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1/2-Transitive Graphs of Order 3*p*

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Abstract. A graph X is called *vertex-transitive, edge-transitive*, or *arc-transitive*, if the automorphism group of X acts transitively on the set of vertices, edges, or arcs of X, respectively. X is said to be *1/2-transitive*, if it is vertex-transitive, edge-transitive, but not arc-transitive.

In this paper we determine all 1/2-transitive graphs with $3p$ vertices, where p is an odd prime. (See Theorem 3.4.)

Keywords: 1/2-transitive graph, metacirculant, factor graph

1. Introduction

Let $X = (V(X), E(X))$ be a graph (that is, no multiple edges or loops). We call an ordered pair of adjacent vertices an *arc* of X. The set of all arcs associated with a graph X is denoted by $A(X)$. Thus, $|A(X)| = 2|E(X)|$. If G is a subgroup of Aut$X$ and G acts transitively on the set of vertices, edges, or arcs of X, then X is said to be *G-vertex-transitive, G-edge-transitive*, or *G-arc-transitive*, respectively. The graph X is said to be *vertex-transitive, edge-transitive*, or *arc-transitive*, if it is Aut$X$-vertex-transitive, Aut$X$-edge-transitive, or Aut$X$-arc-transitive, respectively. We call a graph X *1/2-transitive*, if it is vertex-transitive, edge-transitive, but not arc-transitive.

D. Marušič, L. Nowitz and the first author of this paper studied 1/2-transitive graphs [1] and found several infinite families of such graphs. In [7], R.J. Wang and the second author gave a classification of arc-transitive graphs of order $3p$, where p is a prime. The purpose of this paper is to determine all 1/2-transitive graphs of order $3p$.

We use standard terminology and notation for the most part and refer the reader to [5, 6, 8] if necessary. For $v \in V(X)$, $X_1(v)$ denotes the neighborhood of v in X, that is, the set of vertices adjacent to v in X. If X is a graph and A and B are two vertex-disjoint subsets of the vertex-set $V(X)$ of X, we let $\langle A\rangle$ and $\langle A, B\rangle$ denote the subgraph induced on A and the bipartite subgraph, with bipartition sets A and B, induced on $A \cup B$ by X, respectively. We remind the reader that two representations of a group G as transitive permutation groups are said to be *equivalent* if the pointwise stabilizers of one representation are conjugate in G to the pointwise stabilizers of the other representation.

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Assume that G acts on X imprimitively and that B_0 is a nontrivial block of G. Let $\overline{X} = \{B_0, B_1, \ldots, B_{n-1}\}$ be the complete block system of G containing B_0. We define the *factor graph* of X corresponding to $\overline{X}$, which is still denoted by $\overline{X}$, by

$$V(\overline{X}) = \overline{X},$$
$$E(\overline{X}) = \{B_iB_j\colon \text{ there exist } v_i \in B_i, v_j \in B_j \text{ such that } v_iv_j \in E(X)\}. \quad (1)$$

The group G induces an action on $\overline{X}$. Assume that the kernel of this action is K. Set $\overline{G} = G/K$. Then $\overline{G}$ acts on $\overline{X}$ faithfully. The proof of the following result is immediate and is omitted.

Proposition 1.1 *Let X be a $1/2$-transitive graph such that* Aut$X$ *acts imprimitively on X. Then*

(1) *$\overline{X}$ is $1/2$-transitive or arc-transitive;*

(2) *if X is connected, then so is $\overline{X}$; and*

(3) *if $\langle B_i \rangle$ has an edge, then B_i is a union of several connected components of X.*

Given two graphs X and Y the *wreath product* $X \wr Y$ is defined as the graph with vertex set $V(X) \times V(Y)$ such that $(x, y)(x', y')$ is an edge if and only if either xx' is an edge of X, or $x = x'$ and yy' is an edge of Y.

Informally $X \wr Y$ is obtained by taking $|V(X)|$ copies of Y, labelling these copies with the vertices of X, and, whenever xx' is an edge of X, joining each vertex in the copy of Y labelled x to each vertex in the copy of Y labelled x'. The automorphism group of $X \wr Y$ contains the *wreath product* Aut$Y$ wr Aut X (but may be larger).

