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Automorphisms and Isomorphisms of Symmetric and Affine Designs

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Abstract. Given a finite group G, for all sufficiently large d and for each $q > 3$ there are symmetric designs and affine designs having the same parameters as $PG(d, q)$ and $AG(d, q)$, respectively, and having full automorphism group isomorphic to G.

Keywords: automorphism group of symmetric design

1. Introduction

There are many theorems of the form *every finite group is the full automorphism group of a member of a certain class of combinatorial structures*, such as graphs [4] or Steiner triple systems [11]. Usually these structures are not overly restrictive in appearance, and a construction can be obtained by starting with the result for graphs and applying suitable construction techniques (see [1] for a survey of such results). The purpose of this note is to prove such a theorem for structures that appear to be a bit more constrained: symmetric designs. It should be noted that it is by no means a trivial matter even to construct symmetric designs having no nontrivial automorphisms: some effort was needed in [12] in order to accomplish this for Hadamard designs. Of course, the most desirable theorem of this sort would concern finite projective planes, but there is as yet very little information concerning the structure of the automorphism group of such a plane.

Theorem 1.1. *Let G be a finite group. If $q > 3$ is any prime power, and if d is any integer $\geq 50|G|^2$, then there are designs* $\mathbf{D}$ *and* $\mathbf{A}$ *such that*
(i) $\mathrm{Aut}\mathbf{D} \cong G \cong \mathrm{Aut}\mathbf{A}$,
(ii) $\mathbf{D}$ *is a symmetric design having the same parameters as $PG(d, q)$, and*
(iii) $\mathbf{A}$ *is an affine design having the same parameters as $AG(d, q)$.*

We will see that, for given G, q and d there are at least $[q^{0.8d}]!$ pairwise nonisomorphic designs of this sort. This should be compared with the fact that there are known to be more than $(q^{d-1})!$ symmetric designs having the parameters of $PG(d, q)$ ([5; 7]; cf. (3.2), (4.4)). The Theorem continues to hold when q is 2 or 3, but somewhat different methods seem to be needed [10].

Unlike all previous proofs of this type of result we will not use any variation on the version for graphs as a starting point: there does not appear to be any known construction

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technique that starts with a graph and produces a symmetric or affine design having the stated parameters. (Of course, it would be quite nice to have such a construction, not least in order to simplify the proofs in this paper.) It may be that the proof of the above theorem is more significant than the theorem itself: as we will see, it raises a number of questions concerning symmetric and affine designs. On the other hand, as with other proofs of this type of result, the structure of the group does not enter at all into our arguments; for example, the proof does not distinguish in any way between cyclic and nonsolvable groups.

This paper also describes straightforward construction techniques for symmetric and affine designs (Section 2), together with elementary information concerning isomorphisms and automorphisms (Sections 3, 4). There are unexpected byproducts, relating double cosets to isomorphisms (4.4). Part of this approach was very briefly sketched in [2, pp. 113–114][1] at the same time that isomorphisms and asymptotics were being investigated in detail [7]. The latter remained unpublished due to an inability to control isomorphisms and automorphisms after many successive iterations (cf. (2.6)), and this still seems very difficult (as is readily seen below in Sections 5 and 8). A number of the results in [7] appear here as portions of Sections 2–5; some were obtained independently in [5].

Affine spaces will be visible within most of the designs constructed here. In Section 8 there is a very large chunk of a projective space available to work with: there, we start with a projective space, remove and reglue the hyperplane at infinity in order to obtain a new symmetric design, and then repeat this procedure an additional time by regluing a suitable block of the new symmetric design. This must be accomplished while preserving a given group G as an automorphism group, removing other automorphisms, and ensuring that no unexpected automorphisms arise. Implementing this idea is, however, somewhat delicate. This takes place in Theorems 8.9 and 8.10, which together provide slightly stronger results than (1.1).

Section 6 proves a (corrected version of a) conjecture in [5] concerning the asymptotic behavior of the automorphism groups of the symmetric designs studied in Sections 2–4; this section is not needed for the proof of (1.1). Section 9 contains numerous remarks and conjectures suggested by various results in earlier sections.

Almost all of the difficult portions of this paper reduced to (or were rescued by) results concerning permutations of the points of projective spaces. These have been swept into an Appendix (Section 10). The following is a typical but very special case of what is needed in our approach to (1.1): For any q and d, each finite group of order $< \sqrt{d/20}$ is isomorphic to the stabilizer of some two points in the permutation representation of $S_{(q^d-1)/(q-1)}$ in its action on the cosets of $P\Gamma L(d, q)$. The proofs in Section 10 involve unusual geometric considerations.

Many arguments given in Sections 4, 5 and 8 contain hints of ideas occurring in the proof of the Dembowski-Wagner Theorem [3] and related results. I am indebted to Peter Dembowski for many things, in particular for introducing me to the methods in [3] and for encouragement when the simpler aspects of this paper were being investigated in [7].

[1] The condition $(d, q) \neq (3, 2)$ was omitted from the hypotheses of [2, 2.4.37]. However, apparently it was the brevity of Dembowski's sketch that led to the following conclusion in [6, p. 107]: "We remark that the proof given in {Dembowski's book} is incomplete (if correct)".

Thanks also are due J. H. Dillon for providing the impetus for this paper, and R. A. Liebler for suggesting the use of extension fields in (10.2).

For background concerning symmetric and affine designs see [2]. Blocks of designs will be viewed as sets of points. If $\mathbf{D}$ is any design and B is a block, let $\mathbf{D}^B$ denote the incidence structure whose points are those not in B and whose blocks are the sets $X - B \cap X$, where X is a block $\neq B$. On the other hand, let $\mathbf{D}_{(B)}$ denote the incidence structure *induced* on B, whose points are those of B and whose blocks are the different nonempty intersections of B with the remaining blocks (compare [2, p. 3]).

A block B is called *good* if, for each block $X \neq B$, the blocks containing $B \cap X$ cover all the points of $\mathbf{D}$.

The *line* xy joining 2 different points x, y of a design $\mathbf{D}$ is the intersection of all the blocks containing both of these points [2, p. 65]. Distinct points are always on just one line. Since we will be working with several designs simultaneously, it will often be convenient to use the notation $xy_{\mathbf{D}}$ in place of xy, and we occasionally refer to $\mathbf{D}$-*lines*.

The group Aut$\mathbf{D}$ of automorphisms of $\mathbf{D}$ will be viewed as a group of permutations of the points or the blocks of $\mathbf{D}$, depending upon which is most convenient. If $G \leq \mathrm{Aut}\mathbf{D}$ and S is a point or a set of points, then G_S denotes the set-stabilizer of S.

We will use the same notation $PG(d, q)$ (or $AG(d, q)$) for a projective (or affine) space and its design of points and hyperplanes. The projective space at infinity of an affine space $\mathbf{A}$ is denoted $\underline{\mathbf{A}}$.

2. Gluing

Let $\mathbf{A} = (\mathcal{P}, \mathcal{B}, \in)$ be an affine design with $m = v/k = k/\mu$ blocks per parallel class, so that nonparallel blocks meet in μ points. Let $\underline{B}$ denote the parallel class of the block B, and let $\underline{\mathcal{B}}$ be the set of all these parallel classes. Also, let $\mathbf{D}_\infty = (\mathcal{P}_\infty, \mathcal{B}_\infty, \in)$ denote *any* symmetric design having $v_\infty = r$ and $k_\infty = \lambda$.

Fix a bijection $\alpha: \underline{\mathcal{B}} \to \mathcal{B}_\infty$. Define a new incidence structure $\mathbf{A}(\alpha) = \mathbf{A}(\mathbf{D}_\infty, \alpha)$ using the point set $\mathcal{P} \cup \mathcal{P}_\infty$ and the following subsets as blocks:

$\mathcal{P}_\infty$;

$\overline{B} := B \cup \underline{B}^\alpha$ for each block B of $\mathbf{A}$.

Theorem 2.1 (*Shrikhande* [14]). *$\mathbf{A}(\alpha)$ is a symmetric design with parameters $v(\alpha) = v + v_\infty$, $k(\alpha) = v_\infty$ and $\lambda(\alpha) = k_\infty$.*

Of course, the proof is a straightforward verification, as are the following remarks:

Lemma 2.2.

(i) *$\mathcal{P}_\infty$ is a good block; $\mathbf{A}(\alpha)_{(\mathcal{P}_\infty)} = \mathbf{D}_\infty$ and $\mathbf{A}(\alpha)^{\mathcal{P}_\infty} = \mathbf{A}$.*

(ii) *α can be recovered from $\mathbf{A}$ and $\mathbf{A}(\alpha)$.*

(iii) *If $x, y \in \mathcal{P}$ then $xy_{\mathbf{A}(\alpha)} = xy_{\mathbf{A}} \cup \bigcap\{\underline{B}^\alpha | x, y \in B \in \mathcal{B}\}$. In particular, $|xy_{\mathbf{A}(\alpha)}| \geq |xy_{\mathbf{A}}|$.*

Good blocks will reoccur adnauseam throughout this paper. We begin with a well-known observation:

Lemma 2.3. (i) *If E is a good block of an affine design* $\mathbf{A}$ *then* $\mathbf{A}_{(E)}$ *is an affine design with parameters* $v_{(E)} = k$, $r_{(E)} = \lambda$, $k_{(E)} = \mu$ *and* $m_{(E)} = m$. *Each block of* $\mathbf{A}_{(E)}$ *is contained in exactly* m *blocks* $\neq E$ *of* $\mathbf{A}$. *If* X *and* Y *are parallel blocks of* $\mathbf{A}$ *not parallel to* E, *then* $E \cap X$ *and* $E \cap Y$ *are parallel blocks of* $\mathbf{A}_{(E)}$; *conversely, if* X *and* Y *are blocks of* $\mathbf{A}$ *such that* $E \cap X$ *and* $E \cap Y$ *are parallel in* $\mathbf{A}_{(E)}$, *then* $E \cap Y = E \cap X'$ *for some block* X' *parallel to* X.

(ii) *If* Z *is a good block of a symmetric design* $\mathbf{D}$ *then* $\mathbf{D}_{(Z)}$ *is a symmetric design with parameters* $v_{(Z)} = k$ *and* $k_{(Z)} = \lambda$, *and* $\mathbf{D}^Z$ *is an affine design with parameters* $v^Z = v-k$, $k^Z = k - \lambda$ *and* $m^Z = (v-k)/(k-\lambda)$. *If* $W \neq Z$ *is a block of* $\mathbf{D}$, *then the parallel class of* $\mathbf{D}^Z$ *containing* $W - Z \cap W$ *consists of all the blocks* $\neq Z$ *of* $\mathbf{D}$ *containing* $Z \cap W$.

Proof: (i) Each block $E \cap X$ of $\mathbf{A}_{(E)}$ lies in blocks of $\mathbf{A}$ that intersect pairwise in $E \cap X$ and cover all points; hence, $E \cap X$ lies in $(v-k)/(k-\mu) = m$ blocks $\neq E$. Thus, $\mathbf{A}_{(E)}$ is a design having $v_{(E)} = k$, $r_{(E)} = (r-1)/m = \lambda$, $k_{(E)} = \mu = k/m$ and $\lambda_{(E)} = (\lambda-1)/m$. Disjoint blocks X, Y of $\mathbf{A}$ not parallel to E produce disjoint blocks $E \cap X$, $E \cap Y$ of $\mathbf{A}_{(E)}$. Then $\mathbf{A}_{(E)}$ is a resolvable design for which $r_{(E)} = k_{(E)} + \lambda_{(E)}$, and hence is an affine design by a theorem of Bose [2, p. 72].

It follows that $m_{(E)} = v_{(E)}/k_{(E)} = m$. We have found m blocks of $\mathbf{A}_{(E)}$ parallel to $E \cap X$, arising from the m blocks parallel to X. This implies the final assertion.

(ii) The argument is very similar. □

There are easy converses to both parts of the lemma, essentially by reversing the arguments.

Proposition 2.4. *The following are equivalent for a block* $\overline{E}$ *of* $\mathbf{A}(\alpha)$:

(i) $\overline{E}$ *is good; and*

(ii) E *is a good block of* $\mathbf{A}$, $\underline{E}^\alpha$ *is a good block of* $\mathbf{D}_\infty$, *and if* $E \cap X = E \cap Y \neq \emptyset$ *(for* $X, Y \in \mathcal{B}$*) then* $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E}^\alpha \cap \underline{Y}^\alpha$.

Proof: Note that $\underline{E}^\alpha$ is contained in $m+1$ blocks of $\mathbf{A}(\alpha)$. Assume that (ii) holds. If $\emptyset \neq E \cap X \subseteq Y$ then $\underline{E}^\alpha \cap \underline{X}^\alpha \subseteq \underline{Y}^\alpha$ by hypothesis, so that $\overline{E} \cap \overline{X} \subseteq \overline{Y}$. The m blocks $Y \neq E$ containing $E \cap X$ (cf. (2.3i)) determine m different parallel classes $\underline{Y}$ and hence all m blocks $\underline{Y}^\alpha \neq \underline{E}^\alpha$ of $\mathbf{D}_\infty$ containing $\underline{E}^\alpha \cap \underline{X}^\alpha$ (cf. (2.3ii)). Thus, the m blocks $\overline{Y}$ of $\mathbf{A}(\alpha)$ cover both $\mathcal{P}$ and $\mathcal{P}_\infty$, so that (i) holds. For the other direction, reverse this argument. □

In view of (2.3) and (2.4), if $\overline{E}$ is a good block of $\mathbf{A}(\alpha)$ then we obtain five additional designs to consider: affine designs $\mathbf{A}(\alpha)^{\overline{E}}$, $\mathbf{A}_{(E)}$ and $(\mathbf{D}_\infty)^{\underline{E}^\alpha}$, as well as symmetric designs $\mathbf{A}(\alpha)_{(\overline{E})}$ and $(\mathbf{D}_\infty)_{(\underline{E}^\alpha)}$.

Remark 2.5 (*"Regluing"*). Here is what amounts to a converse of (2.1). Suppose that $\mathbf{D}$ and $\mathbf{D}'$ are two symmetric designs, having good blocks $\mathcal{H}_\infty$ and $\mathcal{P}_\infty$, respectively, such that $\mathbf{D}^{\mathcal{H}_\infty} = \mathbf{D}'^{\mathcal{P}_\infty}$ is the *same* affine design $\mathbf{A}$. Each block X of $\mathbf{A}$ lies in a unique block $X \cup \underline{X}$ of $\mathbf{D}$ and a unique block $X \cup \underline{X}'$ of $\mathbf{D}'$, where $\underline{X}$ and $\underline{X}'$ are blocks of $\mathbf{D}_{(\mathcal{H}_\infty)}$ and $\mathbf{D}_\infty := \mathbf{D}'_{(\mathcal{P}_\infty)}$, respectively. Write $\underline{X}' = \underline{X}^\alpha$, so that α is a bijection from the set $\underline{\mathcal{B}}$ of blocks of $\mathbf{D}_{(\mathcal{H}_\infty)}$ to the set $\mathcal{B}_\infty$ of blocks of $\mathbf{D}_\infty$. Then $\mathbf{D}' \cong \mathbf{A}(\mathbf{D}_\infty, \alpha)$, essentially by definition: we can identify $\underline{\mathcal{B}}$ with the set of parallel classes of $\mathbf{A}$ by identifying $\underline{X}$ with $\{Y \mid \underline{Y} = \underline{X}\}$.

There is also an affine design analogue of (2.5). The regluing process in (2.5) suggests our approach to (1.1):

Construction Procedure 2.6. Start with an affine design $\mathbf{A}$ and a symmetric design $\mathbf{D}_\infty$ with $v_\infty = r$ and $k_\infty = \lambda$. Use (2.1) to glue $\mathbf{D}_\infty$ to $\mathbf{A}$ using α, which is chosen so that $\overline{E}$ is good. Let $\mathbf{A}'$ be the affine design $\mathbf{A}(\mathbf{D}_\infty, \alpha)^{\overline{E}}$, let $\mathbf{D}'_\infty$ be another design having the same parameters as $\mathbf{D}_\infty$, and repeat using $\mathbf{A}'$ and $\mathbf{D}'_\infty$ in place of $\mathbf{A}$ and $\mathbf{D}_\infty$.

This procedure can be repeated, varying the good block chosen—provided goodness can be verified at each stage. As observed in Section 1, it seems very difficult to study these iterations.

We continue with several elementary consequences of (2.3)–(2.5).

Lemma 2.7. (i) *In the notation of* (2.5), *assume that* $E \cup \underline{E}$ *is a good block of* $\mathbf{D}$ *and that* $\underline{E}^\alpha$ *is a good block of* $\mathbf{D}_\infty$. *If* $\underline{E} \cap \underline{X} = \underline{E} \cap \underline{Y}$ *implies that* $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E}^\alpha \cap \underline{Y}^\alpha$, *then* $\overline{E} = E \cup \underline{E}^\alpha$ *is a good block of* $\mathbf{A}(\mathbf{D}_\infty, \alpha)$.

(ii) *Assume that* $\mathbf{A}$ *is an affine space and* $\mathbf{D}_\infty = \underline{\mathbf{A}}$. *If* $\overline{E}$ *is a good block of* $\mathbf{A}(\mathbf{D}_\infty, \alpha)$, *then* $\underline{E} \cap \underline{X} = \underline{E} \cap \underline{Y}$ *implies that* $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E}^\alpha \cap \underline{Y}^\alpha$. *Moreover, if* F *is any hyperplane of* $\mathbf{A}$ *parallel to* E, *then* $\overline{F}$ *is good.*

(iii) *In the notation of* (2.5), *assume that* $E \cup \underline{E}$ *is a good block of* $\mathbf{D}$. *If* $\underline{E}^\alpha \cap \underline{X} = \underline{E}^\alpha \cap \underline{X}^\alpha$ *for all* X, *then* $\overline{E} = E \cup \underline{E}$ *is a good block of* $\mathbf{A}(\mathbf{D}_\infty, \alpha)$; *moreover,* $\mathbf{A}(\mathbf{D}_\infty, \alpha)_{(\overline{E})} = \mathbf{D}_{(\overline{E})}$.

Proof: (i) By (2.4) it suffices to show that, if $E \cap X = E \cap Y \neq \emptyset$, E, then $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E}^\alpha \cap \underline{Y}^\alpha$. By (2.3i), there are m blocks $Y \neq E$ of $\mathbf{A}$ containing $E \cap X$, and m blocks $Y \cup \underline{Y} \neq E \cup \underline{E}$ of $\mathbf{D}$ containing $(E \cup \underline{E}) \cap (X \cup \underline{X})$; the m blocks Y appearing in both of these statements must be the same. Thus, if $E \cap X = E \cap Y \neq \emptyset$, then $\underline{E} \cap \underline{X} = \underline{E} \cap \underline{Y}$, and hence $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E}^\alpha \cap \underline{Y}^\alpha$ by hypothesis.

