}_4$



4 Properties of CFC elements

In this section, we will prove a series of results establishing some basic combinatorial properties of CFC elements. Of particular interest are CFC elements whose powers are not FC, and we give a complete characterization of these elements in any Coxeter group. Unless otherwise stated, (W, S) is assumed to be an arbitrary Coxeter system. Recall that an expression w not being FC means that "w is not a reduced expression for an FC element," i.e., it is either nonreduced, or it is a reduced expression of a non-FC element. By Matsumoto's theorem, if $\mathsf{w} \in S^*$ is a reduced expression for a logarithmic element $w \in W$, then (the group element) w^k is FC if and only if (the expression) w^k is FC.

Proposition 4.1 *If* w *is a reduced expression of a non-CFC element of* W, *then some cyclic shift of* w *is not FC.*

Proof If w is a reduced expression for a non-CFC element of $w \in W$, then by definition, a sequence of i cyclic shifts of some reduced expression $\mathsf{w}' = s_{x_1} \cdots s_{x_m}$ for w produces an expression $\mathsf{u} = s_{x_{i+1}} \cdots s_{x_m} s_{x_1} \cdots s_{x_i}$ that is either not reduced or is a reduced expression for a non-FC element. We may assume that w itself is FC, otherwise the result is trivial. Thus, we can obtain w' from w via a sequence of k commutations, and we may take k to be minimal. The result we seek amounts to proving that $k = 0$. By assumption, the expression u is equivalent via commutations to one containing either (a) ss or (b) $\langle s, t\rangle_{m(s,t)}$ as a consecutive subword, where $m(s, t) \geq 3$. For sake of a contradiction, assume that $k > 0$. If the kth commutation (the one that yields w') does not involve a swap of the letters in the ith and $(i + 1)$th positions, then we can simply remove this commutation from our sequence, because these two letters will be consecutive in u, and they can be transposed after the cyclic shifts. But this contradicts the minimality of k. So, the kth commutation occurs in positions i and $i + 1$, sending an expression w'' to w', that is,

$$\mathsf{w}'' = s_{x_1} \cdots s_{x_{i-1}} s_{x_{i+1}} s_{x_i} s_{x_{i+2}} \cdots s_{x_m} \longmapsto s_{x_1} \cdots s_{x_{i-1}} s_{x_i} s_{x_{i+1}} s_{x_{i+2}} \cdots s_{x_m} = \mathsf{w}'.$$

Similarly, if this commutation does not involve one of the generators in either case (a) or (b), then omitting this commutation before cyclically shifting still yields an expression that is not FC. Again, this contradicts the minimality of k, so it must be the case that the kth commutation involves s in case (a) or, without loss of generality, s in case (b). Moreover, we may assume without loss of generality that $s_{x_i} = s$, which is in the $(i + 1)$th position of w'' (otherwise, we could have considered w^{-1}, which is reduced if and only if w is reduced). Now, apply $i + 1$ cyclic shifts to w'', which yields the element

$$s_{x_{i+2}} \cdots s_{x_m} s_{x_1} \cdots s_{x_{i-1}} s_{x_{i+1}} s_{x_i} = s_{x_{i+2}} \cdots s_{x_m} s_{x_1} \cdots s_{x_{i-1}} s_{x_i} s_{x_{i+1}} \in W.$$

Note that this second expression is a single cyclic shift of u. Since u is commutation equivalent to an expression containing either ss or $\langle s,t\rangle_{m(s,t)}$ as a subword, moving $s_{x_{i+1}}$ (which cannot be s or t) from the front of u to the back does not destroy this property. Thus, we can obtain an expression that is not FC from w by applying $k-1$ commutations before cyclically shifting, contradicting the minimality of k and completing the proof. □

Proposition 4.2 *Let* w *be an expression that is not FC. Then* w *is commutation equivalent to an expression of the form* $\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3$, *where either* $\mathsf{w}_2 = ss$ *for some* $s \in S$, *or* $\mathsf{w}_2 = \langle s,t\rangle_{m(s,t)}$ *for* $m(s,t) \geq 3$.

Proof This is a restatement of Stembridge's [16, Proposition 3.3]. We remark that w_1 or w_3 could be empty. □

Lemma 4.3 *Let* $w \in W$ *be logarithmic. If* w^2 *is FC* (*respectively, CFC*), *then* w^k *is FC* (*respectively, CFC*) *for all* $k > 2$.

Proof Assume without loss of generality that W is irreducible and w has full support. If W has rank 2, then $w = (st)^j$ and $m(s,t) = \infty$, in which case the result is trivial. Thus, we may assume that W has rank $n > 2$, and we will prove the contrapositive. Let w be a reduced expression for w, and suppose that w^k is not FC (it is reduced because w is logarithmic). By Proposition 4.2, w^k is commutation equivalent to some $\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3$ where $\mathsf{w}_2 = \langle s,t\rangle_{m(s,t)}$ with $m(s,t) \geq 3$. Since there is some $u \in \mathrm{supp}(w)$ that does not commute with both s and t, the letters in w_2 can only have come from at most two consecutive copies of w in w^k. Thus, $w^2 \notin \mathrm{FC}(W)$.

If $w^k \notin \mathrm{CFC}(W)$, then by Proposition 4.1, some cyclic shift of w^k is not FC. Since every cyclic shift of w^k is a subword of w^{k+1}, this means that w^{k+1} is not FC. From what we just proved it follows that $w^2 \notin \mathrm{FC}(W)$, and hence $w^2 \notin \mathrm{CFC}(W)$. □

Observe that the assumption that w is logarithmic is indeed necessary—without it, the element $w = s_1s_2$ in $I_2(m)$ for $m \geq 5$ would serve as a counterexample.

Lemma 4.4 *Let* W *be an irreducible Coxeter group of rank* $n \geq 2$. *If* w *is a reduced expression for* $w \in \mathrm{CFC}(W)$ *with full support, then* w^k *is not commutation equivalent to an expression with* ss *as a subword, for any* $s \in S$.

Proof For sake of contradiction, suppose that w^k is commutation equivalent to an expression with ss as a subword. Since w is CFC, these two s's must have come from different copies of w in w^k that we may assume consecutive. Thus, we may write

$$\mathsf{w}^2 = (\mathsf{u}_1 s \mathsf{w}_1)(\mathsf{u}_2 s \mathsf{w}_2), \qquad \mathsf{w} = \mathsf{u}_1 s \mathsf{w}_1 = \mathsf{u}_2 s \mathsf{w}_2,$$

where the word $s\mathsf{w}_1\mathsf{u}_2 s$ is also commutation equivalent to an expression with ss as a subword. There are two cases to consider. If $\ell(\mathsf{u}_1) > \ell(\mathsf{u}_2)$, then $s\mathsf{w}_1\mathsf{u}_2 s$ is a subword of some cyclic shift of w. However, this is impossible because w is CFC. Thus, $\ell(\mathsf{u}_1) \leq \ell(\mathsf{u}_2)$. In this case, some cyclic shift of w is contained in $s\mathsf{w}_1\mathsf{u}_2 s$ as a subword, and since w has full support, every generator appears in this subword. However, in order for commutations to make the two s's consecutive, s must commute with every generator in $\mathsf{w}_1\mathsf{u}_2$, which is the required contradiction. □

There is an analogous result to Lemma 4.3 when w is not logarithmic. However, care is needed in distinguishing between the expression w^2 being FC and the actual element w^2 being FC.