The next obvious proposition gives a method of constructing larger 1/2-transitive graphs from smaller ones.

Proposition 1.2 *If $\overline{X}$ is a $1/2$-transitive graph of order n, then the wreath product $\overline{X} \wr mK_1$ of $\overline{X}$ by mK_1 is a $1/2$-transitive graph of order nm.*

Next we quote two propositions from [1].

Proposition 1.3 *Every vertex- and edge-transitive Cayley graph on an abelian group is also arc-transitive.*

Proposition 1.4 *Every vertex- and edge-transitive graph with p or $2p$ vertices, p a prime, is also arc-transitive.*

Finally we quote a result from [7].

Proposition 1.5 *Let X be an arc-transitive graph of order $3p$ and $A =$ Aut$X$. If A has a block of imprimitivity of length p, then X is a Cayley graph on a cyclic group Z_{3p}.*

2. Reductions

In this section we eliminate some possible types of 1/2-transitive graphs of order $3p$. There is no 1/2-transitive graph of order less than 27 as observed in [1]. All suborbits of a primitive group of degree $3p$ are self-paired [7], which implies there are no vertex-primitive 1/2-transitive graphs of order $3p$. So we may assume that $p \geq 11$ in what follows, and we only need consider those 1/2-transitive graphs with an imprimitive automorphism group.

Let X be a 1/2-transitive graph and $A = \mathrm{Aut}X$. We have two cases: (1) A has a block of length p, and (2) A has a block of length 3, but no blocks of length p. We shall show in the next section that the latter case cannot occur.

Assume that $\overline{X} = \{B_i \mid i \in Z_3\}$ is a complete block system of A, and that K is the kernel of the action of A on $\overline{X}$. Set $\overline{A} = A/K$. We also use $\overline{X}$ to denote the corresponding factor graph. Then $\overline{X} \cong K_3$. Since X is 1/2-transitive, X is not isomorphic to $K_{p,p,p}$.

Lemma 2.1 *The kernel K acts faithfully on each block B_i.*

Proof: We use K_{B_i} to denote the pointwise-stabilizer of B_i in K. If K acts unfaithfully on B_i, we have $K_{B_i}^{B_j}$ is nontrivial for some block B_j adjacent to B_i. Since $K_{B_i} \lhd K$, $K_{B_i}^{B_j} \lhd K^{B_j}$. Since $|B_j| = p$, K^{B_j} is primitive which implies $K_{B_i}^{B_j}$ is transitive. It follows that the induced subgraph $\langle B_i, B_j \rangle \cong K_{p,p}$. The edge-transitivity of $\overline{X}$ implies that $X \cong K_{p,p,p}$. The latter is impossible since $K_{p,p,p}$ is arc-transitive. □

Using the same method as above we can prove the following more general result, which will be used in the next section.

Lemma 2.2 *Suppose that X is a connected $1/2$-transitive graph and $\overline{X} = \{B_1, B_2, \ldots, B_n\}$ is a complete block system of $A = \mathrm{Aut}X$. If K, the kernel of the action of A on $\overline{X}$, acts on B_i unfaithfully and K^{B_i} is primitive, then X is isomorphic to the wreath product $\overline{X} \wr mK_1$ of $\overline{X}$ with mK_1, where $m = |B_i|$.*

By virtue of Lemma 2.1, we may assume that K acts on each B_i faithfully in what follows. Since $|B_i| = p$, we have two cases: (1) K acts on B_i doubly-transitively, and (2) K acts on B_i simply primitively. We shall treat these two subcases next.