(ii) Assume that $\underline{E} \cap \underline{X} = \underline{E} \cap \underline{Y}$. Let $e \in E$. Then $\mathbf{A}$ has blocks $X' \| X$ and $Y' \| Y$ through e. Now $\underline{E} \cap \underline{X}' = \underline{E} \cap \underline{Y}'$ implies that $E \cap X' = E \cap Y'$: this is all taking place inside the projective space $\mathbf{A}(\mathbf{D}_\infty, 1)$.

Now $E \cap X' = E \cap Y'$ implies that $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E}^\alpha \cap \underline{X}'^\alpha = \underline{E}^\alpha \cap \underline{Y}'^\alpha = \underline{E}^\alpha \cap \underline{Y}^\alpha$ by (2.4).

For the final assertion, assume that $F \cap X = F \cap Y \neq \emptyset$. Then $\underline{F} \cap \underline{X} = \underline{F} \cap \underline{Y}$ since $\mathbf{A}$ is an affine space. Then also $\underline{E} \cap \underline{X} = \underline{E} \cap \underline{Y}$, which was just seen to imply that $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E}^\alpha \cap \underline{Y}^\alpha$. Thus, $\underline{F}^\alpha \cap \underline{X}^\alpha = \underline{F}^\alpha \cap \underline{Y}^\alpha$, and hence $\overline{F}$ is good by (2.4) since F and $\underline{E}^\alpha$ certainly are.

(iii) Setting $X = E$ we find that $\underline{E}^\alpha = \underline{E}$, so that $\overline{E} = E \cup \underline{E}$. Consider any block $\overline{X} \neq \overline{E}$. We have $\overline{E} \cap \overline{X} = (E \cap X) \cup (\underline{E}^\alpha \cap \underline{X}^\alpha) = (E \cap X) \cup (\underline{E} \cap \underline{X}) = \overline{E} \cap (X \cup \underline{X})$. Since $\overline{E}$ is a good block of $\mathbf{D}$, these intersections are the blocks of a symmetric design $\mathbf{D}_{(\overline{E})}$. It follows that $\overline{E}$ is also a good block of $\mathbf{A}(\mathbf{D}_\infty, \alpha)$. $\square$

Lemma 2.8. (i) *If $\overline{E}$ is a good block of* $\mathbf{A}(\mathbf{D}_\infty, \alpha)$*, then α induces a bijection $\underline{\alpha}$ from the set of parallel classes of blocks of the affine design $\mathbf{A}_{(E)}$ to the set of blocks of the symmetric design $(\mathbf{D}_\infty)_{(\underline{E}^\alpha)}$, taking the parallel class $\underline{E \cap X}$ of $E \cap X$ to $\underline{E}^\alpha \cap \underline{X}^\alpha$.*

(ii) *In the situation of* (i), $\mathbf{A}(\mathbf{D}_\infty, \alpha)_{(\overline{E})} = \mathbf{A}_{(E)}(\mathbf{D}_{\infty(\underline{E}^\alpha)}, \underline{\alpha})$.

(iii) *In the situation of* (2.5)*, let $E \cup \underline{E}$ be a good block of $\mathbf{D}$ and let E' be any good block of $\mathbf{D}_\infty$. Then every bijection $\underline{\alpha}$ from the set of parallel classes of blocks of $\mathbf{A}_{(E)}$ to the set of blocks of $(\mathbf{D}_\infty)_{(E')}$ extends in exactly $m!^\lambda$ ways to a bijection $\alpha\colon \underline{\mathcal{B}} \to \mathcal{B}_\infty$ such that $\underline{E}^\alpha = E'$, $(\underline{E \cap X})^{\underline{\alpha}} = \underline{E}^\alpha \cap \underline{X}^\alpha$ for all X; and then the block $\overline{E} = E \cup E'$ of $\mathbf{A}(\mathbf{D}_\infty, \alpha)$ is good.*

Proof: Note that E is a good block of $\mathbf{A}$, and $\underline{E}^\alpha$ or E' is a good block of $\mathbf{D}_\infty$ (cf. (2.4)).

(i) By (2.4), if $E \cap X = E \cap Y \neq \emptyset$ then $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E}^\alpha \cap \underline{Y}^\alpha$. If $E \cap X \| E \cap Y$ then, by (2.3i), $E \cap X' = E \cap Y$ for some $X' \| X$. Then $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E}^\alpha \cap \underline{X}'^\alpha = \underline{E}^\alpha \cap \underline{Y}^\alpha$. Thus, if we write $(\underline{E \cap X})^{\underline{\alpha}} = \underline{E}^\alpha \cap \underline{X}^\alpha$ for all X, then $\underline{\alpha}$ is well-defined. Moreover $\underline{\alpha}$ is onto: each block of $(\mathbf{D}_\infty)_{(\underline{E}^\alpha)}$ has the form $\underline{E}^\alpha \cap \underline{X}^\alpha$.

By (2.3), $\mathbf{A}_{(E)}$ has $r_{(E)} = \lambda$ parallel classes while $(\mathbf{D}_\infty)_{(\underline{E}^\alpha)}$ has $v_{\infty(\underline{E}^\alpha)} = k_\infty = \lambda$ blocks. Thus, α is a bijection.

(ii) The blocks of $\mathbf{A}(\mathbf{D}_\infty, \alpha)_{(\overline{E})}$ are the following sets of points:

$$\begin{aligned} \overline{E} \cap \mathcal{P}_\infty &= \underline{E}^\alpha, \\ \overline{E} \cap \overline{X} &= (E \cap X) \cup (\underline{E}^\alpha \cap \underline{X}^\alpha) \\ &= (E \cap X) \cup (\underline{E \cap X})^{\underline{\alpha}} \text{ for } E \text{ and } X \text{ not parallel.} \end{aligned}$$

Therefore, (ii) follows from the definitions preceding (2.1).

(iii) By (2.3ii), $(\mathbf{D}_\infty)_{(E')}$ has $v_{\infty(\underline{E}^\alpha)} = \lambda$ blocks, and $E' \cap \underline{X}^\alpha$ is contained in m blocks $\neq E'$ of $\mathbf{D}_\infty$ whenever $\underline{X}^\alpha \neq E'$. In (2.5) we identified parallel classes of $\mathbf{A}$ with blocks of $\mathbf{D}_\infty$. Any extension of $\underline{\alpha}$ to a map α must send the parallel classes $\underline{X}$ containing $\underline{E} \cap \underline{X}$ to parallel classes containing $E' \cap \underline{X}^\alpha$. This proves the assertion concerning the number of extensions of $\underline{\alpha}$ to a map $\alpha\colon \underline{\mathcal{B}} \to \mathcal{B}_\infty$. Each such extension satisfies the condition in (2.7i): $\underline{E} \cap \underline{X} = \underline{E} \cap \underline{Y}$ implies that $\underline{E}^\alpha \cap \underline{X}^\alpha = (\underline{E \cap X})^{\underline{\alpha}} = (\underline{E \cap Y})^{\underline{\alpha}} = \underline{E}^\alpha \cap \underline{Y}^\alpha$. □

3. Isomorphisms and automorphisms

Let $\mathbf{A}$ and $\mathbf{D}_\infty$ be as in Section 2, and consider another such pair of designs $\mathbf{A}'$, $\mathbf{D}'_\infty$. Let $\mathcal{P}'_\infty$ be the set of points of $\mathbf{D}'_\infty$. Denote by $\underline{\mathrm{Aut}\mathbf{A}}$ the group of permutations of $\underline{\mathcal{B}}$ induced by $\mathrm{Aut}\mathbf{A}$.

The following simple result is the basis for the rest of this paper.

Theorem 3.1. (i) *There is an isomorphism $\mathbf{A}(\mathbf{D}_\infty, \alpha) \to \mathbf{A}'(\mathbf{D}'_\infty, \beta)$ sending the block $\mathcal{P}_\infty$ to the block $\mathcal{P}'_\infty$ if and only if there are isomorphisms $\psi\colon\mathbf{A} \to \mathbf{A}'$ and $\varphi\colon\mathbf{D}_\infty \to \mathbf{D}'_\infty$ such that $\alpha\varphi = \underline{\psi}\beta$, where $\underline{\psi}\colon\underline{\mathcal{B}} \to \underline{\mathcal{B}}'$ denotes the map induced on parallel classes by ψ and both sides of this equation are viewed as acting on $\underline{\mathcal{B}}$.*

(ii) *The group of permutations of $\underline{\mathcal{B}}$ induced by* $(\mathrm{Aut}\mathbf{A}(\mathbf{D}_\infty, \alpha))_{\mathcal{P}_\infty}$ *is* $\mathrm{Aut}\mathbf{D}_\infty \cap (\underline{\mathrm{Aut}\mathbf{A}})^\alpha$ *(where the superscript α refers to conjugation).*

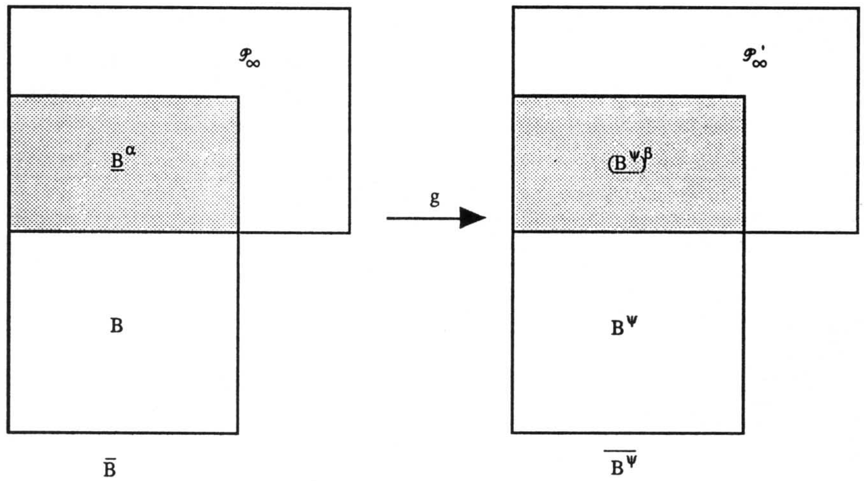

Figure 1.

Proof: Let φ and ψ behave as stated in (i). They define a map g from the sets of points and blocks of $\mathbf{A}(\mathbf{D}_\infty, \alpha)$ to those of $\mathbf{A}'(\mathbf{D}'_\infty, \beta)$: when restricted to $\mathcal{P}_\infty$ and $\mathcal{P}$, g is φ and ψ, respectively, while g sends $\mathcal{P}_\infty$ to $\mathcal{P}'_\infty$ and $\overline{B}$ to $\overline{B^\psi}$. This map is an isomorphism: if $\overline{B} \neq \mathcal{P}_\infty$ is a block of $\mathbf{A}(\mathbf{D}_\infty, \alpha)$ then g certainly preserves incidence with $\overline{B}$ of points of $\mathbf{A}$; and if $u \in \mathcal{P}_\infty$belongs to $\overline{B}$ then $u \in \underline{B}^\alpha$, so that $u^g = u^\varphi \in \underline{B}^{\alpha\varphi} = \underline{B}^{\underline{\psi}\beta} = (\underline{B}^\psi)^\beta \subset \overline{B^\psi} = \overline{B}^g$. (See Figure 1, the left side of which also provides the basic picture used in the study of the designs $\mathbf{A}(\mathbf{D}_\infty, \alpha)$.)

Conversely, suppose that there is an isomorphism $g: \mathbf{A}(\mathbf{D}_\infty, \alpha) \to \mathbf{A}'(\mathbf{D}'_\infty, \beta)$ sending $\mathcal{P}_\infty$ to $\mathcal{P}'_\infty$. Then restricting g induces isomorphisms $\psi: \mathbf{A} \to \mathbf{A}'$ and $\varphi: \mathbf{D}_\infty \to \mathbf{D}'_\infty$ (cf. (2.2i)). Consider any block $\overline{B}$ of $\mathbf{A}(\mathbf{D}_\infty, \alpha)$ other than $\mathcal{P}_\infty$, and let $\overline{B}^g = \overline{C}$. Then $(B \cup \underline{B}^\alpha)^g = C \cup \underline{C}^\beta$, so that $B^\psi = C$ and $(\underline{B}^\alpha)^\varphi = \underline{C}^\beta = (\underline{B^\psi})^\beta = (B^{\underline{\psi}})^\beta$. Thus, $\alpha\varphi = \underline{\psi}\beta$, as required in (i).

In (ii) we have $\mathbf{A} = \mathbf{A}'$, $\mathbf{D}_\infty = \mathbf{D}'_\infty$ and $\alpha = \beta$. Then we just saw that $(\mathrm{Aut}\mathbf{A}(\mathbf{D}_\infty, \alpha))_{\mathcal{P}_\infty}$ can be viewed as the set of all ordered pairs $(\psi, \varphi) \in \mathrm{Aut}\mathbf{A} \times \mathrm{Aut}\mathbf{D}_\infty$ such that $\alpha\varphi = \underline{\psi}\alpha$—i.e., such that $\varphi = \alpha^{-1}\underline{\psi}\alpha$. □

Let $\Gamma[\mathbf{A}]$ be the group of automorphisms of the affine design $\mathbf{A}$ inducing the identity on $\underline{\mathcal{B}}$.

Corollary 3.2. (i) *At least* $v_\infty!/\{(v + v_\infty)|\Gamma[\mathbf{A}]||\mathrm{Aut}\mathbf{A}\,||\mathrm{Aut}\mathbf{D}_\infty|\}$ *pairwise nonisomorphic designs* $\mathbf{A}(\mathbf{D}_\infty, \alpha)$ *are obtained for a fixed choice of* $\mathbf{A}$ *and* $\mathbf{D}_\infty$.

(ii) [5; 7] *There are at least*

$$\frac{\dfrac{q^d-1}{q-1}!}{|P\Gamma L(d+1, q)||P\Gamma L(d, q)|} > (q^{d-1})!$$

pairwise nonisomorphic designs having the same parameters as $PG(d, q)$.

Proof: Fix $\mathbf{A}(\mathbf{D}_\infty, \alpha)$, and consider how many β there might be such that there is an isomorphism $g: \mathbf{A}(\mathbf{D}_\infty, \alpha) \to \mathbf{A}(\mathbf{D}_\infty, \beta)$. First choose a block of $\mathbf{A}(\mathbf{D}_\infty, \alpha)$ that might be sent to $\mathcal{P}_\infty$; there are $v(\alpha) = v + v_\infty$ choices. Once this block is chosen, the number of designs $\mathbf{A}(\mathbf{D}_\infty, \beta)$ that can arise is the number that can arise from one of them by an isomorphism fixing $\mathcal{P}_\infty$, and this is at most $|\underline{\mathrm{Aut}\mathbf{A}}|\ |\mathrm{Aut}\mathbf{D}_\infty|$ by (3.1). Thus, a given design $\mathbf{A}(\mathbf{D}_\infty, \alpha)$ is isomorphic to at most $v(\alpha)|\underline{\mathrm{Aut}\mathbf{A}}||\mathrm{Aut}\mathbf{D}_\infty|$ others. Since there are $v_\infty!$ choices for α, the total number of isomorphism classes is at least $v_\infty!/\{(v+v_\infty)|\underline{\mathrm{Aut}\mathbf{A}}||\mathrm{Aut}\mathbf{D}_\infty|\}$. This proves (i).

For (ii), choose $\mathbf{A} = AG(d, q)$ and $\mathbf{D}_\infty = PG(d-1, q)$, and note that $|P\Gamma L(d+1, q)| = |\Gamma[\mathbf{A}]||\underline{\mathrm{Aut}\mathbf{A}}|$. □

The bound in (ii) is the same as the one in [5]. For a marginal improvement when $q > 2$, see (4.4iii).

Corollary 3.3. *Assume that* $\mathbf{A}$ *is an affine space and* $\mathbf{D}_\infty = \underline{\mathbf{A}}$. *Let* 0 *be a point of* $\mathcal{P}$, *and let* $G \leq (\mathrm{Aut}\mathbf{A})_0$. *If the restriction of* G *to the set* $\underline{\mathcal{B}}$ *of parallel classes of* $\mathbf{A}$ *commutes with* α, *then* G *is naturally isomorphic to a group of automorphisms of* $\mathbf{A}(\alpha)$.

Proof: If $\psi \in G$ then $\underline{\psi} \in \mathrm{Aut}\mathbf{D}_\infty$. By (3.1), the ordered pair $(\psi, \underline{\psi})$ "is" an automorphism of $\mathbf{A}(\alpha)$. The set of such automorphisms clearly is isomorphic to G. □

Corollary 3.4. *If* $\mathbf{A}(\mathbf{D}_\infty, \alpha) \cong PG(d, q)$ *then* $\mathbf{A} \cong AG(d, q)$ *and* α *is induced by an isomorphism* $\underline{\mathbf{A}} \to \mathbf{D}_\infty$. *Conversely, if* $\mathbf{A} = AG(d, q)$ *and* α *is induced by an isomorphism* $\underline{\mathbf{A}} \to \mathbf{D}_\infty$, *then* $\mathbf{A}(\mathbf{D}_\infty, \alpha) \cong PG(d, q)$.

Proof: If $\mathbf{A}(\alpha) \cong PG(d, q)$ then $\mathbf{A} \cong AG(d, q)$ by (2.2i). Thus, throughout this proof we may assume that $\mathbf{A} = AG(d, q)$. Since $PG(d, q) = \mathbf{A}(\underline{\mathbf{A}}, 1)$, the condition for isomorphism in (3.1i) is $\alpha\varphi = \underline{\psi}\beta = \underline{\psi}$. Thus, if there is an isomorphism $\mathbf{A}(\mathbf{D}_\infty, \alpha) \to \mathbf{A}(\underline{\mathbf{A}}, 1)$ then α is induced by the isomorphism $\underline{\psi}\varphi^{-1}$; while if α is induced by an isomorphism φ^{-1}, say, then $\psi = 1$ satisfies the required condition. □

Corollary 3.5. *Let* $d \geq 3$ *and* $\mathbf{A} = AG(d, q)$.

(i) *The number of isomorphism classes of designs* $\mathbf{A}(\alpha) = \mathbf{A}(\underline{\mathbf{A}}, \alpha)$, *each having exactly one good block, is at least*

$$\frac{\dfrac{q^d-1}{q-1}! - \left(\dfrac{q^d-1}{q-1}\right)^2 \dfrac{q^{d-1}-1}{q-1}!_q!^{(q^{d-1}-1)/(q-1)}}{|P\Gamma L(d+1, q)||P\Gamma L(d, q)|} > (q^{d-1})!.$$

(ii) *The proportion of isomorphism classes in* (i), *among all of the isomorphism classes of designs* $\mathbf{A}(\alpha)$, *approaches* 1 *as* $dq \to \infty$.