Lemma 4.5 *Let W be an irreducible Coxeter group of rank $n > 2$. If w is a reduced expression for a nonlogarithmic element $w \in \mathrm{CFC}(W)$ with full support, then $\mathsf{w}^2 \notin \mathrm{FC}(W)$.*

Proof Pick k so that $\ell(w^k) < k \cdot \ell(w)$. By Proposition 4.2, w^k is commutation equivalent to some $\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3$ where either $\mathsf{w}_2 = ss$, or $\mathsf{w}_2 = \langle s, t\rangle_{m(s,t)}$ with $m(s,t) \geq 3$. However, the former is impossible by Lemma 4.4. Moreover, there is another generator $u \in S$ appearing in w that does not commute with both s and t. Therefore, the letters in w_2 can only have come from at most two consecutive copies of w in w^k. Thus, $\mathsf{w}^2 \notin \mathrm{FC}(W)$. □

Proposition 4.6 *Let W be an irreducible Coxeter group of rank $n > 2$. If w is a reduced expression for $w \in \mathrm{CFC}(W)$ with full support, then $\mathsf{w}^k \in \mathrm{CFC}(W)$ for all $k \in \mathbb{N}$.*

Proof Let w be a reduced expression for $w \in \mathrm{CFC}(W)$. Since $\mathsf{w}^2 \in \mathrm{FC}(W)$, Lemma 4.5 tells us that w is logarithmic. Suppose for sake of contradiction, that $\mathsf{w}^k \notin \mathrm{CFC}(W)$ for some $k \geq 2$. By Lemma 4.3, we know that $\mathsf{w}^2 \notin \mathrm{CFC}(W)$, and by Proposition 4.1, some cyclic shift of w^2 is not FC. Every cyclic shift of w^2 is a subword of w^3, thus $\mathsf{w}^3 \notin \mathrm{FC}(W)$. Applying Lemma 4.3 again gives $\mathsf{w}^2 \notin \mathrm{FC}(W)$, the desired contradiction. □

If w is a reduced expression of a CFC element and w^k is FC for all k, then w is clearly logarithmic. Thus, we want to understand which CFC elements have the property that powers of their reduced expressions are not FC. Theorem 4.9 gives necessary and sufficient conditions for this to happen, but first we need more terminology. If a vertex s in Γ has degree 1, we call it an *endpoint*. An endpoint vertex (or generator) s has a unique $t \in S$ for which $m(s,t) \geq 3$, and we call $m(s,t)$ the *weight* of the endpoint. If this weight is greater than 3, we say that the endpoint is *large*. In the remainder of this paper, we will pay particular attention to "large odd endpoints," that is, endpoints $s \in S$ for which $m(s,t)$ is odd and at least 5. (We will say that $m(s,t) = \infty$ is large but not odd.) As we shall see, groups with large odd endpoints have CFC elements with a feature called a "large band," and these elements have properties not shared by other CFC elements.

Definition 4.7 Let $w \in \mathrm{CFC}(W)$ and say that (W', S') is the Coxeter system generated by $\mathrm{supp}(w)$. We say that w has an *st-band* if for some reduced expression w and distinct generators $s, t \in S'$, exactly one of which is an odd endpoint of (W', S'), the following two conditions hold:

1. some cyclic shift of w is commutation equivalent to a reduced expression containing $\langle s, t\rangle_{m(s,t)-1}$ as a subword;
2. neither s nor t appears elsewhere in w.

We analogously define an *ts-band* (i.e., some cyclic shift of w is commutation equivalent to a reduced expression containing $\langle t, s\rangle_{m(s,t)-1}$ as a subword). If we do not care to specify whether s or t comes first, then we will simply say that w has a *band*. An st-band is called *small* if $m(s,t) = 3$ and *large* otherwise.

Remark 4.8 Note that w has an st-band if and only if w^{-1} has a ts-band. If w has a band, then we may assume, without loss of generality, that w has an st-band, where s is the odd endpoint.

The following result highlights the importance of bands and is essential for establishing our main results on CFC elements.

Theorem 4.9 *Let W be an irreducible Coxeter group of rank $n > 2$, and let* w *be a reduced expression for $w \in \mathrm{CFC}(W)$ with full support. Then w^k is FC for all $k \in \mathbb{N}$ if and only if w has no bands.*

Proof Suppose that w^k is not FC for some $k > 2$. If w is logarithmic, then Lemma 4.3 tells us that w^2 is not FC. However, even if w is not logarithmic, we can still conclude that w^2 is not FC, by Lemma 4.5. Thus, to prove the theorem, it suffices to show that w^2 is not FC if and only if w has a band.

First, suppose that w^2 is not FC. We will prove that w has a band by establishing the following properties:

(i) W has an odd endpoint s (say $m(s,t) \geq 3$) for which the word w^2 is commutation equivalent to an expression of the form $\mathsf{w}_1\langle s, t\rangle_{m(s,t)}\mathsf{w}_3$;
(ii) some cyclic shift of w is commutation equivalent to a reduced expression containing $\langle s, t\rangle_{m(s,t)-1}$ or $\langle t, s\rangle_{m(s,t)-1}$ as a subword;
(iii) neither s nor t appears elsewhere in w.

Since w^2 is not FC, Proposition 4.2 implies that w^2 is commutation equivalent to an expression of the form $\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3$ in which $\mathsf{w}_2 = \langle s, t\rangle_{m(s,t)}$. (Note that $\mathsf{w}_2 = ss$ is forbidden by Lemma 4.4.) To prove (i), we will first show that s must be an endpoint and then show that $m(s,t)$ must be odd.

First, we claim that because w is CFC, two occurrences of s in w_2 must correspond to the same letter of w. To see why, consider the subword of w^2 from the original position of the initial s in w_2 to the original position of the final letter (which is either s or t). Clearly, the instances of s and t in this subword must alternate. If no two occurrences of s correspond to the same letter of w, then this subword is a subword of a cyclic shift of w, contradicting the assumption that w is CFC and establishing our claim. In particular, we can write $\mathsf{w}^2 = (\mathsf{w}'_1 s \mathsf{w}'_2)(\mathsf{w}'_1 s \mathsf{w}'_2)$, where both instances of s occur in w_2, and the first instance of s is the initial letter of w_2. This implies that the letters in w'_2 and w'_1 are either other occurrences of s or t, or commute with s. Since $\mathsf{w} = \mathsf{w}'_1 s \mathsf{w}'_2$ and has full support and W is irreducible, it must be the case that s commutes with every other generator of S except t, and so s is an endpoint.

It remains to show that $m(s,t)$ is odd. For sake of a contradiction, suppose otherwise, so that w_2 ends in t. The argument in the previous paragraph using w^{-1} in place of w and t in place of s implies that t must be an endpoint as well. However, we assumed that W is irreducible, and hence W has rank 2. This contradicts our assumption that W has rank $n \geq 3$, and therefore, $m(s,t)$ is odd.

To prove (ii), we first prove that the instance of s sandwiched between w_1' and w_2' in $\mathsf{w}_1' s \mathsf{w}_2'$ is also the terminal letter of w_2. Toward a contradiction, suppose otherwise. That is, assume that $\mathsf{w}^2 = (\mathsf{w}_1' s \mathsf{u}_1 s \mathsf{u}_2)(\mathsf{w}_1' s \mathsf{u}_1 s \mathsf{u}_2)$, where the fourth instance of s is the terminal letter of w_2. Then it must be the case that every letter between the initial and terminal s in w_2 is either s, t, or a generator that commutes with both s and t. However, this includes the supports of w_1', u_1, and u_2, and since $\mathsf{w} = \mathsf{w}_1' s \mathsf{u}_1 s \mathsf{u}_2$, we conclude that every letter in w is either s, t, or commutes with s and t. Again, this contradicts the assumption of W being irreducible and of rank $n \geq 3$, so it follows that the two instances of s in $(\mathsf{w}_1' s \mathsf{w}_2')(\mathsf{w}_1' s \mathsf{w}_2')$ are the initial and terminal letters of w_2, respectively. Now, (ii) follows from the observations that $s\mathsf{w}_2'\mathsf{w}_1'$ is a cyclic shift of w, and every t occurring in w_2 must occur in $\mathsf{w}_2'\mathsf{w}_1'$. Finally, (iii) follows from the easy observation that every letter of w is contained in the word $s\mathsf{w}_2'\mathsf{w}_1's$, which has precisely $m(s,t)$ letters from the set $\{s,t\}$. Together, (i), (ii), and (iii) imply that w has an st-band.