Lemma 2.3 *Let X be a connected vertex- and edge-transitive graph of order $3p$, and $\overline{X} = \{B_0, B_1, B_2\}$ be a complete block system of $A = \mathrm{Aut}X$. If K, the kernel of the action of A on $\overline{X}$, acts doubly transitively on a block, then X is arc-transitive. That is, there are no $1/2$-transitive graphs of order $3p$ for which K acts doubly transitively on a block.*

Proof: By the classification of 2-transitive groups (see [4], for example), it is easy to check that every 2-transitive group has at most two non-equivalent 2-transitive representations. So, without loss of generality, we may assume that K^{B_0} and K^{B_1} are equivalent. Since $\overline{X}$ is edge-transitive, all three groups K^{B_0}, K^{B_1} and K^{B_2} are equivalent to each other. Hence, for any vertex $v_0 \in B_0$, the stabilizer K_{v_0} must fix a vertex v_1 in B_1, and a vertex

v_2 in B_2. By the transitivity of K_{v_1} on $B_1 - \{v_1\}$, either v_0 is adjacent to every vertex in B_1, every vertex in $B_1 - \{v_1\}$, or only v_1. Thus, the induced bipartite graph $\langle B_0, B_1\rangle$ is either $K_{p,p}$, $K_{p,p}$ minus a 1-factor, or a 1-factor. By the edge-transitivity of $\overline{X}$, X is either $K_{p,p,p}$, $K_{p,p,p}$ minus pK_3, or of degree 2. In all these cases, X is arc-transitive. □

We now consider the case that K acts on B_i simply primitively. Then $K^{B_i} < AGL(1,p)$ is solvable. So K has only one transitive representation of degree p, and for any $v \in B_i$, $K_v < Z_{p-1}$ is semiregular on $B_i - \{v\}$. Furthermore, the Sylow p-subgroup of K^{B_i} is normal in K^{B_i}. Since K acts on B_i faithfully, the Sylow p-subgroup P of K is normal, and then is characteristic, in K, implying $P \trianglelefteq A$. Since $|P| = p$, P is cyclic. We will use this information later.

In the next proposition, we determine the factor group $\overline{A} = A/K$.

Proposition 2.4 *If X is a 1/2-transitive graph of order $3p$ with three blocks of length p, then $\overline{A} \cong Z_3$.*

Proof: By Proposition 1.4, there is no 1/2-transitive graph of prime order, which implies that $\overline{X}$ is connected. By Proposition 1.1, $\overline{X} \cong K_3$ so that $\overline{A} \cong S_3$ or Z_3. Assume that $\overline{A} \cong S_3$. Let P be the Sylow p-subgroup of K. Then P is normal in A and is cyclic of order p by the information above. Put $C = C_A(P)$. We have A/C is isomorphic to a subgroup of $\mathrm{Aut}P = Z_{p-1}$, so that $A' \leq C$. Since $A/K \cong S_3$, $A'K/K \cong Z_3$. Hence 3 divides $|A'|$, and then 3 divides $|C|$. Assume that $P = \langle g\rangle$. Take $h \in C$ with $o(h) = 3$. Set $H = \langle g, h\rangle$. Since $h \in C = C_A(P)$, h and g commute. Then $H \cong Z_3 \times Z_p \cong Z_{3p}$ is a regular subgroup of A. By [3, Lemma 16.3], X is a Cayley graph of Z_{3p}. Finally, by Proposition 1.3, X is arc-transitive, a contradiction. □

Now we give an example via the next theorem. We need the concept of metacirculant defined in [2].

Let $n \geq 2$. A permutation on a finite set is said to be (m,n)-*semiregular* if it has m cycles of length n in its disjoint cycle decomposition. We shall be sloppy and refer to the orbits of the group $\langle\alpha\rangle$ generated by α as the orbits of α. A graph X is an (m,n)-*metacirculant* if it has an (m,n)-semiregular automorphism α together with another automorphism β normalizing α and cyclically permuting the orbits of α. Therefore, we may partition the vertex-set of an (m,n)-metacirculant into the orbits $B_0, B_1, \ldots, B_{m-1}$ of α, where $B_i^\beta = B_{i+1}$ for all $i \in Z_m$. We shall refer to the orbits of α as the *blocks* of the metacirculant graph. It should be pointed out that the blocks of a metacirculant graph need not be blocks of imprimitivity of the automorphism group of the graph.