Proof: By (2.8), it suffices to avoid bijections α such that, for some E, E' and some bijection $\underline{\alpha}$ from the set of parallel classes of blocks of the affine space $\mathbf{A}_{(E)}$ to the set of blocks of the projective space $(\mathbf{D}_\infty)_{(\underline{E}^\alpha)}$, we have $\underline{E}^\alpha = \underline{E}'$ and $(\underline{E \cap X})^\alpha = \underline{E}^\alpha \cap \underline{X}^\alpha$

for all X. There are v_∞^2 choices of a pair of blocks $\underline{E}$, $\underline{E}'$ of $\mathbf{D}_\infty$, then $k_\infty! = \lambda!$ bijections $\underline{\alpha}$, and finally $q!^{k_\infty} = q!^\lambda$ extensions of each such $\underline{\alpha}$ to a bijection α by (2.8iii). Thus, there are at most $v_\infty^2 \lambda! q!^\lambda$ "bad" bijections α.

In view of (3.2ii) this proves (i), and (ii) follows from the fact that $v_\infty^2 \lambda! q!^\lambda / \{v_\infty! / |P\Gamma L(d+1, q)||P\Gamma L(d, q)|\} \to 0$ as $qd \to \infty$. □

We include yet another elementary observation for future reference:

Proposition 3.6. (i) *If Z is a good block of a symmetric design $\mathbf{D}$ such that $\mathbf{A} = \mathbf{D}^Z$ is an affine space, then $\mathbf{D} \cong \mathbf{A}(\mathbf{D}_{(Z)}, \alpha)$ for some α, and there is an automorphism group $\Gamma(Z)$ of $\mathbf{D}$ that acts trivially on Z and induces the group of all perspectivities of $\mathbf{A}$ with axis at infinity.*

(ii) *If there are blocks Z behaving as in* (i), *then* Aut$\mathbf{D}$ *is transitive on the set of such blocks. More precisely, any two such blocks can be interchanged by an element of* Aut$\mathbf{D}$. *Moreover, if Z_1 and Z_2 are two such blocks then so is every block $Z_3 \supset Z_1 \cap Z_2$, and if $Z_3 \neq Z_1$, Z_2 then $\Gamma(Z_3)$ has an element moving Z_1 to Z_2.*

Proof: (i) The first assertion is (2.5). If $\mathbf{D} = \mathbf{A}(\mathbf{D}_{(Z)}, \alpha)$ then, for each perspectivity ψ of $\mathbf{A}$ with axis at infinity, $\alpha 1 = 1\alpha = \underline{\psi}\alpha$. Thus $(\psi, 1) \in$ Aut$\mathbf{A}(\mathbf{D}_{(Z)}, \alpha)$ by (3.1); $\Gamma(Z)$ is the set of all such automorphisms $\overline{(\psi, 1)}$.

(ii) Since $\mathbf{D}^{Z_i}$ is an affine space, $\Gamma(Z_i)$ is transitive on the blocks $\neq Z_i$ containing $Z_1 \cap Z_2$ for $i = 1, 2$. Then $\langle \Gamma(Z_1), \Gamma(Z_2) \rangle$ acts 2-transitively on the blocks containing $Z_1 \cap Z_2$, and this implies the desired transitivity. □

4. Gluing and lines

We now use lines in order to get information that is more precise than in the preceding section. Let $\mathbf{A} = AG(d, q), d \geq 3$, and let $\mathbf{D}_\infty = \underline{\mathbf{A}} = (\mathcal{P}_\infty, \mathcal{B}_\infty, \in)$ be its hyperplane at infinity, so that $\mathcal{B}_\infty = \underline{\mathcal{B}}$. Let $\alpha: \mathcal{B}_\infty \to \mathcal{B}_\infty$ be any bijection. Each of the symmetric designs $\mathbf{A}(\alpha)$ has the same parameters as $PG(d, q)$.

By (2.2iii), each $\mathbf{A}(\alpha)$-line containing 2 points not in $\mathcal{P}_\infty$ contains exactly q such points. The following lemma is concerned with lines meeting $\mathcal{P}_\infty$. This type of geometric lemma will be used in the study of $\mathbf{A}(\alpha)$ and of other designs considered later.

Lemma 4.1. (i) *Let $u \in \mathcal{P}_\infty$. Then some $\mathbf{A}(\alpha)$-line meeting $\mathcal{P}_\infty$ at u has size > 2 if and only if the blocks in $\{\underline{X} \in \mathcal{B}_\infty \mid u \in \underline{X}^\alpha\}$ have a nonempty intersection (which is then a point of $\mathcal{P}_\infty$).*

(ii) *Let $u \in \mathcal{P}_\infty$. If $|xu| > 2$ for some $x \in \mathcal{P}$ then the same is true for all $x \in \mathcal{P}$, and when all of these $\mathbf{A}(\alpha)$-lines are intersected with $\mathcal{P}$ the result is a parallel class of $\mathbf{A}$-lines.*

Proof: (i) Let $x \in \mathcal{P}$. Clearly $|xu| > 2$ if and only if $xu = xy$ for some $y \in \mathcal{P}$; and this occurs if and only if $xy \cap \mathcal{P}_\infty = u$.

Therefore, consider distinct points $x, y \in \mathcal{P}$, and let $\underline{xy_{\mathbf{A}}}$ denote the parallel class of $\mathbf{A}$-lines containing $xy_{\mathbf{A}}$; view $\underline{xy_{\mathbf{A}}}$ as a point of $\mathcal{P}_\infty$. Note that $xy \cap \mathcal{P}_\infty = \cap\{\underline{X}^\alpha \mid x, y \in$

$X\} = \cap\{\underline{X}^\alpha \mid xy_{\mathbf{A}} \subseteq X\} = \cap\{\underline{X}^\alpha \mid \underline{xy}_{\mathbf{A}} \in \underline{X}\}$ since any hyperplane $\underline{X}$ of $\underline{\mathbf{A}}$ is on a unique hyperplane of $\mathbf{A}$ through x. Thus, $xy \cap \mathcal{P}_\infty = u$ if and only if α maps the hyperplanes of $\underline{\mathbf{A}}$ containing $\underline{xy}_{\mathbf{A}}$ to those containing u.

(ii) Let $x' \in \mathcal{P}$. There is a unique $\mathbf{A}$-line $x'y'_{\mathbf{A}}$ through x' parallel to $xy_{\mathbf{A}}$, where y is as in (i). As above we see that $x'y' \cap \mathcal{P}_\infty = \cap\{\underline{X}^\alpha \mid \underline{x'y'}_{\mathbf{A}} \in \underline{X}\} = \cap\{\underline{X}^\alpha \mid \underline{xy}_{\mathbf{A}} \in X\} = u$, as required. □

Proposition 4.2. *Assume that $q \geq 3$ and $\mathbf{A}(\alpha)$ is not a projective space.*

(i) *$\mathcal{P}_\infty$ is the only block of $\mathbf{A}(\alpha)$ whose complement meets no $\mathbf{A}(\alpha)$-line in exactly 2 points. In particular,* $\operatorname{Aut}\mathbf{A}(\alpha)$ *fixes $\mathcal{P}_\infty$.*

(ii) *For each block $\overline{E} \neq \mathcal{P}_\infty$ there is a point $u \in \mathcal{P}_\infty - \underline{E}^\alpha$ such that each line xu, $x \in \mathcal{P} - (E \cup \{u\})$, has size 2.*

Proof: By a remark prior to (4.1), the complement of $\mathcal{P}_\infty$ meets each $\mathbf{A}(\alpha)$-line in 0, 1 or exactly $q > 2$ points. If $\overline{E}$ is any other block of $\mathbf{A}(\alpha)$ satisfying this condition, and if $u \in \mathcal{P}_\infty - \underline{E}^\alpha$, then for each $x \in \mathcal{P} - (E \cup \{u\})$ the line xu contains a third point not in $\overline{E}$. Thus, (i) will follow from (ii).

Assume that (ii) fails for some block $\overline{E}$. Then for each $u \in \mathcal{P}_\infty - \underline{E}^\alpha$, there is some point $x \in \mathcal{P} - (E \cup \{u\})$ such that xu has at least 3 points, and hence at least 2 points not in $\mathcal{P}_\infty$ (since $xu \cap \mathcal{P}_\infty = u$). For each $u \in \mathcal{P}_\infty - \underline{E}\alpha$, (4.1i) produces a point of $\mathcal{P}_\infty$, which will be called u^β, such that $u^\beta = \cap\{\underline{X} \mid u \in \underline{X}^\alpha\}$. This defines a map β from the points of $\mathcal{P}_\infty - \underline{E}^\alpha$ into $\mathcal{P}_\infty$ such that $u^\beta \in X$ if $u \in \underline{X}^\alpha$. There are k_∞ blocks on u, and k_∞ on u^β. Since $u \notin \underline{E}^\alpha$, it follows that $u^\beta \notin \underline{E}^\alpha$. That is, $(\mathcal{P}_\infty - \underline{E}^\alpha)^\beta = \mathcal{P}_\infty - \underline{E}$.

Thus, if we let β also act on the blocks in $\mathcal{B}_\infty - \{E^\alpha\}$ by having it coincide with α^{-1} on them, then β becomes an incidence-preserving map $\underline{\mathbf{A}}^{\underline{E}^\alpha} \to \underline{\mathbf{A}}^{\underline{E}}$ of affine spaces. It follows that β arises from an isomorphism $\underline{\mathbf{A}} \to \underline{\mathbf{A}}$ of projective spaces. Consequently, α is induced by an isomorphism $\underline{\mathbf{A}} \to \underline{\mathbf{A}}$. This contradicts (3.4). □

The hypothesis that $\underline{\mathbf{A}} = \mathbf{D}_\infty$ is a projective space was used in order to extend the isomorphism $\underline{\mathbf{A}}^{\underline{E}^\alpha} \to \underline{\mathbf{A}}^{\underline{E}}$ to an isomorphism $\underline{\mathbf{A}} \to \underline{\mathbf{A}}$. This assumption is essential for the validity of (4.2) (cf. Section 9, Remark 1). The result is false when $q = 2$, but there is a substitute:

Proposition 4.3. *If $q = 2$ then the following are equivalent for a hyperplane E of $\mathbf{A}$:*

(i) $\operatorname{Aut}\mathbf{A}(\alpha)$ *has an element moving $\mathcal{P}_\infty$ to $\overline{E}$;*

(ii) *$A(\alpha)^{\overline{E}}$ is an affine space;*

(iii) *There is an automorphism σ of $\underline{\mathbf{A}}$ such that $(\underline{E} \cap \underline{X})^\sigma = \underline{E}^\alpha \cap \underline{X}^\alpha$ for all $X \in \mathcal{B}$; and*

(iv) *$\mathbf{A}(\alpha) \cong \mathbf{A}(\beta)$ by an isomorphism sending $\mathcal{P}_\infty$ to $\mathcal{P}_\infty$ and $E \cup \underline{E}^\alpha$ to $E \cup \underline{E}^\beta$, where β fixes $\underline{E}$ and has the following property: $\underline{E}^\beta \cap \underline{X}^\beta = \underline{E} \cap \underline{X}$ for every hyperplane X of* $\mathbf{A}$.

Proof: Throughout this proof let F denote the hyperplane of $\mathbf{A}$ disjoint from E, so that $\underline{F} = \underline{E}$.

(i)⇔(ii): This is an immediate consequence of (3.6).

(i)⇒(iii): By (3.6ii) there is an element of $\Gamma(\overline{E})$ sending $\mathcal{P}_\infty$ to $\overline{F}$. Then each line of the projective space $\mathbf{D}_\infty$ is mapped to a line of $\mathbf{A}(\alpha)_{(\overline{F})}$ such that the two lines have the same intersection with $\mathcal{P}_\infty \cap \overline{F} = \underline{F}^\alpha = \underline{E}^\alpha$. It follows that each $\mathbf{A}(\alpha)$-line ux with $u \in \underline{E}^\alpha$, $x \in F$, has $q+1=3$ points. By (4.1i), $\cap\{\underline{X} \in \mathcal{B} \mid u \in \underline{X}^\alpha\}$ is a point we will call u^β; here $u^\beta \in \underline{E}$ (use $\underline{X} = \underline{E}$). Thus, β sends points of $\underline{E}^\alpha$ to points of $\underline{E}$, and the map $u \mapsto u^\beta$, $\underline{Y} \mapsto \underline{Y}^{\alpha^{-1}}$ preserves incidence. Then the map $u \mapsto u^\beta$, $\underline{E}^\alpha \cap \underline{Y} \mapsto \underline{E} \cap \underline{Y}^{\alpha^{-1}}$ (for $u \in \underline{E}^\alpha$, $\underline{Y} \neq \underline{E}^\alpha$) also does, and hence is an isomorphism of projective spaces $(\mathbf{D}_\infty)_{(\underline{E}^\alpha)} \to (\mathbf{D}_\infty)_{(\underline{E})}$. Any such isomorphism is induced by some automorphism τ of $\mathbf{D}_\infty = \underline{\mathbf{A}}$ sending $\underline{E}^\alpha$ to $\underline{E}$. Then $(\underline{E}^\alpha \cap \underline{X}^\alpha)^\tau = \underline{E} \cap \underline{X}$ for all $\underline{X}$, so that $\sigma = \tau^{-1}$ behaves as required.

(iii)⇒(iv): Let $\beta = \alpha\sigma^{-1}$. Then $\underline{E}^\beta \cap \underline{X}^\beta = \underline{E} \cap \underline{X}$ for every X. Since $\alpha\sigma^{-1} = \underline{1}\beta$, (3.1i) produces an isomorphism $g\colon \mathbf{A}(\alpha) \to \mathbf{A}(\beta)$ such that $(X \cup \underline{X}^\alpha)^g = X \cup \underline{X}^\beta$.

(iv)⇒(i): After replacing α by β, we may assume that $\underline{E}^\alpha = \underline{E}$ and $\underline{E}^\alpha \cap \underline{X}^\alpha = \underline{E} \cap \underline{X}$ for every block X. Equivalently, $\underline{X}^\alpha = \underline{X}$ or $\underline{X} + Q$ for each $\underline{X} \in \underline{\mathcal{B}}$, where $+$ denotes *symmetric difference* and $Q := \mathcal{P}_\infty - \underline{E}$.

The group $\Gamma(\mathcal{P}_\infty)$ in (3.6) has an element interchanging $\overline{E}$ and $\overline{F}$, so it suffices to produce an element of $\mathrm{Aut}\mathbf{A}(\alpha)$ interchanging $\mathcal{P}_\infty$ and $\overline{F}$. There is an automorphism h of the projective space $\mathbf{A}(1)$ that fixes $E \cup \underline{E}$ pointwise while interchanging $\mathcal{P}_\infty$ and $F \cup \underline{F} = \overline{F}$. *We will show that* $h \in \mathrm{Aut}\mathbf{A}(\alpha)$. First of all, if $X \in \mathcal{B}$ then, since $\underline{X}^\alpha = \underline{X}$ or $\underline{X} + Q$,

$$\overline{X} + X \cup \underline{X} = X + \underline{X}^\alpha + X + \underline{X} = \emptyset \text{ or } Q.$$

Then $\overline{X}^h + (X \cup \underline{X})^h + \overline{X} + X \cup \underline{X} = \emptyset$ or Q (since $Q^h = F$), and hence

$$\overline{X}^h + (X \cup \underline{X})^h + \overline{X} + X \cup \underline{X} = \emptyset \text{ or } F + Q.$$

Also, $(X \cup \underline{X})^h + X \cup \underline{X} = \emptyset$ or $F + Q$: this takes place inside the projective space $\mathbf{A}(1)$, where $E \cup \underline{E}$, $X \cup \underline{X}$ and $X \cup \underline{X} + F + Q$ are the three blocks containing $(E \cup \underline{E}) \cap (X \cup \underline{X})$. Thus,

$$\overline{X}^h + \overline{X} = \emptyset \text{ or } F + Q.$$

However, $\overline{E}$ is good by (2.7iii), so that $\overline{E} \cap \overline{X}$ is contained in three blocks of $\mathbf{A}(\alpha)$. Two of these are $\overline{E}$ and $\overline{X}$; the third one must be the complement $\overline{X} + F + Q$ of $\overline{E} + \overline{X}$. It follows that $\overline{X}^h$ is a block $\overline{X}$ or $\overline{X} + F + Q$ of $\mathbf{A}(\alpha)$. Thus, h is indeed an automorphism of $\mathbf{A}(\alpha)$. □

Theorem 4.4. (i) $\mathbf{A}(\alpha) \cong \mathbf{A}(\beta)$ *if and only if there are automorphisms* ρ *and* φ *of* $\underline{\mathbf{A}}$ *such that* $\alpha\varphi = \rho\beta$.

(ii) *The group* $[\mathrm{Aut}\mathbf{A}(\alpha)]_{\mathcal{P}_\infty}/\Gamma[\mathbf{A}]$ *induced by* $[\mathrm{Aut}\mathbf{A}(\alpha)]_{\mathcal{P}_\infty}$ *on* $\underline{\mathcal{B}} = \mathcal{B}_\infty$ *is isomorphic to* $\underline{\mathrm{Aut}\mathbf{A}} \cap (\underline{\mathrm{Aut}\mathbf{A}})^\alpha$ (*the group* $\Gamma[\mathbf{A}]$ *was defined just before* (3.2)).

(iii) *If* $q > 2$ *then there is a natural bijection* $\mathbf{A}(\alpha) \mapsto P\Gamma L(d, q)\alpha P\Gamma L(d, q)$ *between the isomorphism classes of designs* $\mathbf{A}(\alpha) = \mathbf{A}(\underline{\mathbf{A}}, \alpha)$ *and the* $P\Gamma L(d, q)$, $P\Gamma L(d, q)$ *double cosets in the symmetric group on* $\underline{\mathcal{B}}$. *In particular, the number of isomorphism classes is greater than* $\{(q^d-1)/(q-1)\}!/|P\Gamma L(d, q)|^2$.

Proof: (i) By (3.6), any isomorphism $\mathbf{A}(\alpha) \to \mathbf{A}(\beta)$ can be followed by an automorphism of $\mathbf{A}(\beta)$ so as to guarantee that $\mathcal{P}_\infty$ is sent to $\mathcal{P}_\infty$. By (3.1i), there is such an isomorphism if and only if $\alpha\varphi = \underline{\psi}\beta$ for some isomorphisms $\psi\colon \mathbf{A} \to \mathbf{A}$ and $\varphi\colon \underline{\mathbf{A}} \to \underline{\mathbf{A}}$. Now (i) follows from the fact that $\underline{\mathrm{Aut}\mathbf{A}} \cong \mathrm{Aut}\underline{\mathbf{A}}$.

(ii) This is immediate by (3.1) since $\Gamma[\mathbf{A}]$ is just the kernel of the homomorphism sending ψ to $\underline{\psi}$.

