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Galois groups of multivariate Tutte polynomials

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Abstract The multivariate Tutte polynomial $\hat{Z}_M$ of a matroid M is a generalization of the standard two-variable version, obtained by assigning a separate variable v_e to each element e of the ground set E. It encodes the full structure of M. Let $\mathbf{v} = \{v_e\}_{e\in E}$, let K be an arbitrary field, and suppose M is connected. We show that $\hat{Z}_M$ is irreducible over $K(\mathbf{v})$, and give three self-contained proofs that the Galois group of $\hat{Z}_M$ over $K(\mathbf{v})$ is the symmetric group of degree n, where n is the rank of M. An immediate consequence of this result is that the Galois group of the multivariate Tutte polynomial of any matroid is a direct product of symmetric groups. Finally, we conjecture a similar result for the standard Tutte polynomial of a connected matroid.

Keywords Tutte polynomial · Multivariate Tutte polynomial · Matroids · Graphs · Galois theory

Let M be a finite matroid on the set E. The rank of M is denoted by $r(M)$, and r_M is the rank function on M. With this notation we have $r(M) = r_M(E)$. To avoid degenerate examples and exceptions, a connected matroid will be assumed throughout to have positive rank (our results are trivial for a matroid having zero rank). Following the usual notation in matroid theory, we will write $E \setminus e$ instead of $E \setminus \{e\}$ for $e \in E$, and denote by $M|A$ the restriction of M to some $A \subset E$.

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For each $e \in E$ let v_e be a variable, and let $\mathbf{v}$ be the collection of these variables. If A is a subset of E, we will denote by $\mathbf{v}_A$ the set $\{v_e\}_{e\in A}$. In [7], Sokal defines the following multivariate version of the Tutte polynomial of a matroid M.[1]

For another variable q set

$$\tilde{Z}_M(q,\mathbf{v}) = \sum_{A\subseteq E} q^{-r_M(A)} \prod_{e\in A} v_e.$$

Then $\tilde{Z}_M(q,\mathbf{v})$ is a polynomial in $\frac{1}{q}$ with coefficients in $\mathbb{Z}[\mathbf{v}]$.

For our purpose it is more convenient to use the following minor modification:

$$\hat{Z}_M(q,\mathbf{v}) = \sum_{A\subseteq E} q^{r(M)-r_M(A)} \prod_{e\in A} v_e.$$

Then

$$\hat{Z}_M(q,\mathbf{v}) = q^{r(M)}\tilde{Z}_M(q,\mathbf{v}),$$

and $\hat{Z}_M(q,\mathbf{v})$ is a polynomial of degree $r(M)$ in q, which is monic if M contains no loops. In particular, if M is connected then $\hat{Z}_M(q,\mathbf{v})$ is monic. Combinatorially, $\hat{Z}_M(q,\mathbf{v})$ is a generating function for the content and rank of the subsets of E, and thus encodes all of the information about M.

By making the substitutions

$$q \leftarrow (x-1)(y-1),$$
$$v_e \leftarrow y-1$$

for each $e \in E$, and multiplying by a prefactor $(y-1)^{-r(M)}$, we obtain the standard bivariate Tutte polynomial:

$$T_M(x,y) = \sum_{A\subseteq E} (x-1)^{r(M)-r_M(A)}(y-1)^{|A|-r_M(A)}.$$

Thus T_M is essentially equivalent to a special case of $\hat{Z}_M$ in which the same variable is assigned to every element of E.

Theorem 1 *Let M be a finite connected matroid with positive rank $n = r(M)$, and let $\hat{Z}_M(q,\mathbf{v})$ be as defined above. Let K be an arbitrary field. Then the Galois group of $\hat{Z}_M(q,\mathbf{v})$ over $K(\mathbf{v})$ is the symmetric group on the n roots of $\hat{Z}_M(q,\mathbf{v})$.*

For $e \in E$, let $M \setminus e$ be the deletion of e, and M/e the contraction of e. Note that $M \setminus e$ and M/e are matroids on the set $E \setminus e$. The essential tool for our first proof is a theorem of Tutte (see [6, Theorem 4.3.1]), which says that connectivity of M implies that at least one of the matroids $M \setminus e$ or M/e is connected. Since M is connected, e

[1] The multivariate Tutte polynomial for matroids has in fact been discovered a number of times; it appears, for example, in [2] as the “Tugger polynomial”.

is not a coloop, so $r(M\setminus e)=r_M(E\setminus e)=r_M(E)=r(M)$. By [6, Proposition 3.1.6], we have that $r(M/e)=r_M(E)-r_M(e)$. Now $r_M(e)=1$, since e is not a loop. So $r(M/e)=r(M)-1$.

