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Spin Models on Finite Cyclic Groups

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Abstract. The concept of spin model is due to V. F. R. Jones. The concept of nonsymmetric spin model, which generalizes that of the original (symmetric) spin model, is defined naturally. In this paper, we first determine the diagonal matrices T satisfying the modular invariance or the quasi modular invariance property, i.e., $(PT)^3 = \sqrt{m}P^2$ or $(PT)^3 = m^{\frac{3}{2}}I$ (respectively), for the character table P of the group association scheme of a cyclic group G of order m. Then we show that a (symmetric or nonsymmetric) spin model on G is constructed from each of the matrices T satisfying the modular or quasi modular invariance property.

Keywords: spin model, association scheme, cyclic group, modular invariance property, link invariant

0. Introduction

0.1. Spin models

The definition of spin model is due to V. F. R. Jones [9]. In his definition symmetric conditions are required. Kawagoe, Munemasa, and Watatani [13] generalized it by dropping the symmetric conditions.

Definition 1 (Generalized spin model). (X, w_+, w_-, D) *is called a (generalized) spin model if* X *is a finite set,* w_+ *and* w_- *are complex valued functions on* $X \times X$ *satisfying the following axioms* (1), (2) *and* (3) *(for all* $\alpha, \beta, \gamma \in X$*):*

(1) $w_+(\beta, \alpha)w_-(\alpha, \beta) = 1,$

(2) $\sum_{x\in X} w_-(\alpha, x)w_+(x, \beta) = |X|\delta_{\alpha,\beta},$

(3) *(star-triangle relation)*

$$\sum_{x\in X} w_+(\alpha, x)w_+(x, \beta)w_-(x, \gamma) = Dw_+(\alpha, \beta)w_-(\beta, \gamma)w_-(\alpha, \gamma),$$

where $D^2 = |X|$.
(X, w_+, w_-, D) *is called symmetric if the following condition is satisfied:*

(0) $w_+(\alpha, \beta) = w_+(\beta, \alpha)$, $w_-(\alpha, \beta) = w_-(\beta, \alpha)$ *for any* α *and* β *in* X.

Remark 1. According to Watatani, Jones suggested that he consider spin models without the symmetric conditions. It is proved in [2] that the concepts of symmetric spin models and of generalized spin models (in Definition 1) can be further generalized by using four functions $w_i (i = 1, 2, 3, 4)$ on $X \times X$.

Remark 2. It is easy to check that a (generalized) spin model (in the sense of Definition 1) gives an invariant of oriented links in a similar way as a symmetric one (see [13], [2]). This confirms that Definition 1 given above is a right definition of spin models.

Let W_+ and W_- be the matrices defined by $W_+ = \left(w_+(\alpha, \beta)\right)_{\alpha \in X, \beta \in X}$ and $W_- = \left(w_-(\alpha, \beta)\right)_{\alpha \in X, \beta \in X}$. Let I be the identity matrix and J be the matrix whose entries are all 1. Let $Y_{\alpha\gamma}$ be the column vector defined by $Y_{\alpha\gamma} = \left(w_+(\alpha, x) w_-(x, \gamma)\right)_{x \in X}$. Let $\circ$ denote the Hadamard product (i.e., the entry-wise product) of two square matrices of the same size. Then the conditions (1), (2) and (3) in Definition 1 are expressed in the following forms.

(1) ${}^tW_+ \circ W_- = J$,

(2) $W_+ W_- = |X| I$,

(3) ${}^tW_+ Y_{\alpha\gamma} = D w_-(\alpha, \gamma) Y_{\alpha\gamma}$, where $D^2 = |X|$.

In what follows we will denote a spin model simply by (X, w_+, w_-) or (X, W_+, W_-) without mentioning D when there is no confusion.

0.2. Cyclic group association scheme and the modular and quasi modular invariance properties

Let $G = G_m$ be a cyclic group of order m generated by g. Then the group association scheme $\mathfrak{X}(G)$ is a pair consisting of the finite set $X = G$ and the set of relations $\{R_i\}_{0 \le i \le m-1}$ on X defined for $x, y \in G$ by

$$(x, y) \in R_i \text{ if and only if } yx^{-1} = g^i \ (0 \le i \le m-1) \text{ (see [3])}.$$

Note that the adjacency matrix A_i with respect to the relation R_i is given by

$$A_i = {A_1}^i \text{ where } A_1 \text{ has } (x, y) - \text{entry equal to } \delta_{gx, y} \text{ for } x, y \in G.$$

Let $\mathfrak{A} = \langle A_0, A_1, \ldots, A_{m-1} \rangle = \langle A_1 \rangle$ be the Bose-Mesner algebra of the group association scheme $\mathfrak{X}(G)$. Then the primitive idempotents $E_0, E_1, \ldots, E_{m-1}$ of $\mathfrak{A}$ are given by

$$A_i = \sum_{l=0}^{m-1} \zeta^{il} E_l$$

where ζ is a primitive m-th root of unity. The matrix $P = (\zeta^{ij})_{\substack{0\leq i\leq m-1\\ 0\leq j\leq m-1}}$ is the character table of the group association scheme $\mathfrak{X}(G)$ (and also the character table of the group G) and the (i, j)-entry of P^2 is m for $i+j \equiv 0 \pmod m$ and 0 otherwise. The second eigenmatrix $Q = (\overline{\zeta^{ij}})_{\substack{0\leq i\leq m-1\\ 0\leq j\leq m-1}}$ of $\mathfrak{X}(G)$ corresponds to the linear transformation $mE_i = \sum_{l=0}^{m-1} \overline{\zeta^{il}} A_l$ and satisfies $PQ = mI,\ Q = \overline{P}$, so that $\mathfrak{X}(G)$ is self dual.