(iii) If $\alpha \in \underline{\mathrm{Aut}\mathbf{A}}$ then α is induced by an automorphism of $\mathbf{A}$ and hence $\mathbf{A}(\alpha)$ is a projective space by (3.4). Now consider any double coset $\underline{\mathrm{Aut}\mathbf{A}}\,\alpha\,\underline{\mathrm{Aut}\mathbf{A}} \neq \underline{\mathrm{Aut}\mathbf{A}}\,1\,\underline{\mathrm{Aut}\mathbf{A}}$. By (4.2i), $\mathrm{Aut}\mathbf{A}(\alpha)$ fixes $\mathcal{P}_\infty$. As in (i) we see that $\mathbf{A}(\alpha) \cong \mathbf{A}(\beta)$ if and only if there are elements ρ, $\sigma \in \underline{\mathrm{Aut}\mathbf{A}} = P\Gamma L(d, q)$ such that $\beta = \rho^{-1}\alpha\varphi \in \underline{\mathrm{Aut}\mathbf{A}}\,\alpha\,\underline{\mathrm{Aut}\mathbf{A}}$. Finally, $|\underline{\mathrm{Aut}\mathbf{A}}\,\alpha\,\underline{\mathrm{Aut}\mathbf{A}}| \le |P\Gamma L(d, q)|^2$. □

There are two iterative ways to improve the bound in (4.4iii). One assumes that $d \ge 4$, in which case $\mathbf{D}_\infty$ could have been chosen to be any design having the parameters of $PG(d-1, q)$, including one of those obtained previously (see Section 9, Remark 1 for an example of this); note that, if $d = 3$, then $\mathbf{D}_\infty$ can also be a nondesarguesian projective plane. The other iterative procedure uses (2.6) repeatedly.

5. The geometry of $\mathbf{A}' = \mathbf{A}(\alpha)^{\overline{E}}$: almost an affine space

In this section we will study the geometry of a more restricted class of affine designs, obtained as in (2.6) and needed in Section 8. Let $\mathbf{A} = AG(d, q)$ with $d \ge 3$, and let $\mathbf{D}_\infty = \underline{\mathbf{A}}$ and $\mathbf{A}(\alpha)$ be as in Section 4. *Assume that* $\underline{E}^\alpha = \underline{E}$ *and that* $\overline{E} = E \cup \underline{E}$ *is a good block* of $\mathbf{A}(\alpha)$, and consider the affine design $\mathbf{A}' := \mathbf{A}(\alpha)^{\overline{E}}$ (cf. (2.3ii)). Let $\mathcal{P}_\infty^- := \mathcal{P}_\infty - \underline{E}$ denote the block of $\mathbf{A}'$ determined by $\mathcal{P}_\infty$. Every other block of $\mathbf{A}'$ has the form $\overline{X} - \overline{E} \cap \overline{X}$ for some hyperplane X of $\mathbf{A}$. Note that $\overline{X} - \overline{E} \cap \overline{X}$ and $\overline{Y} - \overline{E} \cap \overline{Y}$ are parallel if and only if $\overline{E} \cap \overline{X} = \overline{E} \cap \overline{Y}$, and hence if and only if $E \cap X = E \cap Y$.

Let S be the set of points of $\mathbf{A}'$ not in $\mathcal{P}_\infty^-$, so that S is just $\mathcal{P} - E$. If x and y are distinct points of S then the "S-line" xy_S is defined to be $xy_S = xy_{\mathbf{A}'} \cap S$. By (2.2iii), we also have $xy_S = xy_{\mathbf{A}}(\alpha) \cap S$, and this is part of the line $xy_{\mathbf{A}}$ of the affine space $\mathbf{A}$; in particular, $|xy_S| \ge q - 1$.

Proposition 5.1. *Suppose that α is not induced by a collineation* (*so that* $\mathbf{A}(\alpha)$ *is not a projective space by* (3.4)), *and* either

(i) $q \ge 3$, *and there is a point* $u \in \mathcal{P}_\infty^-$ *such that* $\cap\{\underline{X} \in \underline{\mathcal{B}} \mid u \in \underline{X}^\alpha\} \neq \emptyset$; or

(ii) $q \ge 4$, *and, for any hyperplane* $B \neq E$ *of* $\mathbf{A}$, *there is a point* $v \in \mathcal{P}_\infty - \{\underline{E}^\alpha \cup \underline{B}^\alpha\}$ *such that* $\cap\{\underline{X} \in \underline{\mathcal{B}} \mid v \in \underline{X}^\alpha\} = \emptyset$.

Then $\mathrm{Aut}\mathbf{A}'$ *fixes* $\mathcal{P}_\infty^-$.

Proof: (i) By (4.1ii), all $\mathbf{A}(\alpha)$-lines through u but not contained in $\mathcal{P}_\infty$ have size $q + 1$. Since $\mathbf{A}'_{(\mathcal{P}_\infty^-)} = (\mathbf{D}_\infty)^{\underline{E}}$ is an affine space, it follows that *all* $\mathbf{A}'$*-lines through* u *have size* q. Suppose that some point $x \in S$ has this same property. Then, for any $v \in \mathcal{P}_\infty^-$, $|xv_{\mathbf{A}'}| \ge q \ge 3$, and hence $\cap\{\underline{X} \in \underline{\mathcal{B}} \mid v \in \underline{X}^\alpha\} \neq \emptyset$ by (4.1i). This contradicts (4.2ii).

Thus, *each point of* $\mathbf{A}'$ *lying only on* q*-point lines is contained in* $\mathcal{P}_\infty^-$. One such point is u. Hence, it suffices to show that Aut$\mathbf{A}'$ fixes the parallel class of $\mathcal{P}_\infty^-$.

Any two points of $\mathbf{A}'$ contained in a block parallel to $\mathcal{P}_\infty^-$ belong to a q-point $\mathbf{A}'$-line (this is clear for points of the affine space $(\mathbf{D}_\infty)^{\underline{E}}$, as well as for the points lying in what is left of the affine space $\mathbf{A}$ after E was removed; cf. (2.2i)). We will show that *no other parallel class of blocks of* $\mathbf{A}'$ *has this property*. By (4.2ii), there is a point v of $\mathcal{P}_\infty^-$ lying on no q-point line of $\mathbf{A}'$. Consider any block $\overline{B} - \overline{E} \cap \overline{B}$ of $\mathbf{A}'$ not parallel to $\mathcal{P}_\infty^-$ and containing v. Since $\overline{B} - \overline{E} \cap \overline{B} \not\subseteq \mathcal{P}_\infty^-$, there is a point $x \in \overline{B} - \overline{E} \cap \overline{B}$ with $x \notin \mathcal{P}_\infty^-$. Then $vx_{\mathbf{A}'}$ is a line of $\mathbf{A}'$ contained in $\overline{B} - \overline{E} \cap \overline{B}$, but it cannot have q points in view of our choice of v.

Thus Aut$\mathbf{A}'$ fixes the parallel class of $\mathcal{P}_\infty^-$ and hence also fixes $\mathcal{P}_\infty^-$.

(ii) We noted above that each S-line has size $\geq q-1 \geq 3$. Thus, $\mathcal{P}_\infty^-$ is a block having the property that *every line with at least* 2 *points not in this block has at least* 3 *points not in it* (compare (4.2i)). We will show that $\mathcal{P}_\infty^-$ is the only block having this property.

Let $\overline{B} - \overline{E} \cap \overline{B}$ be any other block of $\mathbf{A}'$, and assume that it also has the above property. Let v be the point whose existence is hypothesized in (ii). Let $x \in S$ with $x \notin \overline{B}$. By hypothesis, $xv_{\mathbf{A}'}$ has at least 3 points not in $\overline{B} - \overline{E} \cap \overline{B}$, and hence has at least 2 points not in $\mathcal{P}_\infty^-$. Then $\{\underline{X} \in \underline{\mathcal{B}} \mid v \in \underline{X}^\alpha\} \neq \emptyset$ by (4.1i), and this contradicts the choice of v. □

Lemma 5.2. $\mathcal{P}_\infty^-$ *is a good block of* $\mathbf{A}'$.

Proof: Consider a block $\overline{X} - \overline{E} \cap \overline{X}$ of $\mathbf{A}'$. If $Y \| X$ in $\mathbf{A}$, then $(\overline{Y} - \overline{E} \cap \overline{Y}) \cap \mathcal{P}_\infty^- = \underline{Y}^\alpha - \underline{E} \cap \underline{Y}^\alpha = \underline{X}^\alpha - \underline{E} \cap \underline{X}^\alpha = (\overline{X} - \overline{E} \cap \overline{X}) \cap \mathcal{P}_\infty^-$. As Y varies, the blocks $\overline{Y} - \overline{E} \cap \overline{Y}$ clearly cover all of the points of $\mathbf{A}'$ not in $\mathcal{P}_\infty^-$, while the sets $(\overline{Y} - \overline{E} \cap \overline{Y}) \cap \mathcal{P}_\infty^-$ form a parallel class of $\mathbf{A}'_{(\mathcal{P}_\infty^-)} = (\mathbf{D}_\infty)^{\underline{E}}$ if E and X are not parallel in $\mathbf{A}$. (N.B.—This also follows from (2.7i), whose proof is essentially the same as the above one. However, it is faster to prove this lemma directly than it is to match up the notation with (2.7i)!) □

Proposition 5.3. *Assume that either of the conditions in* (5.1) *holds. Then* Aut$\mathbf{A}'$ *is (isomorphic to the restriction to the points and blocks of* $\mathbf{A}'$ *of the group)* $[\mathrm{Aut}\mathbf{A}(\alpha)]_{\overline{E}}$.

Proof: The nonempty intersections of blocks of $\mathbf{A}'$ with S will be called "S-blocks". The S-line xy_S is just the intersection of the S-blocks containing the distinct points $x, y \in S$. The incidence structure whose points are those of S and whose blocks are the S-blocks is canonically associated with $\mathbf{A}'$ by (5.1). So is the set $\mathcal{L}'$ consisting of the following sets of points of $\mathbf{A}'$:

xy_S whenever $x, y \in S$ and $|xy_S| = q-1$, and

$uv_{\mathbf{A}'}$ for $u, v \in \mathcal{P}_\infty^-$, $u \neq v$

(recall that $q - 1 \geq 2$). Define a "parallelism" $\|_E$ among the members of $\mathcal{L}'$, as follows (compare [2, p. 74]):

$xy_S \|_E x'y'_S \Leftrightarrow$ every block of $\mathbf{A}'$ containing x and y is parallel in $\mathbf{A}'$ to some block containing x' and y'; and

$uv_{\mathbf{A}'} \|_E u'v'_{\mathbf{A}'} \Leftrightarrow$ every block of $\mathbf{A}'$ containing u and v is parallel in $\mathbf{A}'$ to some block containing u' and v'.

This defines an equivalence relation: $xy_S \|_E x'y'_S \Leftrightarrow xy_{\mathbf{A}} \cap E = x'y'_{\mathbf{A}} \cap E$, while the relation $uv_{\mathbf{A}'} \|_E u'v'_{\mathbf{A}'}$ is nothing other than parallelism in the affine space $\mathbf{A}'_{(\mathcal{P}_\infty^-)} = (\mathbf{D}_\infty)^{\underline{E}}$. Call the corresponding equivalence classes $\underline{xy}_S$ and $\underline{uv}_{\mathbf{A}'}$, respectively. We will view $\underline{xy}_S$ or $\underline{uv}_{\mathbf{A}'}$ as *incident* with an S-block $S \cap (\overline{X} - \overline{E} \cap \overline{X})$ if and only if some member of $\underline{xy}_S$ or $\underline{uv}_{\mathbf{A}'}$ is contained in $\overline{X} - \overline{E} \cap \overline{X}$.

Define an incidence structure $\mathbf{D}'$ as follows: its points are the points x of S, the points u of $\mathcal{P}_\infty^-$, the parallel classes $\underline{xy}_S$ and the parallel classes $\underline{uv}_{\mathbf{A}'}$; its blocks are the S-blocks $S \cap \overline{X} = S \cap (\overline{X} - \overline{E} \cap \overline{X})$ as well as two further ones: $\mathcal{P}_\infty^-$ and $\underline{\mathcal{L}'}$. Those incidences not defined in the preceding paragraph are the obvious ones.

Now define a map $\mu: \mathbf{D}' \to \mathbf{A}(\alpha)$ as follows (for $x, y \in S$, $u, v \in \mathcal{P}_\infty^-$, and $X \neq E$ a hyperplane of $\mathbf{A}$):

$$x \mapsto x; \quad u \mapsto u; \quad \underline{xy}_S \mapsto xy_{\mathbf{A}} \cap E;$$
$$\underline{uv}_{\mathbf{A}'} \mapsto uv_{\mathbf{D}_\infty} \cap \underline{E}; \quad S \cap \overline{X} \mapsto \overline{X}; \quad \mathcal{P}_\infty^- \mapsto \mathcal{P}_\infty^-; \quad \underline{\mathcal{L}'} \mapsto \overline{E}.$$

(Here $uv_{\mathbf{D}_\infty} \cap \underline{E}$ can be thought of as the point at infinity produced by the line $uv_{\mathbf{A}'}$ of the affine space $\mathbf{A}'_{(\mathcal{P}_\infty^-)}$.) Then μ preserves incidence.

Thus, by (5.1) we have canonically recovered $\mathbf{A}(\alpha)$ from $\mathbf{A}'$. This implies the Proposition. □

See the proof of (8.10) for a related reconstruction. This type of result is a very special case of the Embedding Lemma in [8].

Theorem 5.4. *If $d \geq 4$ and $q \geq 3$ then there are at least $(q^{d-3})!$ pairwise nonisomorphic affine designs, not $AG(d,q)$ but having the same parameters as $AG(d,q)$, and having a parallel class of good blocks on each of which $AG(d-1,q)$ is induced.*

Proof: By (5.2), $\mathcal{P}_\infty^-$ is good. By (2.8ii), $\overline{F}$ is a good block of $\mathbf{A}(\alpha)$ for each hyperplane $F \neq E$ parallel to E in $\mathbf{A}$, and hence F is a good block of $\mathbf{A}'$ by (2.4ii). Moreover, affine spaces are induced on both $\mathcal{P}_\infty^-$ and F (e.g., using (2.2)).

It remains to estimate the number of nonisomorphic designs $\mathbf{A}(\alpha)$ satisfying the conditions needed in this section: $\overline{E}$ must be good, and we want to have a point u behaving as in (5.1i). There are v_∞^2 choices for the pair $\underline{E}$, $\underline{E}^\alpha$, and then $k_\infty!$ bijections $\underline{\alpha}$ from the set of hyperplanes of the projective space $(\mathbf{D}_\infty)_{(\underline{E})}$ to the set of hyperplanes of $(\mathbf{D}_\infty)_{(\underline{E}^\alpha)}$. Now pick points $u \in \mathcal{P}_\infty - \underline{E}$ and $u' \in \mathcal{P}_\infty - \underline{E}^\alpha$, and extend $\underline{\alpha}$ first by requiring that, for each block J of $(\mathbf{D}_\infty)_{(\underline{E})}$, α sends the hyperplane of $\mathbf{D}_\infty$ containing J and u to the one containing $J^{\underline{\alpha}}$ and u'. Finally, complete the extension of $\underline{\alpha}$ to all blocks of $\mathbf{D}_\infty$ in any of $(q-1)!^{k_\infty}$ ways (cf. the proof of (3.5)). The total number of permutations α obtained in this manner is $v_\infty^2 k_\infty!(v_\infty - k_\infty)^2(q-1)!^{k_\infty}$. As in (4.4iii), it follows that the number of isomorphism classes of these particular designs $\mathbf{A}(\alpha)$ is at least $v_\infty^2 k_\infty!(v_\infty - k_\infty)^2\,(q-1)!^{k_\infty}/|P\Gamma L(d+1,q)||P\Gamma L(d,q)|$, and this is $\geq (q^{d-3})!$ if $d \geq 4$. □

Next we turn to the case $q = 2$, where a more concrete description of the blocks of $\mathbf{A}'$ will be helpful. This time $S = \mathcal{P} - E$ is another block of $\mathbf{A}'$, so that $\mathbf{A}'_{(S)}$ is an affine space by (2.2i); so is $\mathbf{A}'_{(\mathcal{P}_\infty^-)} = (\mathbf{D}_\infty)^{(\underline{E})}$. Each block $\neq S, \mathcal{P}_\infty^-$ of $\mathbf{A}'$ is the union of a hyperplane of $\mathbf{A}'_{(S)}$ and a hyperplane of $\mathbf{A}'_{(\mathcal{P}_\infty^-)}$. Each hyperplane of $\mathbf{A}'_{(\mathcal{P}_\infty^-)}$ lies in two such blocks, and hence the same is true for each hyperplane of $\mathbf{A}'_{(S)}$.

If $\theta \in \mathrm{Aut}\mathbf{A}(1)$ fixes $\overline{E}$ pointwise and interchanges S and $\mathcal{P}_\infty^-$, and if J is any hyperplane of $\mathbf{A}'_{(S)} = \mathbf{A}_{(S)}$, then J^θ is contained in a unique hyperplane $J^{\theta'}$ of $\mathbf{D}_\infty$. Then $J^{\alpha'} := J^{\theta'\alpha} \cap \mathcal{P}_\infty^-$ is a hyperplane of $\mathbf{A}'_{(\mathcal{P}_\infty^-)}$, and we have seen that

Each block $\neq S, \mathcal{P}_\infty^-$ of $\mathbf{A}'$ has the form $J \cup J^{\alpha'}$ or $(S - J) \cup J^{\alpha'}$ for some hyperplane J of $\mathbf{A}_{(S)}$. (5.5)

Note that *α' is a parallelism-preserving bijection from the blocks of $\mathbf{A}_{(S)}$ to those of $\mathbf{A}'_{(\mathcal{P}_\infty^-)}$.* This gluing process, which apparently first appeared in [15], is studied more generally in [13, 10]. Note that (5.5) implies that (5.3) is always false when $q = 2$: the pointwise stabilizer in $\mathrm{Aut}\mathbf{A}'$ of either $\mathcal{P}_\infty^-$ or S is transitive on S or $\mathcal{P}_\infty^-$, respectively (inducing the full translation group of the respective affine spaces $\mathbf{A}'_{(S)}$ or $\mathbf{A}'_{(\mathcal{P}_\infty^-)}$; compare (3.6)). In particular, this property of $\mathrm{Aut}\mathbf{A}'$ is shared by all of the designs in the next theorem.

Theorem 5.6. *If $d \geq 5$ then there at least $(2^{d-4})!$ pairwise nonisomorphic affine designs, not $AG(d,2)$ but having the same parameters as $AG(d,2)$, with a parallel pair of good blocks on each of which $AG(d-1,2)$ is induced.*

Proof: Fix a hyperplane E of $\mathbf{A}$, and consider only maps α such that α fixes $\underline{E}$ and induces a permutation $\underline{\alpha}$ of the set of hyperplanes of the projective space $(\mathbf{D}_\infty)^{(\underline{E})}$, but $\underline{\alpha}$ is not induced by a collineation of $(\mathbf{D}_\infty)_{(\underline{E})}$. By (4.3ii, iii), $\mathbf{A}(\alpha)^{\overline{E}}$ is not an affine space. There are more than $\{k_\infty! - |PGL(d-1,2)|\}2!^{k_\infty}$ choices for α.

