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Quasi-Varieties, Congruences, and Generalized Dowling Lattices*

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Abstract. Dowling lattices and their generalizations introduced by Hanlon are interpreted as lattices of congruences associated to certain quasi-varieties of sets with group actions. This interpretation leads, by a simple application of Möbius inversion, to polynomial identities which specialize to Hanlon's evaluation of the characteristic polynomials of generalized Dowling lattices. Analogous results are obtained for a few other quasi-varieties.

Keywords: Dowling lattice, congruence, free algebra, characteristic polynomial, quasi-variety

1. Introduction

We shall compute the characteristic functions and some additional information for several sorts of lattices by interpreting these lattices as lattices of congruence relations in some free algebras. The characteristic function of a finite lattice L, introduced in [3], Chapter 16, is

$$\chi_L(\lambda) = \sum_{x \in L} \mu(0, x)\lambda^{r(1)-r(x)},$$

where μ is the Möbius function of L, 0 and 1 are the bottom and top elements of L, and r is a rank function on L. When L satisfies the chain condition, i.e., when all maximal chains from 0 to x have the same length, then $r(x)$ is usually defined to be this length, but we shall also deal with lattices not satisfying the chain condition, and then there is some arbitrariness in the choice of r.

This paper is based on the following two facts. First, a straightforward application of Möbius inversion leads to formulas for the characteristic functions of the congruence lattices of free algebras in various varieties of algebras. For two particularly simple varieties—sets and vector spaces over a specified finite field—these computations are in [7], Chapter 3, Exercises 44 and 45. (We review them here at the beginning of Section 4.) In fact, the same method applies also to quasi-varieties of algebras and to somewhat more general classes.

Second, the generalized Dowling lattices introduced in [5] can be viewed in a natural way as just such congruence lattices for certain quasi-varieties. This is a more conceptual view of the generalized Dowling lattices, and it leads to a calculation of certain multi-variable polynomials that specialize to the characteristic functions when variables are suitably identified. Thus, as a specialization, we recover the characteristic polynomials computed in [5] by quite different methods.

In Section 2 we present the preliminary information that we need about quasi-varieties, and in Section 3 we show how to identify generalized Dowling lattices and the closely

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related lattices of K digraphs [5] with certain lattices of congruence relations. Sections 4 and 5 contain the calculations of the (generalized) characteristic polynomials for these lattices as well as certain values of the Möbius functions. Section 6 contains the analogous computation for a certain variety of monoid actions. Finally, in Section 7 we extend the method beyond the context of quasi-varieties by applying it to a class of relational structures.

It is reasonable to expect that the same method will provide tractable computations of (generalized) characteristic functions for numerous other sorts of lattices. The specific computations done here are intended as suggestive examples.

2. Quasi-varieties and congruences

We use standard terminology from universal algebra [2]. An *algebra* is a set A together with an indexed family of operations $A^n \to A$ on it, the indexing set and the specification of the number n of argument places for each index being called the *signature* of the algebra. A *quasi-variety* is a class of algebras, all of the same signature, closed under isomorphism, subalgebras, and products. (In particular, the algebra having only one element is in every quasi-variety, being the product of the empty family.) A *variety* is, in addition, closed under homomorphic images. A *congruence* on an algebra is an equivalence relation respected by the operations in the sense that equivalent inputs to the operations produce equivalent outputs. Every homomorphism f of algebras induces a congruence, called the *kernel* of f, on its domain, two elements being equivalent if and only if they have the same f-image. The equivalence classes of any congruence on an algebra A form a *quotient* algebra of the same signature.

If $\mathcal{V}$ is a quasi-variety, then a congruence on an algebra in $\mathcal{V}$ will be called a *$\mathcal{V}$-congruence* if the quotient algebra belongs to $\mathcal{V}$. (Note that all congruences on $\mathcal{V}$-algebras are $\mathcal{V}$-congruences if and only if $\mathcal{V}$ is a variety.) The intersection of two $\mathcal{V}$-congruences x and y is again a $\mathcal{V}$-congruence, because the quotient algebra determined by $x \cap y$ can be embedded in the product of the quotient algebras determined by x and y. The same argument applies to the intersection of any number (even infinite) of $\mathcal{V}$-congruences, and it follows that the $\mathcal{V}$-congruences on any $\mathcal{V}$-algebra form a complete lattice, in which meet is ordinary intersection (but join is in general neither union, which need not even be an equivalence, nor the equivalence relation generated by the union, which is a congruence but need not be a $\mathcal{V}$-congruence).

Quasi-varieties can be characterized logically as follows. A *conditional identity* or *strict universal Horn sentence* is a sentence of the form

$$\forall x_1 \forall x_2 \cdots \forall x_k ((E_1 \text{ and } E_2 \text{ and } \cdots \text{ and } E_n) \Rightarrow F), \tag{1}$$

where the E_i and F are equations between terms built from the variables x_j and operation symbols appropriate for the signature. We allow $n = 0$, in which case (1) is simply an *identity*, namely

$$\forall x_1 \forall x_2 \cdots \forall x_k F. \tag{2}$$

For any set of conditional identities, the algebras in which all of them are true constitute a quasi-variety, and every quasi-variety admits such a description by conditional identities. Similarly, varieties are described by sets of identities of the form (2).

If $\mathcal{V}$ is a quasi-variety and S is any set, then there is a $\mathcal{V}$-algebra $F_{\mathcal{V}}(S)$ freely generated by S. This means that there is a specified function $\eta: S \to F_{\mathcal{V}}(S)$ such that every function from S to any $A \in \mathcal{V}$ factors uniquely through η by a homomorphism $F_{\mathcal{V}}(S) \to A$. The existence of such free algebras follows immediately from the adjoint functor theorem (see [6] Section 5.6). One can also explicitly construct $F_{\mathcal{V}}(S)$ by building terms out of the operation symbols and the elements of S and then identifying two such terms if and only if this identification is required by the conditional identities describing $\mathcal{V}$.

We write $C_n(\mathcal{V})$ for the lattice of $\mathcal{V}$-congruences on the free $\mathcal{V}$-algebra on n generators, $F_{\mathcal{V}}(\{1, 2, \ldots, n\})$. By remarks above, it is a complete lattice. One of the central ingredients of this paper is the observation that the Dowling lattices [4] and their generalizations due to Hanlon [5] are isomorphic to $C_n(\mathcal{V})$ for rather natural quasi-varieties $\mathcal{V}$, to be described in Section 3.

We close this section with two simple examples. First, sets may be regarded as algebras with no operations, and the class $\mathcal{S}$ of all sets is a variety. All equivalence relations are $\mathcal{S}$-congruences. The free set on n generators is simply an n-element set, so the lattice $C_n(\mathcal{S})$ is the partition lattice Π_n.