Definition 2 (Modular invariance property). *Let P be the matrix defined above. A diagonal matrix T is said to satisfy the modular invariance property if the relation*

$$(PT)^3 = \sqrt{m}P^2$$

holds.

Definition 3 (Quasi modular invariance property). *Let P be the matrix defined above. A diagonal matrix T is said to satisfy the quasi modular invariance property if the relation*

$$(PT)^3 = m^{\frac{3}{2}} I$$

holds.

Remark 3. Note that for a finite cyclic (or abelian) group G, the matrix S of the corresponding fusion algebra at algebraic level (*cf.* [1]) satisfies

$$S = \frac{1}{\sqrt{|G|}} P$$

and the modular and quasi modular invariance properties become

$$(ST)^3 = S^2 \quad (S^4 = I)$$

and

$$(ST)^3 = I$$

respectively. The matrix S is symmetric and unitary. For further explanations on why we are led to notice the modular invariance properties for association schemes in connection with spin models, the reader is referred to the following survey article by the first author: Eiichi Bannai, Algebraic Combinatorics—Recent topics on association schemes—, Sugaku (Mathematics) (Math. Soc. of Japan) 45 (1993), 55–75 (in Japanese). An English translation of this article will be published in Sugaku Exposition (Amer. Math. Soc.). Further relations between modular invariance properties and spin models will be treated in a joint paper by Eiichi Bannai, Etsuko Bannai and François Jaeger, which is in preparation.

The first purpose of this paper is to give the complete list of the diagonal matrices T satisfying the modular or quasi modular invariance property for the character table P of $\mathfrak{X}(G_m)$. The results are given by the following two theorems.

Theorem 1. *$\mathfrak{X}(G_m)$ has the modular invariance property with a diagonal matrix*

$$T = diag(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$$

if and only if the following holds:

$$\alpha_i = \eta^{i^2}\alpha_0 \text{ for } i \in \{0, 1, \ldots, m-1\} \text{ and } \alpha_0^3 = \sqrt{m}\Big/\sum_{l=0}^{m-1}\eta^{l^2},$$

where $\eta = \zeta^{\frac{m-1}{2}}$ if m is odd and $\eta^2 = \zeta^{-1}$ if m is even.

Theorem 2. 1) *Let m be odd. Then $\mathfrak{X}(G_m)$ has the quasi modular invariance property with a diagonal matrix*

$$T = diag(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$$

if and only if the following holds:

$$\alpha_i = \zeta^{i(i+2s-1)/2}\alpha_0 \text{ for } i \in \{0, 1, \ldots, m-1\}$$

and

$$\alpha_0^3 = \sqrt{m}\zeta^{(\frac{m+1}{2})^3(2s-1)^2}\Big/\sum_{l=0}^{m-1}\zeta^{(\frac{m+1}{2})l^2}$$

with $s \in \{0, \ldots, m-1\}$.

2) *Let m be even. Then $\mathfrak{X}(G_m)$ has the quasi modular invariance property with a diagonal matrix*

$$T = diag(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$$

if and only if the following holds:

$$\alpha_i = \eta^{i(i+2s)}\alpha_0 \text{ for } i \in \{0, 1, \ldots, m-1\} \text{ and } \alpha_0^3 = \sqrt{m}\eta^{s^2}\Big/\sum_{l=0}^{m-1}\eta^{l^2}$$

with $\eta^2 = \zeta$ and $s \in \{0, \ldots, m-1\}$.

Remark 4. For any positive integer n, it is known (*cf.* [15, 14, 7]) that

(i) $$\Big|\sum_{l=0}^{n-1}\xi^{l^2}\Big| = \sqrt{n} \text{ if } n \equiv 1\ (2),$$

(ii) $$\sum_{l=0}^{n-1}\xi^{l^2} = 0 \text{ if } n \equiv 2\ (4),$$

(iii) $$\left|\sum_{l=0}^{n-1} \xi^{l^2}\right| = \sqrt{2n} \text{ if } n \equiv 0\,(4),$$

where ξ is any primitive n-th root of unity. In Theorem 1 or Theorem 2, if m is odd then $\zeta^{\frac{m-1}{2}}$ and $\zeta^{\frac{m+1}{2}}$ are primitive m-th roots of unity. Therefore $|\alpha_0| = 1$ in Theorem 1 or Theorem 2 if m is odd. In Theorem 1 or Theorem 2, if m is even then η is a primitive $2m$-th root of unity. Since $2m \equiv 0(4)$, by (iii) we have

$$\left|\sum_{l=0}^{2m-1} \eta^{l^2}\right| = \sqrt{4m} = 2\sqrt{m}.$$

On the other hand

$$\sum_{l=0}^{2m-1} \eta^{l^2} = 2\sum_{l=0}^{m-1} \eta^{l^2}.$$

Therefore $|\sum_{l=0}^{m-1} \eta^{l^2}| = \sqrt{m}$ and hence we also have $|\alpha_0| = 1$ for those cases. Moreover we can show that α_0 in Theorem 1 or Theorem 2 is a root of unity using results in Schur [15] or Nagell [14, Section 53].