Recall that a *circulant* graph is a Cayley graph on a cyclic group. Using additive notation for the underlying cyclic group, the *symbol* S of a circulant is defined by $S = \{j\colon u_0u_j$ is an edge of the circulant graph$\}$. If $S_0 \subseteq Z_n\backslash\{0\}$ is the symbol of the subcirculant $\langle B_0\rangle$ and, for all $i \in Z_m\backslash\{0\}$, $T_i \subseteq Z_n$ is the symbol of the bipartite subgraph $\langle B_0, B_i\rangle$, then there exists an $r \in Z_n^*$, where Z_n^* denotes the multiplicative group of units in Z_n, such that for all $j \in Z_m$, the symbol of $\langle B_j\rangle$ is r^jS_0 and the symbol of the bipartite graph $\langle B_j, B_{j+i}\rangle$, $i \in Z_m$, is r^jT_i. Moreover, for all $i \in Z_m$, we have $T_{m-i} = r^{m-i}(-T_i)$. Thus, the metacirculant graph X is completely determined by the $\lfloor(m+4)/2\rfloor$-tuple $(r; S_0, T_1, T_2, \ldots, T_{\lfloor m/2\rfloor})$ which is called a *symbol* of X. (For a more detailed discussion of metacirculants, the reader is referred to [2].)

Now let p be a prime and $p \equiv 1 (\text{mod } 3)$. Assume that u is an element of order 3 in Z_p^*. We use H_{3p} to denote the unique non-abelian group of order $3p$, that is,

$$H_{3p} = \langle \alpha, \beta \mid \alpha^p = 1, \beta^3 = 1, \alpha^\beta = \alpha^u \rangle.$$

Definition Let p be a prime with $p \equiv 1(\text{mod } 3)$. Let $d > 1$ be a divisor of $(p-1)/3$. Let $T = \langle t \rangle$ be the subgroup of Z_p^* of order d. Let $r \in Z_p^* \backslash T$ be a 3-element with $r^3 \in T$. We use $M(d; 3, p)$ to denote the $(3, p)$-metacirculant graph with symbol $(r; \emptyset, T)$.

Theorem 2.5 *If $(d, p) \neq (2, 7)$ or $(3, 19)$, then the graph $M(d; 3, p)$ is a 1/2-transitive graph of order $3p$ and of degree $2d$. This graph is independent of the choice of r. The automorphism group $A = AutM(d; 3, p)$ is isomorphic to a semidirect product of Z_p and Z_{3d}, and A acts regularly on the edge set of $M(d; 3, p)$.*

Proof: Checking the vertex-primitive graphs of order $3p$ listed in [7], we know that $M(2; 3, 7)$ and $M(3; 3, 19)$ are the only vertex-primitive $(3, p)$-metacirculants and both of them are arc-transitive. Suppose now that $p \geq 11$ and $d \neq 3$ if $p = 19$.

Assume that $B_i = \{x_j^i \mid j = 0, 1, \ldots, p-1\}$, $i = 0, 1, 2$, are the three blocks of $X = M(d; 3, p)$ as a metacirculant. It is easy to see that the following mappings α, β and γ are automorphisms of X:

$$\alpha: x_j^i \mapsto x_{j+1}^i$$
$$\beta: x_j^i \mapsto x_{rj}^{i+1}$$
$$\gamma: x_j^i \mapsto x_{tj}^i.$$

Assume that $3^e \| d$. Then $o(\alpha) = p$, $o(\gamma) = d$ and $o(\beta) = 3^{e+1}$. Set $P = \langle \alpha \rangle$, $L = \langle \alpha, \gamma \rangle$, $M = \langle \beta, \gamma \rangle$ and $G = \langle \alpha, \gamma, \beta \rangle$. We can see that $P \lhd G$, G is a semidirect product of P and M, and the centralizer of P in G is P itself. Thus, $M \cong G/P$ is isomorphic to a subgroup of $\mathrm{Aut}P \cong Z_{p-1}$, so that in particular, $M = \langle \beta\gamma \rangle$ is cyclic. Also it is easy to see that X is G-vertex-transitive and G-edge-transitive.

