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Twisted Extensions of Spin Models

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Abstract. A spin model is one of the statistical mechanical models which were introduced by V.F.R. Jones to construct invariants of links. In this paper, we give a new construction of spin models of size $4n$ from a given spin model of size n. The process is similar to taking tensor product with a spin model of size four, but we add some sign exchange. This construction also gives symmetric four-weight spin models of the type introduced by E. Bannai and E. Bannai.

Keywords: spin model, star-triangle relation

1 Introduction

A spin model is one of the statistical mechanical models which were introduced by V.F.R. Jones in [9] in connection with his invariant 'Jones polynomial' [7, 8]. The main importance of spin models comes from the fact that every spin model gives an invariant of knots and links through its partition function. Precise explanations about spin models and corresponding link invariants can be found in [9, 2, 3].

A spin model is essentially an $n \times n$-matrix W, whose entries $W_{\alpha,\beta}$ are non-zero complex numbers, such that (for $\alpha, \beta, \gamma = 1, \ldots, n$)

$$\sum_{x=1}^{n} \frac{W_{\alpha,x}}{W_{\beta,x}} = 0, \quad \text{if } \alpha \neq \beta, \tag{1}$$

$$\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \frac{W_{\alpha,x} W_{x,\beta}}{W_{\gamma,x}} = \frac{W_{\alpha,\beta}}{W_{\gamma,\alpha} W_{\gamma,\beta}}. \tag{2}$$

The second equation is known as the 'star-triangle relation'. Remark that W *may be non-symmetric* in this paper. Kawagoe-Munemasa-Watatani [13] showed that the symmetric condition in [9] is not necessary to construct invariants of oriented links.

As easily shown, the tensor product $W_1 \otimes W_2$ of two spin models W_1 and W_2 becomes a spin model. So we can obtain a spin model of size $4n$ from an arbitrary spin model W by taking its tensor product with a spin model of size four. In this paper, we show that we can add some 'twist' (sign exchange).

Let W be a spin model of size n. For an n-tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $\epsilon_i = \pm 1 (i = 1, \ldots, n)$, let W' be the matrix whose entries are given as

$$W'_{\alpha,\beta} = \epsilon_\alpha \epsilon_\beta W_{\alpha,\beta} \quad (\alpha, \beta = 1, \ldots, n).$$

Using W and W', construct the following matrices U, V of size $4n$:

$$U = \begin{pmatrix} W & W & W' & -W' \\ W & W & -W' & W' \\ W' & -W' & W & W \\ -W' & W' & W & W \end{pmatrix},$$

$$V = \begin{pmatrix} W' & -W' & W & W \\ -W' & W' & W & W \\ W & W & W' & -W' \\ W & W & -W' & W' \end{pmatrix}.$$

Theorem 1 *Let W be a spin model, and let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $\epsilon_i = \pm 1$ $(i = 1, \ldots, n)$. Then both U and V, defined above, become spin models.*

Remark We can replace W' by $\omega W'$ in the matrix U of Theorem 1, where ω is one of the 4-th roots of unity. Clearly U and V become symmetric if W is symmetric.

When $\epsilon_i = +1$ $(i = 1, \ldots, n)$, these models split into the tensor product of W and the following spin models:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

Otherwise, U and V might possibly give new spin models.

Question. Can one find a new spin model by the above construction of U (or V) starting from a known spin model W?

2 Lemmas

Theorem 1 is easily obtained from the following two Lemmas.

Lemma 2 *Let W and W' be matrices of size n, and let U be the following matrix of size $4n$.*

$$U = \begin{pmatrix} W & W & W' & -W' \\ W & W & -W' & W' \\ W' & -W' & W & W \\ -W' & W' & W & W \end{pmatrix}.$$