0.3. Spin models on G_m

Let $A_i (i = 0, 1, \ldots, m-1)$ be the adjacency matrices of the group association scheme $\mathfrak{X}(G_m)$, namely, the A_i as given in Section 0.2. We want to construct a generalized spin model (X, W_+, W_-) on the cyclic group $X = G_m$ with

$$W_+ = \sum_{i=0}^{m-1} t_i A_i \ (t_i \in \mathbf{C}, t_i \neq 0).$$

By the relation (1) in Definition 1 we have

$$W_- = \sum_{i=0}^{m-1} t_i{}^{-1} A_{i'},$$

where ${}^tA_i = A_{i'}$ (that is , $i' \equiv -i(m)$).

The second purpose of this paper is to construct a spin model from each matrix T satisfying the modular or quasi modular invariance property, which was completely characterized in Theorem 1 or Theorem 2 respectively. Our result is summarized in the following two theorems.

Theorem 3. *Let W_+ and W_- be defined by $t_i = \alpha_i t_0/\alpha_0$, $i = 0, 1, \ldots, m-1$, and $t_0^2 = \alpha_0^3$, where α_i are given in Theorem 1. Then (G_m, W_+, W_-) is a symmetric spin model with $D = \sqrt{m}$.*

Theorem 4. *Let W_+ and W_- be defined by $t_i = \alpha_i t_0/\alpha_0$, $i = 0, \ldots, m-1$, and $t_0^2 = \alpha_0^3$, where α_i are given in Theorem 2. Then (G_m, W_+, W_-) is a spin model with $D = \sqrt{m}$. Moreover, (G_m, W_+, W_-) is a symmetric spin model if and only if $2s - 1 \equiv 0(m)$ for m odd or $2s \equiv 0(m)$ for m even.*

Remark 5. It seems that symmetric spin models constructed in the above two theorems are known in some forms (*cf.* [9], or *cf.* [6]. See also [5], [12]). However, nonsymmetric spin models on G_m have not been studied, except for the following result due to Kawagoe, Munemasa and Watatani [13]. They found an example for each of G_3, G_4, and G_5, through a systematic search by computer, namely

$$W_+ = \zeta_{24}\begin{pmatrix} 1 & 1 & \zeta_{24}^{16} \\ \zeta_{24}^{16} & 1 & 1 \\ 1 & \zeta_{24}^{16} & 1 \end{pmatrix} \text{ on } G_3,$$

$$W_+ = \begin{pmatrix} 1 & \zeta_8 & 1 & \zeta_8^5 \\ \zeta_8^5 & 1 & \zeta_8 & 1 \\ 1 & \zeta_8^5 & 1 & \zeta_8 \\ \zeta_8 & 1 & \zeta_8^5 & 1 \end{pmatrix} \text{ on } G_4,$$

and

$$W_+ = \zeta_{20}\begin{pmatrix} 1 & \zeta_{20}^8 & 1 & \zeta_{20}^{16} & \zeta_{20}^{16} \\ \zeta_{20}^{16} & 1 & \zeta_{20}^8 & 1 & \zeta_{20}^{16} \\ \zeta_{20}^{16} & \zeta_{20}^{16} & 1 & \zeta_{20}^8 & 1 \\ 1 & \zeta_{20}^{16} & \zeta_{20}^{16} & 1 & \zeta_{20}^8 \\ \zeta_{20}^8 & 1 & \zeta_{20}^{16} & \zeta_{20}^{16} & 1 \end{pmatrix} \text{ on } G_5.$$

where $\zeta_m = \exp(2\pi\sqrt{-1}/m)$. However, it has not been clear where they came from. Our theorems include as special cases the examples by Kawagoe, Munemasa and Watatani [13], and show how these spin models are constructed in a general context.

Remark 6. The question of what kinds of invariants of links are obtained from nonsymmetric spin models constructed in Theorem 3 and Theorem 4 has not yet been studied. It would be interesting to know whether new invariants of links are obtained from these spin models. (For general informations on link invariants, see [9], [10], [8], [4].) (For the symmetric case these have been studied, say in [6], [11].)

1. Proofs of the theorems

1.1. *Proofs of Theorem 1 and Theorem 2*

First we give the following proposition which will be used several times in the proofs of the theorems.

Proposition 1. *Let $\eta = \zeta^{\frac{m-1}{2}}$ if m is odd and $\eta^2 = \zeta^{-1}$ if m is even, and $l_1 \equiv l_2$ (m). Then we have*

(i) $\eta^{2l_1} = \eta^{2l_2}$,

(ii) $\eta^{l_1^2} = \eta^{l_2^2}$.

Proof: (i) is obvious. If m is odd, then $\eta = \zeta^{\frac{m-1}{2}}$ and (ii) is obvious. If m is even, then by the assumption, $l_1^2 - l_2^2$ is a multiple of $2m$. Since $\eta^{2m} = 1$, we have (ii). □

Let $T = \mathrm{diag}(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ be a diagonal matrix satisfying

$$(PT)^3 = \sqrt{m}P^2 \text{ or } (PT)^3 = m^{\frac{3}{2}}I.$$

Note that T is invertible.Then for any $i, j \in \{0, 1, \ldots, m-1\}$ with $i + \epsilon j \not\equiv 0\ (m)$ we have

$$\sum_{l=0}^{m-1}\sum_{k=0}^{m-1} \zeta^{il+lk+kj}\alpha_l\alpha_k\alpha_j = 0, \tag{1}$$

where $\epsilon = 1$ if $(PT)^3 = \sqrt{m}P^2$ and $\epsilon = -1$ if $(PT)^3 = m^{\frac{3}{2}}I$.
To prove the theorems we need the following propositions.