The proofs will be based on some lemmas.

Lemma 2 *Let M be a finite connected matroid and $e\in E$. Then*

$$\hat{Z}_M=\hat{Z}_{M\setminus e}+v_e\hat{Z}_{M/e}.$$

Proof Since M is connected, e is neither a loop nor a coloop. By [7, (4.18a)], $\tilde{Z}_M=\tilde{Z}_{M\setminus e}+\frac{v_e}{q}\tilde{Z}_{M/e}$, hence

$$\hat{Z}_M=q^{r(M)-r(M\setminus e)}\hat{Z}_{M\setminus e}+q^{r(M)-r(M/e)}\frac{v_e}{q}\hat{Z}_{M/e}.$$

The claim then follows from the previous determination of the ranks of $E\setminus e$ and E/e. □

As an intermediate step in the proof of the theorem, we need to know that $\hat{Z}_M$ is irreducible over $K(\mathbf{v})$. As T_M is essentially a specialization of $\hat{Z}_M$, this would follow from [4] in the case where K has characteristic zero. However, the multivariate case allows for a much simpler proof, and one which holds for any characteristic.

Lemma 3 *Let M be a finite connected matroid. Then $\hat{Z}_M$ is irreducible over $K(\mathbf{v})$.*

Proof The induction proof is most conveniently formulated by considering a counterexample M where $r(M)$ is minimal; among those counterexamples, we pick one where $|E|$ is minimal. Clearly, the result holds for $r(M)=1$, so $r(M)\geq 2$. Pick $e\in E$. By Lemma 2, $\hat{Z}_M=\hat{Z}_{M\setminus e}+v_e\hat{Z}_{M/e}$. Note that v_e does not appear in $\hat{Z}_{M\setminus e}$ and $\hat{Z}_{M/e}$. If $M\setminus e$ is connected, then $\hat{Z}_{M\setminus e}$ is irreducible by minimality of $|E|$. As $\hat{Z}_M$ and $\hat{Z}_{M\setminus e}$ have the same degree, setting $v_e=0$ shows that $\hat{Z}_M$ is irreducible, a contradiction. So $M\setminus e$ is not connected, which by Tutte's theorem means that M/e is connected. So $r(M/e)\geq 1$ (because $r(M)\geq 2$), and $\hat{Z}_{M/e}$ is monic. Note also that because M is loopless, so too is $M\setminus e$, and hence $\hat{Z}_{M\setminus e}$ is also monic.

Now, consider a non-trivial factorization of $\hat{Z}_M$. Since $\hat{Z}_M$ is monic and linear in v_e, we can write $\hat{Z}_M=(U+v_eV)W$, where U, V, W are polynomials in $K[\mathbf{v}][q]$ in which v_e does not appear, and where each factor has positive degree in q.

So $(U+v_eV)W=\hat{Z}_{M\setminus e}+v_e\hat{Z}_{M/e}$. Comparing coefficients with respect to v_e gives $UW=\hat{Z}_{M\setminus e}$ and $VW=\hat{Z}_{M/e}$. By minimality of the counterexample, $\hat{Z}_{M/e}$ is irreducible. But W has positive degree in q, so $V=1$ and $W=\hat{Z}_{M/e}$. Thus $U\hat{Z}_{M/e}=\hat{Z}_{M\setminus e}$. Now, $\hat{Z}_{M/e}$ and $\hat{Z}_{M\setminus e}$ are monic of degrees $r(M)-1$ and $r(M)$, respectively. So $U=q+\beta$ for some $\beta\in K[\mathbf{v}]$. Let $\bar{\mathbf{v}}=\mathbf{v}\setminus\{v_e\}$, and note that

$$\hat{Z}_{M\setminus e}(1,\bar{\mathbf{v}})=\prod_{i\in E\setminus e}(1+v_i)=\hat{Z}_{M/e}(1,\bar{\mathbf{v}}),$$

so $\beta = 0$. Now setting $q = 0$ gives $\hat{Z}_{M\setminus e}(0, \bar{\mathbf{v}}) = 0$. This means that there are no bases in $M \setminus e$, which is only possible if every element of $E \setminus e$ is a loop. So we have a contradiction. □