Second, if we fix a field k, then the vector spaces over k constitute a variety $k\mathcal{V}$, the operations in a vector space being addition and, for each scalar $a \in k$, the unary operation of multiplication by a. A congruence on a vector space can be specified by giving the equivalence class containing 0, as the other equivalence classes are its translates. So the lattice of congruences of a vector space is isomorphic to the lattice of subspaces. The free vector space on n generators is the n-dimensional vector space k^n. So $C_n(k\mathcal{V})$ is the subspace lattice of an n-dimensional vector space over k.

3. G, K-sets and generalized Dowling lattices

In this section, we introduce the quasi-varieties for which the congruence lattices $C_n(\mathcal{V})$ are (canonically isomorphic to) the lattices $\Omega_n(G, K)$ and $D_n(G, K)$ introduced by Hanlon [5].

As in [5], let G be a finite group, and let K be a family of subgroups of G satisfying the following three conditions.

(1) K is closed under intersection.
(2) K is closed under conjugation in G.
(3) K contains the trivial subgroup $\{e\}$ of G.

Among the intersections in (1), we allow the intersection of the empty family, so $G \in K$. It follows from (1) that K is a lattice and that, for every subset $S \subseteq G$, there is a smallest group in K containing S. In the lattice K, meet is intersection and join is the smallest group in K containing the union.

By a *G-set*, we mean a set A with a left action of G. G-sets are algebras with one unary operation for each element of G, and they constitute a variety called $G\mathcal{S}$. The free G-set on a set S of generators is the disjoint union of $|S|$ copies of the regular action of G, i.e., it is $G \times S$ with the action $g \cdot (h, s) = (gh, s)$.

It is well known that every G-set is the disjoint union of *orbits*, which are (up to isomorphism) G-sets of the form G/H, where H is a subgroup of G and where G/H is the set of left cosets gH with the G-action $g \cdot g'H = (gg')H$. Two orbits G/H and G/H' are isomorphic if and only if H and H' are conjugate in G, so we may assume that H

ranges over a system of representatives, chosen once and for all, of the conjugacy classes of subgroups of G. We index these representatives as H_i, and we use the notation

$$\sum_i n_i \cdot G/H_i$$

for a G-set consisting of n_i copies of G/H_i for all i.

The *stabilizer* of an element a in a G-set is the set of elements of G whose action leaves a fixed. The stabilizers of the elements gH of G/H are exactly the conjugates gHg^{-1} of H in G.

We call a G-set a G, K*-set* if the stabilizers of all its elements are in K. Because of requirement (1) on K, the G, K-sets constitute a quasi-variety $G, K\mathcal{S}$.

(Requirements (2) and (3) are less important and amount to normalizations. If (2) failed, then we could replace K with the subfamily of those groups whose conjugates all lie in K; this would not alter $G, K\mathcal{S}$ because, if a group occurs as a stabilizer in a G-set, then so do all its conjugates. If (3) failed, then the intersection of all the groups in K would be a normal subgroup N of G, and G, K-sets would be essentially the same as $G/N, K'$-sets, where $K' = \{H/N \mid H \in K\}$.)

The free G-sets have all stabilizers equal to $\{e\}$, which is in K, so they are G, K-sets, and it is easy to check that they are the free G, K-sets. The congruence lattices $C_n(G\mathcal{S})$ and $C_n(G, K\mathcal{S})$ differ, however, because the former contains all the congruences x on the free G-set, while the latter contains only those x for which the quotient, which we denote by $\hat{x}$, is a G, K-set. Our next task is to relate these congruence lattices to the lattices of digraphs in [5].

Recall that the free G-set $F_n(G\mathcal{S})$ was described above as $G \times [n]$, where $[n] = \{1, 2, \ldots, n\}$ and G acts by left multiplication on the first component. To specify a congruence x on this G-set, it suffices to tell, for each $j \in [n]$, which elements (g, i) are equivalent to (e, j); indeed, as the action of the group respects the congruence x, we have that (g_1, i) is equivalent in x to (g_2, j) if and only if $(g_2^{-1}g_1, i)$ is equivalent to (e, j). Thus, x can be specified by giving the function which, to each ordered pair (i, j) of elements of $[n]$, assigns the subset $\{g \mid (g, i)x(e, j)\}$. Of course, not every function, assigning subsets of G to pairs from $[n]$, corresponds to a congruence in this fashion, but it is easy to compute which ones do and also which ones correspond to $G, K\mathcal{S}$-congruences, i.e., which ones have $\hat{x} \in G, K\mathcal{S}$.

A function assigning subsets of G to pairs from $[n]$ can be viewed as an edge-labeled digraph with vertex set $[n]$. The edges are the pairs to which a nonempty set is assigned, and the labels are those nonempty sets. A routine computation shows that, for any such labeled digraph, the corresponding relation x on $G \times [n]$, namely

$$(g_1, i)x(g_2, j) \Leftrightarrow g_2^{-1}g_1 \text{ is in the set labeling } (i, j),$$

is a $G, K\mathcal{S}$-congruence if and only if the labeled digraph is a K-digraph as defined in [5]. Furthermore, the inclusion relation on these congruences corresponds to the partial ordering of K-digraphs by inclusion of labels on all edges. Thus, the lattice $C_n(G, K\mathcal{S})$ is canonically isomorphic to the lattice $\Omega_n(G, K)$ of Hanlon [5].

It will be useful to relate a K-digraph to the quotient G-set $\hat{x}$ determined by the associated congruence x. Each generator $i \in [n]$ of $F_n(G, K\mathcal{S})$ projects to an element $i^* \in \hat{x}$, and the label of an edge (i, j) is the set of elements $g \in G$ such that $gi^* = j^*$. In particular, i^*

and j^* belong to the same orbit in $\hat{x}$ if and only if i and j belong to the same connected component of the digraph. Also, the stabilizer of i^* is the label of the loop (i, i).

The generalized Dowling lattices $D_n(G, K)$ of [5] consist of those K-digraphs for which at most one connected component has (one of, hence all of) its edges labeled G. In terms of the associated congruence relation x, this means that at most one orbit in the quotient $\hat{x}$ has points stabilized by G, i.e., G has at most one fixed point in $\hat{x}$.

The G, K-sets with at most one fixed point constitute a quasi-variety, which we call G, $K^*\mathcal{S}$. The preceding remarks show that Hanlon's generalized Dowling lattices $D_n(G, K)$ are canonically isomorphic to the congruence lattices $C_n(G, K^*\mathcal{S})$.

4. Möbius identities for G, K-sets

In this section, we compute the characteristic polynomial and a more general several-variable polynomial for the lattices $C_n(G, K\mathcal{S})$. The idea of the computation is based on a well-known computation [7] of the characteristic polynomials of partition lattices and subspace lattices of finite vector spaces.