To prove that X is not arc-transitive, first we claim that A has a block of length p. If not, A is either primitive, or imprimitive but only has blocks of length 3 on the vertex set of X. By the reason mentioned at the beginning of the proof, assuming that $p \geq 11$ and $d \neq 3$ for $p = 19$, we have that X is vertex-imprimitive and A has only blocks of length 3. By a result in the next section, there are no 1/2-transitive graphs having this property, so that X must be arc-transitive. By a result in [7] the only arc-transitive graphs, which are not vertex-primitive and whose automorphism groups do not have a block of length p, have automorphism groups A, with $PSL(2, 2^{2^s}) \leq A \leq P\Gamma L(2, 2^{2^s})$, and $p = 2^{2^s} + 1$ being a Fermat prime. In this case 3 does not divide $p - 1 = 2^{2^s}$, so this case cannot occur.

We have proved that A has a block of length p. Since the only blocks of length p of G, which is a subgroup of A, are B_i, $i = 0, 1, 2$, they must be blocks of A too. Let K be the kernel of A acting on $\overline{X} = \{B_0, B_1, B_2\}$. By the same argument as in the proof of Lemma 2.3, we know that K is not doubly-transitive on B_i. This implies that the Sylow p-subgroup of K, which is P defined above and generated by α, is normal in A. Assume that X is arc-transitive. Noting that X is not isomorphic to the multipartite complete graph $K_{p,p,p}$, by Theorem 3 in [7], $X \cong G(3p, d)$ defined in [7]. By Example 3.4 in [7], $A = \mathrm{Aut}G(3p, d) \cong (Z_p.Z_d).S_3$, where $G.H$ denotes an extension of G by H, and A contains a cyclic subgroup of order $3p$. It follows that the order of a Sylow 3-subgroup of A is 3^{e+1}, where $3^e \| d$, and that the centralizer of the Sylow p-subgroup P contains an element of order 3. Since $o(\beta) = 3^{e+1}$, $\langle\beta\rangle$ is the Sylow 3-subgroup of A. It follows that β^{3^e} and α commute. However, it is not the case, a contradiction. This shows that X is 1/2-transitive as required.

It is not difficult to show that different choices of r correspond to isomorphic graphs. We leave this as an exercise for the reader.

Now we determine the automorphism group $A = \mathrm{Aut}M(d; 3, p)$. Since A is an extension of the kernel K by Z_3, and K is an extension of P by the stabilizer K_v of $v = x_0^0$ in K, it is easy to see that $K_v \cong T$. This shows that $K \cong L$ defined before. Note that β is not in K and is a 3-element, implying that $A = \langle K, \beta\rangle = \langle L, \beta\rangle = G$, as desired. It follows that $A = G \cong Z_p.Z_{3d}$ acts regularly on the edge set of X.

(Note that if $3 \not| \, d$, then $M(d; 3, p)$ is a Cayley graph on H_{3p} with respect to $S = \{\beta\alpha^i \mid i \in T\} \cup \{\beta^2\alpha^{-u^2 i} \mid i \in T\}$, while if $3 \mid d$, $M(d; 3, p)$ is not a Cayley graph.)

□

Theorem 2.6 *Let X be a $1/2$-transitive graph of order $3p$. If* Aut$X$ *acts imprimitively on $V(X)$ and has a block of length p, then X is isomorphic to $M(d; 3, p)$ for some divisor d of $\frac{p-1}{3}$, where $(d, p) \neq (2, 7)$ or $(3, 19)$.*

Proof: Assume that $\overline{X} = \{B_0, B_1, B_2\}$ is a complete block system of A =Aut$X$ and that K is the kernel of A acting on $\overline{X}$.

(1) We claim that X is connected. If not, every connected component has either p or 3 vertices, and is also 1/2-transitive. But by Proposition 1.4, there are no 1/2-transitive graphs with a prime number of vertices.

(2) It follows from Proposition 1.1 and Proposition 2.4 that there are no edges in any induced subgraph $\langle B_i\rangle$, the factor graph $\overline{X}$ is a triangle, and the factor group $\overline{A} = A/K$ is isomorphic to Z_3.