Proposition 2. *For any $u, s \in \{0, 1, \ldots, m-1\}$ we have*

$$m\zeta^{us}\alpha_u\alpha_s - \sum_{l+\epsilon(s-k)-u\equiv 0(m)} \zeta^{lk}\alpha_l\alpha_k = 0.$$

Proof: Since $\alpha_j \neq 0$, $j = 0, \ldots, m-1$, by (1) we have

$$\sum_{\substack{j=0\\ j\not\equiv -\epsilon i(m)}}^{m-1} \zeta^{-sj}\left(\sum_{l=0}^{m-1}\sum_{k=0}^{m-1} \zeta^{il+lk+kj}\alpha_l\alpha_k\right) = 0$$

for any $s \in \{0, 1, \ldots, m-1\}$. Since

$$\sum_{\substack{j=0\\ j\not\equiv -\epsilon i(m)}}^{m-1} \zeta^{tj} = m-1 \text{ if } t \equiv 0\ (m)$$

and

$$\sum_{\substack{j=0\\ j\not\equiv -\epsilon i(m)}}^{m-1} \zeta^{tj} = -\zeta^{-\epsilon it} \text{ if } t \not\equiv 0\ (m),$$

we have

$$m\sum_{l=0}^{m-1} \zeta^{il+ls}\alpha_l\alpha_s - \sum_{l=0}^{m-1}\sum_{k=0}^{m-1} \zeta^{i(l-\epsilon k+\epsilon s)+lk}\alpha_l\alpha_k = 0$$

for any $i \in \{0, 1, \ldots, m-1\}$. Then we have

$$\sum_{i=0}^{m-1} \zeta^{-iu}\left(m \sum_{l=0}^{m-1} \zeta^{il+ls}\alpha_l\alpha_s - \sum_{l=0}^{m-1}\sum_{k=0}^{m-1} \zeta^{i(l-\epsilon k+\epsilon s)+lk}\alpha_l\alpha_k\right) = 0.$$

Since $\sum_{i=0}^{m-1} \zeta^{ti} = 0$ if $t \not\equiv 0(m)$, we have Proposition 2. □

Proposition 3. *For any* $u, s, j \in \{0, 1, \ldots, m-1\}$ *with* $u - \epsilon s \equiv j \ (m)$, *we have*

$$\zeta^{us}\alpha_u\alpha_s = \alpha_j\alpha_0.$$

Proof: Let $s = 0$ and $u = j$ in Proposition 2. Then we have

$$m\alpha_j\alpha_0 = \sum_{l-\epsilon k-j\equiv 0(m)} \zeta^{kl}\alpha_l\alpha_k.$$

On the other hand, let $u - \epsilon s \equiv j \ (m)$ in Proposition 2. Then we have

$$m\zeta^{us}\alpha_u\alpha_s = \sum_{l-\epsilon k-j\equiv 0(m)} \zeta^{kl}\alpha_l\alpha_k$$

for any $u, s \in \{0, 1, \ldots, m-1\}$ with $u - \epsilon s \equiv j \ (m)$. Therefore $\zeta^{us}\alpha_u\alpha_s = \alpha_j\alpha_0$ for any u and s with $u - \epsilon s \equiv j \ (m)$. □

Proposition 4. (i) *If* $\epsilon = 1$, *then*

$$\alpha_j = \eta^{j^2}\alpha_0 \textit{ for } j \in \{0, 1, \ldots, m-1\},$$

where $\eta = \zeta^{\frac{m-1}{2}}$ *if* m *is odd and* $\eta^2 = \zeta^{-1}$ *if* m *is even.*
(ii) *If* $\epsilon = -1$, *then*

$$\alpha_j = \eta^j\zeta^{\frac{j(j-1)}{2}}\alpha_0,$$

where $\eta^m = 1$ *if* m *is odd and* $\eta^m = -1$ *if* m *is even.*

Proof: Let $s = 1$ in Proposition 3. Then $u \equiv j + \epsilon(m)$ and

$$\zeta^{j+\epsilon}\alpha_{j+\epsilon}\alpha_1 = \alpha_j\alpha_0,$$

where indices are to be read modulo m. Let $\alpha_1 = \eta\alpha_0$. Then we have

$$\alpha_j = \eta\zeta^{j+\epsilon}\alpha_{j+\epsilon}. \tag{2}$$

(i) If $\epsilon = 1$, then $\alpha_j = \eta\zeta^{j+1}\alpha_{j+1}$ and

$$\alpha_j = \eta^{-1}\zeta^{-j}\alpha_{j-1} \text{ for } j = 1, \ldots, m-1. \tag{3}$$

Put $j = 1$ in (3), then

$$\alpha_1 = \eta^{-1}\zeta^{-1}\alpha_0.$$

Therefore we get $\eta^2 = \zeta^{-1}$ and

$$\alpha_j = \eta^{2j-1}\alpha_{j-1} \text{ for } j = 1, \ldots, m-1.$$

Hence we have

$$\alpha_j = \eta^{j^2}\alpha_0 \text{ for } j = 1, \ldots, m-1. \tag{4}$$

Moreover by Proposition 3 with $\epsilon = 1$ we obtain

$$\zeta^{m-1}\alpha_1\alpha_{m-1} = \alpha_2\alpha_0 = \zeta^3\alpha_3\alpha_1.$$

Therefore by (4)

$$\alpha_{m-1} = \zeta^4\alpha_3 = \zeta^4\eta^9\alpha_0 = \eta\alpha_0.$$

The equation (4) also gives

$$\alpha_{m-1} = \eta^{(m-1)^2}\alpha_0 = \eta^{m^2+1}\alpha_0.$$

Hence we have $\eta^{m^2} = 1$. Since $\eta^2 = \zeta^{-1}$, if m is odd then $\eta = \zeta^{\frac{m-1}{2}}$. This completes the proof for (i).