The characteristic polynomial $\chi_L(\lambda)$ of a finite lattice L is defined in terms of the Möbius function $\mu_L(x, y)$ of L (see for example [3, 7]) and an integer-valued rank function r on L by

$$\chi_L(\lambda) = \sum_{x \in L} \mu_L(0, x)\lambda^{r(1)-r(x)},$$

where 0 and 1 are the bottom and top elements of L.

When L satisfies the chain condition, i.e., when all maximal chains from 0 to any fixed x have the same length, then this length is the natural choice for $r(x)$. But the lattices under consideration here do not satisfy the chain condition except in very special cases, so there is some arbitrariness in the choice of a rank function.

We describe for future reference a slight generalization of the way rank functions on $\Omega_n(G, K)$ are obtained in [5]. Begin with any weakly increasing, conjugacy-invariant, integer-valued function r_K on the lattice K. (In [5], there are some additional conditions on r_K, but they are never needed.) Now if $x \in C_n(G, K\mathcal{S})$, write the quotient $\hat{x}$ as a disjoint union of orbits,

$$\hat{x} = \sum_i \nu_i(x) \cdot G/H_i,$$

where the H_i constitute a system of representatives of the conjugacy classes of subgroups in K. (The notation $\nu_i(x)$ for the number of copies of G/H_i in $\hat{x}$ will be used from now on.) Then the rank $r(x)$ of x in $C_n(G, K\mathcal{S})$ is defined so that the corank plus 1 is additive, i.e.,

$$r(1) - r(x) + 1 = \sum_i \nu_i(x)\,(r_K(G) - r_K(H_i) + 1)\,.$$

Strictly speaking, this defines r only up to an additive constant, but this ambiguity clearly does not affect characteristic functions. An easy calculation shows that Hanlon's definition 2.1.10 in [5] is the result of choosing this constant so that $r(0) = 0$.

As preparation and motivation for our calculations concerning $C_n(G, K\mathcal{S})$, we review the analogous calculations in the two much simpler cases of $C_n(\mathcal{S})$, the nth partition lattice, and $C_n(k\mathcal{V})$, the lattice of subspaces of an n-dimensional vector space over the field k of q elements. (See [7], Chapter 3, Exercises 44 and 45.)

Since $C_n(\mathcal{S})$ satisfies the chain condition, we use its natural rank function; the number of equivalence classes of x, i.e., the cardinality of $\hat{x}$, is the corank of x plus 1. Fix a positive integer λ, and define, for $x \in C_n(\mathcal{S})$,

$$\begin{aligned} f(x) &= \text{the number of maps } [n] \to [\lambda] \text{ with kernel } \supseteq x \\ &= \text{the number of maps } \hat{x} \to [\lambda] \\ &= \lambda^{r(1)-r(x)+1}, \end{aligned}$$

and

$$\begin{aligned} g(x) &= \text{the number of maps } [n] \to [\lambda] \text{ with kernel } = x \\ &= \text{the number of one-to-one maps } \hat{x} \to [\lambda] \\ &= (\lambda)_{r(1)-r(x)+1}, \end{aligned}$$

where the notation $(\lambda)_p$ means $\lambda(\lambda-1)\cdots(\lambda-p+1)$. Clearly,

$$f(x) = \sum_{y \supseteq x} g(y),$$

so by Möbius inversion

$$g(0) = \sum_x \mu(0, x) f(x),$$

where μ is the Möbius function of the partition lattice. Thus,

$$(\lambda)_n = \sum_x \mu(0, x) \lambda^{r(1)-r(x)+1}.$$

Cancelling λ, we have the characteristic polynomial of the partition lattice,

$$\chi_{C_n(\mathcal{S})}(\lambda) = (\lambda - 1)_{n-1}.$$

Since both sides are polynomials, our proof for positive integers λ suffices to establish this equation as a polynomial identity. Notice that by setting $\lambda = 0$ we obtain the well-known formula $\mu(0, 1) = (-1)^{n-1}(n-1)!$.

As a second preparatory example, we apply the same technique with the variety $\mathcal{S}$ of sets replaced by the variety $k\mathcal{V}$ of vector spaces over a field k. The free k-vector space on n generators is simply the n-dimensional vector space k^n. We fix a positive integer m and let $f(x)$, resp. $g(x)$, be the number of linear transformations (i.e., homomorphisms) from k^n to k^m with kernel at least, resp. exactly, x. (If, as at the end of Section 2, we identify a congruence x with the subspace of points equivalent to 0, then kernels in the sense of universal algebra are identified with kernels in the sense of linear algebra.) Then $f(x) = q^{md}$ where q is the cardinality of k and d is the dimension of the quotient $k^n/x = \hat{x}$, the corank of x in the subspace lattice. (This lattice satisfies the chain condition, so we use the customary notion of rank.) $g(0)$, the number of one-to-one homomorphisms from k^n to k^m, is well known to be

$$(q^m - 1)(q^m - q)(q^m - q^2) \cdots (q^m - q^{n-1}).$$

So Möbius inversion gives us

$$(q^m - 1)(q^m - q)(q^m - q^2) \cdots (q^m - q^{n-1}) = \sum_x \mu(0, x)(q^m)^{r(1)-r(x)} = \chi(q^m),$$

where μ and χ refer, of course, to the lattice $C_n(k\mathcal{V})$ isomorphic to the lattice of subspaces of an n-dimensional vector space over k. Therefore,

$$\chi(\lambda) = \prod_{p=0}^{n-1} (\lambda - q^p)$$

holds as a polynomial identity in λ because we have verified it when λ is a power of q. Again, we can set $\lambda = 0$ to obtain a well-known formula: $\mu(0, 1) = (-1)^n q^{n(n-1)/2}$.

We now apply the same technique, systematically replacing sets and vector spaces with G, K-sets. This example is more complicated than the previous ones because, while a finite set can be completely specified by a single integer (its cardinality) and a finite-dimensional k-vector space can also be specified by a single integer (its dimension), a finite G, K-set requires for its specification as many integers as there are conjugacy classes in K, for the specification must tell how many orbits of each of the possible types G/H_i are in the set. This circumstance, though it complicates the calculations, is also the reason why we obtain not only the characteristic function but a multi-variable generalization of it.