Then U is a spin model if and only if the following conditions hold (for all $\alpha, \beta, \gamma = 1, \ldots, n$):

$$\sum_{x=1}^{n} \frac{W_{\alpha,x}}{W_{\beta,x}} = 0, \quad \sum_{x=1}^{n} \frac{W'_{\alpha,x}}{W'_{\beta,x}} = 0, \quad \text{if } \alpha \neq \beta, \tag{3}$$

$$\frac{1}{\sqrt{n}}\sum_{x=1}^{n}\frac{W_{\alpha,x}W_{x,\beta}}{W_{\gamma,x}} = \frac{W_{\alpha,\beta}}{W_{\gamma,\alpha}W_{\gamma,\beta}}, \tag{4}$$

$$\frac{1}{\sqrt{n}}\sum_{x=1}^{n}\frac{W'_{\alpha,x}W'_{x,\beta}}{W_{\gamma,x}} = \frac{W_{\alpha,\beta}}{W'_{\gamma,\alpha}W'_{\gamma,\beta}}, \tag{5}$$

$$\frac{1}{\sqrt{n}}\sum_{x=1}^{n}\frac{W'_{\alpha,x}W_{x,\beta}}{W'_{\gamma,x}} = \frac{W'_{\alpha,\beta}}{W_{\gamma,\alpha}W'_{\gamma,\beta}}, \tag{6}$$

$$\frac{1}{\sqrt{n}}\sum_{x=1}^{n}\frac{W_{\alpha,x}W'_{x,\beta}}{W'_{\gamma,x}} = \frac{W'_{\alpha,\beta}}{W'_{\gamma,\alpha}W_{\gamma,\beta}}. \tag{7}$$

Lemma 3 *Let W and W' be matrices of size n, and let V be the following matrix of size $4n$.*

$$V = \begin{pmatrix} W' & -W' & W & W \\ -W' & W' & W & W \\ W & W & W' & -W' \\ W & W & -W' & W' \end{pmatrix}.$$

Then V is a spin model if and only if the following conditions hold (for all $\alpha, \beta, \gamma = 1, \dots, n$):

$$\sum_{x=1}^{n}\frac{W_{\alpha,x}}{W_{\beta,x}} = 0, \quad \sum_{x=1}^{n}\frac{W'_{\alpha,x}}{W'_{\beta,x}} = 0 \qquad \text{if } \alpha \neq \beta, \tag{8}$$

$$\frac{1}{\sqrt{n}}\sum_{x=1}^{n}\frac{W_{\alpha,x}W_{x,\beta}}{W_{\gamma,x}} = \frac{W'_{\alpha,\beta}}{W'_{\gamma,\alpha}W'_{\gamma,\beta}}, \tag{9}$$

$$\frac{1}{\sqrt{n}}\sum_{x=1}^{n}\frac{W'_{\alpha,x}W'_{x,\beta}}{W_{\gamma,x}} = \frac{W'_{\alpha,\beta}}{W_{\gamma,\alpha}W_{\gamma,\beta}}, \tag{10}$$

$$\frac{1}{\sqrt{n}}\sum_{x=1}^{n}\frac{W'_{\alpha,x}W_{x,\beta}}{W'_{\gamma,x}} = \frac{W_{\alpha,\beta}}{W'_{\gamma,\alpha}W_{\gamma,\beta}}, \tag{11}$$

$$\frac{1}{\sqrt{n}}\sum_{x=1}^{n}\frac{W_{\alpha,x}W'_{x,\beta}}{W'_{\gamma,x}} = \frac{W_{\alpha,\beta}}{W_{\gamma,\alpha}W'_{\gamma,\beta}}. \tag{12}$$

3 Proof of Lemmas

First let us prove Lemma 2. The conditions (1), (2) for the matrix U become

$$\mathrm{OT}(\lambda,\mu): \sum_{x=1}^{4n}\frac{U_{\lambda,x}}{U_{\mu,x}} = 0 \quad \text{if } \lambda \neq \mu,$$

$$\mathrm{ST}(\lambda,\mu,\nu): \frac{1}{\sqrt{4n}}\sum_{x=1}^{4n}\frac{U_{\lambda,x}U_{x,\mu}}{U_{\nu,x}} = \frac{U_{\lambda,\mu}}{U_{\nu,\lambda}U_{\nu,\mu}}.$$

We write down the above conditions in terms of W and W' for various values of λ, μ, ν. Put $X = \{1, \dots, n\}$, and identify $x \in X$ with $(x, 00)$, $n + x$ with $(x, 01)$, $2n + x$

with $(x, 10)$ and $3n + x$ with $(x, 11)$, where the second components 00, 01, 10 and 11 belong to $(\mathbf{Z}/2\mathbf{Z})^2$. This group acts by translation on the rows and columns of U, V and leaves these matrices invariant. Hence it is enough to consider OT and ST for λ in $X = \{(x, 00) \mid x = 1, \dots, n\}$.

Now choose arbitrary $\alpha, \beta, \gamma \in X$.