(ii) If $\epsilon = -1$ then by (2) we get

$$\alpha_j = \eta\zeta^{j-1}\alpha_{j-1} \text{ for } j = 1, \ldots, m-1.$$

Then we obtain

$$\alpha_j = \zeta^{\frac{(j-1)j}{2}}\eta^j\alpha_0 \text{ for } j = 0, \ldots, m-1. \tag{5}$$

Let $s = 1$ and $u = m - 1$ in Proposition 3. Then

$$\zeta^{m-1}\alpha_{m-1}\alpha_1 = \alpha_0^2.$$

Therefore we have

$$\alpha_{m-1} = \zeta\alpha_0^2\alpha_1^{-1} = \zeta\eta^{-1}\alpha_0.$$

On the other hand by (5) we have

$$\alpha_{m-1} = \zeta^{\frac{(m-2)(m-1)}{2}} \eta^{m-1} \alpha_0.$$

Hence we have $\zeta^{\frac{(m-3)m}{2}} \eta^m = 1$. If m is odd, then $\zeta^{\frac{(m-3)m}{2}} = 1$ and we have $\eta^m = 1$. If m is even, then $\zeta^{\frac{(m-3)m}{2}} = (\zeta^{\frac{m}{2}})^{m-3} = (-1)^{m-3} = -1$. Therefore $\eta^m = -1$. This completes the proof. □

Proposition 5. *For a diagonal matrix $T = diag(\alpha_0, \ldots, \alpha_{m-1})$, satisfying, $\alpha_i = \alpha_j$ for $i, j \in \{0, \ldots, m-1\}$ with $i + j \equiv 0\ (m)$, the modular invariance property $(PT)^3 = \sqrt{m}P^2$ is equivalent to the quasi modular invariance property $(QT)^3 = m^{\frac{3}{2}} I$ with respect to the second eigenmatrix Q of the group association scheme $\mathfrak{X}(G)$.*

Proof: Assume that T satisfies the statement of Proposition 5. Then T commutes with the matrix P^2. Since $PQ = mI$, $P^4 = m^2 I$, we have Proposition 5. □

Now we are ready to prove Theorem 1 and Theorem 2. Let $T = \text{diag}(\alpha_0, \ldots, \alpha_{m-1})$ satisfy the modular invariance property. Then by Proposition 1 and Proposition 4(i) $\alpha_i = \alpha_j$ for any $i, j \in \{0, \ldots, m-1\}$ with $i + j \equiv 0(m)$. Since $Q = \overline{P}$, Proposition 5 says that if T satisfies the modular invariance property, then T satisfies a special case of quasi-modular invariance property for the inverse value of ζ. Actually in Proposition 4 (i) we can express the solution by

$$\alpha_j = \eta^j \zeta^{-\frac{j(j-1)}{2}} \alpha_0 \text{ for } j \in \{0, 1, \ldots, m-1\}$$

for both cases m even and odd and this is indeed a special case of Proposition 4(ii) with ζ replaced by ζ^{-1}. Therefore by Proposition 5 it is enough for us to prove Theorem 2.

We have seen that if T satisfies the quasi modular invariance property, then α_l, $l = 0, 1, \ldots, m-1$, must satisfy the conditions given in Proposition 4(ii). Conversely for such α_l we have

$$\begin{aligned}&(i, j)\text{-entry of } (PT)^3 \\ &\quad = \alpha_0^3 \zeta^{\frac{(j-1)j}{2}} \eta^j \sum_{l=0}^{m-1} \sum_{k=0}^{m-1} \zeta^{i(l+k)+k(j-i)+\frac{(l+k)^2-(l+k)}{2}} \eta^{l+k}.\end{aligned}$$

If m is odd, then, by Proposition 4, $\eta = \zeta^s$ with some $s \in \{0, 1, \ldots, m-1\}$. Therefore in this case