In order to prove the theorem, we need more precise information about how Galois groups behave under specializations of parameters. The next result is well-known, it follows, for instance, from [3, Theorem IX.2.9].

Proposition 4 *Let R be an integral domain which is integrally closed in its quotient field F. Let $f \in R[X]$ be monic and irreducible over F. Let $R \to k$, $r \mapsto \bar{r}$ be a homomorphism to a field k. If $\bar{f} \in k[X]$ is separable, then* $\mathrm{Gal}(\bar{f}/k)$ *is a subgroup of* $\mathrm{Gal}(f/F)$.

The following two lemmas can be obtained through applications of this proposition.

Lemma 5 *Let A be a subset of E. Then* $\mathrm{Gal}(\hat{Z}_{M|A}/K(\mathbf{v}_A))$ *is a subgroup of* $\mathrm{Gal}(\hat{Z}_M/K(\mathbf{v}))$.

Proof Let B be such that $A \subset B \subseteq E$, and let e be an element of $B \setminus A$. Note that removing e from B corresponds to specializing v_e to zero in $\hat{Z}_{M|B}$. Let $R = K(\mathbf{v}_{B\setminus e})[v_e]$, and let I be the maximal ideal of R generated by v_e. The image of $\hat{Z}_M$ in the canonical homomorphism $R \to R/I$ is either $q\hat{Z}_{M|(B\setminus e)}$ or $\hat{Z}_{M|(B\setminus e)}$, depending on whether or not e is a coloop. In both cases, we have a separable polynomial, as the presence of a repeated irreducible factor would contradict the fact that $\hat{Z}_{M|(B\setminus e)}$ is linear in the elements of $\mathbf{v}_{B\setminus e}$. Furthermore, R is integrally closed in its quotient field $K(\mathbf{v})$. So we have that $\mathrm{Gal}(\hat{Z}_{M|(B\setminus e)}/K(\mathbf{v}_{B\setminus e})) \le \mathrm{Gal}(\hat{Z}_{M|B}/K(\mathbf{v}_B))$ by Proposition 4, and the result follows by induction. □

Lemma 6 *Let y be a variable over the field k, and $U, V \in k[X]$ with* $\deg V = n - 1$, *and U monic of degree n (where $n \ge 2$). Suppose that $f(X) = U(X) + yV(X)$ is irreducible over $k(y)$ (which is equivalent to U and V being relatively prime). If* $\mathrm{Gal}(U/k) = S_n$ *or* $\mathrm{Gal}(V/k) = S_{n-1}$, *then* $\mathrm{Gal}(f/k(y)) = S_n$.

Proof First suppose that $\mathrm{Gal}(U/k) = S_n$. Then the assertion follows immediately from Proposition 4 by setting $R = k[y]$ and considering the homomorphism $R \to k$, $h(y) \mapsto h(0)$.

Now assume that $\mathrm{Gal}(V/k) = S_{n-1}$. Set $t = 1/y$ and replace $f(X) = U(X) + yV(X) = U(X) + \frac{1}{t}V(X)$ with t times the reciprocal of $f(X)$, that is, set $\hat{f}(X) = X^n(tU(1/X) + V(1/X))$. Clearly, $k(t) = k(y)$ and $\mathrm{Gal}(f/k(y)) = \mathrm{Gal}(\hat{f}/k(t))$. The coefficient of X^n in $\hat{f}$ is $tu + v$, where u and v are the constant terms of U and V. If $v = 0$, then V has the root 0. However, V is irreducible since $\mathrm{Gal}(V/k) = S_{n-1}$. So $n = 2$. The result clearly holds in this case because f is then irreducible of degree 2.