First we consider the equations $\mathrm{OT}(\lambda, \mu)$. For $\alpha, \beta \in X$, $\mathrm{OT}(\alpha, \beta)$ becomes

$$\sum_{x\in X}\frac{W_{\alpha,x}}{W_{\beta,x}}+\sum_{x\in X}\frac{W_{\alpha,x}}{W_{\beta,x}}+\sum_{x\in X}\frac{W'_{\alpha,x}}{W'_{\beta,x}}+\sum_{x\in X}\frac{-W'_{\alpha,x}}{-W'_{\beta,x}}=0,$$
$$\sum_{x\in X}\frac{W_{\alpha,x}}{W_{\beta,x}}+\sum_{x\in X}\frac{W'_{\alpha,x}}{W'_{\beta,x}}=0, \tag{13}$$

and $\mathrm{OT}(\alpha, n + \beta)$ becomes

$$\sum_{x\in X}\frac{W_{\alpha,x}}{W_{\beta,x}}+\sum_{x\in X}\frac{W_{\alpha,x}}{W_{\beta,x}}+\sum_{x\in X}\frac{W'_{\alpha,x}}{-W'_{\beta,x}}+\sum_{x\in X}\frac{-W'_{\alpha,x}}{W'_{\beta,x}}=0,$$
$$\sum_{x\in X}\frac{W_{\alpha,x}}{W_{\beta,x}}-\sum_{x\in X}\frac{W'_{\alpha,x}}{W'_{\beta,x}}=0. \tag{14}$$

Also $\mathrm{OT}(\alpha, 2n + \beta)$ becomes

$$\sum_{x\in X}\frac{W_{\alpha,x}}{W'_{\beta,x}}+\sum_{x\in X}\frac{W_{\alpha,x}}{-W'_{\beta,x}}+\sum_{x\in X}\frac{W'_{\alpha,x}}{W_{\beta,x}}+\sum_{x\in X}\frac{-W'_{\alpha,x}}{W_{\beta,x}}=0, \tag{15}$$

and $\mathrm{OT}(\alpha, 3n + \beta)$ becomes

$$\sum_{x\in X}\frac{W_{\alpha,x}}{-W'_{\beta,x}}+\sum_{x\in X}\frac{W_{\alpha,x}}{W'_{\beta,x}}+\sum_{x\in X}\frac{W'_{\alpha,x}}{W_{\beta,x}}+\sum_{x\in X}\frac{-W'_{\alpha,x}}{W_{\beta,x}}=0. \tag{16}$$

Remark that (15) and (16) always hold. If $\mathrm{OT}(\lambda, \mu)$ holds for all λ, μ with $\lambda \neq \mu$, then (13) and (14) hold for all $\alpha, \beta \in X$ with $\alpha \neq \beta$, so we get (3). Conversely, if (3) holds for all $\alpha, \beta \in X$ with $\alpha \neq \beta$, then (13) and (14) hold for $\alpha \neq \beta$. Since (14) holds also for $\alpha = \beta$, we get OT (λ, μ) for all λ, μ with $\lambda \neq \mu$.

Next we consider $\mathrm{ST}(\lambda, \mu, \nu)$. $\mathrm{ST}(\alpha, \beta, \gamma)$ becomes

$$\sum_{x\in X}\frac{W_{\alpha,x}W_{x,\beta}}{W_{\gamma,x}}+\sum_{x\in X}\frac{W_{\alpha,x}W_{x,\beta}}{W_{\gamma,x}}+\sum_{x\in X}\frac{W'_{\alpha,x}W'_{x,\beta}}{W'_{\gamma,x}}$$
$$+\sum_{x\in X}\frac{(-W'_{\alpha,x})(-W'_{x,\beta})}{(-W'_{\gamma,x})}=\frac{2\sqrt{n}W_{\alpha,\beta}}{W_{\gamma,\alpha}W_{\gamma,\beta}},$$

i.e.