$$\begin{aligned}
&(i, j)\text{-entry of } (PT)^3 \\
&= \alpha_0^3 \zeta^{\frac{(j-1)j}{2}} \zeta^{sj} \sum_{k=0}^{m-1} \zeta^{k(j-i)} \sum_{l=0}^{m-1} \zeta^{il+\frac{l^2-l}{2}+sl} \\
&= \delta_{i,j} \alpha_0^3 \zeta^{\frac{(j-1)j}{2}} \zeta^{sj} m \sum_{l=0}^{m-1} \zeta^{il+\frac{l^2-l}{2}+sl} \\
&= \delta_{i,j} m \alpha_0^3 \sum_{l=0}^{m-1} \zeta^{\frac{(l+i)^2-(l+i)}{2}+s(l+i)} \\
&= \delta_{i,j} m \alpha_0^3 \sum_{l=0}^{m-1} \zeta^{\frac{l^2-l}{2}+sl} \\
&= \delta_{i,j} m \alpha_0^3 \sum_{l=0}^{m-1} \zeta^{(\frac{m+1}{2})(l^2-l+2sl)} \\
&= \delta_{i,j} m \alpha_0^3 \sum_{l=0}^{m-1} \zeta^{(\frac{m+1}{2})\{(l+\frac{(m+1)}{2}(2s-1))^2-(\frac{m+1}{2})^2(2s-1)^2\}} \\
&= \delta_{i,j} m \alpha_0^3 \sum_{l=0}^{m-1} \zeta^{(\frac{m+1}{2})l^2-(\frac{m+1}{2})^3(2s-1)^2}.
\end{aligned}$$

Therefore we have $(PT)^3 = m\sqrt{m}I$ if and only if

$$\alpha_0^3 = \sqrt{m}\zeta^{(\frac{m+1}{2})^3(2s-1)^2} \Big/ \sum_{l=0}^{m-1} \zeta^{(\frac{m+1}{2})l^2}.$$

This completes the proof for m odd. (To obtain Theorem 1 with m odd for Q instead of P take $s = \frac{m+1}{2}$.)

If m is even, then, by Proposition 4, $\eta = {\eta_0}^{1+2s}$ with some $s \in \{0, 1, \ldots, m-1\}$ and η_0 satisfying ${\eta_0}^2 = \zeta$. Then by Proposition 4,

$$\begin{aligned}
&(i, j)\text{-entry of } (PT)^3 \\
&= \alpha_0^3 {\eta_0}^{j^2+2sj} \sum_{l=0}^{m-1} \sum_{k=0}^{m-1} \zeta^{i(l+k)+k(j-i)+s(l+k)} {\eta_0}^{(l+k)^2}.
\end{aligned}$$

By Proposition 1,

$$\begin{aligned}
&(i, j)\text{-entry of } (PT)^3 \\
&= \alpha_0^3 \eta_0^{j^2+2sj} \sum_{k=0}^{m-1} \zeta^{k(j-i)} \sum_{l=0}^{m-1} \zeta^{il+sl} \eta_0^{l^2} \\
&= \delta_{i,j} m \alpha_0^3 {\eta_0}^{-s^2} \sum_{l=0}^{m-1} \eta_0^{l^2}.
\end{aligned}$$

Therefore we have $(PT)^3 = m\sqrt{m}I$ if and only if

$$\alpha_0^3 = \sqrt{m}\eta_0^{s^2} \Big/ \sum_{l=0}^{m-1} \eta_0^{l^2}.$$

This completes the proof for m even, and Theorem 2 has been completely proved. (To obtain Theorem 1 with m even for Q instead of P take $s = \frac{m}{2}$ or $s = 0$.)

□

1.2. Proofs of Theorem 3 and Theorem 4

Let α_i, $i = 0, 1, \ldots, m-1$ be the complex numbers defined either in Theorem 1 or Theorem 2. Let t_0 be a complex number satisfying $t_0^2 = \alpha_0^3$ and $t_i = \alpha_i t_0 \alpha_0^{-1}$, $i = 0, 1, \ldots, m-1$. Let $W_+ = \sum_{i=0}^{m-1} t_i A_i$ and $W_- = \sum_{i=0}^{m-1} t_{i'}^{-1} A_i$ where $A_{i'} = {}^t A_i$. In this section we will show that (G_m, W_+, W_-) is a spin model.

Let $\mathbf{T}$ and $\mathbf{T}_-$ be m-dimensional column vectors whose i-th entries are t_i and t_i^{-1} ($i = 0, 1, \ldots, m-1$) respectively.

Proposition 6.

(i) $P\mathbf{T} = \sqrt{m}\mathbf{T}_-$,

(ii) $\overline{P}\mathbf{T}_- = \sqrt{m}\mathbf{T}$.

Proof: Since $\overline{P}P = mI$, (ii) is obtained from (i) immediately. In order to prove (i), first let $\mathbf{T}$ and $\mathbf{T}_-$ be defined using α_i ($i = 0, 1, \ldots, m-1$) given in Theorem 1. In this case we have

$$\alpha_i = \alpha_{i'} \text{ for } i = 0, 1, \ldots, m-1. \tag{6}$$

Since

$$(PT)^3 = \sqrt{m}P^2$$

for any $i, j \in \{0, 1, \ldots, m-1\}$ we have

$$\sum_{l=0}^{m-1}\sum_{k=0}^{m-1} \zeta^{il+lk+kj} \alpha_l \alpha_k \alpha_j = \delta_{ij'} m^{\frac{3}{2}}.$$

Since $\alpha_j \neq 0$, we get

$$\sum_{j=0}^{m-1}\sum_{l=0}^{m-1}\sum_{k=0}^{m-1} \zeta^{il+lk+kj} \alpha_l \alpha_k = \sum_{j=0}^{m-1} \delta_{ij'} \alpha_j^{-1} m^{\frac{3}{2}}$$

and then we have

$$m \sum_{l=0}^{m-1} \zeta^{il} \alpha_l \alpha_0 = \alpha_{i'}^{-1} m^{\frac{3}{2}}.$$

Therefore by (6) we have

$$\sum_{l=0}^{m-1} \zeta^{il} \alpha_l \alpha_0 = \sqrt{m} \alpha_i^{-1}$$

and then

$$\sum_{l=0}^{m-1} \zeta^{il} t_l = \sqrt{m} t_i^{-1}$$

by the definition of t_l ($l = 0, 1, \ldots, m-1$). The left hand side of the last equation is the i-th entry of the vector $P\mathbf{T}$.