So assume $v \neq 0$. Let $R \subset k(t)$ be the localization of $k[t]$ with respect to the ideal (t), so R consists of the fractions $p(t)/q(t)$ with $q(0) \neq 0$. Note that $\frac{1}{tu+v}\hat{f}$ is monic with coefficients in R. Also, R (as a local ring) is integrally closed in $k(t)$. Let

$R \to k$ be the homomorphism given by $p(t)/q(t) \mapsto p(0)/q(0)$. Proposition 4 then gives $\mathrm{Gal}(\hat{f}/k(t)) \geq \mathrm{Gal}(X^n V(1/X)/k) = S_{n-1}$. Because $\mathrm{Gal}(\hat{f}/k(t))$ is transitive on the n roots of $\hat{f}$, we must have $\mathrm{Gal}(\hat{f}/k(t)) = S_n$. □

We are now ready to prove Theorem 1.

First proof of Theorem 1 Again assume that the matroid M is a counterexample with $r_M(E)$ minimal, and among these cases pick one with $|E|$ minimal. Note that the statement is trivially true if $r(M) = 1$, thus $r(M) \geq 2$ in the minimal counterexample.

Pick $e \in E$. By Lemma 2, $\hat{Z}_M = \hat{Z}_{M\setminus e} + v_e \hat{Z}_{M/e}$. Let $\bar{\mathbf{v}} = \mathbf{v} \setminus \{v_e\}$, and set $k = K(\bar{\mathbf{v}})$. Recall that $\hat{Z}_M$ is irreducible over $k(v_e)$ by Lemma 3. We have seen above that $r(M \setminus e) = r(M) = n$ and $r(M/e) = n - 1$. As established previously, either $M \setminus e$ or M/e is connected. By assuming a minimal counterexample, we have $\mathrm{Gal}(\hat{Z}_{M\setminus e}/k) = S_n$ or $\mathrm{Gal}(\hat{Z}_{M/e}/k) = S_{n-1}$. Theorem 1 then follows from Lemma 6. □

We will now present an alternative proof of Theorem 1. While it is less efficient than the above proof, it uses a group-theoretical inductive process which is perhaps more intuitive. We will need to first prove that the theorem holds for circuits.

Lemma 7 *Let $C \subseteq E$ be a circuit of a finite matroid M. Then* $\mathrm{Gal}(\hat{Z}_{M|C}/K(\mathbf{v}_C)) = S_{r_M(C)}$.

Proof The rank of any proper subset of C is the same as its cardinality, and $r_M(C) = |C| - 1$, so:

$$\hat{Z}_{M|C}(q, \mathbf{v}) = q^n + \sigma_1 q^{n-1} + \sigma_2 q^{n-2} + \cdots + \sigma_{n-1} q + (\sigma_n + \sigma_{n+1}),$$

where σ_i is the ith elementary symmetric polynomial in the $\{v_e\}_{e\in C}$ for each i. The elementary symmetric polynomials are algebraically independent, and thus so too are the coefficients of $\hat{Z}_{M|C}(q, \mathbf{v})$. It is well known that the Galois group of a polynomial with algebraically independent coefficients is the full symmetric group. □

Second proof of Theorem 1 Let C be a circuit of maximum cardinality in M. By Lemma 7, $\mathrm{Gal}(\hat{Z}_{M|C}/K(\mathbf{v}_C)) = S_{r_M(C)}$. This will serve as the base case for the induction.

Now, let A be any proper subset of E such that $C \subseteq A$ and $M|A$ is connected, and suppose that $\mathrm{Gal}(\hat{Z}_{M|A}/K(\mathbf{v}_A)) = S_{r_M(A)}$. Identify a non-empty independent set $B \subseteq E \setminus A$ of minimal size such that $M|(A \cup B)$ is connected, and let $A' = (A \cup B)$. We will show that $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'})) = S_{r_M(A')}$.

By [6, Lemma 1.3.1], $r_M(A') \leq r_M(A) + r_M(B)$. By maximality of C, any circuit of $M|A'$ has rank at most $r_M(C)$. By minimality of B, any circuit of $M|A'$ not contained in $M|A$ must include at least one element of A, so $r_M(B) \leq r_M(C) - 1$, and we have $r_M(A') \leq r_M(A) + r_M(C) - 1$.