We proceed in analogy with the preceding sample calculations, using the quasi-variety $G, K\mathcal{S}$ in place of $\mathcal{S}$ and $k\mathcal{V}$. Thus, x is now an element of $C_n(G, K\mathcal{S})$, and $[\lambda]$ and k^m are replaced by a G, K-set, say

$$\Lambda = \sum_i \lambda_i \cdot G/H_i.$$

We define $f(x)$, resp. $g(x)$ to be the number of homomorphisms (i.e., maps preserving the G-action) from the free G, K-set $F_n(G, K\mathcal{S}) = G \times [n]$ to Λ that have kernel including, resp. equal to x. Notice that, since Λ is a G, K-set, the kernel of any homomorphism to Λ is a $G, K\mathcal{S}$-congruence. So we have again

$$f(x) = \sum_{y \supseteq x} g(y),$$

and by Möbius inversion

$$g(0) = \sum_x \mu(0, x) f(x),$$

where now x and y range over $C_n(G, K\mathcal{S})$ and μ is the Möbius function of this lattice.

To continue, we need formulas for $f(x)$ and $g(0)$. Assume, for convenience, that the representatives H_i of the conjugacy classes in K have been indexed in order of non-decreasing size. In particular, the trivial subgroup $\{e\}$ is H_1, so λ_1 is the number of orbits in Λ where the stabilizers are trivial. The homomorphisms counted by $g(0)$ are just the one-to-one homomorphisms from $G \times [n]$ into Λ. For such a homomorphism, each of the n generators (e, i) of $G \times [n]$ must map to a point with trivial stabilizer (otherwise distinct points (e, i) and (h, i) with h in the stabilizer, would map to the same point in Λ), and distinct generators must map into distinct components of Λ. Any map of the generators satisfying these requirements extends to a one-to-one homomorphism. So to compute $g(0)$, we need only observe that we have $\lambda_1|G|$ possible images for the first generator, that once such an image is chosen we have $(\lambda_1 - 1)|G|$ possible images for the second generator, and so forth. Therefore,

$$g(0) = (\lambda_1)_n |G|^n.$$

To evaluate $f(x)$, we first observe that it is the number of homomorphisms from

$$\hat{x} = \sum_i \nu_i(x) \cdot G/H_i$$

into Λ. Since such a homomorphism can be defined independently on each of the orbits in $\hat{x}$, we have

$$f(x) = \prod_i (\text{number of homomorphisms } G/H_i \to \Lambda)^{\nu_i(x)}.$$

Now a homomorphism from an orbit G/H_i into Λ must map into a single orbit in Λ. Therefore,

$$\begin{aligned} f(x) &= \prod_i \left(\sum_j \lambda_j (\text{number of homomorphisms } G/H_i \to G/H_j)\right)^{\nu_i(x)} \\ &= \prod_i \left(\sum_j M_{ij}\lambda_j\right)^{\nu_i(x)}. \end{aligned}$$

Here M_{ij} is the number of homomorphisms from G/H_i to G/H_j, also known as the *mark* of H_i in G/H_j (see [1], Section 180). Since such a homomorphism is determined by where it sends one element, say eH_i, of G/H_i, and since that image can be any element of G/H_j with stabilizer $\supseteq H_i$, the mark M_{ij} is also the number of H_i-fixed points in G/H_j. As the stabilizer of any $gH_j \in G/H_j$ is gH_jg^{-1}, it follows that

$$M_{ij} = \frac{\text{number of } g \in G \text{ with } H_i \subseteq gH_jg^{-1}}{|H_j|}.$$

In particular, since we have indexed the H_i in order of non-decreasing size, $M_{ij} = 0$ when $j < i$, i.e., $M = (M_{ij})$ is a triangular matrix. Its diagonal entries are the indices of the H_i in their normalizers, hence are non-zero. So M is non-singular.

Summarizing the preceding computation, we have

$$(\lambda_1)_n |G|^n = \sum_x \mu(0, x) \prod_i \left(\sum_j M_{ij}\lambda_j\right)^{\nu_i(x)}, \tag{1}$$

where x ranges over $C_n(G, KS)$. As before, this equation must hold as a polynomial identity in the variables λ_i because it holds for all non-negative integer values of these variables.

We can simplify our result a bit by using the fact that M is non-singular. Because of the non-singularity, we can regard the linear combinations of the λ_i that occur on the right side of (1) as independent variables,

$$\zeta_i = \sum_j M_{ij}\lambda_j;$$

then the λ_1 on the left side is to be regarded as a linear combination of these new variables,

$$\lambda_1 = \sum_i (M^{-1})_{1i}\zeta_i.$$

Now (1) reads

$$(\lambda_1)_n |G|^n = \sum_x \mu(0, x) \prod_i \zeta_i^{\nu_i(x)}, \tag{2}$$

We can get a more useful expression for λ_1 if we notice that the relation between the λ_i and the ζ_j depends only on G and K, not on x or n. In particular, we can put $n = 1$ in (2) to get

$$\lambda_1 |G| = \sum_x \mu(0, x) \prod_i \zeta_i^{\nu_i(x)},$$

where now x ranges over $C_1(G, K\mathcal{S})$. But this lattice is isomorphic to K, because a congruence on $G \times [1] \cong G$ is specified by saying what the stabilizer of 1^* in the quotient shall be. Furthermore, if x is the congruence corresponding to the subgroup $H \in K$, then the $\nu_i(x)$ are all zero except for one which equals 1, namely the one for which H_i and H are conjugate. If we write ζ_H to mean ζ_i for this i, then we have

$$\lambda_1 |G| = \sum_{H \in K} \mu_K(\{e\}, H)\zeta_H. \tag{3}$$

Equation (2) with λ_1 defined by (3) is our generalization of the characteristic polynomial of $C_n(G, K\mathcal{S})$. We show next that it specializes to Hanlon's computation, in Theorem 2.1.14 of [5], of the characteristic polynomial.

Let rank functions r_K on K and r on $C_n(G, K\mathcal{S})$ be as described earlier. Specialize the variables ζ_i to the following functions of a single variable ξ:

$$\zeta_i = \xi^{r_K(G) - r_K(H_i) + 1}.$$

Then (3) becomes

$$\lambda_1 = |G|^{-1}\xi\chi_K(\xi),$$

and so the left side of (2) becomes

$$\prod_{p=0}^{n-1} (\xi\chi_K(\xi) - p|G|).$$

On the right side of (2), the product becomes

$$\prod_i (\xi^{r_K(G) - r_K(H_i) + 1})^{\nu_i(x)} = \xi^{r(1) - r(x) + 1},$$

and so the whole right side of (2) becomes $\xi\chi(\xi)$, where χ is the characteristic polynomial of $C_n(G, K\mathcal{S})$. Equating these specializations of the two sides of (2), we find

$$\chi_{C_n(G,K\mathcal{S})}(\xi) = \xi^{-1}\prod_{p=0}^{n-1}(\xi\chi_K(\xi) - p|G|),$$

which agrees with Theorem 2.1.14 in [5]. After canceling ξ^{-1} against the ξ in the $p = 0$ factor, we can set $\xi = 0$ and recover Hanlon's evaluation, [5] Theorem 2.1.12, of $\mu(0, 1)$.