We now turn to the converse. Let w be a CFC element with full support and a band. By Remark 4.8, we may assume, without loss of generality, that w has an st-band, where s is the endpoint. That is, some cyclic shift of w is commutation equivalent to an expression containing $\langle s,t\rangle_{m(s,t)-1}$ as a subword. Suppose that $\mathsf{w} = \mathsf{w}_1\mathsf{w}_2$ and the cyclic shift $\mathsf{w}_2\mathsf{w}_1$ is commutation equivalent to a word $\mathsf{u} = \mathsf{u}_1\langle s,t\rangle_{m(s,t)-1}\mathsf{u}_3$, with $\{s,t\} \cap \operatorname{supp}(\mathsf{u}_1\mathsf{u}_3) = \emptyset$. Clearly, u^2 is not FC, and so $(\mathsf{w}_2\mathsf{w}_1)^2$ is not FC either. However, $(\mathsf{w}_2\mathsf{w}_1)^2$ is a subword of w^3, and so w^3 is not FC and hence not CFC. By Proposition 4.6, w^2 is not FC. ☐

Lemma 4.10 *Let W be an irreducible Coxeter group with graph Γ, and let $w \in$* CFC(W). *Let $s,t \in S$ satisfy $m(s,t) \geq 3$, and let Γ' be the graph obtained from Γ by removing the edge $\{s,t\}$. Suppose that w is a reduced expression for w in which t occurs exactly once and that Γ' is disconnected. Let w' be the expression obtained from w by deleting all occurrences of generators corresponding to the connected component Γ_s' of Γ' containing s. Then w' is a reduced expression for a CFC element of W.*

Proof Suppose for a contradiction that w' is not a reduced expression for a CFC element. Then either w' is not a reduced expression, or w' is a reduced expression for a non-CFC element. In the former case, w' is commutation equivalent to an expression w'' containing either (a) a subword of the form aa or (b) a subword of the form $\langle a,b\rangle_{m(a,b)}$ with $m(a,b) \geq 3$. In the latter case, Proposition 4.1 implies that w' can be cyclically shifted to yield a non-FC expression. By Proposition 4.2, this expression is commutation equivalent to one with a subword equal to either aa or $\langle a,b\rangle_{m(a,b)}$ as in cases (a) and (b) above. Regardless, by applying a sequence of commutations or cyclic shifts to w', we can obtain a word w'' containing either aa or $\langle a,b\rangle_{m(a,b)}$ (but *not* $\langle b,a\rangle_{m(a,b)}$).

Since w does not contain such a subword, it follows in case (a) that $a = t$, which is a contradiction because w contains a unique occurrence of t. A similar contradiction arises in case (b), except possibly if $b = t$ and $m(a, b) = 3$. However, in this case, a commutes with all generators in Γ'_s, and so w would be commutation equivalent to an expression with subword of the form aba. This contradicts the hypothesis that w is FC, completing the proof. □

Lemma 4.10 has an important corollary: if a CFC element has a small band, then the corresponding endpoint can be removed to create a shorter CFC element.

Corollary 4.11 *Let* w *be a reduced expression for* $w \in \mathrm{CFC}(W)$. *If* w *has a small band, then removing the corresponding endpoint from* w *yields a reduced expression for a CFC element* w'. *Moreover, if* w *has no large bands, then neither does* w'.

Proof Suppose that w has a small st-band where s is the endpoint. By definition, s and t occur uniquely in w. Deleting the edge $\{s, t\}$ disconnects the Coxeter graph, and the connected component containing s is $\Gamma'_s = \{s\}$. We may now apply Lemma 4.10 to conclude that the word w' formed from deleting the (unique) instance of s is CFC in W.

If w has no large bands, the only way that w' could have a large band is if it involved t. That is, it would have to be a tu-band or a ut-band for some u where $m(t, u) \geq 5$. However, this is impossible because t occurs uniquely in w and hence in w'. □

It is important to note that Corollary 4.11 does not generalize to large bands. For example, suppose that s is an endpoint with $m(s, t) = 3$ and $\mathsf{w} = \mathsf{w}_1 st \mathsf{w}_2$ (reduced) is a CFC element with a small st-band. By Corollary 4.11, we can infer that $\mathsf{w}_1 t \mathsf{w}_2$ is CFC. In contrast, suppose that $m(s, t) = 5$ and w has a large st-band, e.g., $\mathsf{w} = \mathsf{w}_1 stst \mathsf{w}_2$ (reduced). Now, it is *not* necessarily the case that $\mathsf{w}_1 t \mathsf{w}_2$, or even $\mathsf{w}_1 st \mathsf{w}_2$, is CFC. Indeed, it may happen that the last letter of w_1 and the first letter of w_2 are both a common generator u with $m(t, u) = 3$. This peculiar quirk has far-reaching implications—in Sect. 7, we will use this deletion property inductively to give a complete characterization of the logarithmic CFC elements with no large bands.

5 Enumeration of CFC elements

In this section, we will enumerate the CFC elements in all Coxeter groups. In the groups that contain finitely many, we will also completely determine the structure of the CFC elements. Once again, there is a dichotomy between the groups without large odd endpoints and those with, as the latter class of groups contain CFC elements with large bands. In [16], J. Stembridge classified the Coxeter groups that contain finitely many FC elements, calling them the *FC-finite groups*. In a similar vein, the *CFC-finite groups* can be defined as the Coxeter groups that contain only finitely many CFC elements. Our next result shows that a group is CFC-finite if and only if it is

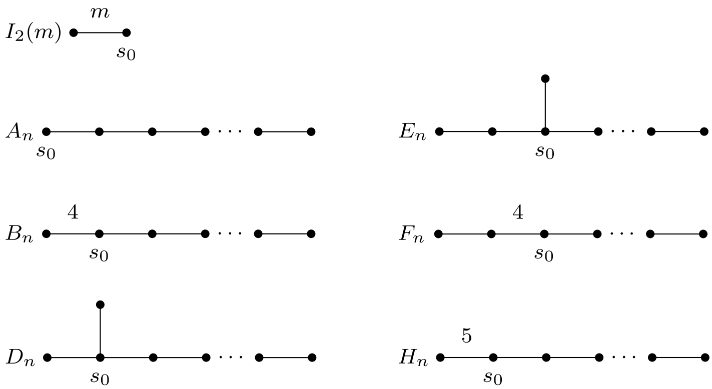


Fig. 3 Coxeter graphs of the irreducible CFC-finite groups

FC-finite. The Coxeter graphs of these (irreducible) groups are shown in Fig. 3, and they comprise seven infinite families. (The vertex labeled s_0 is called the *branch vertex* and will be defined later.) Though it is a slight abuse of notation, we will for clarity use the same symbol for the group type (e.g., A_n) and the actual group (e.g., $W(A_n)$) in this section.

Theorem 5.1 *The irreducible CFC-finite Coxeter groups are* A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(n \geq 6)$, F_n $(n \geq 4)$, H_n $(n \geq 3)$, *and* $I_2(m)$ $(5 \leq m < \infty)$. *Thus, a Coxeter group is CFC-finite if and only if it is FC-finite.*

Proof The "if" direction is immediate since $\mathrm{CFC}(W) \subseteq \mathrm{FC}(W)$, so it suffices to show that every CFC-finite group is FC-finite. Stembridge classified the FC-finite groups in [16] by classifying their Coxeter graphs. In particular, he gave a list of ten forbidden properties that an FC-finite group cannot have. The list of FC-finite groups is precisely those that avoid all ten of these obstructions. The first five conditions are easy to state and are listed below.

1. Γ cannot contain a cycle.
2. Γ cannot contain an edge of weight $m(s,t) = \infty$.
3. Γ cannot contain more than one edge of weight greater than 3.
4. Γ cannot have a vertex of degree greater than 3 or more than one vertex of degree 3.
5. Γ cannot have both a vertex of degree 3 and an edge of weight greater than 3.