Consider two such choices α and β, and assume that there is an isomorphism $\mathbf{A}(\alpha)^{\overline{E}} \to \mathbf{A}(\beta)^{\overline{E}}$ sending $\mathcal{P}_\infty^-$ to itself. Define bijections α' and β' from the blocks of $\mathbf{A}_{(S)}$ to those of $(\mathbf{D}_\infty)^{\underline{E}} = \mathbf{A}'_{(\mathcal{P}_\infty^-)}$ as above. As in the proof of (3.1), we find that there are automorphisms ψ and φ of the affine spaces $\mathbf{A}'_{(S)}$ and $\mathbf{A}'_{(\mathcal{P}_\infty^-)}$, respectively, such that $\underline{\alpha'\varphi} = \underline{\psi\beta'}$ for the maps $\underline{\alpha'}$ and $\underline{\beta'}$ induced by α and β on the parallel classes of $\mathbf{A}'_{(S)}$ (cf. [12]).

Since α' determines α, it follows as in the proof of (3.2) that any one of these affine designs $\mathbf{A}(\alpha)^{\overline{E}}$ is isomorphic to at most $|PGL(d+1,2)||PGL(d,2)|$ others. Consequently, there are at least $\{k_\infty! - |PGL(d-1,2)|\}2!^{k_\infty}/|PGL(d+1,2)||PGL(d,2)|$ pairwise nonisomorphic designs of the type being considered, and this is $\geq (2^{d-4})!$ for $d \geq 5$. □

6. AutA(α): asymptotics

In [5, p. 177] "it is conjectured that most of the examples constructed here are indeed automorphism-free". Those examples include (among others) the designs $\mathbf{A}(\mathbf{D}_\infty, \alpha)$, where the initial affine design $\mathbf{A}$ is an affine space and the design $\mathbf{D}_\infty$ is allowed to vary.

The conjecture is false for every such symmetric design $\mathbf{A}(\mathbf{D}_\infty, \alpha)$—and it is also false for all of the other examples considered in [5] (except in the case $q = 2$ of what are called there "biaffine divisible designs")—since there are always nontrivial perspectivities of the underlying affine space that automatically act on the new design, just as $\Gamma[\mathbf{A}]$ did in (3.6). Nevertheless, there is a version of the conjecture that is correct. We will only consider the case of symmetric designs, but analogues of the following result are easily proved for the other situations examined in [5].

Proposition 6.1. *The proportion of those isomorphism classes of designs* $\mathbf{A}(\alpha) = \mathbf{A}(\underline{\mathbf{A}}, \alpha)$, *for which* $\mathbf{A}$ *is an affine space of dimension at least* 3 *and* $\mathrm{Aut}\mathbf{A}(\alpha) = \Gamma[\mathbf{A}]$, *approaches* 1 *as the number of points* $\to \infty$.

Proof: By (3.5ii), we may restrict to designs having just one good block. By (4.4ii), $\mathrm{Aut}\mathbf{A}(\alpha) = \Gamma[\mathbf{A}]$ if and only if $(\mathrm{Aut}\mathbf{D}_\infty)^\alpha \cap \mathrm{Aut}\mathbf{D}_\infty = 1$. We will show that there are relatively few triples (σ, τ, α) with $\sigma, \tau \in P\Gamma L(d, q)$ of prime order $p, \alpha \in S_{v_\infty}$, and $\sigma\alpha = \alpha\tau$. There are $|P\Gamma L(d, q)|^2$ choices of two elements σ, τ of $P\Gamma L(d, q)$. If some α conjugates σ to τ then there are exactly $c(\sigma) := |C_{S_{v_\infty}}(\sigma)|$ such elements α. Therefore, we will need an upper bound for $c(\sigma) = f! p^{(v_\infty - f)/p}[(v_\infty - f)/p]!$, where f is the number of fixed points of σ. The following possibilities for f will be treated somewhat differently: (i) $k_\infty + 1 \geq f > \lambda_\infty = (k_\infty - 1)/q$, in which event it is easy to check that $|\sigma| = p \leq q$; (ii)$(k_\infty - 1)/q \geq f > 0$ and $p \leq k_\infty$; and (iii) $f = 0$. In (i) and (ii), the number of nontrivial cycles of σ is $(v_\infty - f)/p \geq 2$.

Write $Q := v_\infty!/\{f![(v_\infty - f)/p]! p^{(v_\infty - f)/p}\}$. *We claim* that $Q \geq (1.3)^{q^{d-2}-1}$. In all cases, since $p^{-1/p} \geq 2.1^{-1/2}$ we have

$$\begin{aligned} Q &\geq (v_\infty - f)!/\{[(v_\infty - f)/p]! p^{(v_\infty - f)/p}\} \\ &\geq [1 + (v_\infty - f)/p]^{(v_\infty - f)(1-1/p)}(2.1)^{-(v_\infty - f)/2} \\ &\geq [\{1 + (v_\infty - f)/p\}/2.1]^{(v_\infty - f)/2}. \end{aligned}$$

In (i) or (ii) we now see that $Q \geq (1.3)^{(v_\infty - k_\infty - 1)/2} = (1.3)^{(q^{d-1}-1)/2} \geq (1.3)^{q^{d-2}-1}$. In (iii), Q is at least $(v_\infty - 1)!$ or $((1+2)/2.1)^{v_\infty/2}$ according to whether $p = v_\infty$ or $p < v_\infty$. This proves the claim in all cases.

Consequently, $c(\sigma) = f! p^{(v_\infty - f)/p}[(v_\infty - f)/p]! \leq v_\infty!/(1.3)^{q^{d-2}-1}$ for each σ. Then the number of elements of S_{v_∞} conjugating some nontrivial element of $P\Gamma L(d, q)$ to another one is at most $|P\Gamma L(d, q)|^2 v_\infty!/(1.3)^{q^{d-2}-1}$. By (3.5ii), the proportion of those isomorphism classes of designs $\mathbf{A}(\underline{\mathbf{A}}, \alpha)$ for which $\mathbf{A}(\underline{\mathbf{A}}, \alpha) \neq \Gamma[\mathbf{A}]$ is at most

$$\frac{|P\Gamma L(d,\,q)|^2 \dfrac{q^{d-1}-1}{q-1}!}{(1.3)^{q^{d-2}-1}} \Bigg/ \frac{\dfrac{q^d-1}{q-1}! - \left(\dfrac{q^d-1}{q-1}\right)^2 \dfrac{q^{d-1}-1}{q-1}! q!^{(q^{d-1}-1)/(q-1)}}{|P\Gamma L(d,\,q)|} \to 0$$

as $qd \to \infty$. □

7. Some representations

Throughout the proof of (1.1) in the next section we will always use a simple type of representation of a finite group G:

Notation 7.1. Assume that d and ℓ are positive integers such that $d \geq \ell|G| + 2 > 2$. Let V be a $d+1$-dimensional vector space over $GF(q)$ on which G acts as a group of linear transformations, and assume that there is a basis (the "standard basis") $v_1, \ldots, v_{d+1}$ such that G permutes $v_1, \ldots, v_{\ell|G|}$ via ℓ copies of its right regular representation while fixing all remaining basis vectors. Then G also acts on the corresponding projective space $\mathbf{P} := PG(d,\,q)$. It is easy to see that the subgroup of $PGL(d+1,\,q)$ induced by G is isomorphic to G; we will identify these two groups.

Lemma 7.2. (i) *The representation of G on the dual space of V is equivalent to that on V.*

(ii) *Let j be 0 or 1, and write $U_j := \langle v_i \mid 1 \leq i \leq d+1,\ i \neq d+1-j\rangle$. Then G acts on U_j, permuting the basis $\{v_i \mid 1 \leq i \leq d+1,\ i \neq d+1-j\}$ via ℓ copies of its right regular representation while fixing all remaining basis vectors. Moreover, $V = \langle v_{d+1}\rangle \oplus U_0$.*

(iii) *No nontrivial element of G fixes $U_0 \cap U_1$ pointwise.*

(iv) *If G fixes a hyperplane of $AG(d+1,\,q)$ then it fixes every parallel hyperplane.*

(v) *G commutes with the involutory linear transformation σ of V defined by $v_i \mapsto v_i$ for $i < d$, and $v_d \leftrightarrow v_{d+1}$.*

Proof: (i) G preserves the usual dot-product with respect to the standard basis.

(ii) G fixes v_{d+1} and v_d, and hence acts on U_j. The representation is clear, as is the assertion (iii).

(iv) If $G = 1$ this is clear. If $G \neq 1$ then every fixed hyperplane W of V contains $v_1, \ldots, v_{\ell|G|}$; and every hyperplane of $AG(d+1,\,q)$ fixed by G is parallel to one through 0 fixed by G. Choose $i > \ell|G|$ such that $v_i \notin W$. Then G fixes each translate $W + cv_i$, $c \in GF(q)$.

Finally, (v) is clear. □

8. Proof of (1.1)

We are given a group G, a prime power $q > 3$, and an integer $d \geq 50|G|^2$. The design $\mathbf{D}$ in (1.1) is defined below in (8.1). First we need some notation.

Start with $\mathbf{P} = PG(d,\,q)$ and the representation of G appearing in (7.1), using $\ell := \max\{4,\ |G|\}$. Let $\mathcal{P}_\infty$ denote the hyperplane of $\mathbf{P}$ corresponding to the subspace

U_0 in (7.2ii), and write $\mathbf{A} = \mathbf{P}^{U_0}$ and $\mathbf{D}_\infty = \underline{\mathbf{A}} = (\mathcal{P}_\infty, \mathcal{B}_\infty, \in)$. Then $\mathbf{P} = \mathbf{A}(1)$ in the notation of Section 2: its hyperplanes are $\mathcal{P}_\infty$ and $X \cup \underline{X}$, where X ranges over the hyperplanes of $\mathbf{A}$. Let E denote the hyperplane of $\mathbf{A}$ such that $\overline{E} = E \cup \underline{E}$ corresponds to the subspace U_1 in (7.2ii). The hyperplanes of $\mathbf{D}_\infty$ have the form $X = \mathcal{P}_\infty \cap (X \cup \underline{X})$, while those of $\mathbf{P}_{(\overline{E})}$ have the form

$$\begin{aligned}&\underline{E} = \overline{E} \cap \mathcal{P}_\infty \quad \text{or} \\ &\underline{\underline{X}} := \overline{E} \cap (X \cup \underline{X})\end{aligned}$$

for a hyperplane X of $\mathbf{A}$ not parallel to E. Note that $\underline{E} \cap \underline{\underline{X}} = \underline{E} \cap \underline{X}$.

By (7.2i), G acts on the *dual* of the projective space $\mathbf{D}_\infty$ as it does on $\mathbf{D}_\infty$. Apply (10.4) to the points of this dual space, choosing notation so that E is the *dual* of the point $\langle w \rangle$ appearing in (10.4ii). This produces a permutation α of the hyperplanes of $\mathbf{D}_\infty$. Write $\zeta = \alpha^{-1}$. Let σ denote the involutory collineation of $\mathbf{P}$ defined in (7.2v). Then σ fixes $\underline{E}$ pointwise, interchanges $\mathcal{P}_\infty$ and $\overline{E}$, and commutes with G. Moreover,

α *is a permutation of the hyperplanes of* $\mathbf{D}_\infty$, and
$\beta := \sigma^{-1}\zeta\sigma$ *is a permutation of the hyperplanes of* $\mathbf{P}_{(\overline{E})}$.

(N.B.—Many choices for permutations ζ behaving as in (10.4), other than α^{-1}, could have been used here in order to define β. The present choice simplifies the proof, while producing a pleasant additional property (8.9iii) of the designs in (1.1ii). However, it also leads to an unreasonably poor bound on the number of nonisomorphic designs we construct.)

By (10.4ii), α induces a permutation $\underline{\alpha}$ of the hyperplanes of $(\mathbf{D}_\infty)_{(\underline{E})}$ (the projective space at infinity of $\mathbf{A}_{(E)}$). Namely, if $\underline{E}$, $\underline{X}$ and $\underline{Y}$ are distinct hyperplanes of $\mathbf{D}_\infty$ such that $\underline{E} \cap \underline{X} = \underline{E} \cap \underline{Y}$, then $\underline{E} \cap \underline{X}^\alpha = \underline{E} \cap \underline{Y}^\alpha$ by (10.4ii) (dualized and recalling that $\langle w \rangle$ in (10.4) "is" our $\underline{E}$), so we define

$$(\underline{E} \cap \underline{X})^{\underline{\alpha}} = \underline{E} \cap \underline{X}^\alpha.$$

In other words, $(\underline{E} \cap \underline{X})^{\underline{\alpha}}$ can be viewed as the image under α of the parallel class of hyperplanes of $\mathbf{A}_{(E)}$ determined by $E \cap X$, as in (2.8i) (at this point we have not yet left ordinary projective geometry). There are similar definitions for $\underline{\zeta}$ and $\underline{\beta}$, where in fact $\underline{\beta} = \underline{\zeta} = \underline{\alpha}^{-1}$ since $\sigma = 1$ on $\underline{E}$.

The incidence structure $\mathbf{D}$ is defined as follows. Its points are those of $\mathbf{P}$. Its blocks are the following sets of points:

$$\begin{aligned}&\mathcal{P}_\infty, \\ &\overline{E} = E \cup E, \text{ and} \\ &\tilde{X} := \left(\overline{X} - \{(E \cap X) \cup (\underline{E} \cap \underline{X}^\alpha)\}\right) \\ &\qquad \cup \left(E \cap \{\overline{E} \cap (X \cup \underline{X})\}^\beta\right) \cup \left(\underline{E} \cap \{\overline{E} \cap (X \cup \underline{X})\}^\beta\right)^{\underline{\alpha}},\end{aligned} \tag{8.1}$$

where X runs through the hyperplanes of $\mathbf{A}$ other than E. Since $\sigma = 1$ on $\underline{E}$,

$$\left(\underline{E} \cap \{\overline{E} \cap (X \cup \underline{X})\}^\beta\right)^{\underline{\alpha}} = (\underline{E} \cap \underline{\underline{X}}^\beta)^{\underline{\alpha}} = (\underline{E} \cap \underline{\underline{X}}^{\sigma\zeta\sigma})^{\underline{\alpha}} = \left((\underline{E} \cap \underline{\underline{X}}^{\sigma\zeta})^\sigma\right)^{\underline{\alpha}}$$

$$= \left(\underline{E} \cap (\underline{\underline{X}}^{\sigma})^{\zeta}\right)^{\underline{\alpha}} = (\underline{E} \cap \underline{\underline{X}}^{\sigma})^{\underline{\zeta\alpha}} = (\underline{E} \cap \underline{\underline{X}})^{\underline{\zeta\alpha}}.$$

Since $\underline{\zeta\alpha} = 1$ and $\underline{E} \cap \underline{\underline{X}} = \underline{E} \cap \underline{X}$, we have

$$\tilde{X} = \left(\overline{X} - \{(E \cap X) \cup (\underline{E} \cap \underline{X}^{\alpha})\}\right) \cup \left(\overline{E} \cap \{\overline{E} \cap (X \cup \underline{X})\}^{\beta}\right) \cup (\underline{E} \cap \underline{X}). \tag{8.1'}$$

If we write $\mathcal{P}_{\infty}^{-} = \mathcal{P}_{\infty} - \underline{E}$ as in Section 5, then we also have

$$\tilde{X} = (X - E \cap X) \cup (\mathcal{P}_{\infty}^{-} \cap \underline{X}^{\alpha}) \cup (E \cap \underline{\underline{X}}^{\beta}) \cup (\underline{E} \cap \underline{X}). \tag{8.1''}$$

This definition of $\mathbf{D}$ is certainly opaque. In order to see that $\mathbf{D}$ is, indeed, a symmetric design, and in order to study its structure, we will need to unravel the definition using Section 2. For now we note that each hyperplane $X \neq E$ of $\mathbf{A}$ determines a set $E - E \cap X$ that uniquely determines the block $\tilde{X}$.

Let $\mathbf{A}(\alpha) = \mathbf{A}(\mathbf{D}_{\infty}, \alpha)$ be the symmetric design obtained in (2.1). One of its blocks is $E \cup \underline{E}^{\alpha} = E \cup \underline{E} = \overline{E}$. Note that this is a good block of $\mathbf{A}(\alpha)$. For, since $E \cup \underline{E}$ is a good block of the projective space $\mathbf{P}$, by (2.7i) it suffices to check that $\underline{E} \cap \underline{X} = \underline{E} \cap \underline{Y}$ implies that $\underline{E}^{\alpha} \cap \underline{X}^{\alpha} = \underline{E}^{\alpha} \cap \underline{Y}^{\alpha}$ (for all hyperplanes X, Y of $\mathbf{A}$); and this is precisely the condition in (10.4ii) used above.

Let $\mathbf{A}'$ denote the affine design $\mathbf{A}(\alpha)^{\overline{E}}$ (cf. (2.3i)).

Also, let $\mathbf{D}'_{\infty}{:=}\mathbf{A}(\alpha)_{(\overline{E})}$. By (2.8ii), this symmetric design is obtained by gluing:

$$\mathbf{D}'_{\infty}{:=}\mathbf{A}(\alpha)_{(\overline{E})} = \mathbf{A}_{(E)}(\underline{\alpha})$$

using the permutation $\underline{\alpha}$ described above. That is, the blocks of $\mathbf{D}'_{\infty}$ have the form

$$\underline{E} \quad \text{or}$$

$$\begin{aligned}\overline{E} \cap \overline{X} &= (E \cap X) \cup (\underline{E} \cap \underline{X}^{\alpha}) = (E \cap \underline{\underline{X}}) \cup (\underline{E} \cap \underline{X})^{\underline{\alpha}} \\ &= (E \cap \underline{\underline{X}}) \cup (\underline{E} \cap \underline{\underline{X}})^{\underline{\alpha}},\end{aligned}$$

where X runs through the hyperplanes of $\mathbf{A}$ not parallel to E. (Thus, in (2.1), $\underline{E}$ is playing the role of $\mathcal{P}_{\infty}$, while $\underline{E} \cap \underline{\underline{X}}$ is playing the role of $\underline{X}$.)