In the preceding calculations, we used Möbius inversion to express only $g(0)$ in terms of f, because we were interested in the characteristic function, in which $\mu(0, x)$ occurs. But Möbius inversion also expresses $g(a)$ for $a \neq 0$. Thus, we have, for any $a \in C_n(G, K\mathcal{S})$,

$$g(a) = \sum_{x\geq a}\mu(a, x)f(x).$$

To use this equation, we express $f(x)$ in terms of the variables ζ_i as before, and we compute another expression for $g(a)$ as follows. The homomorphisms with kernel exactly a that are counted by $g(a)$ amount to one-to-one homomorphisms from the quotient $\hat{a}$ to Λ. Such a homomorphism must send each of the $\nu_i(a)$ orbits of type G/H_i in $\hat{a}$ bijectively to an orbit of the same type in Λ. (This is because every homomorphism of G-sets sends every orbit onto an orbit.) Furthermore, distinct orbits in $\hat{a}$ must map to distinct orbits in Λ. Finally, the number of G-homomorphisms from one orbit of type G/H_i onto another is the mark M_{ii}. Combining these observations, we find that

$$g(a) = \prod_i(\lambda_i)_{\nu_i(a)}M_{ii}^{\nu_i(a)}.$$

Therefore,

$$\sum_{x\geq a}\mu(a, x)\prod_i\zeta_i^{\nu_i(x)} = \prod_i(\lambda_i)_{\nu_i(a)}M_{ii}^{\nu_i(a)}.$$

Here, as in the previous calculation, the λ_i are to be regarded as linear functions of the ζ_i. As before, we obtain a one-variable result, describing the characteristic polynomial of the part of $C_n(G, K\mathcal{S})$ above a, by specializing ζ_i to $\xi^{r_K(G)-r_K(H_i)+1}$ and dividing by ξ.

$$\sum_{x\geq a}\mu(a, x)\xi^{\mathrm{corank}(x)} = \xi^{-1}\prod_i(\lambda_i)_{\nu_i(a)}M_{ii}^{\nu_i(a)}, \tag{4}$$

where now the λ_i are regarded as functions of ξ.

Some information about the Möbius function μ can be easily extracted from this equation by considering the constant terms on both sides. On the left, the constant term is simply $\mu(a, 1)$ provided $r_K(G) > r_K(H_i)$ for all $H_i \neq G$ (so that only 1 has corank 0). On the right, we notice that every λ_i is divisible by ξ (because it is a linear combination of the ζ_j, which are all divisible by ξ), so if two or more of the $\nu_i(a)$ are non-zero then the product on the right will be divisible by ξ^2 and the right side will not have a constant term. This proves that, if two orbits in $\hat{a}$ are not isomorphic, then $\mu(a, 1) = 0$.

Let us now consider the case that all the orbits in $\hat{a}$ are isomorphic, say of type G/H_i (so i is now fixed), and let us write simply ν for the number $\nu_i(a)$ of these orbits. Now (4) simplifies to

$$\sum_{x \geq a} \mu(a, x)\xi^{\mathrm{corank}(x)} = \xi^{-1}(\lambda_i)_\nu M_{ii}^{\nu}. \tag{5}$$

As before, we can obtain a useful formula for λ_i by considering (5) in the very special case that $n = \nu = 1$ and $\hat{a} = G/H_i$. We find

$$\begin{aligned}\xi^{-1}\lambda_i M_{ii} &= \sum_{H \supseteq H_i} \mu_K(H_i, H)\xi^{\mathrm{corank}(H)} \\ &= \chi_{K,i}(\xi),\end{aligned}$$

so

$$\lambda_i = M_{ii}^{-1}\xi\chi_{K,i}(\xi), \tag{6}$$

where $\chi_{K,i}$ means the characteristic function of the part of the lattice K lying above H_i. Formulas (5) and (6) express the characteristic function of the part of $C_n(G, K\mathcal{S})$ above a in terms of that of the part of K above H_i. Specifically, we have

$$\sum_{x \geq a} \mu(a, x)\xi^{\mathrm{corank}(x)} = \chi_{K,i}(\xi)\prod_{p=1}^{\nu-1}(\xi\chi_{K,i}(\xi) - pM_{ii}).$$

By considering the constant terms on both sides (i.e., by setting $\xi = 0$) and assuming as before that only G has corank 0 in K, we find

$$\mu(a, 1) = \mu_K(H_i, G)(-M_{ii})^{\nu-1}(\nu - 1)!.$$

5. Möbius identities for generalized Dowling lattices

In this section, we compute the characteristic polynomials of Hanlon's generalized Dowling lattices, $D_n(G, K)$. We saw in Section 3 that $D_n(G, K)$ is isomorphic to $C_n(G, K^*\mathcal{S})$, where $G, K^*\mathcal{S}$ is the quasi-variety of G, K-sets with at most one fixed point. We apply to this quasi-variety the same technique that we used for $G, K\mathcal{S}$ in Section 4, and for this purpose we retain notations like $\hat{x}$ and $\nu_i(x)$ from there.

Let c be the number of conjugacy classes of groups in K. By our convention that the conjugacy class representatives H_i are listed in order of non-decreasing size, $H_c = G$. Thus, if x is a $G, K^*\mathcal{S}$-congruence on the free algebra $G \times [n]$, then $\nu_c(x) \leq 1$.

Exactly the same reasoning as in Section 4 shows that, for any $\Lambda \in G, K^*\mathcal{S}$,

$$(\lambda_1)_n|G|^n = \sum_x \mu(0, x)\prod_i \left(\sum_j M_{ij}\lambda_j\right)^{\nu_i(x)}, \tag{1}$$

where now x ranges over $G, K^*\mathcal{S}$-congruences, and where the λ_i are no longer quite arbitrary non-negative integers since λ_c, the number of fixed points in Λ, is at most 1. As in Section 4, we introduce the ζ_i as new variables. Of course, the constraint that $\lambda_c = 0$ or 1 now becomes a constraint on the ζ_i, which is easily seen to be simply that $\zeta_c = 0$ or 1. (The reason is that the marks M_{cj} are 0 for $j \neq c$ while $M_{cc} = 1$, so $\zeta_c = \lambda_c$.) Thus, we have

$$(\lambda_1)_n |G|^n = \sum_x \mu(0, x) \prod_i \zeta_i^{\nu_i(x)}, \tag{2}$$

as a polynomial identity in the variables ζ_i for $i \neq c$, where ζ_c is 0 or 1. Here, as before,

$$\lambda_1 |G| = \sum_{H \in K} \mu_K(\{e\}, H) \zeta_H. \tag{3}$$

Formulas (2) and (3) constitute our generalization of the characteristic polynomial of the Dowling lattice.