$$2\sum_{x\in X}\frac{W_{\alpha,x}W_{x,\beta}}{W_{\gamma,x}}=\frac{2\sqrt{n}W_{\alpha,\beta}}{W_{\gamma,\alpha}W_{\gamma,\beta}},$$

this coincides with (4). $\mathrm{ST}(\alpha, \beta, 2n+\gamma)$ becomes

$$\sum_{x\in X}\frac{W_{\alpha,x}W_{x,\beta}}{W'_{\gamma,x}}+\sum_{x\in X}\frac{W_{\alpha,x}W_{x,\beta}}{(-W'_{\gamma,x})}+\sum_{x\in X}\frac{W'_{\alpha,x}W'_{x,\beta}}{W_{\gamma,x}}+\sum_{x\in X}\frac{(-W'_{\alpha,x})(-W'_{x,\beta})}{W_{\gamma,x}}=\frac{2\sqrt{n}W_{\alpha,\beta}}{W'_{\gamma,\alpha}W'_{\gamma,\beta}},$$

i.e.

$$2\sum_{x\in X}\frac{W'_{\alpha,x}W'_{x,\beta}}{W_{\gamma,x}}=\frac{2\sqrt{n}W_{\alpha,\beta}}{W'_{\gamma,\alpha}W'_{\gamma,\beta}},$$

this is (5). $\mathrm{ST}(\alpha, 2n+\beta, \gamma)$ becomes

$$\sum_{x\in X}\frac{W_{\alpha,x}W'_{x,\beta}}{W_{\gamma,x}}+\sum_{x\in X}\frac{W_{\alpha,x}(-W'_{x,\beta})}{W_{\gamma,x}}+\sum_{x\in X}\frac{W'_{\alpha,x}W_{x,\beta}}{W'_{\gamma,x}}+\sum_{x\in X}\frac{(-W'_{\alpha,x})W_{x,\beta}}{(-W'_{\gamma,x})}=\frac{2\sqrt{n}W'_{\alpha,\beta}}{W_{\gamma,\alpha}W'_{\gamma,\beta}},$$

i.e.

$$2\sum_{x\in X}\frac{W'_{\alpha,x}W_{x,\beta}}{W'_{\gamma,x}}=\frac{2\sqrt{n}W'_{\alpha,\beta}}{W_{\gamma,\alpha}W'_{\gamma,\beta}},$$

this is (6). $\mathrm{ST}(\alpha, 2n+\beta, 2n+\gamma)$ becomes

$$\sum_{x\in X}\frac{W_{\alpha,x}W'_{x,\beta}}{W'_{\gamma,x}}+\sum_{x\in X}\frac{W_{\alpha,x}(-W'_{x,\beta})}{(-W'_{\gamma,x})}+\sum_{x\in X}\frac{W'_{\alpha,x}W_{x,\beta}}{W_{\gamma,x}}+\sum_{x\in X}\frac{(-W'_{\alpha,x})W_{x,\beta}}{W_{\gamma,x}}=\frac{2\sqrt{n}W'_{\alpha,\beta}}{W'_{\gamma,\alpha}W_{\gamma,\beta}},$$

i.e.

$$2\sum_{x\in X}\frac{W_{\alpha,x}W'_{x,\beta}}{W'_{\gamma,x}}=\frac{2\sqrt{n}W'_{\alpha,\beta}}{W'_{\gamma,\alpha}W_{\gamma,\beta}},$$

this is (7). Thus, if $\mathrm{ST}(\lambda, \mu, \nu)$ holds for all λ, μ, ν, then W and W' satisfy (4), (5), (6), (7). For the proof of the converse, we must show $\mathrm{ST}(\lambda, \mu, \nu)$ for all λ, μ, ν. We may assume $\lambda \in X$ by the above remark on $(\mathbf{Z}/2\mathbf{Z})^2$-symmetry. Moreover if we perform a 01-translation on μ, this amounts to change of the sign of $W'_{x,\beta}$, $W'_{\alpha,\beta}$, $W'_{\gamma,\beta}$ whenever these matrices appear in $\mathrm{ST}(\lambda, \mu, \nu)$. It is easy to see that these changes of sign do not modify the four equations given above. A similar argument can be applied to a 01-translation on ν, and clearly using these translations we may reduce all the remaining cases to the above four cases.

Let us now prove Lemma 3. Observe that V can be obtained from U by performing a 10-translation on the columns, and can also be obtained from U by 10-translation on the rows. Thus if we proceed as in the proof of Lemma 2 (with the same notations), we shall obtain $\mathrm{OT}(\lambda, \mu)$ for V from $\mathrm{OT}(\lambda, \mu)$ for U by permuting the summands in the left-hand side, and this yields the same equation (3) (i.e. (8)). Similarly, the left-hand sides of the four cases of $\mathrm{ST}(\lambda, \mu, \nu)$ are the same for U and for V up to permutation of terms. The right-hand sides are easily seen to be those of (9)–(12). Finally the symmetry arguments used for U can also be applied to V.