Define a permutation γ of the blocks of $\mathbf{D}'_{\infty}$ as follows:

$$\begin{aligned}&\underline{E}^{\gamma} = \underline{E}; \quad \text{and} \\ &\textit{If } X \textit{ is not parallel to } E \textit{ in } \mathbf{A} \textit{ then } (\overline{E} \cap \overline{X})^{\gamma} = (E \cap \underline{\underline{X}}^{\beta}) \cup (\underline{E} \cap \underline{\underline{X}}^{\beta})^{\underline{\alpha}} \\ &(\textit{this is the block of } \mathbf{D}'_{\infty} \textit{ containing } E \cap \underline{\underline{X}}^{\beta}).\end{aligned} \tag{8.2}$$

This is well-defined: if $\overline{E} \cap \overline{X} = \overline{E} \cap \overline{Y}$ then $E \cap X = E \cap Y$ (as is seen by considering the set of points not in $\mathcal{P}_{\infty}$), so that $(E \cup \underline{E}) \cap (X \cup \underline{X}) = (E \cup \underline{E}) \cap (Y \cup \underline{Y})$ (this takes place inside $\mathbf{P}$), which states that $\underline{\underline{X}}^{\beta} = \underline{\underline{Y}}^{\beta}$. By (8.1),

$$\tilde{X} = (\overline{X} - \overline{E} \cap \overline{X}) \cup (\overline{E} \cap \overline{X})^{\gamma}. \tag{8.3}$$

We can now show that $\mathbf{D}$ is a symmetric design, and at the same time identify it in two ways:

Lemma 8.4. (i) $\mathbf{D} = \mathbf{A}'(\mathbf{D}'_\infty, \gamma) = \mathbf{A}(\alpha)^{\overline{E}}\left(\mathbf{A}(\alpha)_{(\overline{E})}, \gamma\right)$.

(ii) $\mathbf{D} \cong \mathbf{A}(\alpha^{-1})^{\overline{E}}\left(\mathbf{A}(\alpha^{-1})_{(\overline{E})}, \gamma^*\right)$ *by an isomorphism interchanging* $\mathcal{P}_\infty$ *and* $\overline{E}$, *where* γ^* *is defined as in* (8.2) *with* α *and* β *replaced by* α^{-1} *and* β^{-1}, *respectively.*

Proof: (i) Since $\mathcal{P}_\infty = \{\mathcal{P}_\infty - (\overline{E} \cap \mathcal{P}_\infty)\} \cup (\overline{E} \cap \mathcal{P}_\infty)^\gamma$, (8.3) and the definition preceding (2.1) imply (i).

(ii) We will show that σ produces an isomorphism. Write $(X \cup \underline{X})^\sigma = Y \cup \underline{Y}$ (where $X \cup \underline{X}$ and $Y \cup \underline{Y}$ are hyperplanes of $\mathbf{P}$ other than $\mathcal{P}_\infty$ and $\overline{E}$). Since σ interchanges $\overline{E}$ and $\mathcal{P}_\infty$,

$$\begin{aligned}
(X - E \cap X)^\sigma &= \left(X \cup \underline{X} - [\mathcal{P}_\infty \cup \overline{E}] \cap [X \cup \underline{X}]\right)^\sigma \\
&= Y \cup \underline{Y} - [\overline{E} \cup \mathcal{P}_\infty] \cap [Y \cup \underline{Y}] = Y - E \cap Y, \\
\underline{X}^\sigma &= \left(\mathcal{P}_\infty \cap (X \cup \underline{X})\right)^\sigma = \overline{E} \cap (Y \cup \underline{Y}) = \underline{\underline{Y}}, \\
(\mathcal{P}^-_\infty \cap \underline{X}^\alpha)^\sigma &= E \cap \underline{X}^{\sigma\cdot\sigma^{-1}\alpha\sigma} = E \cap \underline{\underline{Y}}^{\beta^{-1}}, \\
(E \cap \underline{\underline{X}}^\beta)^\sigma &= \mathcal{P}^-_\infty \cap \underline{\underline{X}}^{\sigma\alpha^{-1}\sigma\sigma} = \mathcal{P}^-_\infty \cap \underline{Y}^{\alpha^{-1}}, \quad \text{and} \\
\underline{E} \cap \underline{X} &= \underline{E} \cap \underline{Y} \text{ since } \sigma = 1 \text{ on } \underline{E}.
\end{aligned}$$

By two applications of (8.1″), it follows first that

$$\begin{aligned}
\tilde{X}^\sigma &= (X - E \cap X)^\sigma \cup (\mathcal{P}^-_\infty \cap \underline{X}^\alpha)^\sigma \cup (E \cap \underline{\underline{X}}^\beta)^\sigma \cup (\underline{E} \cap \underline{X})^\sigma \\
&= (Y - E \cap Y) \cup \left(E \cap \underline{\underline{Y}}^{\beta^{-1}}\right) \cup \left(\mathcal{P}^-_\infty \cap \underline{Y}^{\alpha^{-1}}\right) \cup (\underline{E} \cap \underline{Y}),
\end{aligned} \tag{8.5}$$

and then that $\mathbf{D}^\sigma$ is obtained from α^{-1} and β^{-1} in the same manner that $\mathbf{D}$ was obtained from α and β. Now (i) completes the proof. □

Part (i) says that $\mathbf{D}$ is obtained by "regluing" $\mathbf{D}'_\infty$ to $\mathbf{A}'$ "at infinity" (i.e., within $\overline{E}$) using the map γ appearing in (8.2), as in (2.5). Note, however, that this has led us to a notational irritation: we have had to change notation slightly from Section 2, using $\tilde{X}$ to denote blocks of $\mathbf{A}'(\mathbf{D}'_\infty, \gamma)$ since $\overline{X}$ is already defined in terms of $\mathbf{A}(\alpha)$. Part (ii) implicitly suggests additional confusing notation.

Write $\mathbf{A}_\infty := \mathbf{A}_{(E)} \cong AG(d-1, q)$; its projective space at infinity is $(\mathbf{D}_\infty)_{(\underline{E})}$, which arises here reglued to $\mathbf{A}_\infty$ in three different ways:

Lemma 8.6. (i) *The good blocks of* $\mathbf{D}$ *are precisely the blocks containing* $\underline{E}$.

(ii) $\mathbf{D}_{(\mathcal{P}_\infty)} \cong \mathbf{A}_\infty(\underline{\alpha}^{-1})$.

(iii) $\mathbf{D}_{(\overline{E})} = \mathbf{D}'_\infty = \mathbf{A}(\alpha)_{(\overline{E})} = \mathbf{A}_{(E)}(\underline{\alpha}) = A_\infty(\underline{\alpha})$.

(iv) *If* $F \neq E$ *is any hyperplane of* $\mathbf{A}$ *parallel* E, *then* $\tilde{F} = \overline{F}$ *and* $\mathbf{D}_{(\overline{F})} \cong \mathbf{A}_\infty(1) \cong PG(d-1, q)$.

(v) *No two of the three designs* $\mathbf{D}_{(\mathcal{P}_\infty)}$, $\mathbf{D}_{(\overline{E})}$ *and* $\mathbf{D}_{(\overline{F})}$ *are isomorphic.*

Proof: (i) By (8.4i) and (2.2i), $\overline{E}$ is good. The same is true of $\mathcal{P}_\infty$ by (8.4ii); alternatively, this will follow once we prove (ii). Similarly, we will show in (iv) that $\overline{F}$ is good.

Any good block of $\mathbf{D}$, other than $\overline{E}$, must meet $\overline{E}$ in a good block of $\mathbf{D}'_\infty$, by (2.4ii). Therefore, it suffices to show that $\underline{E}$ *is the only good block of* $\mathbf{D}'_\infty$.

We know that $\mathbf{D}'_\infty = \mathbf{A}_\infty(\underline{\alpha}) = \mathbf{A}_{(E)}\big((\mathbf{D}_\infty)_{(\underline{E})}, \underline{\alpha}\big)$ (cf. (2.8ii)). By (2.2i), $\underline{E}$ is a good block of $\mathbf{D}'_\infty$. Suppose that there is another good block of $\mathbf{A}_{(E)}(\underline{\alpha})$, and hence one arising from some hyperplane K of $\mathbf{A}_{(E)}$. Let $\underline{K}$ denote the hyperplane at infinity of K. By (2.7ii), if I and J are any hyperplanes of $\mathbf{A}_{(E)}$ such that $\underline{I} \cap \underline{K} = \underline{J} \cap \underline{K}$, then $\underline{I}^{\underline{\alpha}} \cap \underline{K}^{\underline{\alpha}} = \underline{J}^{\underline{\alpha}} \cap \underline{K}^{\underline{\alpha}}$. Here, $\underline{I}$, $\underline{J}$ and $\underline{K}$ are the hyperplanes at infinity of I, J and K, respectively, and hence are just hyperplanes of $(\mathbf{D}_\infty)_{(\underline{E})}$. Consequently, we are now dealing with a property of α taking place entirely within $(\mathbf{D}_\infty)_{(\underline{E})}$: the hyperplane $\underline{K}$ is such that, if $\underline{I} \cap \underline{K} = \underline{J} \cap \underline{K}$ then $\underline{I}^{\underline{\alpha}} \cap \underline{K}^{\underline{\alpha}} = \underline{J}^{\underline{\alpha}} \cap \underline{K}^{\underline{\alpha}}$. By (the dual of) (10.4vii), there is no hyperplane $\underline{K}$ of $(\mathbf{D}_\infty)_{(\underline{E})}$ behaving in this manner. This contradiction shows that $\mathbf{D}'_\infty$ has exactly one good block, and hence proves (i).

(ii) If X is a hyperplane of $\mathbf{A}$ not parallel to E, and if $(X \cup \underline{X})^\sigma = Y \cup \underline{Y}$, then (8.5) implies that $\overline{E} \cap \tilde{X}^\sigma = (E \cap \underline{\underline{Y}}^{\beta^{-1}}) \cup (\underline{E} \cap \underline{Y})^{\underline{\beta}^{-1}\underline{\alpha}^{-1}} = (E \cap \underline{\underline{Y}}^{\beta^{-1}}) \cup (\underline{E} \cap \underline{\underline{Y}}^{\beta^{-1}})^{\underline{\alpha}^{-1}}$ since $\underline{E} \cap \underline{Y} = \underline{E} \cap \underline{\underline{Y}}$. By (2.1), this proves (ii), as well as the fact that $\mathcal{P}_\infty$ is good (cf. (2.3ii)).

(iii) This was noted earlier.

(iv) By (8.1″), $\tilde{F} = F \cup \underline{E} = \overline{F}$, and $\overline{F} \cap \tilde{X} = (F \cap X) \cup (\underline{E} \cap \underline{X}) = (F \cap X) \cup (\underline{F} \cap \underline{X})$ is a hyperplane of $\mathbf{P}_{(\overline{F})}$ whenever X is not parallel to F. Thus, $\mathbf{D}_{(\overline{F})} = \mathbf{P}_{(\overline{F})} \cong \mathbf{A}_\infty(1)$ and $\overline{F}$ is good.

(v) By (the dual of) (10.4x), $\underline{\alpha}^{-1}$, $\underline{\alpha}$ and 1 lie in different $P\Gamma L(d-1, q)$, $P\Gamma L(d-1, q)$ double cosets. Thus, (4.4iii) together with the preceding parts (ii–iv) imply (v). □

Lemma 8.7. (i) $\mathrm{Aut}\mathbf{D}$ *fixes* $\mathcal{P}_\infty$ *and* $\overline{E}$.

(ii) $G \le \mathrm{Aut}\mathbf{D}$.

(iii) $\mathrm{Aut}\mathbf{D}$ *is isomorphic to a subgroup of* $\mathrm{Aut}\mathbf{A}' \cong (\mathrm{Aut}\mathbf{A}(\alpha))_{\overline{E}} = \Gamma(\mathcal{P}_\infty)_{\overline{E}} \rtimes G$, *where* $\Gamma(\mathcal{P}_\infty)_{\overline{E}}$ *is the group of perspectivities of* $\mathbf{P}$ *with axis* $\mathcal{P}_\infty$ *and center in* $\overline{E}$.

(iv) *No nontrivial element of* $\mathrm{Aut}\mathbf{D}$ *induces the identity on* $\mathcal{P}_\infty$.

Proof: (i) This is immediate by (8.6v), since $\mathrm{Aut}\mathbf{D}$ must permute the blocks containing $\underline{E}$.

(ii) By (the dual of) (10.4iii), G commutes with α. Since G commutes with σ it also commutes with $\sigma^{-1}\alpha^{-1}\sigma = \beta$. If $g \in G$ and X is a hyperplane of $\mathbf{A}$, then $\underline{X^g} = \underline{X}^g$ and $\underline{\underline{X^g}} = \underline{\underline{X}}^g$. By (8.1″), $\tilde{X}^g = (X^g - E \cap X^g) \cup (\mathcal{P}_\infty^- \cap \underline{X}^{g\alpha}) \cup (E \cap \underline{\underline{X}}^{g\beta}) \cup (\underline{E} \cap \underline{X}^g)$, so that $G \le \mathrm{Aut}\mathbf{D}$. (N.B.—While (3.1i) could have been used here, it was easier to proceed directly since G is *given* as a group of permutations of the points of $\mathbf{P}$ and hence of $\mathbf{D}$.)

(iii) By (i) and (8.4i), $\mathrm{Aut}\mathbf{D}$ is isomorphic to a subgroup of $\mathrm{Aut}\mathbf{A}'$. By (5.3), $\mathrm{Aut}\mathbf{A}' \cong [\mathrm{Aut}\mathbf{A}(\alpha)]_{\overline{E}}$. By (4.4ii), together with (the dual of) (10.4vi), $\mathrm{Aut}\mathbf{A}(\alpha)/\Gamma(\mathcal{P}_\infty) \cong G$. As in (ii), $G \le \mathrm{Aut}\mathbf{A}(\alpha)$; and $\Gamma(\mathcal{P}_\infty)_{\overline{E}} \cap G = 1$ by (7.2iii). Thus, $\mathrm{Aut}\mathbf{A}(\alpha) = \Gamma(\mathcal{P}_\infty)_{\overline{E}}G = \Gamma(\mathcal{P}_\infty)_{\overline{E}} \rtimes G$.

(iv) By (7.2iii), no nontrivial element of G fixes $\underline{E}$ pointwise. Then $\Gamma(\mathcal{P}_\infty)$ is the pointwise stabilizer of $\underline{E}$ in $\Gamma(\mathcal{P}_\infty)G$. Since $\mathrm{Aut}\mathbf{D} \le \Gamma(\mathcal{P}_\infty)_{\overline{E}}G$ by (iii), no nontrivial element of $\mathrm{Aut}\mathbf{D}$ induces the identity on $\overline{E}$.

In view of (8.4ii), we can interchange the roles of $\mathcal{P}_\infty$ and $\overline{E}$, and hence (iv) holds. □

In (iii) we saw that $\mathrm{Aut}\mathbf{A}(\alpha) = \Gamma(\mathcal{P}_\infty)_{\overline{E}} \rtimes G$, so that $\mathbf{A}(\alpha)$ "almost" behaves as in (1.1). We obtained $\mathbf{D}$ by modifying $\mathbf{A}(\alpha)$ in order to kill the group $\Gamma(\mathcal{P}_\infty)_{\overline{E}}$ appearing in (8.7iii).

Lemma 8.8. $G = \mathrm{Aut}\mathbf{D}$.

Proof: By (8.7ii, iii), $G \le \mathrm{Aut}\mathbf{D} \le \Gamma(\mathcal{P}_\infty)_{\overline{E}} \rtimes G$, and $\Gamma(\mathcal{P}_\infty)_{\overline{E}} \cap \mathrm{Aut}\mathbf{D} = 1$ by (8.7iv), so that $G = \mathrm{Aut}\mathbf{D}$. □

Theorem 8.9. *Given a finite group G, a prime power $q > 3$, and an integer $d \ge 50|G|^2$, there are at least $[q^{0.8d}]!$ pairwise nonisomorphic symmetric designs $\mathbf{D}$ having the parameters of $PG(d, q)$ such that*

(i) $\mathrm{Aut}\mathbf{D} \cong G$;

(ii) *The incidence structure induced by the removal of a suitable pair of good blocks is isomorphic to an incidence structure obtained in the same manner from $PG(d, q)$; and*

(iii) *The intersection of the two blocks in* (ii) *is contained in $q - 1$ other good blocks $\overline{F}$, and on each of these $\mathbf{D}$ induces a projective space $\mathbf{D}_{(\overline{F})}$.*

Proof: Part (ii) is clear from the construction (cf. (8.1)), while (iii) is just (8.6iv).

It remains to obtain a lower bound on the number of designs $\mathbf{D}$ just obtained. By (8.6), the pair $\{\mathbf{A}_\infty(\underline{\alpha}), \mathbf{A}_\infty(\underline{\alpha}^{-1})\}$, of designs is canonically associated with $\mathbf{D}$.

By (the dual of) (10.5), we can choose among at least $[q^{0.8d}]!$ permutations α such that the corresponding permutations $\underline{\alpha}$ and $\underline{\alpha}^{-1}$ all lie in at least $2[q^{0.8d}]!$ different $P\Gamma L(d-1, q)$, $P\Gamma L(d-1, q)$ double cosets. By (4.4iii), the associated symmetric designs $\mathbf{A}_\infty(\underline{\alpha})$ and $\mathbf{A}(\underline{\alpha}^{-1})$ are all nonisomorphic. Hence, the same is true for at least $[q^{0.8d}]!$ symmetric designs $\mathbf{D}$ arising from these choices of α. □

Theorem 8.10. *Given a finite group G, a prime power $q > 3$, and an integer $d \ge 50|G|^2$, there are at least $[q^{0.8d}]!$ pairwise nonisomorphic affine designs $\mathbf{A}''$ having the parameters of $AG(d, q)$ such that* $\mathrm{Aut}\mathbf{A}'' \cong G$ *and such that the incidence structure induced by the removal of a suitable pair of parallel good blocks is isomorphic to an incidence structure obtained in the same manner from $AG(d, q)$.*

Proof: By (8.6), $\overline{F}$ is a good block of $\mathbf{D}$. This leads us to consider the affine design $\mathbf{A}'' := \mathbf{D}^{\overline{F}}$. Since G fixes F by (7.2iv), it acts on $\mathbf{A}''$. We will show that $\mathrm{Aut}\mathbf{A}'' \cong G$ by recovering $\mathbf{D}$ from the geometry of $\mathbf{A}''$. Our approach parallels that of (5.3).

If $X \ne E$, F is a hyperplane of $\mathbf{A}$, let X'' denote the corresponding block $\tilde{X} - \overline{F} \cap \tilde{X}$ of $\mathbf{A}''$; there are two further blocks $\overline{E}$, $\mathcal{P}_\infty^-$ of $\mathbf{A}''$. By (8.1''),

$$X'' = [X - (E \cup F) \cap X] \cup (\mathcal{P}_\infty^- \cap \underline{X}^\alpha) \cup (E \cap \underline{\underline{X}}^\beta).$$

Then $\mathcal{P}_\infty^- \cap X'' = \mathcal{P}_\infty^- \cap \underline{X}^\alpha$ is a hyperplane of $(\mathbf{D}_\infty)^{\underline{E}}$, $E \cap X'' = D \cap \underline{\underline{X}}^\beta$ is a hyperplane of $\mathbf{A}_{(E)}$, and $F_1 \cap X'' = F_1 \cap X$ for any hyperplane $F_1 \ne E$, F of $\overline{\mathbf{A}}$ parallel to E. It follows from (2.3i) that each member of the parallel class of $\mathcal{P}_\infty^-$ is a good block, with an affine space induced on it. In particular, each $\mathbf{A}''$*-line contained in such a block has size* q.