To specialize from $c-1$ variables down to one variable and obtain the ordinary characteristic polynomial, we must proceed slightly differently than in Section 4. This is because the definition of rank in Dowling lattices $D_n(G, K)$ is slightly different from that in $\Omega_n(G, K)$. Given a weakly increasing, conjugacy-invariant, integer-valued function r_K on K, as before, use it to define (up to an irrelevant additive constant) a notion of rank on $C_n(G, K^*\mathcal{S})$ by making the corank (not the corank plus one as in Section 4) additive:

$$r(1) - r(x) = \sum_i \nu_i(x) \, (r_K(G) - r_K(H_i)).$$

When the additive constant is chosen to make $r(0) = 0$, this agrees with Definition 2.2.2 of [5].

To match this notion of rank, we specialize our ζ_i to functions of one variable ξ by setting

$$\zeta_i = \xi^{r_K(G) - r_K(H_i)}.$$

Notice that, since $H_c = G$, we have $\zeta_c = 1$, so (2) is valid for this specialization.

The specialization gives in (3)

$$\lambda_1 |G| = \chi_K(\xi),$$

(without the factor ξ that was present in Section 4), so the left side of (2) becomes

$$\prod_{p=0}^{n-1} (\chi_K(\xi) - p|G|).$$

The right side of (2) becomes simply the characteristic polynomial of $C_n(G, K^*\mathcal{S})$ (again without the previously present factor ξ), and so we have

$$\chi_{C_n(G, K^*\mathcal{S})}(\xi) = \prod_{p=0}^{n-1} (\chi_K(\xi) - p|G|),$$

in agreement with Theorem 2.2.4 of [5].

As in Section 4, we can extend the preceding results by replacing 0 by an arbitrary $a \in C_n(G, K^*\mathcal{S})$. To do so, we must compute $g(a)$, the number of homomorphisms from the free algebra to Λ with kernel exactly a, or, equivalently, the number of one-to-one homomorphisms from $\hat{a}$ to Λ. Under such a homomorphism, for each i, each of the $\nu_i(a)$ copies of G/H_i in $\hat{a}$ maps to one of the λ_i copies of G/H_i in Λ. There are $(\lambda_i)_{\nu_i(a)}$ ways to choose these copies, since they must be distinct, and, once the copies are chosen, there are M_{ii} ways to choose the mapping in each copy. Therefore,

$$\sum_{x \geq a} \mu(a, x) \prod_i \zeta_i^{\nu_i(x)} = g(a) = \prod_i (\lambda_i)_{\nu_i(a)} M_{ii}^{\nu_i(a)}. \tag{4}$$

If we specialize by setting $\zeta_i = \xi^{\mathrm{corank}(H_i)}$ as before, and thus regard the λ_i as polynomials in ξ, then the left side of (4) becomes the characteristic function of the part of $C_n(G, K^*S)$ above a.

Let us compare the constant terms here, assuming as before that $r_K(H) = r_K(G)$ only when $H = G$. Then $\xi^{\mathrm{corank}(x)} = 1$ only when all orbits of $\hat{x}$ have stabilizer G; but as $\hat{x} \in C_n(G, K^*S)$ there can only be one such orbit, so $x = 1$. Thus, the constant term on the left side is simply $\mu(a, 1)$. On the right, since $\lambda_i = \sum_j (M^{-1})_{ij} \zeta_j$ and since we have just seen that setting $\xi = 0$ annihilates all the ζ_j except ζ_c which becomes 1, we find that the constant term in λ_i is $(M^{-1})_{ic}$. Inserting this into the right side of (4), we obtain

$$\mu(a, 1) = \prod_i ((M^{-1})_{ic})_{\nu_i(a)} \cdot (M_{ii})^{\nu_i(a)}. \tag{5}$$

(It may be reassuring to note that, since M is triangular and M_{ii} divides M_{ki} for all i and k, each product $(M^{-1})_{ic} M_{ii}$ is an integer, and therefore so is the right side of (5).)

6. A monoid example

In this section, we apply the same method to another variety, the variety of sets with an idempotent self-map. This is the simplest case of a variety of G-sets where G is not a group but a monoid, i.e., a set with an associative, binary operation with unit e (but without inverses in general). An action of a monoid G on a set A is defined just as for groups: a map $G \times A \to A: (g, a) \mapsto ga$ with $g(g'a) = (gg')a$ and $ea = a$.

For our example, we take for G the smallest monoid that is not a group, namely $\{0, 1\}$ with the operation of multiplication. An action of this monoid on a set A is specified by telling how 0 acts (since 1 must act as the identity map), and the action of 0 on A is a function $0: A \to A$ satisfying $0^2 = 0$. That is, 0 fixes all points in its range. A G-set A can be specified by giving a set A of points, a partition P of A, and a choice of one *special* element from each block of P; the action of 0 on A takes each element to the special element in the same block.

For each non-negative integer j, we write j^+ for the G-set consisting of one block of size $j + 1$ (so the label j indicates the number of *non-special* elements), i.e., the G-set of size $j + 1$ with 0 acting as a constant map. Any finite G-set is a disjoint union of copies of such j^+s, and we write $\sum_j n_j \cdot j^+$ to indicate such a disjoint union with n_j copies of each j^+; here the n_j are non-negative integers, and all but finitely many of them are zero. In this notation, the free G-set on n generators is $n \cdot 1^+$, the generators being the n non-special elements.

A congruence on this free algebra can be regarded as consisting of the following data. First, there is an equivalence relation R on $[n]$ indicating which of the n copies of 1^+ are to be in the same blocks in the quotient; this also determines which of the special elements have equal images in the quotient. Second, there is an equivalence relation E indicating which of the non-special elements are identified with each other in the quotient. As an identification between two non-special elements forces the identification of their special images under the action of 0, we must have $E \subseteq R$. Finally, within each R-equivalence class, one E-equivalence class may be singled out, to indicate that the non-special elements in the latter have been identified not only with each other but also with the special elements of the former. Thus, a congruence of the free G-set on n generators can be regarded as a triple (E, R, M) where $E \subseteq R$ are equivalence relations on $[n]$ and M is a set of E-classes containing at most one E-class in each R-class.

The covers of (E, R, M) in the lattice $C_n(GS)$ of congruences of the free algebra are obtained from (E, R, M) in the following three ways.

(1) Enlarge E by merging into a single equivalence class two E-classes within the same R-class. Leave R and M unchanged.
(2) Enlarge R by merging two of its equivalence classes into one. If both contained members of M then these two members of M are to be merged, the result being in the new M. Apart from this, E and M are unchanged.
(3) Add to M an E-class (in an R-class that doesn't already contain a member of M). Leave E and R unchanged.