(3) By the information preceding Proposition 2.4 we have that the Sylow p-subgroup P of K is cyclic and normal in A.

(4) We claim that X is a $(3, p)$-metacirculant. Let $P = \langle\alpha\rangle$. Then α is a $(3, p)$-semiregular automorphism of X. Since $A/K \cong Z_3$, any element $\beta \in A \backslash K$ permutes $\overline{X} = \{B_0, B_1, B_2\}$ cyclically. Replacing it by its suitable power, we may assume that β is a 3-element. Hence, by definition, X is a $(3, p)$-metacirculant. Assume that $\alpha^\beta = \alpha^r$. Then r is a 3-element in $Z_p^* \cong Z_{p-1}$.

Now we may label the vertices of X as follows: for $i = 0, 1, 2$, let $B_i = \{x_0^i, x_1^i, \ldots, x_{p-1}^i\}$, and we may assume that ${x_j^i}^\alpha = x_{j+1}^i$ and ${x_j^i}^\beta = x_{rj}^{i+1}$ for all i and j.

(5) Finally, we claim that $X \cong M(d;3,p)$ for a divisor $d > 1$ of $\frac{p-1}{3}$. Since there are no edges in $\langle B_i \rangle$ for any i, X has a symbol of the form $(r;\emptyset,S)$. Since X has an odd number of vertices, the degree of X is even, say $2d$. Fix a vertex $v = x_0^0$. The neighborhood of v in X is $X_1(v) = X_1^{B_1}(v) \cup X_1^{B_2}(v)$, where $X_1^{B_i}(v) = X_1(v) \cap B_i$.

Consider the stabilizer A_v of v in A. Since $A/K \cong Z_3$, A_v fixes B_i setwise for each i. So $A_v = K_v$. Since K is solvable, K has only one permutation representation of degree p, and K is a Frobenius group or $K = P$. So K_v must fix one vertex in B_1 and one vertex in B_2. Without loss of generality, we may assume that K_v fixes $v_1 = x_0^1$ in B_1 and $v_2 = x_0^2$ in B_2. By the edge-transitivity of X, A_v has two orbits in $X_1(v)$, which must be $X_1^{B_1}(v) = \{x_j^1 \mid j \in S\}$ and $X_1^{B_2}(v) = \{x_j^2 \mid j \in -r^2S\}$. Since $A_v = K_v$, the action of A_v on B_1 is equivalent to the action of K_v on B_1, and then to the action of K_{v_1} on B_1. Since K^{B_1} is a Frobenius group, the subscripts of the vertices in $X_1^{B_1}(v)$, which is an orbit of K_{v_1}, is a coset of a subgroup of Z_{p-1} of order d, say aT, where $T \leq Z_{p-1}$, $|T| = d$ and $a \neq 0$. So we have proved d is a divisor of $p-1$. If $d = 1$, then X has degree 2, contradicting the fact that X is not arc-transitive. So $d > 1$. If d does not divide $\frac{p-1}{3}$, then $r \in T$. Set $\nu: x_j^i \mapsto x_{r^{-1}j}^i$ for all i and j. Then $\nu \in A$, and $\beta\nu$ maps x_j^i to x_j^{i+1}. Thus $\beta\nu$ is an automorphism of X of order 3 which commutes with α. This implies that $\langle \alpha\beta\nu \rangle$ is a regular subgroup of A which is isomorphic to Z_{3p}. By Proposition 1.5, X is arc-transitive which is a contradiction. So we have $d | \frac{p-1}{3}$. Finally, noticing that two metacirculants with symbols $(r;\emptyset,T)$ and $(r;\emptyset,aT)$ are isomorphic, we have the desired result. □

3. *A* has a block of length 3

The results of the previous section leave us with the case where A has a block of length 3 and no blocks of length p. We assume that $\overline{X} = \{B_i \mid i \in Z_p\}$ is a complete block system of A, and that K is the kernel of the action of A on $\overline{X}$. Set $\overline{A} = A/K$. We also use $\overline{X}$ to denote the corresponding block graph.