Consider the set T of points of $\mathbf{A}$ not in $E \cup F$; this is just the set of points of $\mathbf{A}''$ not in $E \cup \mathcal{P}_\infty^-$. The nonempty intersections of the blocks of $\mathbf{A}''$ with T will be called "T-blocks"; together with T they produce an incidence structure $\mathbf{T}$ which could also have been obtained from $\mathbf{A}$ by the removal of E, F and all of their points.

Lemma 8.11. $\mathbf{T}$ *is determined by the geometry of* $\mathbf{A}''$.

Proof: There are two special points u and e of $\mathbf{A}''$. Namely, by (the dual of) (10.4viii) there is a unique point u of $\mathcal{P}_\infty^-$ such that α sends the hyperplanes of $\mathbf{D}_\infty$ on u to the hyperplanes on some point of $\mathcal{P}_\infty^-$ (namely, to hyperplanes containing u). By symmetry (cf. (8.4ii)), there is a unique point $e \in E$ such that β sends the hyperplanes of $\mathbf{P}_{(\overline{E})}$ on e to the hyperplanes on some point of E (namely, to hyperplanes containing e). By (4.1), each $\mathbf{A}(\alpha)$-line through u but not contained in $\mathcal{P}_\infty$ has size $q+1$. We already noted that each $\mathbf{A}''$-line lying in a block parallel to $\mathcal{P}_\infty^-$ has size q. Then each $\mathbf{A}''$*-line through* u (or e) *has size* $\geq q-1 \geq 3$, by (2.2iii).

On the other hand, by (2.2iii) and (4.1), any $\mathbf{A}''$-line containing a point of $\mathcal{P}_\infty^-$ as well as two points of T must contain u.

Now we can show that the *parallel class of* $\mathcal{P}_\infty^-$ *is determined by the geometry of* $\mathbf{A}''$. For, consider any q-point $\mathbf{A}''$-line L not lying in any block parallel to $\mathcal{P}_\infty^-$. Then L meets each block parallel to E, and hence in particular meets both $\mathcal{P}_\infty^-$ and E, and $|L \cap T| \geq q-2 \geq 2$. As noted above, this implies that L contains u and, by symmetry, also e. Thus, all but one q-point $\mathbf{A}''$-line lies in a block parallel to $\mathcal{P}_\infty^-$. This shows that the parallel class of $\mathcal{P}_\infty^-$ is uniquely determined.

Next, we claim that $\{\mathcal{P}_\infty^-, E\}$ *is also determined by the geometry of* $\mathbf{A}''$. Namely, we will show that any point of $\mathbf{A}''$ lying only on $\mathbf{A}''$-lines of size $\geq q-1$ must be inside $\mathcal{P}_\infty^- \cup E$; recall that both u and e behave in this manner. Suppose that x is such a point not in $\mathcal{P}_\infty^- \cup E$, and hence lying in T. Then choose a point $y \in T$ as follows: y does not lie in the block through x parallel to $\mathcal{P}_\infty^-$, and $y \notin ux_{\mathbf{A}''} \cup ex_{\mathbf{A}''}$. Then $xy_{\mathbf{A}''}$ cannot meet $\mathcal{P}_\infty^- \cup E$ (as noted above), and hence has size $\leq q-2$. This proves our claim.

In particular, we have now shown that the geometry of $\mathbf{A}''$ determines T and hence also $\mathbf{T}$. □

We now return to the proof of (8.10). The set of all intersections of T-blocks is a lattice (under set inclusion) that is "locally a projective space". It is straightforward to reconstruct a projective space $\mathbf{P}'$ isomorphic to $\mathbf{P}$ from T (as in Section 5, this is again a very special case of the Embedding Lemma of [8]). More precisely, each point w of $\mathbf{P}$ determines the set $[w]_T$ of T-blocks each of which is in a hyperplane of $\mathbf{P}$ containing w; the points of $\mathbf{P}'$ are defined to be the sets $w^\theta := [w]_T$. Similarly, each hyperplane $H \neq \mathcal{P}_\infty, \overline{E}, \overline{F}$ of $\mathbf{P}$ determines a T-block H^θ; the hyperplanes of $\mathbf{P}'$ are defined to be these T-blocks H^θ as well as the sets $\mathcal{P}_\infty^\theta = \{w^\theta \mid w \in \mathcal{P}_\infty\}$, $\overline{E}^\theta = \{w^\theta \mid w \in \overline{E}\}$ and $\overline{F}^\theta = \{w^\theta \mid w \in \overline{F}\}$. In this way we obtain an isomorphism $\theta \colon \mathbf{P} \to \mathbf{P}'$. (Note that all of this used $\mathbf{P}$ and T but not $\mathbf{A}''$.)

Now consider any block $X'' \neq \mathcal{P}_\infty^-$, E of $\mathbf{A}''$. This determines a T-block $X'' \cap T$, hence a hyperplane H' of $\mathbf{P}'$, and then also a hyperplane $H'^{\theta^{-1}}$ of $\mathbf{P}$. This produces a subset $(\overline{F} \cap H'^{\theta^{-1}})^\theta$ of $\mathbf{P}'$. If $\mathbf{P}'$ is now identified with $\mathbf{P}$, in which case θ becomes the identity, we see that we have just determined $\tilde{X} = X'' \cup (\overline{F} \cap H')$. In other words, we have indeed recovered $\mathbf{D}$ from $\mathbf{A}''$, as claimed in the first paragraph of the proof of the Theorem.

Finally, if two designs $\mathbf{A}''$ constructed in this manner from different maps α are isomorphic, then the same must hold for the corresponding symmetric designs $\mathbf{D}$. Consequently, by (8.9) there are at least $[q^{0.8d}]!$ pairwise nonisomorphic affine designs $\mathbf{A}''$. □

Theorem 1.1 follows from Theorems 8.9 and 8.10.

9. Concluding remarks

Remark 1. Let $\underline{\mathcal{P}}$ denote a proper, nonempty set of points of a symmetric design $\mathbf{D}$ having the following property: (∗) *Each block of* $\mathbf{D}$ *either contains* $\underline{\mathcal{P}}$ *or meets* $\underline{\mathcal{P}}$ *in exactly* $\underline{k}$ *points for some constant* $\underline{k}$, where $1 \leq \underline{k} < k - \lambda$. Let $\mathcal{P}$ consist of the remaining points of $\mathbf{D}$, let $\mathcal{H}_\infty$ and $\mathcal{B}$ be the sets of intersections with $\mathcal{P}$ of blocks of $\mathbf{D}$ containing or not containing $\underline{\mathcal{P}}$, respectively. If $X \in \mathcal{B}$ let $X \cup \underline{X}$ be the unique block of $\mathbf{D}$ containing it, where $\underline{X} \subseteq \underline{\mathcal{P}}$ (uniqueness follows from the hypothesis $\underline{k} < k - \lambda$). We now have an analogue $\mathbf{E}:=(\mathcal{P}, \mathcal{B}, \in)$ of the affine design $\mathbf{A}$. There is also an analogue of $(\underline{\mathcal{P}}, \underline{\mathcal{B}}, \in)$: let $\underline{\mathcal{B}}$ be the set of intersections with $\underline{\mathcal{P}}$ of the blocks of $\mathbf{D}$ not containing $\underline{\mathcal{P}}$. We will *assume* that this incidence structure $\underline{\mathbf{E}}$ is a symmetric design.

Let $\mathbf{D}_\infty = (\mathcal{P}_\infty, \mathcal{B}_\infty, \in)$ be any symmetric design having the same parameters as $\underline{\mathbf{E}}$. Fix a bijection $\alpha: \underline{\mathcal{B}} \to \mathcal{B}_\infty$. Define a new incidence structure $\mathbf{E}(\mathbf{D}_\infty, \alpha)$ using the point set $\underline{\mathcal{P}} \cup \underline{\mathcal{P}}_\infty$ and the following subsets as blocks:

$$\begin{array}{ll} \widetilde{Z} := Z \cup \mathcal{P}_\infty & \text{for each } Z \in \mathcal{H}_\infty, \text{ and} \\ X \cup \underline{X}^\alpha & \text{for each } X \in \mathcal{B}. \end{array}$$

It is straightforward to check that $\mathbf{E}(\mathbf{D}_\infty, \alpha)$ *is a symmetric design having the same parameters as* $\mathbf{D}$.

The case of special interest is that of a subspace $\underline{\mathcal{P}}$ of $\mathbf{D} = PG(d,q)$. If $\underline{\mathcal{P}}$ is a hyperplane we are back in our old situation. In the general case $\mathcal{H}_\infty$ produces good blocks $\widetilde{Z}$ of $\mathbf{E}(\mathbf{D}_\infty, \alpha)$, and $\mathbf{E}(\mathbf{D}_\infty, \alpha)^{\widetilde{Z}}$ is just an affine space (it has nothing to do with $\underline{\mathcal{P}}$ or α!). It follows from (2.5) that each example here is isomorphic to one in (2.1). More precisely, if $\underline{\mathcal{P}}$ is not a hyperplane then the design $\mathbf{E}(\mathbf{D}_\infty, \alpha)_{(\widetilde{Z})}$ is obtained by exactly the same process as $\mathbf{E}(\mathbf{D}_\infty, \alpha)$ was; and this is exactly the design being glued to $\mathbf{E}(\mathbf{D}_\infty, \alpha)^{\widetilde{Z}}$ in (2.1). Thus, these designs $\mathbf{E}(\mathbf{D}_\infty, \alpha)$ arise in a recursive manner. The advantage of the present construction is that it makes a large group of automorphisms evident: the pointwise stabilizer of $\mathcal{P}$ in $P\Gamma L(d+1, q)$ acts on $\mathbf{E}(\mathbf{D}_\infty, \alpha)$. In particular, $\mathrm{Aut}\mathbf{E}(\mathbf{D}_\infty, \alpha)$ is 2-transitive on the above set $\mathcal{H}_\infty$ of blocks: the analogue of (4.2) is false here. On the other hand, it is not hard to push the methods of Section 4 further in order to characterize the designs $\mathbf{E}(\mathbf{D}_\infty, \alpha)$.

Remark 2. All of the designs studied in this paper have very large chunks of affine spaces nicely embedded inside them (the same is also true of those examined in [6, 2]). It may be worthwhile pursuing a better understanding of this situation.

A variation on this can be used to handle the missing cases $q \leq 3$ of (1.1) [10]. This makes fuller use of the notion, visible in Section 8, that the good blocks determine a well-behaved partition of the set of all points, and can be reglued using permutations behaving like α and β—but one can arrange to use more than just two permutations.

Remark 3. The bounds used throughout this paper were cruder than needed. Theorem 1.1 is still true for $d > 100|G|\log_2\log_2(4|G|)$, using essentially the proof given earlier but being more careful with estimates. In the opposite direction, the estimates producing the number $[q^{0.8d}]!$ would have produced a somewhat larger constant than 0.8 if we had allowed d to be larger relative to $|G|$.

Remark 4. In the proof of (1.1), the representation on $\mathcal{P}_\infty$ used for G within $P\Gamma L(d+1, q)$ was very special. It seems to be difficult to find a symmetric design $\mathbf{D}$ having all of the following properties: the parameters are the same as $PG(d, q)$; Aut$\mathbf{D}$ fixes a block B; $\mathbf{D}_{(B)} \cong PG(d-1, q)$; and the action of Aut$\mathbf{D}$ on B is that of an *arbitrarily given* subgroup of $P\Gamma L(d, q)$ isomorphic to G—assuming that d is sufficiently large relative to G.

Remark 5. What is really going on in the Appendix? Why did the construction (2.1) lead so "naturally" to the type of question appearing in the Appendix? Other arguments lead to the same general type of question concerning projective spaces [10]. Are there different proofs of (1.1) avoiding such seemingly foreign considerations? On the other hand, is there a wider framework in which technical lemmas such as (10.3) appear?

Remark 6. There should be further variations on (1.1), for example constructing symmetric designs admitting a null polarity preserved by G.

Little seems to be known about infinite families $\mathcal{F}$ of symmetric designs such that each finite group is isomorphic to a *subgroup* of the automorphism group of one of the designs. One family consists of those designs with v a power of 2 arising from the tensor powers of the Hadamard matrix of order 2 [9]. One can tensor these with arbitrary Hadamard matrices to get further families; and there is no doubt that one can obtain an analogue of (1.1) using such designs. The only known families $\mathcal{F}$ arise from Hadamard matrices or have the parameters of projective spaces. Many more such families undoubtedly exist.

Remark 7. We conclude by outlining a modification of our proof of (1.1ii) that applies when $q = 3$. For this we define the following strange notion: if x and y are distinct points of a design, then a *pseudoline* through them is an intersection of $\lambda - 1$ of the blocks containing x and y. Each line is contained in as many as λ different pseudolines, but in any event the set of all pseudolines is canonically associated with the design.

Now suppose that we have chosen the bijection α used in Section 8 so that conditions (10.4i–vii,ix,x) appearing later all hold, and so that (10.4viii) is replaced by the following condition: there are unique hyperplanes H, H' not containing $\langle w \rangle$ such that δ maps all points of H to points of H' (moreover, $H = H'$). This allows the removal of the obstacle noted in the proof of (10.4vi). (In order to make δ behave in this manner, change the construction in (10.4) in just one place: have δ induce a 3-cycle on the points $\neq \langle w\rangle$ of $\langle w, w_1'\rangle$, interchange $\langle w_1' + w_2'\rangle$ and $\langle w_1' - w_2'\rangle$, and induce a suitable 4-cycle on the remaining points of $\langle w, w_1', w_2'\rangle$ not in $\langle w, w_2'\rangle$. Now the modified version of (10.4viii) is proved as in the original proof; and (10.4vi) also holds since the homology obtained in the course of that proof must commute with the 3-cycle on $\langle w, w_1'\rangle$ and hence must be 1.)

Proceeding as in Sections 4 and 5, we obtain a uniquely determined point $u \in \mathcal{P}_\infty^-$. Here, u is the intersection of all but one of those hyperplanes $\underline{X}^\alpha$ of $\mathbf{D}_\infty$ such that $u \in \underline{X}$. The arguments in (4.1), (4.2), (5.1) and (5.3) go through using pseudolines. (For example, u is the only point of $\mathcal{P}_\infty^-$ such that any point of $\mathcal{P}$ is on *some* $q+1$-point pseudoline of $\mathbf{A}(\alpha)$ containing u.) At that stage, the remainder of the proof of (8.9) goes through with no changes whatsoever.

10. Appendix: Permutations of a projective space

The proof of (1.1) ultimately depends upon permutations of the hyperplanes of a projective space. In this Appendix *we will consider the dual situation*, which is easier to visualize. We begin with an example.

Example 10.1. Let $\ell \geq 4$. Write $N = (q^\ell - 1)/(q-1)$. Basic ingredients in this section are permutations π of the points of $PG(\ell-1,q)$ such that $C_{P\Gamma L(\ell,q)}(\pi) = 1$ and π has a cycle of length $N-q$. Large numbers of these can be constructed as follows.

Let y and L be an incident point and line of $PG(\ell-1,q)$. Define a permutation π of the points of $PG(\ell-1,q)$ as follows:

> *π induces an arbitrary permutation on the points of $L - \{y\}$, and*
> *π induces a cycle π' of length $N-q$ on the complement of $L - \{y\}$.*

(i) *Claim:* $C_{P\Gamma L(\ell,q)}(\pi) = 1$. For, suppose that φ is a collineation commuting with π. Then φ commutes with the unique longest cycle π' of π and fixes the subspace L spanned by all of the remaining cycles, and hence also fixes the intersection y of that subspace with the support of π'. Thus, φ fixes every point of that support, and hence fixes every point of the projective space, as claimed.

(ii) Now restrict π slightly further: require that $L, y, y^\pi, y^{\pi^{-1}}$ and y^{π^2} all lie in some plane E, while $y^{\pi^{-2}}$ does not belong to E.

Claim: No element of $P\Gamma L(\ell,q)$ can conjugate π to π^{-1}. For, suppose that $\varphi \in P\Gamma L(\ell,q)$ and $\varphi^{-1}\pi\varphi = \pi^{-1}$. As above, φ fixes L and y. Also, y^π and $(y^\pi)^\varphi = y^{\varphi\pi^{-1}} = y^{\pi^{-1}}$ lie in E, so that φ fixes E. However, this contradicts the fact that $(y^{\pi^2})^\varphi = y^{\varphi\pi^{-2}} = y^{\pi^{-2}}$ does not lie in E while y^{π^2} does.

(iii) Let P be the number of permutations π obtained in this manner from a given incident point x, line L and plane E and a given permutation of $L - \{y\}$. Then we obtain $2P$ permutations π and π^{-1} (note that we cannot interchange π and π^{-1} in the first sentence of (ii), so we obtain $2P$ permutations). This produces at least $2P/|P\Gamma L(\ell,q)|$ permutations, no two of which are conjugate under $P\Gamma L(\ell, q)$, and hence a set of at least $P/|P\Gamma L(\ell, q)|$ permutations π as in (i) and (ii) such that none is conjugate under $P\Gamma L(\ell, q)$ to any other nor to the inverse of any other. Here $P/|P\Gamma L(\ell, q)| > (N - q^2)!/|P\Gamma L(\ell, q)| > 2(q^{\ell-1})!$ if $q^{\ell-1} > 8$.

(iv) Note that there is no hyperplane H such that π sends all points of H back into H, since the cycles of π have length $N - q$ or at most q.

Of course, there are many other permutations exhibiting behaviors similar to that seen in (i) and (ii).

We are now ready for the main technical lemmas of this paper. We start with a result that is much less precise than what is actually needed, but which gives the flavor of the question considered in this Appendix:

Proposition 10.2. *Any finite group of order $<\sqrt{d/20}$ is isomorphic to the stabilizer of some two points in the permutation representation of $S_{(q^d-1)/(q-1)}$ in its action on the cosets of $P\Gamma L(d, q)$.*

This should be compared with what was proved in Section 6: the stabilizer of "almost every" pair of points is trivial. We will need more precise versions of (10.2), including the fact that there are more than $[q^{0.8d}]!$ orbits of ordered pairs of points behaving as in the Proposition (which follows from (10.5)). The next result is a first approximation, and certainly implies (10.2).