By Lemma 5, $S_{r_M(A)} = \mathrm{Gal}(\hat{Z}_{M|A}/K(\mathbf{v}_A)) \leq \mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$. So $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$ must contain at least one transposition. Let H be the group generated by all of the transpositions in $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$; then H is a direct product of symmetric groups. As $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$ is transitive, each of these symmetric groups must have the same degree i, which must therefore divide the degree of $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$. By Lemma 3, $\hat{Z}_{M|A'}$ is irreducible, and its Galois group must therefore be transitive of degree $r_M(A')$. So we have that $ji = r_M(A')$ for some positive integer j.

Now, $S_{r_M(A)}$ contains at least one of the transpositions of H, so must be a subgroup of one of the S_i, which means $r_M(A) \le i$. So we have:

$$jr_M(A) \le ji = r_M(A') \le r_M(A) + r_M(C) - 1.$$

Suppose that $j \ge 2$. Then $2r_M(A) \le r_M(A) + r_M(C) - 1$, and so $r_M(A) \le r_M(C) - 1$. This is impossible, as $C \subset A$. So $j = 1$, and hence $i = r_M(A')$. This means that H is a direct product of symmetric groups of degree $r_M(A')$. But H is a subgroup of $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'}))$, which is transitive of degree $r_M(A')$, and so $\mathrm{Gal}(\hat{Z}_{M|A'}/K(\mathbf{v}_{A'})) = H = S_{r_M(A')}$. □

Now, in view of the proof of Lemma 7, one might wonder if the coefficients of $\hat{Z}_M(q, \mathbf{v})$ are algebraically independent for *any* finite connected matroid. This does indeed turn out to be the case, leading us to our third and final proof of Theorem 1.

Third proof of Theorem 1 Let M be a finite connected matroid of rank $r(M) = n \ge 1$, and write $\hat{Z}_M(q, \mathbf{v}) = q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + a_0 \in K[\mathbf{v}][q]$, where K is an arbitrary field. It suffices to show that the coefficients $a_0, a_1, \dots, a_{n-1}$ are algebraically independent over K.

If $n = 1$, then $\hat{Z}_M(q, \mathbf{v}) = q - 1 + \prod_{e\in E}(v_e + 1)$, so the claim clearly holds. Thus we may assume $n \ge 2$.

Assume that M is a counterexample in which $|E|$ is minimal. We will use the deletion–contraction identity $\hat{Z}_M = \hat{Z}_{M\setminus e} + v_e \hat{Z}_{M/e}$ of Lemma 2. First consider the case that $M \setminus e$ is connected. By the assumption of a minimal counterexample, the coefficients of $\hat{Z}_{M\setminus e}$ (excluding the leading coefficient 1) are algebraically independent over K. However, these coefficients arise from the coefficients $a_0, a_1, \dots, a_{n-1}$ upon setting $v_e = 0$. Of course, an algebraic dependency relation of $a_0, a_1, \dots, a_{n-1}$ over K remains an algebraic dependency relation upon setting $v_e = 0$, a contradiction.

Thus $M \setminus e$ is not connected, so we may assume that M/e is connected. For each $0 \le i \le n-1$, write $a_i = b_i + v_e c_i$, where b_i and c_i are polynomials in the elements of $\mathbf{v}_{E\setminus e}$. Each c_j is then the coefficient of q^j in $\hat{Z}_{M/e}$, so $c_{n-1} = 1$ (as $r(M/e) = n - 1$) and $c_0, c_1, \dots, c_{n-2}$ are algebraically independent over K. As $a_0, a_1, \dots, a_{n-1}$ are algebraically dependent, there is a non-zero polynomial P in n variables over K such that

$$P(b_0 + v_e c_0, \dots, b_{n-2} + v_e c_{n-2}, b_{n-1} + v_e) = 0.$$

Let Q be the expansion of P with respect to v_e, so that Q is a polynomial in v_e with coefficients in $K[\mathbf{v}_{E\setminus e}]$. As the elements of $\mathbf{v}$ are algebraically independent, these coefficients must be identically zero. Let d be the total degree of P. Then Q has degree d in v_e, and the v_e^d term must arise from a K-linear sum of products of the form:

$$(b_0 + v_e c_0)^{d_0} \cdots (b_{n-2} + v_e c_{n-2})^{d_{n-2}} (b_{n-1} + v_e)^{d_{n-1}},$$

where $d_0, \ldots, d_{n-1}$ are non-negative integers which sum to d. This means that the coefficient of v_e^d in Q is a K-linear combination of monomials of the form $c_0^{d_0} \cdots c_{n-2}^{d_{n-2}}$, where $d_i \geq 0$ for each i, and $d_0 + \cdots + d_{n-2} \leq d$. The vanishing of this coefficient then implies that the set of such monomials is linearly dependent over K, which contradicts our assertion that $c_0, \ldots, c_{n-2}$ are algebraically dependent over K. □

Remark 8 Sokal showed that the multivariate Tutte polynomial for matroids factorizes over summands (see [7, (4.4)]). That is, if M is the direct sum of connected matroids M_1, M_2 on the sets E_1, E_2, respectively (where E_1 and E_2 are disjoint and $E = E_1 \cup E_2$) then:

$$\hat{Z}_M(q, \mathbf{v}) = \hat{Z}_{M_1}(q, \mathbf{v}_{E_1})\hat{Z}_{M_2}(q, \mathbf{v}_{E_2}).$$

As $\mathbf{v}_{E_1}$ and $\mathbf{v}_{E_2}$ are disjoint, there are clearly no algebraic dependencies between the roots of $\hat{Z}_{M_1}$ and $\hat{Z}_{M_2}$, so we have that

$$\mathrm{Gal}\big(\hat{Z}_M/K(\mathbf{v})\big) = \mathrm{Gal}\big(\hat{Z}_{M_1}/K(\mathbf{v}_{E_1})\big) \times \mathrm{Gal}\big(\hat{Z}_{M_2}/K(\mathbf{v}_{E_2})\big).$$

Theorem 1 then implies that the Galois group of the multivariate Tutte polynomial of any matroid is a direct product of symmetric groups corresponding to the connected direct summands.

Finally, we computed the Galois group of the bivariate Tutte polynomial $T_G(x, y)$ over $\mathbb{Q}(y)$ for every biconnected graph G of order $n \leq 10$, and found that all were the symmetric group of degree $n - 1$. As the Tutte polynomial of any connected matroid is irreducible over fields of characteristic zero (as noted in [4], this is not necessarily the case for fields of positive characteristic), this would seem to suggest the following:

Conjecture 9 *Let M be a finite connected matroid with positive rank $n = r(M)$, and let K be a field of characteristic zero. Then the Galois group of the Tutte polynomial $T_M(x, y)$ over $K(y)$ is the symmetric group of degree n.*

As remarked previously, the bivariate Tutte polynomial is essentially a specialization of the multivariate version. This means that Theorem 1 would follow from a proof of Conjecture 9 for fields of characteristic zero.

Interestingly, specializing the Tutte polynomial further produces a range of different Galois groups. For example, it was shown in [1] that all of the transitive permutation groups of degree at most 5 apart from C_5 appear as Galois groups of just one family of chromatic polynomials. Furthermore, Morgan [5] showed that a range of transitive groups of higher degree occur for chromatic polynomials of graphs on up to 10 vertices.

References

1. Cameron, P.J., Morgan, K.: Algebraic properties of chromatic roots. Submitted
2. Kung, J.P.S.: Twelve views of matroid theory. In: Combinatorial & Computational Mathematics: Present and Future, Pohang, the Republic of Korea, 15–17 February 2000, p. 56 (2001)

3. Lang, S.: Algebra. Addison-Wesley, Menlo Park (1984)
4. Merino, C., de Mier, A., Noy, M.: Irreducibility of the Tutte polynomial of a connected matroid. J. Comb. Theory **83**, 298–304 (2001)
5. Morgan, K.: Algebraic aspects of the chromatic polynomial. PhD thesis
6. Oxley, J.G.: Matroid Theory, Oxford Science Publications. Clarendon/Oxford University Press, New York (1992)
7. Sokal, A.D.: The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In: Surveys in Combinatorics, vol. 327, pp. 173–226 (2005)