The remaining five conditions all require the definition of a heap, and in the interest of space, will not be stated here. For each of the ten conditions, including the above five, Stembridge shows that if it fails, one can produce a word $\mathsf{w} \in W$ such that w^k is FC for all $k \in \mathbb{N}$. This, together with Proposition 4.6, implies that if W is CFC-finite, then it is FC-finite, and the result follows immediately. □

We now turn our attention to enumerating the CFC elements in the CFC-finite groups. The following lemma is well known, but we are not aware of a suitable reference, so we provide a proof here.

Lemma 5.2 *Let W be a Coxeter group of type A_n, and let s be an endpoint generator of A_n. If w is a reduced expression for $w \in \mathrm{FC}(W)$, then s occurs at most once in w.*

Proof We may assume that s occurs in w, and by symmetry, we may assume that $s = s_n$.

In type A_n, a well-known reduced expression for the longest element w_0 is

$$s_1(s_2s_1)(s_3s_2s_1)\cdots(s_ns_{n-1}\cdots s_1).$$

Every element of w satisfies $w \leq w_0$ with respect to the Bruhat order, which means that any such w may be written as a subexpression of the given expression. In particular, any element w has a reduced expression containing at most one occurrence of s_n. This applies to the case where $w \in \mathrm{FC}(W)$, in which case one (and hence all) reduced expressions for w contain at most one occurrence of s_n. □

Lemma 5.3 *Let W be a Coxeter group of type H_n. Label the elements of S as $s_1, s_2, \ldots, s_n$ in the obvious way such that $m(s_1, s_2) = 5$. Let w be a reduced expression for an element $w \in \mathrm{CFC}(H_n)$ having full support. Then the following all hold*:

(i) w *contains precisely one occurrence of each generator s_i for $i \geq 3$;*
(ii) w *contains precisely j occurrences of each generator s_1 and s_2, where $j \in \{1, 2\}$;*
(iii) *if w is not a Coxeter element, then it has a large band.*

Proof We prove (i) and (ii) by induction on n. For both, the base case is $n = 2$, which follows by a direct check of $W(I_2(5))$. We will prove (i) first and will assume that $n > 2$. From Theorem 5.1 we know that W has finitely many CFC elements. It follows that for some $k \in \mathbb{N}$ (actually, $k = 2$ works, but this is unimportant), w^k is not FC, and so by Theorem 4.9, w has a band. Thus, w has a reduced expression w that can be cyclically shifted to a word that is commutation equivalent to an expression u containing either $s_1s_2s_1s_2$ or $s_{n-1}s_n$ as a subword (by Remark 4.8, we can disregard the other two cases, $s_2s_1s_2s_1$ and s_ns_{n-1}).

First, suppose w has an s_1s_2-band, so $\mathsf{u} = \mathsf{u}_1s_1s_2s_1s_2\mathsf{u}_2$, and $\{s_1, s_2\} \cap \mathrm{supp}(\mathsf{u}_1\mathsf{u}_2) = \emptyset$. Since w is CFC, $\mathsf{u}_2\mathsf{u}_1$ is FC. This element sits inside a type A_{n-2} parabolic subgroup of W of which s_3 is an endpoint. By Lemma 5.2, s_3 occurs uniquely in $\mathsf{u}_2\mathsf{u}_1$. Now consider the word $\mathsf{u}_1\mathsf{u}_2$. By Lemma 4.10 applied to w and the pair of generators $\{s_2, s_3\}$, we see that $\mathsf{u}_1\mathsf{u}_2$ is CFC, and we already know that it contains a unique instance of s_3. By repeated applications of Corollary 4.11 and the fact that type A is finite, we deduce that $\mathsf{u}_1\mathsf{u}_2$ contains precisely one occurrence of each generator in the set $\{s_3, s_4, \ldots, s_n\}$, and this proves (i).

For (ii), assume again that $n > 2$ and suppose that w has no large band, meaning it must have an $s_{n-1}s_n$-band. We may use Corollary 4.11 to delete the (unique) occurrence of s_n from w to obtain a CFC element of $W(H_{n-1})$ also having full support and no large band. The result now follows by induction.

For (iii), assume that w is CFC but not a Coxeter element, and $n > 2$. By (i) and (ii), s_1 and s_2 must occur in w twice each, and s_3 can only occur once. Clearly, w is a cyclic shift of a CFC element beginning with s_3, and since this is the only occurrence of s_3 (the only generator that does not commute with both s_1 and s_2), this element is commutation equivalent to one containing either $s_1s_2s_1s_2$ or $s_2s_1s_2s_1$ as a subword. Therefore, w has a large band. □

Suppose that Γ is the Coxeter graph for an irreducible CFC-finite Coxeter group. Define Γ_0 to be the type A subgraph of Γ consisting of (a) the generator s_0 as labeled in Fig. 3 and (b) everything to the right of it. We call Γ_0 the *branch* of Γ and refer to the distinguished vertex s_0 as the *branch vertex*.

The FC elements in the FC-finite groups can be quite complicated to describe (see [16, 17]). In contrast, the CFC elements have a very restricted form. The following result shows that except in types H_n and $I_2(m)$, they are just the Coxeter elements.

Proposition 5.4 *Let W be an irreducible CFC-finite group. Suppose that $w \in \mathrm{CFC}(W)$ has full support and that some generator $s \in S$ appears in w more than once. Then one of the following situations occurs.*

(i) *$W = I_2(m)$, and $w = stst \cdots st$ has even length and satisfies $0 \le \ell(w) < m$, or*
(ii) *$W = H_n$ for $n > 2$, and w has a large band.*

Proof The proof is by induction on $|S| = n$, the case with $n = 1$ being trivial. If $n = 2$, then $W = I_2(m)$. In this case, it is easily checked that the CFC elements are those of the form $w = stst \cdots st$, where s and t are distinct generators, $\ell(w)$ is even, and $0 \le \ell(w) < m = m(s,t)$.

Suppose now that $n > 2$. The case where $W = H_n$ follows from Lemma 5.3. For all other cases, Theorem 5.1 tells us that W has no large odd endpoints. Let w be a reduced expression for w. Since W is CFC-finite, there exists $k \in \mathbb{N}$ such that w^k is not FC. In this case, it follows by induction on rank and Corollary 4.11 that w is a Coxeter element, which is a contradiction. □

Remark 5.5 If $w \in \mathrm{CFC}(W)$ with full support such that $W \ne I_2(m), H_n$, then w must be a Coxeter element.

Finally, we can drop the restriction that w should have full support.

Corollary 5.6 *Let W be an irreducible CFC-finite group. Suppose that $w \in \mathrm{CFC}(W)$ and that some generator $s \in S$ appears in w more than once. Then there exists a unique generator $t \in S$ with $m(s,t) \ge 5$. Furthermore, the generators s and t occur j times each, in alternating order (but not necessarily consecutively), where $2j < m(s,t)$.*

Proof This follows from Proposition 5.4 by considering the parabolic subgroup corresponding to $\mathrm{supp}(w)$ and by considering each connected component of the resulting Coxeter graph. □

Corollary 5.6 allows us to enumerate the CFC elements of the CFC-finite groups. Let W_n denote a rank-n irreducible CFC-finite group of a fixed type, where $n \ge 3$, and let W_{n-1} be the parabolic subgroup generated by all generators except the rightmost generator of the branch of W_n.