4 Spin models coming from Hadamard matrices

We have constructed new spin models from Hadamard graphs [14]. Let H be a Hadamard matrix of order $n = 4m$, i.e. an $n \times n$-matrix with ± 1 entries such that $H\,{}^tH = nI$. Let A be the Potts model of size n;

$$A = (a - b)I + bJ,$$

where J denotes the $n \times n$-matrix with all entries 1, and a, b are complex numbers such that

$$a^2 = \frac{\sqrt{n}}{1 + (n-1)\theta}, \quad b = a\theta,$$

where θ is one of the roots of $\theta + \theta^{-1} + n - 2 = 0$. Using H and A, construct the following matrix;

$$W_H = \begin{pmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \omega\,{}^tH & -\omega\,{}^tH & A & A \\ -\omega\,{}^tH & \omega\,{}^tH & A & A \end{pmatrix},$$

where ω denotes one of the 4-th roots of unity. As shown in [14], the above matrix becomes a symmetric spin model.

The partition function of the above model was obtained by F. Jaeger. At first he proved that the partition function does not depend on the choice of the Hadamard matrix H [4], and secondly he obtained an explicit formula [5]. In fact, the partition function for a link L can be written in terms of Jones polynomials [7] of sublinks of L.

The link invariant associated with the above model is somewhat stronger than Jones polynomial. In fact, Jones [12] constructed a pair (L_1, L_2) of two links which can be distinguished by the above invariant but not by the Jones polynomial. In the construction, he used his recent method of commuting transfer matrices [11].

Jones informed us [12] that the above invariant is similar to the invariant given by Rolfsen [15]. In fact, Rolfsen's invariant and our invariant are the same for links having at most two components, and so the examples of links given in [6] also give examples of links which are distinguished by our invariant but not by the Jones polynomial.

Here we give a variation of Lemma 2 which covers the above spin model W_H.

Lemma 4 *Let W and W' be matrices of size n, and let U be the following matrix of size $4n$.*

$$U = \begin{pmatrix} W & W & W' & -W' \\ W & W & -W' & W' \\ {}^tW' & -{}^tW' & W & W \\ -{}^tW' & {}^tW' & W & W \end{pmatrix}.$$

Then U is a spin model if and only if the conditions (17)–(24) *hold (for all α, β, $\gamma = 1, \dots, n$);*

$$\sum_{x=1}^{n} \frac{W_{\alpha,x}}{W_{\beta,x}} = 0, \quad \sum_{x=1}^{n} \frac{W'_{\alpha,x}}{W'_{\beta,x}} = 0, \quad \text{if } \alpha \neq \beta, \tag{17}$$

$$\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \frac{W_{\alpha,x} W_{x,\beta}}{W_{\gamma,x}} = \frac{W_{\alpha,\beta}}{W_{\gamma,\alpha} W_{\gamma,\beta}}, \tag{18}$$

$$\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \frac{W'_{\alpha,x} W'_{\beta,x}}{W_{\gamma,x}} = \frac{W_{\alpha,\beta}}{W'_{\alpha,\gamma} W'_{\beta,\gamma}}, \tag{19}$$

$$\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \frac{W'_{\alpha,x} W_{x,\beta}}{W'_{\gamma,x}} = \frac{W'_{\alpha,\beta}}{W_{\gamma,\alpha} W'_{\gamma,\beta}}, \tag{20}$$

$$\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \frac{W_{\alpha,x} W'_{x,\beta}}{W'_{x,\gamma}} = \frac{W'_{\alpha,\beta}}{W'_{\alpha,\gamma} W_{\gamma,\beta}}, \tag{21}$$

$$\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \frac{W_{\alpha,x} W'_{\beta,x}}{W'_{\gamma,x}} = \frac{W'_{\beta,\alpha}}{W'_{\gamma,\alpha} W_{\gamma,\beta}}, \tag{22}$$

$$\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \frac{W'_{x,\alpha} W_{x,\beta}}{W'_{x,\gamma}} = \frac{W'_{\beta,\alpha}}{W_{\gamma,\alpha} W'_{\beta,\gamma}}, \tag{23}$$

$$\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \frac{W'_{x,\alpha} W'_{x,\beta}}{W_{\gamma,x}} = \frac{W_{\alpha,\beta}}{W'_{\gamma,\alpha} W'_{\gamma,\beta}}. \tag{24}$$