Now let $\mathbf{T}$ and $\mathbf{T}_-$ be defined using α_i ($i = 0, 1, \ldots, m-1$) given in Theorem 2. Since $(PT)^3 = m^{\frac{3}{2}} I$ in this case, we have

$$\sum_{l=0}^{m-1} \sum_{k=0}^{m-1} \zeta^{il+lk+kj} \alpha_l \alpha_k \alpha_j = \delta_{i,j} m^{\frac{3}{2}}$$

for any $i, j \in \{0, 1, \ldots, m-1\}$. By a similar argument as above we obtain $P\mathbf{T} = \sqrt{m}\mathbf{T}_-$. □

Proposition 7. *W_+ and W_- satisfy the conditions* (1) *and* (2) *of Definition* 1 *for spin models.*

Proof: By the definition of W_+ and W_-, (1) of Definition 1 is clear. For (2)

$$A_i = \sum_{l=0}^{m-1} \zeta^{il} E_l, \quad A_{j'} = \sum_{l=0}^{m-1} \zeta^{-jl} E_l, \quad E_l E_k = \delta_{l,k} E_l,$$

implies

$$W_+ W_- = \sum_{l=0}^{m-1} \sum_{i=0}^{m-1} t_i \zeta^{il} \sum_{j=0}^{m-1} t_j{}^{-1} \zeta^{-jl} E_l.$$

Hence by Proposition 6 we have

$$W_+ W_- = mI.$$ □

To show that (G_m, W_+, W_-) is a spin model we only need to show the star-triangle relation.

Proposition 8. *The star-triangle relation ((3) of Definition 1) is equivalent to the following equations.*

$$(t_{i'} - t_{a'})E_i Y_{\alpha\gamma} = \mathbf{0}$$

for any $i \in \{0, 1, \dots, m-1\}$ *and* $\alpha, \gamma \in G_m$ *with* $(\alpha, \gamma) \in R_a$.

Proof: In Section 0.1 we mentioned that the star-triangle relation (with $D = \sqrt{m}$) is equivalent to

$$^tW_+ Y_{\alpha\gamma} = \sqrt{m}\, w_-(\alpha, \gamma) Y_{\alpha\gamma} \tag{7}$$

for any $\alpha, \gamma \in G_m$. (Note that $w_-(\alpha, \gamma) = t_{a'}{}^{-1}$ for $(\alpha, \gamma) \in R_a$.) By the definition of W_+ and (i) of Proposition 6 (multiplied on the left by $m^{-1}P^2$)

$$\begin{aligned} {}^tW_+ Y_{\alpha\gamma} &= \sum_{i=0}^{m-1} t_{i'} A_i Y_{\alpha\gamma} \\ &= \sum_{i=0}^{m-1} t_{i'} \sum_{l=0}^{m-1} \zeta^{il} E_l Y_{\alpha\gamma} \\ &= \sqrt{m} \sum_{l=0}^{m-1} t_{l'}{}^{-1} E_l Y_{\alpha\gamma}. \end{aligned}$$

Therefore (7) is equivalent to

$$\left(\sum_{l=0}^{m-1} t_{l'}{}^{-1} E_l - t_{a'}{}^{-1} \right) Y_{\alpha\gamma} = \mathbf{0}. \tag{8}$$

Since $E_i E_l = \delta_{i,l} E_i$ for any $i \in \{0, 1, \dots, m-1\}$ and $\sum_{l=0}^{m-1} E_l = I$, (8) is equivalent to

$$(t_{i'}{}^{-1} - t_{a'}{}^{-1}) E_i Y_{\alpha\gamma} = \mathbf{0}, \quad \text{for any } i \in \{0, 1, \dots, m-1\}.$$

This proves Proposition 8. □

Proposition 9. *Let* W_+ *be as in Theorem 3 or Theorem 4. Then* $E_i Y_{\alpha\gamma} = 0$ *if and only if* $a \neq i$, *where* $(\alpha, \gamma) \in R_a$.

Proof: By Proposition 5 it is enough to prove the case of Theorem 4. Since $E_i = \frac{1}{m} \sum_{j=0}^{m-1} \zeta^{-ij} A_j$, $E_i Y_{\alpha\gamma} = \mathbf{0}$ if and only if $\sum_{j=0}^{m-1} \zeta^{-ij} A_j Y_{\alpha\gamma} = \mathbf{0}$. Let $(A_j)_{x,y}$ and $(\mathbf{v})_y$ be the (x, y)-entry of A_j and y-entry of $\mathbf{v}$ respectively, where $\mathbf{v}$ is a column vector of dimension m and $x, y \in G_m$. Then

$$(A_j Y_{\alpha\gamma})_x = \sum_{y \in G_m} (A_j)_{x,y} (Y_{\alpha\gamma})_y$$

$$= \sum_{y \in G_m} (A_j)_{x,y} w_+(\alpha, y) w_-(y, \gamma).$$

Since $w_+(\alpha, y) = t_k$ with $(\alpha, y) \in R_k$ and $w_-(y, \gamma) = t_{l'}{}^{-1}$ with $(y, \gamma) \in R_l$ we have