Corollary 5.7 *Let* $n \geq 4$. *If* $\alpha_n = |\mathrm{CFC}(W_n)|$, *then* α_n *satisfies the recurrence*

$$\alpha_n = 3\alpha_{n-1} - \alpha_{n-2}. \tag{5.1}$$

Proof The base cases can be easily checked by hand for each type. Every CFC element in W_{n-1} is also CFC in W_n, and there are α_{n-1} of these. Let s be the rightmost generator of the branch of W_n, and consider the CFC elements that contain s. By Proposition 5.4, s and the unique generator t such that $m(s,t) \geq 3$ occur at most once each. This implies that every element can be written as sw or ws (both reduced), and thus we need to compute the cardinality of

$$\{sw \mid w \in \mathrm{CFC}(W_{n-1})\} \cup \{ws \mid w \in \mathrm{CFC}(W_{n-1})\}.$$

Each of these two sets has size α_{n-1}, and $sw = ws$ if and only if $s_{n-1} \notin \mathrm{supp}(w)$. Thus, their intersection has size $|\mathrm{CFC}(W_{n-2})| = \alpha_{n-2}$, and their union has size $2\alpha_{n-1} - \alpha_{n-2}$. In summary, there are $2\alpha_{n-1} - \alpha_{n-2}$ CFC elements that contain s, and α_{n-1} CFC elements that do not, so $\alpha_n = 3\alpha_{n-1} - \alpha_{n-2}$. $\square$

Remark 5.8 If one restricts attention to CFC elements with full support, then there is a version of Corollary 5.7 for which the recurrence relation is $\alpha_n = 2\alpha_{n-1}$ for sufficiently large n.

By Corollary 5.7, to enumerate the CFC elements in W_n for each type, we just need to count them in the smallest groups of that family. We will denote the number of CFC elements in the rank-n Coxeter group of a given type by the corresponding lowercase letter, e.g., $b_n = |\mathrm{CFC}(B_n)|$. Table 1 contains a summary of the results of each (nondihedral) type, up to $n = 9$. It also lists the number of FC elements in each type, which was obtained in [17]. It is interesting to note that the enumeration of the FC elements is quite involved and uses a variety of formulas, recurrences, and generating functions. In contrast, the CFC elements in these groups can all be described by the same simple recurrence (except in type $I_2(m)$, which is even easier).

5.1 Type A

The elements of $A_1 = \{1, s\}$ have orders 1 and 2, respectively, and the set of CFC elements in $A_2 = I_2(3)$ is $\{1, s, t, st, ts\}$. It follows that $a_1 = 2$ and $a_2 = 5$. The odd-index Fibonacci numbers satisfy the recurrence in (5.1) as well as the initial seeds (see [13, A048575]). Therefore, $a_n = \mathrm{Fib}_{2n-1}$, where Fib_k denotes the kth Fibonacci number. By Corollary 5.6, the CFC elements in A_n are precisely those that have no repeat generators. In the language of [18], these are the Boolean permutations and are characterized by avoiding the patterns 321 and 3412. (A permutation π avoids 3412 if there is no set $\{i, j, k, l\}$ with $i < j < k < \ell$ and $\pi(k) < \pi(\ell) < \pi(i) < \pi(j)$.) The following result is immediate.

Table 1 The number of FC and CFC elements in the CFC-finite groups, by their rank n

	Type	$n=1$	2	3	4	5	6	7	8	9
#FC	A	2	5	14	42	132	429	1430	4862	16796
#FC	B	2	7	24	83	293	1055	3860	14299	53481
#FC	F	2	5	24	106	464	2003	8560	36333	153584
#CFC	A, B, F	2	5	13	34	89	233	610	1597	4181
#FC	D	2	4	14	48	167	593	2144	7864	29171
#CFC	D	2	4	13	35	92	241	631	1652	4325
#FC	E			10	42	167	662	2670	10846	44199
#CFC	E			10	34	92	242	634	1660	4346
#FC	H	2	9	44	195	804	3185	12368	47607	182720
#CFC	H	2	7	21	56	147	385	1008	2639	6909

Corollary 5.9 *An element $w \in A_n$ is CFC if and only if w is* 321- *and* 3412-*avoiding.*

It is worth noting that Fib_{2n-1} also counts the 1324-avoiding *circular* permutations on $[n+1]$ (see [3]). Roughly speaking, a circular permutation is a circular arrangement of $\{1, \dots, n\}$ up to cyclic shift. Though Fib_{2n-1} counts the circular permutations that avoid 1324, these are set-wise not the same as the CFC elements in $W(A_n) = \mathrm{S}\textsc{ym}_{n+1}$. As a simple example, the permutation $(2, 3) = s_2 \in W(A_3)$ does not avoid 1324 since it equals [1324] in 1-line notation, but it is clearly CFC. Also, the element $s_2s_3s_1s_2s_4s_3 \in W(A_4)$ (or $(1, 3, 5, 2, 4)$ in cycle notation) has no (circular) occurrence of 1324, but it is not CFC.

5.2 Type B

The two elements of B_1 have orders 1 and 2. In $B_2 = I_2(4)$, the elements sts and tst are not cyclically reduced. All remaining elements other than the longest element are CFC, so we have $b_1 = 2$ and $b_2 = 5$.

5.3 Type D

The group D_1 is isomorphic to A_1, D_2 has two commuting Coxeter generators, and D_3 is isomorphic to A_3. Therefore, $d_1 = 2$, $d_2 = 4$, and $d_3 = 13$.

5.4 Type E

The groups E_4 and E_5 are isomorphic to A_4 and D_5, respectively, and so $e_4 = 34$ and $e_5 = 92$. We note that if we define E_3 by removing the branch vertex from the Coxeter graph of E_4, leaving an edge and singleton vertex, then is readily checked that $e_3 = 10$, and so $e_5 = 3e_4 - e_3$.

5.5 Type F

The groups F_2 and F_3 are isomorphic to A_2 and B_3, respectively, and so $f_2 = 5$ and $f_3 = 13$. As in Type E, if we define F_1 as having a singleton Coxeter graph, then $f_1 = 2$, and $f_3 = 3f_2 - f_1$. Thus, these are also counted by the odd-indexed Fibonacci numbers with a "shifted" seed, yielding $f_n = \mathrm{Fib}_{2n+1}$.

5.6 Type H

The group H_1 has order 2, and in $H_2 = I_2(5)$, the elements sts and tst are not cyclically reduced. All other elements except the longest element are CFC, so $h_1 = 2$ and $h_2 = 7$.

6 The root automaton

In order to prove our main result, Theorem 7.1, we will induct on the size of the generating set S. A key part in the inductive step is Lemma 6.2, which shows that in certain circumstances, one can insert occurrences of a new generator into an existing reduced expression in such a way as to make a new reduced expression. To do this, we use the *root automaton*. This technique is described in [1, Chaps. 4.6–4.9] and has recently been used to tackle problems similar to ours by H. Eriksson and K. Eriksson [6]. We formalize it differently, though, in a way that is useful for our purposes, and should be of general interest in its own right.

For a Coxeter system (W, S) on n generators, let V be an n-dimensional real vector space with basis $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n\}$, and equip V with a symmetric bilinear form B such that $B(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j) = -\cos(\pi/m_{i,j})$. The action of W on V by $s_i : \mathbf{v} \mapsto \mathbf{v} - 2B(\mathbf{v}, \boldsymbol{\alpha}_i)\boldsymbol{\alpha}_i$ is faithful and preserves B, and the elements of the set $\Phi = \{w\boldsymbol{\alpha}_i \mid w \in W\}$ are called *roots*. The map

$$W \longrightarrow \mathrm{GL}(V), \qquad s_i \longmapsto \big(\mathbf{v} \overset{F_i}{\mapsto} \mathbf{v} - 2B(\mathbf{v}, \boldsymbol{\alpha}_i)\boldsymbol{\alpha}_i\big)$$

is called the standard geometric representation of W. Henceforth, we will let $\boldsymbol{\alpha}_i = \mathbf{e}_i \in \mathbb{R}^n$, the standard unit basis vector, hereby identifying roots of W with vectors in $\mathbb{R}^n$. Partially ordering the roots by $\le$ componentwise yields the *root poset* of W. For any $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$, the action of W on Φ is given by

$$\mathbf{z} \overset{s_i}{\longmapsto} \mathbf{z} + \sum_{j=1}^{n} 2\cos(\pi/m_{i,j}) z_j \mathbf{e}_i. \tag{6.1}$$