By Lemma 2.2, if K acts on B_i unfaithfully, then $X \cong \overline{X} \wr 3K_1$, where $\overline{X}$ is an arc-transitive graph of order p. Thus, X is also arc-transitive, so that we may assume that K acts on each B_i faithfully. Thus, we have $K \cong S_3$, or $K \cong Z_3$, or $K = 1$. We also know that there are no edges inside any B_i.

Lemma 3.1 *The group $\overline{A}$ is insolvable, and the graph $\overline{X}$ is isomorphic to K_p.*

Proof: If $\overline{A}$ is solvable, then $\overline{A}$ has a normal subgroup $\overline{H} = H/K$ of order p since $\overline{A}$ is of degree p. Let $P \in \mathrm{Syl}_p(H)$. Then it is easy to check that $P \lhd H$, and hence $P \lhd A$. So A has a block of length p, contradicting our assumption.

Since $\overline{A}$ is insolvable and transitive of degree p, the well known theorem of Burnside implies that it is doubly transitive. Since there is no 1/2-transitive graphs of order 3, $\overline{X}$ is connected. Hence, $\overline{X}$ must be isomorphic to K_p. □

Lemma 3.2 *The group $K = 1$.*

Proof: If $K \neq 1$, either $K \cong S_3$ or $K \cong Z_3$. Hence, K has a characteristic subgroup N of order 3 which is normal in A. Put $C = C_A(N)$. Then A/C is isomorphic to a subgroup of Aut$K \cong Z_2$. This implies that every element of order p in A is contained in C. It follows that A has a subgroup isomorphic to Z_{3p}, and this subgroup must be regular. Thererefore, X is a Cayley graph on an abelian group. By Proposition 1.3, X is arc-transitive, which is a contradiction. □

We may now assume that $K = 1$. In this case, $A \cong \overline{A}$ as abstract groups. But as permutation groups, $\overline{A}$ is a group of degree p and A is of degree $3p$. Since $\overline{A}$ is insolvable, it is doubly-transitive as observed above. Then $\overline{A}$ is known by the finite simple group classification.

If G is a doubly-transitive group of degree p, one necessary condition for G to be the automorphism group of a 1/2-transitive graph of order $3p$ (as abstract groups) is that the point stabilizer G_α has a subgroup of index 3. A table of 2-transitive groups of degree p with simple socle is given in [7], and after checking all (insolvable) doubly-transitive groups of degree p listed there, the only possible groups have socle either $PSL(3,2)$, $p = 7$, or $PSL(2, 2^{2^s})$, where $s > 0$ and $p = 2^{2^s} + 1$ is a Fermat prime.

There are no 1/2-transitive graphs with fewer than 27 vertices [1], so we only need to consider the latter case, where the socle is $PSL(2, 2^{2^s})$ and $p = 2^{2^s} + 1$ is a Fermat prime. In this case, noting that $|P\Gamma L(2, 2^{2^s}) : PSL(2, 2^{2^s})| = 2^s$, the stabilizer of $\overline{A}$ having a subgroup of index 3 implies that the stabilizer of $PSL(2, 2^{2^s})$ also has such a subgroup. This is true since $2^{2^s} - 1$ is divisible by 3. Hence $PSL(2, 2^{2^s})$ is vertex-transitive on X.

Lemma 3.3 *If $PSL(2, 2^{2^s}) \leq A \leq P\Gamma L(2, 2^{2^s})$, A is not the automorphism group of any $1/2$-transitive graph.*

Proof: As noted above $PSL(2, 2^{2^s})$ is vertex-transitive. Then since all orbitals are self-paired (see [7]) it follows that any edge-transitive graph X admitting the group is arc-transitive. □

Summarizing the result of Section 2 and Section 3, we get the main theorem of this paper.

Theorem 3.4 *A graph of order $3p$ is $1/2$-transitive if and only if it is a $(3, p)$-metacirculant graph of the form $M(d; 3, p)$, where $(d, p) \neq (2, 7)$ or $(3, 19)$.*

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