It follows easily from this description of the covering relation that the lattice $C_n(GS)$ satisfies the chain condition. So we have a natural rank function, assigning to each congruence x the rank $2n - |\hat{x}|$, so that the corank of x plus 1 is the cardinality $|\hat{x}|$ of the quotient. We shall calculate the characteristic function of $C_n(GS)$ with this rank function; in fact, we shall obtain a bit more information. Let us write $\kappa(x)$ for the number of blocks in $\hat{x}$ (i.e., the number of points in the range of the action of 0) and $\rho(x)$ for the number of points in $\hat{x}$ moved by the action of 0. Thus, $\kappa + \rho - 1$ is the corank of x under the natural rank function described above. We shall evaluate the two-variable characteristic polynomial that keeps track of these two parts of the corank separately, namely,

$$\sum_x \mu(0, x)\lambda^{\kappa(x)}\xi^{\rho(x)}. \tag{4}$$

Then the ordinary characteristic polynomial

$$\sum_x \mu(0, x)\xi^{\mathrm{corank}(x)}$$

is obtained from (4) by setting $\lambda = \xi$ and dividing by ξ.

To begin the computation of (4), temporarily fix two non-negative integers λ and q, and let $\Lambda = \lambda \cdot q^+$. We shall, as in previous calculations, use Möbius inversion to relate the number of one-to-one homomorphisms of G-sets $n \cdot 1^+ \to \Lambda$ and the numbers of homomorphisms $n \cdot 1^+ \to \Lambda$ whose kernels include specific congruences x.

For a one-to-one homomorphism $n \cdot 1^+ \to \Lambda$, each of the n generators (non-special elements) a of $n \cdot 1$ must map to a non-special element of Λ, for otherwise a would map to the same image as $0a$. Furthermore, distinct generators a and b must map into distinct blocks in Λ, for otherwise the distinct elements $0a$ and $0b$ would have the same image. Therefore, if we specify images for the generators of $n \cdot 1^+$ one at a time, we have λq possibilities for the first (any non-special element of Λ), $(\lambda - 1)q$ for the second (any non-special element not in the same block as the image of the first generator), etc. Thus, the number of one-to-one homomorphisms $n \cdot 1^+ \to \Lambda$ is $(\lambda)_n q^n$.

The number of homomorphisms $n \cdot 1^+ \to \Lambda$ whose kernels include a specific congruence x is the number of homomorphisms $\hat{x} \to \Lambda$. We write $\nu_j(x)$ for the number of j^+ blocks in $\hat{x}$. Since homomorphisms $\hat{x} \to \Lambda$ can be specified independently on each block, the number of such homomorphisms is

$$\prod_j (\text{number of homomorphisms } j^+ \to \Lambda)^{\nu_j(x)}.$$

A homomorphism $j^+ \to \Lambda$ must map into a single block of Λ, and there are λ blocks to choose from. Once one of them is specified, the special point in j^+ must map to the special point in this block, and each of the j other points in j^+ can map to any of the $q + 1$ points in this block. Thus, the product above becomes

$$\prod_j (\lambda \cdot (q+1)^j)^{\nu_j(x)}.$$

But $\sum_j \nu_j(x)$ is just the number of blocks in $\hat{x}$, which we called $\kappa(x)$, and $\sum_j j\nu_j(x)$ is the number of non-special points in $\hat{x}$, which we called $\rho(x)$. So the product simplifies to

$$\lambda^{\kappa(x)}(q+1)^{\rho(x)}.$$

By Möbius inversion, we obtain

$$(\lambda)_n q^n = \sum_x \mu(0, x)\lambda^{\kappa(x)}(q+1)^{\rho(x)}.$$

To obtain the two-variable characteristic function (4), we substitute $\xi - 1$ for q, obtaining $(\lambda)_n(\xi - 1)^n$. As indicated earlier, the one-variable characteristic function is obtained by setting $\lambda = \xi$ and dividing by ξ, so it is $(\xi - 1)_{n-1}(\xi - 1)^n$.

In Sections 4 and 5, we obtained generalized characteristic functions of many variables, the number of variables (c or $c - 1$) being the number of arbitrary non-negative integer parameters needed to specify a finite algebra in the quasi-variety. In the present context, since infinitely many integer parameters n_j (albeit almost all zero) are needed to specify a finite G-set $\sum_j n_j \cdot j^+$, we might expect to obtain more than the two-variable characteristic functions calculated above. It is indeed possible in principle to obtain more general results, by using a Λ whose components are not all the same size. Unfortunately, for general Λ, the computation of $g(0)$ appears intractable.

7. A non-algebraic example

Although we have presented calculations of characteristic functions in the context of algebraic structures, i.e., sets with operations, the technique is also applicable to more general structures of the sort studied in first-order logic, namely sets with both operations and relations.

The basic concepts used when studying algebras extend naturally to these first-order structures. A substructure of a structure is determined by any subset closed under the specified operations; both the operations and the relations are then simply restricted to that subset. The product of a family of structures (of the same signature) is the product of the sets, with operations and relations defined componentwise; for relations this means that a tuple of elements of the product satisfies a relation if and only if all the tuples of corresponding components satisfy the relation in the factor structures. A homomorphism is a function from one structure to another that commutes with the operations and preserves the relations; preservation means that, if a relation holds of certain elements in the domain structure, then the corresponding relation holds of their images in the target structure, but not necessarily conversely.

This "not necessarily conversely" implies that the target of a surjective homomorphism is in general not determined by the domain and the kernel congruence. For example, if a structure is modified by enlarging some of its relations (without changing the operations), then the identity function is a homomorphism from the original structure to the modified one. (It is not an isomorphism, as its inverse is not a homomorphism.) Because of this, it is natural to replace the congruence lattices considered in universal algebra with lattices of "generalized congruences" which describe not only which elements of a structure are to be identified in a quotient but also what the relations on the quotient structure are. A simple way to make this precise is to define a *generalized congruence* on a structure A to be a congruence x together with relations on the quotient $A/x = \hat{x}$ making the canonical projection $A \twoheadrightarrow \hat{x}$ a homomorphism. The generalized congruences on A form a lattice if we define $x \leq y$ to mean that the canonical projection from A to $\hat{y}$ factors (by a homomorphism, of course) through the one to $\hat{x}$.