In summary, the action of s_i flips the sign of the ith entry and adds each neighboring entry z_j weighted by $2\cos(\pi/m_{i,j})$. It is convenient to view this as the image of s_i under the standard geometric representation $W \to \mathrm{GL}(\mathbb{R}^n)$, which is a linear map $F_i : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F_i\colon \quad (z_1, \dots, z_n) \longmapsto \left(z_1, \dots, z_{i-1}, z_i + \sum_{j=1}^{n} 2\cos(\pi/m_{i,j}) z_j, z_{i+1}, \dots, z_n\right). \tag{6.2}$$

Similarly, for any $\mathsf{w} = s_{x_1} \cdots s_{x_k} \in S^*$, let $F_{\mathsf{w}} = F_{s_{x_k}} \circ \cdots \circ F_{s_{x_1}}$. It is well known that for every root, all nonzero entries have the same sign, thus the root poset consists of positive roots Φ^+ and negative roots Φ^-, with $\Phi = \Phi^+ \cup \Phi^-$. In 1993, Brink and Howlett [2] proved that Coxeter groups are automatic, guaranteeing the existence of an automaton for detecting reduced expressions (see also [1, 5]). This *root automaton* has vertex set Φ and edge set $\{(\mathbf{z}, s_i\mathbf{z}) \mid \mathbf{z} \in \Phi,\ s_i \in S\}$. For convenience, label each edge $(\mathbf{z}, s_i\mathbf{z})$ with the corresponding generator s_i. It is clear that upon disregarding loops and edge orientations (all edges are bidirectional anyways), we are left with the Hasse diagram of the root poset. We represent a word $\mathsf{w} = s_{x_1}s_{x_2}\cdots s_{x_m}$ in the root automaton by starting at the unit vector $\mathbf{e}_{x_1} \in \Phi^+$ and traversing the edges labeled $s_{x_2}, s_{x_3}, \ldots, s_{x_m}$ in sequence. Denote the root reached in the root poset upon performing these steps by $\mathbf{r}(\mathsf{w})$. The sequence

$$\mathbf{e}_{x_1} = \mathbf{r}(s_{x_1}), \mathbf{r}(s_{x_1}s_{x_2}), \ldots, \mathbf{r}(s_{x_1}s_{x_2}\cdots s_{x_m}) = \mathbf{r}(\mathsf{w})$$

is called the *root sequence* of w. If $\mathbf{r}(s_{x_1}s_{x_2}\cdots s_{x_i})$ is the first negative root in the root sequence for w, then a shorter expression for w can be obtained by removing s_{x_1} and s_{x_i}. By the exchange property of Coxeter groups (see [1]), every nonreduced word $\mathsf{w} \in S^*$ can be made into a reduced expression by iteratively removing pairs of letters in this manner. Clearly, the word $\mathsf{w} = s_{x_1}\cdots s_{x_m} \in S^*$ is reduced if and only if $\mathbf{r}(s_{x_i}s_{x_{i+1}}\cdots s_{x_j}) \in \Phi^+$ for all $i < j$.

We say that a Coxeter system (W', S) *dominates* (W, S) if each bond strength in (W', S) is at least as large as the corresponding bond strength in (W, S).

Lemma 6.1 *Suppose that* (W', S) *dominates* (W, S) *and let* w *be a reduced expression for* $w \in W$. *Then* w *is reduced in* W' *as well.*

Proof This is a consequence of Matsumoto's theorem. □

The following lemma is reminiscent of [6, Proposition 3.3].

Lemma 6.2 *Suppose that* W' *is obtained from* W *by adding a new generator* s *to* S, *setting* $m(s,t) \geq 3$ *for some* $t \in S$ *and* $m(s,s') = 2$ *for all* $s' \neq t$. *Let* w_i *be a reduced expression for* $w_i \in W$, *and suppose that* $\mathsf{w}_1\mathsf{w}_2\cdots\mathsf{w}_n$ *is reduced and that each of* $\mathsf{w}_2, \ldots, \mathsf{w}_{n-1}$ *contains at least one occurrence of* t. *Then* $\mathsf{w}_1 s\mathsf{w}_2 s\mathsf{w}_3\cdots s\mathsf{w}_n$ *is a reduced expression for an element of* W'.

Proof It suffices to show that $\mathbf{r}(\mathsf{w}_1 s\mathsf{w}_2 s\mathsf{w}_3\cdots s\mathsf{w}_n)$ is a positive root, and we will induct on n. Moreover, by Lemma 6.1, we only need to prove it for the case where $m(s,t) = 3$.

The base case is where $n = 3$, because this guarantees at least one instance of t in $\mathsf{w}_1 s\mathsf{w}_2 s\mathsf{w}_3$. First, observe that $s\mathsf{w}_2 s$ is reduced because $s \notin D_R(s\mathsf{w}_2)$. Also, note that $\mathbf{r}(\mathsf{w}_1 s) = \mathbf{r}(\mathsf{w}_1) + c_1\mathbf{e}_s = \mathbf{r}(\mathsf{w}_1) + c_1\mathbf{r}(s)$ for some nonnegative constant c_1. By linearity,

$$\begin{aligned}
\mathbf{r}(\mathsf{w}_1 s\mathsf{w}_2 s\mathsf{w}_3) &= F_{\mathsf{w}_3} \circ F_s \circ F_{\mathsf{w}_2}\big[\mathbf{r}(\mathsf{w}_1 s)\big] \\
&= F_{\mathsf{w}_3} \circ F_s \circ F_{\mathsf{w}_2}\big[\mathbf{r}(\mathsf{w}_1) + c_1\mathbf{r}(s)\big] \\
&= \mathbf{r}(\mathsf{w}_1\mathsf{w}_2 s\mathsf{w}_3) + c_1\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3).
\end{aligned}$$

It suffices to show that both of these roots are positive or, equivalently, that the corresponding words are reduced. First off, $\mathsf{w}_1\mathsf{w}_2 s\mathsf{w}_3$ is clearly reduced in the Coxeter group formed by setting $m(s,t)=2$, and so it is reduced in W' by Lemma 6.1. We now turn our attention to $\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3)$. Suppose that $\mathsf{w}_2=\mathsf{u}_0 t\mathsf{u}_1 t\mathsf{u}_2\cdots t\mathsf{u}_k$ with $t\notin\operatorname{supp}(\mathsf{u}_i)$ for each i (by assumption, $i\geq 1$). Since s is disjoint from all vertices in each u_i, we have $\mathbf{r}(s\mathsf{u}_i)=\mathbf{r}(s)$. Thus, we may omit u_0 from w_2 when computing $\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3)$. Since $m(s,t)=3$, we have $\mathbf{r}(st)=\mathbf{r}(t)+\mathbf{r}(s)$, and so

$$\begin{aligned}\mathbf{r}(s\mathsf{w}_2 s)=\mathbf{r}(st\mathsf{u}_1 t\mathsf{u}_2\cdots t\mathsf{u}_k s)&=F_{\mathsf{u}_1 t\mathsf{u}_2\cdots t\mathsf{u}_k s}\big[\mathbf{r}(t)+\mathbf{r}(s)\big]\\&=\mathbf{r}(t\mathsf{u}_1 t\mathsf{u}_2\cdots t\mathsf{u}_k s)+\mathbf{r}(s\mathsf{u}_1 t\mathsf{u}_2\cdots t\mathsf{u}_k s)\\&=\mathbf{r}(t\mathsf{u}_1 t\mathsf{u}_2\cdots t\mathsf{u}_k s)+\mathbf{r}(st\mathsf{u}_2\cdots t\mathsf{u}_k s).\end{aligned}$$

Applying this same technique to $\mathbf{r}(st\mathsf{u}_2\cdots t\mathsf{u}_k s)$ yields

$$\mathbf{r}(st\mathsf{u}_2\cdots t\mathsf{u}_k s)=F_{\mathsf{u}_2 t\mathsf{u}_3\cdots t\mathsf{u}_k s}\big[\mathbf{r}(t)+\mathbf{r}(s)\big]=\mathbf{r}(t\mathsf{u}_2 t\mathsf{u}_3\cdots t\mathsf{u}_k s)+\mathbf{r}(st\mathsf{u}_3\cdots t\mathsf{u}_k s).$$