Proof: We shall proceed as in the proof of Lemma 2. A 01-translation on the columns corresponds as before to a change of sign of W'. But now a simultaneous 10-translation on rows and columns amounts to the transposition of W', and similarly for 11-translations. So provided we close our set of equations by transposition of W', we can restrict our attention to the same cases studied for Lemma 2 by applying the same symmetry arguments. We get (17) as we got (3) in Lemma 2 (transposing W' in the second equation yields an equivalent equation). We also obtain (18), (19), (20), (21) from the proof of (4), (5), (6), (7) by transposing suitable occurrences of W'. Then (22), (23), (24) are obtained from (19), (20), (21) by transposition of every occurrence of W'. □

In [14], we proved that the matrix W_H given above becomes a spin model. Here we give an alternating proof of this fact by using Lemma 4. The matrix U in Lemma 4 becomes W_H for $W = A$, $W' = \omega H$. We would like to show that the matrices W and W' satisfy (17)–(24). We may assume $\omega = 1$ since all equations are invariant under multiplication of W' by ω. Since $W = A$ is a spin model and since $W' = H$ is a Hadamard matrix, the equations (17) and (18) are clearly satisfied.

Let us prove (19). If $\alpha = \beta$, then the left hand side of (19) becomes

$$\text{l.h.s.} = \frac{1}{\sqrt{n}} \sum_{x \in X} \frac{1}{A_{\gamma,x}} = \frac{1}{\sqrt{n}} \left(\frac{1}{a} + \frac{n-1}{b} \right),$$

and the right hand side becomes r.h.s. $= A_{\alpha,\beta} = a$, so (19) holds by our choice of a and b.

Now fix $\alpha, \beta \in X$ with $\alpha \neq \beta$. We partition $X = \{1, \dots, n\}$ into two subsets X_+ and X_-:

$$X_+ = \{x \in X \mid H_{\alpha,x} H_{\beta,x} = +1\},$$
$$X_- = \{x \in X \mid H_{\alpha,x} H_{\beta,x} = -1\}.$$

Since H is a Hadamard matrix, we have $|X_+| = |X_-| = n/2$ and $X = X_+ \cup X_-$. The left hand side of (19) becomes

$$\begin{aligned}\text{l.h.s.} &= \frac{1}{\sqrt{n}} \sum_{x \in X_+} \frac{H_{\alpha,x} H_{\beta,x}}{A_{\gamma,x}} + \frac{1}{\sqrt{n}} \sum_{x \in X_-} \frac{H_{\alpha,x} H_{\beta,x}}{A_{\gamma,x}} \\ &= \frac{1}{\sqrt{n}} \sum_{x \in X_+} \frac{1}{A_{\gamma,x}} - \frac{1}{\sqrt{n}} \sum_{x \in X_-} \frac{1}{A_{\gamma,x}}.\end{aligned}$$

When $\gamma \in X_+$, this becomes

$$\text{l.h.s.} = \frac{1}{\sqrt{n}} \left(\frac{1}{a} + \left(\frac{n}{2} - 1 \right) \frac{1}{b} - \frac{n}{2} \cdot \frac{1}{b} \right) = \frac{1}{\sqrt{n}} \left(\frac{1}{a} - \frac{1}{b} \right),$$

and the right hand side of (19) becomes

$$\text{r.h.s.} = \frac{A_{\alpha,\beta}}{H_{\alpha,\gamma} H_{\beta,\gamma}} = b.$$

So (19) holds. The above computation also works well in the case $\gamma \in X_-$ (in this case signs of both sides are changed).

Let us prove (20). If $\alpha = \gamma$, then (20) becomes

$$\frac{1}{\sqrt{n}}(a + (n-1)b) = \frac{1}{a},$$

so (20) holds. Now fix $\alpha, \gamma \in X$ with $\alpha \neq \gamma$. We partition X into subsets X_+ and X_-:

$$X_\pm = \{x \in X \mid H_{\alpha,x} H_{\gamma,x} = \pm 1\}.$$

The left hand side of (20) becomes

$$\text{l.h.s.} = \frac{1}{\sqrt{n}} \sum_{x \in X_+} A_{x,\beta} - \frac{1}{\sqrt{n}} \sum_{x \in X_-} A_{x,\beta}.$$

When $\beta \in X_+$, this becomes $(1/\sqrt{n})(a - b)$, and the right hand side becomes b^{-1}, so (20) holds. In the same way, we can show (20) when $\beta \in X_-$.