$$(A_j Y_{\alpha\gamma})_x = \sum_{l=0}^{m-1} \sum_{k=0}^{m-1} p_{jkl}(x, \alpha, \gamma) t_k t_{l'}^{-1},$$

where $p_{jkl}(x, \alpha, \gamma) = \sharp\{y \mid (x, y) \in R_j,\ (\alpha, y) \in R_k,\ (y, \gamma) \in R_l\}$. Let $(x, \alpha) \in R_b$. Then by the definition of the relations $\{R_l\}_{0 \le l \le m-1}$ we have

$$\begin{aligned} p_{jkl}(x, \alpha, \gamma) &= 1 \text{ if } k \equiv j - b\,(m),\ l \equiv a + b - j\,(m), \\ p_{jkl}(x, \alpha, \gamma) &= 0 \text{ otherwise.} \end{aligned} \tag{9}$$

Hence

$$(A_j Y_{\alpha\gamma})_x = t_k t_{l'}^{-1} \text{ with } k \equiv j - b\,(m) \text{ and } l \equiv a + b - j\,(m).$$

Therefore

$$\left(\sum_{j=0}^{m-1} \zeta^{-ij} A_j Y_{\alpha\gamma} \right)_x = \sum_{j=0}^{m-1} \zeta^{-ij} t_k t_{l'}^{-1}, \tag{10}$$

where $k \equiv j - b(m)$ and $l \equiv a + b - j(m)$.

Now we are ready to show Proposition 9.

(i) If W_+ is as in Theorem 4 and m is odd, then $t_l = \zeta^{\frac{l(l+2s-1)}{2}} t_0$, $0 \le l \le m-1$ with some $s \in \{0, 1, \ldots, m-1\}$. Therefore by (9) we have

$$\left(\sum_{j=0}^{m-1} \zeta^{-ij} A_j Y_{\alpha\gamma} \right)_x = \zeta^{-bi} \zeta^{\frac{-a^2 + a(2s-1)}{2}} \sum_{j=0}^{m-1} \zeta^{j(a-i)}$$

for any $x \in G_m$, where $(x, \alpha) \in R_b$. The right hand side equals 0 if and only if $(a - i) \not\equiv 0\,(m)$.

(ii) If W_+ is as in Theorem 4 and m is even, then $t_l = \eta^{l(l+2s)} t_0$ with $\eta^2 = \zeta$ and $s \in \{0, 1, \ldots, m-1\}$. Therefore by (9) we have

$$\left(\sum_{j=0}^{m-1} \zeta^{-ij} A_j Y_{\alpha\gamma} \right)_x = \zeta^{-bi} \eta^{2sa - a^2} \sum_{j=0}^{m-1} \zeta^{(a-i)j}$$

for any $x \in G_m$, where $(x, \alpha) \in R_b$. The right hand side of this equation equals 0 if and only if $a - i \not\equiv 0\,(m)$. Since $a, i \in \{0, \ldots, m-1\}$, $a \equiv i(m)$ is equivalent to $a = i$. Hence in any of the cases, $E_i Y_{\alpha\gamma} = 0$ if and only if $a \neq i$. This completes the proof of Proposition 9.

□

Proof of Theorem 3 and Theorem 4. By Proposition 9, clearly we obtain $(t_{i'} - t_{a'}) E_i Y_{\alpha\gamma} = \mathbf{0}$ for any $i \in \{0, 1, \ldots, m-1\}$ and $\alpha, \gamma \in G_m$ with $(\alpha, \gamma) \in R_a$. Therefore, by Proposition 8, (G_m, W_+, W_-) satisfies the star-triangle equation. Together with Proposition 7 we can complete the proof of Theorem 3 and Theorem 4.

2. Concluding remarks

Remark 7. Constructions of spin models for finite cyclic groups given in Theorem 3 and Theorem 4 have obvious generalizations for constructions of spin models for finite abelian groups. Generally, let $(X_i, (W_i)_+, (W_i)_-)$, for $i = 1, 2$, be (generalized) spin models. Then it is easy to see that the triple $(X_1 \times X_2, (W_1)_+ \otimes (W_2)_+, (W_1)_- \otimes (W_2)_-)$ is also a spin model which is called the tensor product of the two previous models. (This is well known, see e.g., Kawagoe, Munemasa and Watatani [13] or de la Harpe [4].) Since any finite abelian group is a direct product of cyclic groups, we shall obtain a spin model by assigning one of the spin models constructed in Theorem 3 and Theorem 4 to each cyclic factor of the abelian group. It seems to be an interesting question to know how far, in general, the spin models for an abelian group are different from the ones obtained this way, i.e., as the tensor product of spin models constructed in Theorem 3 and Theorem 4 for each of the cyclic factors.

Remark 8. Although Theorem 1 and Theorem 2 give the complete characterization of the matrices T satisfying the modular invariance property or quasi modular invariance property, the complete characterization of spin models on the cyclic groups $G = G_m$ with $W_+ = \sum_{i=0}^{m-1} t_i A_i$ is not yet determined, even for odd primes m. It would be interesting to know the answer to this question. Also, it would be interesting to study constructions (or determinations) of more general types of spin models (*cf.* [2]) on finite cyclic groups.

Remark 9. The idea of using association schemes to construct spin models is due to Jaeger [8]. We remark that we owe Jaeger [8] for some ideas of the proofs in the present paper.

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