We can continue this process and successively pick off roots of the form $\mathbf{r}(t\mathsf{u}_i\cdots t\mathsf{u}_k s)$ for $i=1,2,\ldots$. At the last step, we get

$$\mathbf{r}(st\mathsf{u}_k s)=F_{\mathsf{u}_k s}\big[\mathbf{r}(t)+\mathbf{r}(s)\big]=\mathbf{r}(t\mathsf{u}_k s)-\mathbf{r}(s)=\big[\mathbf{r}(t\mathsf{u}_k)+\mathbf{r}(s)\big]-\mathbf{r}(s)=\mathbf{r}(t\mathsf{u}_k).$$

Putting this together, we have

$$\begin{aligned}\mathbf{r}(s\mathsf{w}_2 s)&=\mathbf{r}(s\mathsf{u}_0 t\mathsf{u}_1\cdots t\mathsf{u}_k s)\\&=\mathbf{r}(st\mathsf{u}_1\cdots t\mathsf{u}_k s)\\&=\big[\mathbf{r}(t\mathsf{u}_1\cdots t\mathsf{u}_k s)+\cdots+\mathbf{r}(t\mathsf{u}_{k-1}t\mathsf{u}_k s)+\mathbf{r}(t\mathsf{u}_k s)\big]-\mathbf{r}(s)\\&=\big[\mathbf{r}(t\mathsf{u}_1\cdots t\mathsf{u}_k s)+\cdots+\mathbf{r}(t\mathsf{u}_{k-1}t\mathsf{u}_k s)\big]+\mathbf{r}(t\mathsf{u}_k).\end{aligned}$$

Finally, we get $\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3)$ from this by applying the map F_{w_3} to each term, yielding

$$\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3)=\big[\mathbf{r}(t\mathsf{u}_1\cdots t\mathsf{u}_k s\mathsf{w}_3)+\cdots+\mathbf{r}(t\mathsf{u}_{k-1}t\mathsf{u}_k s\mathsf{w}_3)\big]+\mathbf{r}(t\mathsf{u}_k\mathsf{w}_3).\tag{6.3}$$

Each of the roots on the right-hand side of (6.3) are roots of expressions that are subwords of $\mathsf{w}_2 s\mathsf{w}_3$ or $\mathsf{w}_2\mathsf{w}_3$, both of which are reduced. Thus, $\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3)$ is a positive root, and this establishes the base case.

For the inductive step, we need to show that $\mathbf{r}(\mathsf{w}_1 s\mathsf{w}_2\cdots s\mathsf{w}_n)$ is positive. By linearity,

$$\begin{aligned}\mathbf{r}(\mathsf{w}_1 s\mathsf{w}_2 s\mathsf{w}_3\cdots s\mathsf{w}_n)&=F_{\mathsf{w}_n}\circ F_s\circ\cdots\circ F_{\mathsf{w}_3}\circ F_s\circ F_{\mathsf{w}_2}\big[\mathbf{r}(\mathsf{w}_1)+c_1\mathbf{r}(s)\big]\\&=\mathbf{r}(\mathsf{w}_1\mathsf{w}_2 s\mathsf{w}_3\cdots s\mathsf{w}_n)+c_1\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3\cdots s\mathsf{w}_n).\end{aligned}$$

The first root is positive by the induction hypothesis, so to prove the lemma, it suffices to show that $\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3\cdots s\mathsf{w}_n)$ is positive. Using (6.3), we get

$$\begin{aligned}\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3\cdots s\mathsf{w}_n) &= F_{s\mathsf{w}_4\cdots s\mathsf{w}_n}\big[\mathbf{r}(s\mathsf{w}_2 s\mathsf{w}_3)\big]\\ &= \big[\mathbf{r}(t\mathsf{u}_1\cdots t\mathsf{u}_k s\mathsf{w}_3 s\mathsf{w}_4\cdots s\mathsf{w}_n)+\cdots\\ &\quad + \mathbf{r}(t\mathsf{u}_{k-1}t\mathsf{u}_k s\mathsf{w}_3 s\mathsf{w}_4\cdots s\mathsf{w}_n)\big] + \mathbf{r}(t\mathsf{u}_k\mathsf{w}_3 s\mathsf{w}_4\cdots s\mathsf{w}_n).\end{aligned}$$

Each of these are roots of expressions that are subwords of either $\mathsf{w}_2 s\mathsf{w}_3 s\mathsf{w}_4\cdots s\mathsf{w}_n$ or $\mathsf{w}_2\mathsf{w}_3 s\mathsf{w}_4\cdots s\mathsf{w}_n$, both of which are reduced by the induction hypothesis. □

7 Logarithmic CFC elements

Recall Theorem 3.1, which said that Coxeter elements are logarithmic if and only if they are torsion-free. The following theorem generalizes this to CFC elements without large bands.

Theorem 7.1 *Let w be a CFC element of W with no large bands. Then w is logarithmic if and only if w is torsion-free.*

Proof The forward direction is trivially handled by Proposition 2.3, so we will only consider the reverse direction. Moreover, it suffices to consider the case where W is irreducible and w has full support. This means that either $|S|\geq 3$, or W is the free Coxeter group on two generators (i.e., $m(s_1,s_2)=\infty$). The latter case is trivial, and so we will ignore it and assume that $|S|\geq 3$.

Let w be a reduced expression for w. If w^k is FC for all k, then we are done. Assume otherwise. By Theorem 4.9, with the assumption that w has no large bands, w must have a small st-band for some $s,t\in S$, meaning that the occurrences of s and t in w are both unique. Assume without loss of generality that s (and not t) is the endpoint, and let W' be the parabolic subgroup of W obtained by removing s. By Corollary 4.11, deleting the unique occurrence of s from w yields a reduced expression w' for a CFC element w' of W' that has no large bands. From here we have two potential ways to show that w is logarithmic. If W' is infinite and w' is a Coxeter element, then w is a Coxeter element of W and hence logarithmic by Theorem 3.1. Alternatively, if w' is logarithmic, then $(\mathsf{w}')^k$ is reduced for all k, and so by Lemma 6.2, w^k is reduced as well.

We will proceed by induction on $|S|$. For the base case, suppose that $|S|=3$, meaning that W' is of type $I_2(m)$. Since t occurs exactly once in w, the remaining generator of $I_2(m)$ occurs precisely once. Thus, w' is a Coxeter element, and we are done.

For the inductive step, assume that $|S|\geq 4$. If W' is infinite, then by induction, $(\mathsf{w}')^k$ is reduced in W', and so w must be logarithmic. Thus, suppose that W' is finite. We have two cases. If W' has no large odd endpoints, then it follows from Corollary 5.6 that w' is a Coxeter element. Now, suppose that W' has a large odd endpoint. Since W' is finite and of rank at least 3, it must be of type H_3 or H_4. In this case, the only possibilities for the Coxeter graph of W are shown in Fig. 4. For each of these six Coxeter graphs, we may assume that s and t are the indicated vertices. (Note that any other choice would result in either an isomorphic copy of W'

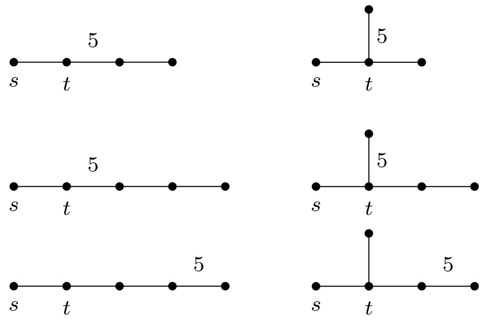


Fig. 4 The last remaining obstructions to Theorem 7.1

or an infinite group.) These six graphs fall into two cases. In the top four graphs, t is involved in a strength 5 bond, and so the uniqueness of the occurrence of t forces w' to be a Coxeter element (of H_3 or H_4) because we have $j = 1$ in Lemma 5.3(ii). In the bottom two graphs, t is not involved in a strength 5 bond, so w' has a large band if and only if w does, and by Lemma 5.3(iii), w' is a Coxeter element. In either case, it follows that w is also a Coxeter element, and hence w is logarithmic. □

Corollary 7.2 *Let (W, S) be a Coxeter system without large odd endpoints. An element $w \in \mathrm{CFC}(W)$ is logarithmic if and only if it is torsion-free.*

Proof The forward direction is handled by Proposition 2.3. For the converse, let w be torsion-free with reduced expression w. We may assume that it has full support and W is irreducible. Since W has no large odd endpoints, w has no large bands and hence is logarithmic by Theorem 7.1. □

The class of Coxeter groups without large odd endpoints includes all affine Weyl groups and simply laced Coxeter groups. In fact, we can say even more about CFC elements in affine Weyl groups. The following corollary says that the only logarithmic CFC elements with bands in an affine Weyl group are the Coxeter elements.