The proof of (21) is the same as that of proof (20), but in this case $X_\pm$ are defined with respect to β and γ. Now the proof of (19), (20) and (21) also gives a proof of (24), (23) and (22), since the transpose of $W' = H$ is also a Hadamard matrix.

5 Symmetric four-weight spin models

E. Bannai and E. Bannai [1] introduced four-weight spin models, which generalize the ordinary spin models given by Jones. In this section, we describe a relation between our construction and four-weight spin models.

A four-weight spin model is defined in [1] as follows. Let X be a set of n elements, and w_i $(i = 1, 2, 3, 4)$ be functions on $X \times X$ to the complex numbers. Then (X, w_1, w_2, w_3, w_4) is a four-weight spin model if the following conditions are satisfied (for all $\alpha, \beta, \gamma \in X$):

(i) $w_1(\alpha, \beta)w_3(\beta, \alpha) = 1$, $w_2(\alpha, \beta)w_4(\beta, \alpha) = 1$,

(ii) $\sum_{x\in X} w_1(\alpha, x)w_3(x, \beta) = n\delta_{\alpha,\beta}$, $\sum_{x\in X} w_2(\alpha, x)w_4(x, \beta) = n\delta_{\alpha,\beta}$,

(iii) $\frac{1}{\sqrt{n}} \sum_{x\in X} w_1(\alpha, x)w_1(x, \beta)w_4(\gamma, x) = w_1(\alpha, \beta)w_4(\gamma, \alpha)w_4(\gamma, \beta)$,

(iii′) $\frac{1}{\sqrt{n}} \sum_{x\in X} w_1(x, \alpha)w_1(\beta, x)w_4(x, \gamma) = w_1(\beta, \alpha)w_4(\alpha, \gamma)w_4(\beta, \gamma)$.

Lemma 5 *Let W and W' be symmetric matrices of size n with non-zero complex entries. Put $X = \{1, \dots, n\}$ and define w_i $(i = 1, 2, 3, 4)$ as*

$$w_1(\alpha, \beta) = W'_{\alpha,\beta}, \qquad w_3(\alpha, \beta) = \frac{1}{W'_{\alpha,\beta}},$$

$$w_2(\alpha, \beta) = W_{\alpha,\beta}, \qquad w_4(\alpha, \beta) = \frac{1}{W_{\alpha,\beta}}.$$

Then (X, w_1, w_2, w_3, w_4) is a four-weight spin model if and only if the following matrix V is a spin model.

$$V = \begin{pmatrix} W' & -W' & W & W \\ -W' & W' & W & W \\ W & W & W' & -W' \\ W & W & -W' & W' \end{pmatrix}.$$

Proof: If V is a spin model, Lemma 3 implies

$$\sum_{x\in X} \frac{W_{\alpha,x}}{W_{\beta,x}} = n\delta_{\alpha,\beta}, \quad \sum_{x\in X} \frac{W'_{\alpha,x}}{W'_{\beta,x}} = n\delta_{\alpha,\beta},$$

$$\frac{1}{\sqrt{n}} \sum_{x\in X} \frac{W'_{\alpha,x} W'_{x,\beta}}{W_{\gamma,x}} = \frac{W'_{\alpha,\beta}}{W_{\gamma,\alpha} W_{\gamma,\beta}}.$$

Therefore (X, w_1, w_2, w_3, w_4) becomes a four-weight spin model.

Conversely, assume (X, w_1, w_2, w_3, w_4) is a four-weight spin model. Then conditions (8) and (10) in Lemma 3 hold. But the conditions (9), (10), (11) and (12) in Lemma 3 are equivalent to each other under the assumption (8) by [1] Theorem 1. So V becomes a spin model. □

Remark Theorem 1 and Lemma 5 imply a construction of symmetric four-weight spin models from an ordinary spin model. Moreover we can conclude that every symmetric four-weight spin model comes from an ordinary symmetric spin model.

Problem. Give a formula for the partition function of the spin model U, V in Theorem 1, in terms of the partition function of W.

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