Lemma 10.3. *Let G be a finite group and let $d - 1 \geq 20|G|^2$. Let $\ell = \max\{4, |G|\}$, and let q be any prime power. Then there is a permutation α of the points of a $d-1$-dimensional vector space W over $GF(q)$ such that the following all hold:*

(i) *G is (isomorphic to) a subgroup of $PGL(d-1, q)$, acting on W as in (7.1) (with ℓ as just defined and $d+1$ replaced by $d-1$);*

(ii) *α commutes with G;*

(iii) *α moves fewer than q^{d-4} points;*

(iv) *$P\Gamma L(d-1, q)^\alpha \cap P\Gamma L(d-1, q) = G$; and*

(v) *For each point z there are points x and y such that z, x, y are collinear but $z^\alpha, x^\alpha, y^\alpha$ are not.*

Proof: Write $d - 1 = \ell(|G| + 1) + \ell'$. Since $\ell \leq |G| + 3$, we have $\ell' \geq 20|G|^2 - (|G| + 3)(|G| + 1) \geq 15|G| - 3 \geq \ell + 2$. Let $F = GF(q^\ell)$. Let $\{f_i \mid 1 \leq i \leq \ell\}$ be a basis of F over $GF(q)$, where we assume that *no* f_i is 1, and let $\{w_i' \mid 1 \leq i \leq \ell'\}$ be a basis of an ℓ'-dimensional $GF(q)$-space W'.

Write $W = (\oplus_g F u_g) \oplus Fu \oplus W'$, where $(\oplus g F u_g) \oplus Fu$ can be viewed as an $(|G|+1)$-dimensional F-space with basis $\{u_g, u \mid g \in G\}$. Let each $h \in G$ act on this F-space by sending u_g to u_{gh} while fixing u; also let h fix every vector in W'. Note that this yields the representation of G indicated in (i), and we will identify G both with this group of linear transformations and the corresponding subgroup of $PGL(d-1, q)$.

We will use permutations $\pi_1, \pi_2, \pi_3, \pi_4, \pi_{g,i}$ (where $g \in G$ and $1 \leq i \leq \ell$) of the points of subspaces of W specified below (of dimensions $\ell + 1, \ell', \ell, \ell, \ell$, respectively). These permutations are chosen so that each behaves as in (10.1) while no two are conjugate under the action of $P\Gamma L(d-1,\, q)$. By (10.1iii), since $\ell \geq 4$ there are large numbers of permutations satisfying these conditions (in particular, there are at least $2 + \ell|G|$ permutations of $PG(\ell - 1,\, q)$ constructed in (10.1, iii)). Recall that each of these permutations has the property that it is centralized by no nontrivial collineation of the subspace spanned by its support.

Define α as follows:

α induces π_1 on $\langle w_1', Fu\rangle$, fixing $\langle w_1'\rangle$;

α induces π_2 on the points of W', fixing $\langle w_1'\rangle$, inducing a $q-1$-cycle on the points $\neq \langle w_1'\rangle, \langle w_2'\rangle$ of $\langle w_1', w_2'\rangle$, and moving $\langle w_2'\rangle$ to a point of W' not in $\langle w_1', w_2'\rangle$;

α induces $h^{-1}\pi_3 h$ on the points of $(Fu_1)^h$ whenever $h \in G$;

α induces $h^{-1}\pi_4 h$ on the points of $(F(u_1+u))^h$ whenever $h \in G$;

α induces $h^{-1}\pi_{g,i} h$ on the points of $(F(u_1 + f_i u_g + u))^h$ whenever $g, h \in G, g \neq 1$, and $1 \leq i \leq \ell$;

α fixes every other point of W.

(Expressions such as $h^{-1}\pi_3 h$ should be interpreted to mean that h^{-1} is being restricted to the subspace $(Fu_1)^h$. If $G = 1$ then no permutations $\pi_{g,i}$ are needed.)

This is well-defined. For, the definitions on $\langle w_1', Fu\rangle$ and W' do not conflict. The only other conceivable overlap in parts of the definition might occur if $\{F(u_1 + f_i u_g + u)\}^h = \{F(u_1 + f_{i'} u_{g'} + u)\}^{h'}$ for some g, h, g', h', i, i'. Then $u_h + f_i u_{gh} + u = c(u_{h'} + f_{i'} u_{g'h'} + u)$ for some $c \in F$. Linear independence over F implies that $c = 1$, then (since $f_i \neq 1, f_{i'} \neq 1$) that $h = h', g = g'$, and finally that $i = i'$.

It remains to verify properties (i–v).

(i, ii) These are clear.

(iii) The number of points moved by α is less than

$$q^{\ell+1} + q^{\ell'} + |G|q^{\ell} + |G|q^{\ell} + |G|(|G|-1)\ell q^{\ell} \leq q^{\ell' + |G|\ell + \ell - 3} = q^{d-4}.$$

(iv) Since G centralizes α it lies in $P\Gamma L(d-1,q)^{\alpha} \cap P\Gamma L(d-1,q)$.

Let $\tau = \alpha^{-1}\sigma\alpha \in P\Gamma L(d-1,q)^{\alpha} \cap P\Gamma L(d-1,q)$; we must show that $\tau = \sigma \in G$. We have $x^{\alpha\tau} = x^{\sigma\alpha}$ for each point x. If x is chosen so that α fixes both x and x^{σ}, then $x^{\tau} = x^{\sigma}$. Hence, if x is chosen so that it is fixed by both α and $\sigma\alpha\sigma^{-1}$, then $x^{\tau\sigma^{-1}} = x$. By (iii), $\tau\sigma^{-1}$ is an element of $P\Gamma L(d-1,q)$ fixing more than $(q^{d-1}-1)/(q-1) - 2q^{d-4} > (q^{d-2}-1)/(q-1)$ points, so that $\tau = \sigma$.

Thus, α commutes with σ, so that σ permutes the subspaces spanned by the nontrivial cycles of α. In particular, in view of our assumption that no two of the permutations $\pi_1, \pi_2, \pi_3, \pi_4, \pi_{g,i}$ are conjugate under the action of $P\Gamma L(d-1,q)$, it follows that σ fixes each of $\langle w_1', Fu\rangle, W', (Fu_1)^G, (F(u_1+u))^G$ and $(F(u_1 + f_i u_g + u))^G$ for all i and all $g \neq 1$. Since α and σ commute and act on $\langle w_1', Fu\rangle$ and W', (10.1i) implies that σ fixes these subspaces pointwise. Let $\psi \in \Gamma L(W)$ induce σ. After following ψ by a scalar transformation we may assume that ψ fixes every vector in $\langle w_1', Fu\rangle$; in particular, ψ is linear. Then ψ also fixes every vector in $\langle Fu, W'\rangle$.

By replacing ψ by ψg for some $g \in G$, we may assume that ψ also fixes Fu_1. As above, it follows that ψ fixes every point in this subspace. If $f \in F$ then $(fu_1)^\psi = cfu_1$ for some $c \in GF(q)$, so that $cfu_1 + fu = (f(u_1+u))^\psi \in F(u_h+u)$ for some h. By linear independence, $cf = f$, and hence ψ fixes every vector in Fu_1.

Similarly, if $g \in G$ and $g \neq 1$, then ψ sends Fu_g to some subspace of the form $Fu_{g'}$ with $g' \in G$. Let $1 \le i \le \ell$. Then $(f_iu_g)^\psi = f'u_{g'}$ for some $f' \in F$. Also, since ψ acts on $(F(u_1+f_iu_g+u))^G$ we have $(F(u_1+f_iu_g+u))^\psi = (F(u_1+f_iu_g+u))^h$ for some $h \in G$, and hence

$$\begin{aligned} u_1 + f'u_{g'} + u &= (u_1+f_iu_g+u)^\psi = c(u_1+f_iu_g+u)^h \\ &= c(u_h+f_iu_{gh}+u) \end{aligned}$$

for some $c \in F$. Then $1 = c$ and $u_1 + f'u_{g'} = u_h + f_iu_{gh}$. Recall that $f_i \neq 1$. Consequently, $1 = h, g' = gh = g$, and $f' = f_i$, so that $(f_iu_g)^\psi = f_iu_g$ for all i and all $g \neq 1$. Thus, $\psi = 1$, as required in (iv).

(v) By (iii), there is a line L on z such that L-$\{z\}$ consists of fixed points of α; if $z^\alpha \neq z$, let x and y be any distinct points of L-$\{z\}$. Suppose that $z^\alpha = z$. If $z \notin W'$, let $x = \langle w_2'\rangle$; then α fixes every point $y \neq z$, x of $\langle z, x\rangle$, and $x^\alpha \in W' - \{x\}$, so that $x^\alpha \notin \langle z, x\rangle$. If $z = z^\alpha \in W'$, let x be any point of $F(u_1+u)$ moved by α and let y be any point $\neq z$, x of $\langle z, x\rangle$ (so that y is fixed by α). □

The $q-1$-cycle in the definition of α was not needed in (10.3) but will arise in the proof of the next result.

Lemma 10.4. *Let G be a finite group and let $d > 20|G|^2$. Let $\ell = max\{4, |G|\}$ and $q > 3$. Then there is a permutation δ of the points of a d-dimensional vector space V over $GF(q)$ such that the following all hold:*

(i) *G is (isomorphic to) a subgroup of $PGL(d, q)$, acting on V as in* (7.1) *(with ℓ as just defined and $d+1$ replaced by d);*

(ii) *There is a vector w in the standard basis (cf.* (7.1)*) such that G and δ fix the point $\langle w\rangle$, no nontrivial element of G fixes all points of $V/\langle w\rangle$, and δ maps points collinear with $\langle w\rangle$ to points collinear with $\langle w\rangle$, inducing a permutation $\underline{\delta}$ of the points of $V/\langle w\rangle$;*

(iii) *δ commutes with G;*

(iv) *$\underline{\delta}$ moves fewer than q^{d-4} points of $V/\langle w\rangle$, while δ moves fewer than q^{d-3} points of V;*

(v) *$P\Gamma L(d-1, q)^{\underline{\delta}} \cap P\Gamma L(d-1, q) \cong G$;*

(vi) *$P\Gamma L(d, q)^\delta \cap P\Gamma L(d, q) = G$;*

(vii) *For each point z of $V/\langle w\rangle$ there are points x and y of $V/\langle w\rangle$ such that z, x, y are collinear but $z^{\underline{\delta}}$, $x^{\underline{\delta}}$, $y^{\underline{\delta}}$ are not;*

(viii) *There are unique hyperplanes H, H' not containing $\langle w\rangle$ such that δ maps all points of H to points of H' (moreover, $H = H'$);*

(ix) *δ^{-1} satisfies* (i–viii) *with the same $\langle w\rangle$; and*

(x) *1, $\underline{\delta}$ and $\underline{\delta}^{-1}$ are in different $P\Gamma L(d-1, q)$, $P\Gamma L(d-1, q)$ double cosets.*

Proof: We will repeat parts of the proof of (10.3), taking $\langle w\rangle$ into account. As in that proof, let $d-1=\ell(|G|+1)+\ell'$. As before, $\ell'\geq \ell+2$. Write $V=\langle w\rangle\oplus W$, and let $u_g, u, \{f_i\}, W'$ and $\{w_i'\}$ be as in the proof of (10.3); all of these are inside W. We may assume that $W=U_0$ and $w=v_d$ in the notation of (7.2ii) (with d in place of $d+1$). Also let $\pi_1,\pi_2,\pi_3,\pi_4,\pi_{g,i}$ be permutations behaving as in the proof of (10.3).

Let δ induce on W the permutation α appearing in the proof of (10.3). We need to define δ on the remaining points of V.

Let δ fix $\langle w\rangle$. For each $k>1$, every k-cycle of δ on W determines a k-cycle on the lines through $\langle w\rangle$: just join the points of the cycle to $\langle w\rangle$. This produces a set of $k(q-1)$ points not in $W\cup\{\langle w\rangle\}$ partitioned by the k lines. Let δ induce on these points a $k(q-1)$-cycle that induces the k-cycle we already have on the k lines, *except for one instance:* the $q-1$-cycle on $\langle w_1', w_2'\rangle$ in the construction given in (10.3). In the latter case we assume that δ fixes all points of $\langle w, w_1'\rangle$, induces a $q-1$-cycle on the points $\neq\langle w_1'\rangle,\langle w_2'\rangle$ of $\langle w_1', w_2'\rangle$ and a $q-1$-cycle on the points $\neq\langle w_1'\rangle, \langle w+w_2'\rangle$ of $\langle w_1', w+w_2'\rangle$, as well as a $(q-1)(q-2)$-cycle on the remaining points of $\langle w, w_1', w_2'\rangle$, so as to permute the lines $\neq\langle w, w_2'\rangle$ of $\langle w, w_1', w_2'\rangle$ through $\langle w\rangle$ in the same manner as the points $\neq\langle w_2'\rangle$ of $\langle w_1', w_2'\rangle$ are permuted. (Recall that in (10.1) one of the points of the distinguished line L is part of the long cycle. In the present situation, $\langle w_1', w_2'\rangle$ is the distinguished line of W', and $\langle w_2'\rangle$ is moved outside that line.)

Finally, let δ fix every point of V not already known to be moved. It remains to verify properties (i–ix).

(i), (ii), (iii) These are clear.

(iv) The first assertion is (10.3iii). The second follows from the fact that each moved point lies on a line through $\langle w\rangle$ containing a moved point of W.

(v) This follows from (10.3iv).

(vi) We will repeat part of the argument in (10.3iv). Let $\tau=\delta^{-1}\sigma\delta\in P\Gamma L(d,q)^{\delta}\cap P\Gamma L(d,q)$. Counting fixed points shows that $\tau\sigma^{-1}=1$, as in (10.3iv).

Next, as before σ fixes each of $\langle w_1', Fu\rangle, W', (Fu_1)^G, (F(u_1+u))^G, (F(u_1+f_iu_g+u))^G, \langle w, w_1', Fu\rangle$ and $\langle w, W'\rangle$. Replace σ by σg for some $g\in G$ so that σ fixes Fu_1. By repeating the proof of (10.3) we find that σ fixes every point of W. Also, σ fixes $\langle w, W'\rangle\cap\langle w, Fu_1\rangle=\langle w\rangle$. By (v), σ induces the identity on $V/\langle w\rangle$. Thus, σ is a homology with center $\langle w\rangle$ and axis W.

There is only one plane of V containing a $q(q-2)$-cycle of δ, namely $\langle w, w_1', w_2'\rangle$. Then σ must fix this plane, as well as its unique line $\langle w_1', w+w_2'\rangle$ on $\langle w_1'\rangle$, not lying in W, that contains a $q-1$-cycle of δ. Since σ is a homology, we finally have $\sigma=1$. (N.B.—The uniqueness of the aforementioned line required our assumption that $q>3$.)

(vii) This is immediate by (10.3v).

(viii) Certainly σ sends W to itself. Consider a pair H, H' as in (viii). Since δ moves at most q^{d-3} points of H (by (iv)), the points of H it fixes must span H, so that $H'=H$. Then δ permutes the points in the intersections of H with $\langle w_1', Fu\rangle, W'$ and $(Fu_1)^h$ for all h, i. By (10.1iv), it follows that H contains $(\oplus_g Fu_g)\oplus Fu\oplus W'=W$.

(ix) This is clear.

(x) Certainly $\underline{\delta}$ and $\underline{\delta}^{-1}$ are not in $P\Gamma L(d-1,q)$. Suppose that they are in the same $P\Gamma L(d-1,q), P\Gamma L(d-1,q)$ double coset, so that $\tau=\underline{\delta}\sigma\underline{\delta}$ for some $\tau,\sigma\in P\Gamma L(d-1,q)$.

If x is any point of $V/\langle w\rangle$ fixed by both $\underline{\delta}$ and $\sigma\underline{\delta}\sigma^{-1}$, then $x^{\tau} = x^{\underline{\delta}\sigma\underline{\delta}} = x^{\sigma\underline{\delta}} = x^{\sigma}$. By (iv) there are more than $(q^{d-1}-1)/(q-1) - 2q^{d-4}$ such points x, so that $\tau\sigma^{-1} = 1$. Then $\underline{\delta}$ and $\underline{\delta}^{-1}$ are conjugate in $P\Gamma L(d-1,q)$. We chose the permutation π_2 as in (10.1). Since $\ell' > \ell + 1$, the construction in (10.3) shows that π_2 contains the unique longest cycle of $\underline{\delta}$, and its inverse contains the unique longest cycle of $\underline{\delta}^{-1}$. Each of these longest cycles spans W'. Thus, σ must fix W', and hence it induces an element of $\Gamma L(W')$ conjugating π_2 to π_2^{-1}. This contradicts (10.1ii). □

Lemma 10.5. *Assume that* $d \geq 50|G|^2$ *in* (10.4). *Then there is a set* $\mathcal{S}$ *of permutations* δ *of the points of* V *that behave as in* (10.4) *such that* $|\mathcal{S}| \geq [q^{0.8d}]!$ *and such that* $\{P\Gamma L(d-1,q)\underline{\delta}P\Gamma L(d-1,q), P\Gamma L(d-1,q)\underline{\delta}^{-1}P\Gamma L(d-1,q) \mid \delta \in \mathcal{S}\}$ *consists of* $2|\mathcal{S}|$ *different* $P\Gamma L(d-1,q), P\Gamma L(d-1,q)$ *double cosets.*

Proof: Let ℓ and ℓ' be as before. Note that $\ell' - 1 - 8d/10 = 2d/10 - 2 - \ell(|G|+1) \geq 10|G|^2 - 2 - |(|G|+3)(|G|+1) \geq 0$.

In (10.1) use ℓ' in place of ℓ. Start with any of at least $(q^{\ell'-1})!$ permutations π in (10.1iii). Construct a set $\mathcal{S}$ of permutations δ in (10.4) by letting π_2 vary over these permutations π, while using the *same* vectors w, u, w_i', u_g, the *same* decomposition $W = (\oplus_g Fu_g) \oplus Fu \oplus W'$, and the *same* permutations $\pi_1, \pi_3, \pi_4, \pi_{g,i}$.

Suppose that δ and ε are two permutations of the points of V arising in this manner, and suppose that $\underline{\delta}$ and $\underline{\varepsilon}^{\pm 1}$ lie in the same $P\Gamma L(d-1,q), P\Gamma L(d-1,q)$ double coset. Then $\underline{\delta}\tau = \sigma\underline{\varepsilon}^{\pm 1}$ for some $\sigma, \tau \in P\Gamma L(d-1,q)$. As in (10.4x), by counting the number of points fixed by both $\underline{\delta}$ and $\sigma\underline{\varepsilon}^{\pm 1}\sigma^{-1}$ we find that $\tau\sigma^{-1} = 1$. Thus, $\underline{\delta}$ and $\underline{\varepsilon}^{\pm 1}$ are conjugate under $P\Gamma L(d-1,q)$.

Since $\ell' > \ell + 1$, the construction in (10.3) shows that π_2 is the unique longest cycle of $\underline{\delta}$, and this cycle spans W'. Thus, σ must fix W'. But then σ must conjugate the choice of π_2 yielding δ to that yielding $\varepsilon^{\pm 1}$. This contradicts (10.1iii), and shows that $\mathcal{S}$ behaves as required. □

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