Corollary 7.3 *Let W be an affine Weyl group, and w a reduced expression for $w \in \mathrm{CFC}(W)$ with full support. Then w is logarithmic, and either*

(i) *w is a Coxeter element, or*
(ii) *$\mathsf{w}^k \in \mathrm{FC}(W)$ for all $k \in \mathbb{N}$.*

Proof Since W is an affine Weyl group, each $m(s,t) \in \{1, 2, 3, 4, 6, \infty\}$, which means that W has no large odd endpoints, and none of its CFC elements have large bands. The proof of Theorem 7.1 carries through, except that the only situation where (i) and (ii) do not occur is the case where it is possible to remove an element of S and still be left with an infinite Coxeter group. The proof follows from a well-known (and easily checked) property of affine Weyl groups, which is that all of their proper parabolic subgroups are finite. □

Example 7.4 Here are some examples of CFC elements in affine Weyl groups, and what our results tell us about their properties.

(i) Consider the affine Weyl group of type $\widetilde{A}_n$ for $n \geq 2$. The corresponding Coxeter graph is an $(n+1)$-gon, all of whose edges have bond strength three. Let c be a Coxeter element of $W(\widetilde{A}_n)$. Then c is CFC and is logarithmic by Theorem 3.1. Since $\widetilde{A}_n$ has no endpoints and c has full support, c cannot have any bands. By Theorem 4.9, c^k is FC for all k, and now we can use Proposition 4.6 to deduce that c^k is CFC for all k.

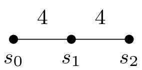
(ii) Consider the affine Weyl group of type $\widetilde{E}_8$, or in other words, type E_9, and let c be a Coxeter element of $W(\widetilde{E}_8)$. Again, by Theorem 3.1, c is logarithmic. However, $\widetilde{E}_8$ is FC-finite, so it cannot be the case that c^k is FC (and hence CFC) for all k. By Lemma 4.3, c^2 is not FC, and by Theorem 4.9, c must have a band.

(iii) Recall from Example 3.5(iv) that $w = s_1s_3s_2s_4s_3s_5s_4s_6s_0s_3s_2s_6$ is a CFC element in the affine Weyl group of type $\widetilde{E}_6$. Though the Coxeter graph has three odd endpoints, w has no bands, which is easily verified from the observation that each generator adjacent to an endpoint occurs twice in w. By Theorem 4.9, w^k is FC for all k, and by Proposition 4.6, w^k is CFC for all k.

(iv) As in Example 3.5(v), let $w_1 = s_0s_2s_4s_1s_3$ and $w_2 = s_0s_1s_2s_3s_4s_3s_2s_1$ be elements in $W(\widetilde{C}_4)$. Since w_1 and w_2 are CFC elements with full support, by Corollary 7.2, both are logarithmic. Moreover, since $W(\widetilde{C}_4)$ has no odd endpoints, CFC elements with full support in $W(\widetilde{C}_4)$ have no bands, so powers of w_1 and w_2 remain FC (Theorem 4.9) and CFC (Proposition 4.6).

8 Conclusions and future work

Our motivation for defining and studying the CFC elements arose from recent work on Coxeter elements described in Sect. 3, in which the source-to-sink operation arose. It seemed that certain properties of Coxeter elements were not due to the fact that every generator appears once, but rather that conjugation is described combinatorially by this source-to-sink operation. Thus, CFC elements seemed like the natural generalization, because they are the largest class of elements for which the source-to-sink operation extends. Indeed, we showed that for any CFC element w (without large bands), w is logarithmic iff w is torsion-free. This generalizes Speyer's recent result that says the same for the special case of Coxeter elements. If the source-to-sink operation is indeed crucial to this logarithmic property, then there should be a simple example of a cyclically reduced non-CFC element that fails to be logarithmic. The following example of this was pointed out recently by Dyer [4], where W is the affine Weyl group $\widetilde{C}_2$, and w the following non-CFC element:



$$w = s_0s_1s_0s_1s_2.$$

Clearly, w is cyclically reduced and torsion-free, but

$$w^2 = (s_0s_1s_0s_1s_0)(s_2s_1s_0s_1s_2) = (s_1s_0s_1s_0s_0)(s_2s_1s_0s_1s_2) = (s_1s_0s_1)(s_2s_1s_0s_1s_2),$$

and so $\ell(w^2) < 2\ell(w)$. Obviously, such a counterexample works for any $m(s_1, s_2) \geq 4$. Thus, being cyclically reduced and torsion-free together are *not* sufficient for a non-CFC element to be logarithmic. So, what are the necessary and sufficient conditions for an arbitrary element in a Coxeter group to be logarithmic? In this paper, we formalized the root automaton of a Coxeter group in a new way, and it led to a new technique for proving reducibility. We expect this approach to be useful for other questions about reducibility. However, new geometric tools would need to be developed to attack this general question for non-CFC elements. In [9], D. Krammer defines the "axis" of an element, which generalizes the property of being logarithmic (which Krammer calls *straight*). Krammer proves some results on the axis but does not use these to draw conclusions about combinatorial properties of logarithmic elements. We do not know yet whether these techniques will help, but it remains a possibility.

Another natural question is whether torsion-free CFC elements with large bands are necessarily logarithmic. Consider the following sets of elements:

$$\left\{\begin{matrix}\text{Coxeter}\\ \text{elements}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{CFC elements}\\ \text{w/o large bands}\end{matrix}\right\} \subset \left\{\text{CFC elements}\right\} \subset \left\{\begin{matrix}\text{cyclically reduced}\\ \text{elements}\end{matrix}\right\}$$

The source-to-sink operation holds for these first three sets but breaks down for the fourth. Being torsion-free implies being logarithmic for elements in the first two sets but not for elements in the fourth. Is it also sufficient for elements in the third set? If so, that would imply that in any Coxeter group, a CFC element is logarithmic if and only if it is torsion-free (recall that in Corollary 7.2, we proved that this is true for all Coxeter groups without large odd endpoints), and this would give even more evidence that the combinatorics behind the source-to-sink operation is governing the logarithmic property. It is tempting to conjecture this for purely aesthetic reasons, and it may in fact be true. However, we do not have any firm mathematical evidence.

As mentioned earlier, we expect that these results will be useful in better understanding the conjugacy problem in Coxeter groups. Since the logarithmic property was key to establishing the cyclic version of Matsumoto's theorem (as mentioned in the introduction) for Coxeter elements, we expect that it will be necessary for CFC elements. We conjecture that the cyclic version of Matsumoto's theorem holds for at least the CFC elements (and likely much more), and once again, the combinatorial techniques involving the source-to-sink operation should play a central role. But does it hold for general torsion-free cyclically reduced elements? If there is a counterexample, it is certainly not obvious. In the meantime, progress toward this goal should lead to valuable new developments in the combinatorial understanding of reducibility and conjugacy. Understanding any obstacles to this conjecture would also be of considerable interest, and even if it were shown to be false, understanding when it fails (and proving a modified version) would surely bring new insight.

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