The role of conditional equations in universal algebra is played in the present context by strict universal Horn sentences, i.e., sentences of the form

$$\forall x_1 \, \forall x_2 \cdots \forall x_k \, ((E_1 \text{ and } E_2 \text{ and } \cdots \text{ and } E_n) \Rightarrow F), \tag{1}$$

where the E_i are now either equations (as before) or atomic assertions about the relations, i.e., assertions to the effect that a particular relation holds of certain terms. For any collection of such sentences (1), the class $\mathcal{V}$ of structures satisfying them is closed under isomorphisms, substructures and products, and conversely all classes with these closure properties can be axiomatized by strict universal Horn sentences, just as in the algebraic situation. (The main difference is terminological; "quasi-variety" is traditionally used only in the algebraic case, while for general structures one refers to "strict universal Horn classes.")

For a structure A in a strict universal Horn class $\mathcal{V}$, we call a generalized congruence x on A a $\mathcal{V}$-congruence if the quotient structure $\hat{x}$ is in $\mathcal{V}$. Because $\mathcal{V}$ is closed under substructures and products, the meet of any family of $\mathcal{V}$-congruences on A is again a $\mathcal{V}$-congruence. So the $\mathcal{V}$-congruences on A form a complete lattice.

For any strict universal Horn class $\mathcal{V}$ and any set S, there is a free $\mathcal{V}$-structure $F_{\mathcal{V}}(S)$ on the set S of generators. Its members are given by terms built from elements of S by means of the operations of $\mathcal{V}$, two terms being identified if and only if this is required by the strict universal Horn sentences axiomatizing $\mathcal{V}$, and relations holding of tuples of (equivalence classes of) terms if and only if this is required by those same axioms. We write $C_n(\mathcal{V})$ for the lattice of $\mathcal{V}$-congruences of the free $\mathcal{V}$-structure on n generators. This is the analog in the present context of the congruence lattices associated to quasi-varieties in the preceding sections.

The rest of this section is devoted to one example, in which a familiar set of strict universal Horn axioms defines a class $\mathcal{V}$ for which the associated generalized congruence lattices have a pleasant combinatorial interpretation, related to but slightly more natural than the congruence lattices obtained from actions of the two-element monoid in the preceding section.

The axioms for our example are simply the definition of an equivalence relation. That is, $\mathcal{V}$ is the class of structures that consist of a set A together with an equivalence relation on it. The conditions of reflexivity, symmetry, and transitivity that define the notion of equivalence relation are of the form (1) (with $n = 0, 1$, and 2, respectively), so the preceding discussion applies to $\mathcal{V}$. The free $\mathcal{V}$-structure on n generators is simply an n-element set $[n]$ with the equality relation. (Recall that a relation holds between elements of a free structure only when this is required by the axioms, so the relation here is as small as possible. The same set with a larger equivalence relation would be a proper quotient of the free algebra.) A generalized congruence x on any structure in $\mathcal{V}$ amounts to specifying, first, which elements of the given structure are to become equal in the quotient $\hat{x}$ (a congruence) and, second, which elements are to become equivalent in the (equivalence relation that is part of the) quotient. In particular, if the structure we began with is the free one on n generators, then an element x of $C_n(\mathcal{V})$ amounts to a pair (E, R) of equivalence relations on $[n]$ with $E \subseteq R$. The lattice structure on $C_n(\mathcal{V})$ is given by ordering these pairs componentwise.

As in previous sections, we intend to use Möbius inversion to compute a generalized characteristic polynomial. The lattice $C_n(\mathcal{V})$ satisfies the chain condition, so we use the standard rank function, which can be written as

$$r(x) = 2n - \varepsilon(x) - \rho(x),$$

where $\varepsilon(x)$ and $\rho(x)$ are the numbers of elements and of equivalence classes, respectively, in $\hat{x}$. Thus, if $x = (E, R)$, then they are the numbers of equivalence classes of E and R, respectively. We shall obtain a generalized characteristic polynomial in which the partial coranks ε and ρ are treated separately.

Let Λ be the $\mathcal{V}$-structure consisting of a set of size $\lambda\xi$ with an equivalence relation dividing it into λ equivalence classes of size ξ. For $x \in C_n(\mathcal{V})$, let (by analogy with previous computations)

$$\begin{aligned} f(x) = {} & \text{number of homomorphisms } F_n(\mathcal{V}) \to \Lambda \\ & \text{that factor through the canonical projection to } \hat{x} \end{aligned}$$

and

$$\begin{aligned} g(x) = {} & \text{number of homomorphisms } F_n(\mathcal{V}) \to \Lambda \\ & \text{that factor through the canonical projection to } \hat{x} \\ & \text{via an isomorphism from } \hat{x} \text{ to a substructure of } \Lambda. \end{aligned}$$

Then we have

$$f(x) = \sum_{y \geq x} g(y)$$

and therefore by Möbius inversion

$$g(0) = \sum_{x} \mu(0, x) f(x), \tag{2}$$

where μ is the Möbius function of $C_n(\mathcal{V})$.

For any $x \in C_n(\mathcal{V})$ and any positive integer i, let $\nu_i(x)$ be the number of equivalence classes of size i in $\hat{x}$. Thus,

$$\sum_i \nu_i(x) = \rho(x) \quad \text{and} \quad \sum_i i\nu_i(x) = \varepsilon(x). \tag{3}$$

Furthermore, $f(x)$, the number of homomorphisms $\hat{x} \to \Lambda$, can be expressed in terms of the ν_i as follows. To specify such a homomorphism, one must specify, for each i and each of the $\nu_i(x)$ equivalence classes P of size i in $\hat{x}$, one equivalence class in Λ for P to map into (λ possible choices) and where in this class each of the i members of P is to be mapped (ξ^i possible choices). Therefore,

$$\begin{aligned} f(x) &= \prod_i (\lambda \cdot \xi^i)^{\nu_i(x)} \\ &= \lambda^{\rho(x)} \xi^{\varepsilon(x)}, \end{aligned}$$

by (3).

$g(0)$ is the number of functions $h\colon [n] \to \Lambda$ for which all the values $h(k)$ are inequivalent. Thus, there are $\lambda\xi$ choices for $h(1)$, $(\lambda-1)\xi$ choices for $h(2)$, etc. Therefore, $g(0) = (\lambda)_n \xi^n$. Inserting the formulas for $f(x)$ and $g(0)$ into (2), we obtain our generalized characteristic polynomial for $C_n(\mathcal{V})$,

$$\sum_x \mu(0, x) \lambda^{\rho(x)} \xi^{\varepsilon(x)} = (\lambda)_n \xi^n.$$

To obtain the ordinary characteristic polynomial, we set $\xi = \lambda$ and divide by λ^2, since the corank of x is

$$r(1) - r(x) = \varepsilon(x) + \rho(x) - 2.$$

We obtain

$$\sum_x \mu(0, x) \lambda^{r(1)-r(x)} = (\lambda - 1)_{n-1} \lambda^{n-1}.$$

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