Commutative Association Schemes Whose Symmetrizations Have Two Classes*

SUNG Y. SONG   sung@pollux.math.iastate.edu
Department of Mathematics, Iowa State University, Ames, IA 50011

Abstract. If a symmetric association scheme of class two is realized as the symmetrization of a commutative association scheme, then it either admits a unique symmetrizable fission scheme of class three or four, or admits three fission schemes, two of which are class three and one is of class four. We investigate the classification problem for symmetrizable (commutative) association schemes of two-class symmetric association schemes. In particular, we give a classification of association schemes whose symmetrizations are obtained from completely multipartite strongly regular graphs in the notion of wreath product of two schemes. Also the cyclotomic schemes associated to Paley graphs and their symmetrizable fission schemes are discussed in terms of their character tables.

Keywords: cyclotomic association scheme, strongly regular graph, wreath product

1. Introduction and preliminary

Although every nonsymmetric commutative association scheme essentially has one symmetrization, there are many symmetric schemes that are realized as symmetrizations for many commutative schemes while others that do not admit any symmetrizable fission schemes at all. As a continuation of the work in [13] on classification of symmetrizable commutative association schemes, in this paper we investigate four-class symmetrizable fission schemes for two- or three-class symmetric schemes.

In this section, we give some terminology and the notion of the wreath product of two schemes, and recall two basic lemmas related to fusions and fissions of association schemes. In Section 2, we give a method for obtaining symmetrizable fission schemes for completely multipartite strongly regular graphs and classify such schemes of class four. Finally in Section 3 we discuss the four-class fission schemes of Paley graphs.

1.1. Character tables

Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be a commutative association scheme. Let $A_0, A_1, \ldots, A_d$ be the adjacency matrices of the scheme $\mathfrak{X}$ and let $E_0, E_1, \ldots, E_d$ be the primitive idempotents of the Bose-Mesner algebra $\mathfrak{A} = \langle A_0, A_1, \ldots, A_d \rangle = \langle E_0, E_1, \ldots, E_d \rangle$ of $\mathfrak{X}$. The character table $P = (p_j(i))$ of $\mathfrak{X}$ is the $(d+1) \times (d+1)$ matrix given by

$$A_j = \sum_{i=0}^{d} p_j(i) E_i.$$

The (i, j)-entry $p_j(i)$ of P is characterized by the relation $A_j E_i = p_j(i) E_i$. We have $p_0(i) = 1$ and $p_j(0) = k_j = |\{y \in X \mid (x, y) \in R_j\}|$ for all i, j in $\{0, 1, \dots, d\}$. The rank of E_i is called the multiplicity m_i, $i = 0, 1, \dots, d$.

We can calculate the character tables of fusion schemes using the following lemma, which is due to Bannai [1], Johnson & Smith [10], and Muzichuk [12], if the fission tables are known.

Lemma 1.2 *Let* $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ *be a commutative scheme, and* $\{\wedge_\alpha\}_{0 \le \alpha \le \delta}$ *be a partition of* $\{0, 1, \dots, d\}$ *such that* $\wedge_0 = \{0\}$. *Suppose for every* $\alpha \in \{0, 1, \dots, \delta\}$, $\bigcup_{i \in \wedge_\alpha} R_{i'} = \bigcup_{j \in \wedge_{\alpha'}} R_j$ *for some* $\alpha' \in \{0, 1, \dots, \delta\}$. *Then the partition gives rise to a fusion scheme* $\tilde{\mathfrak{X}} = (X, \{\tilde{R}_\alpha\}_{0 \le \alpha \le \delta})$ *with* $\tilde{R}_\alpha = \bigcup_{i \in \wedge_\alpha} R_i$ *if and only if there exists a partition* $\{\wedge_\alpha^*\}_{0 \le \alpha \le \delta}$ *of* $\{0, 1, \dots, d\}$ *with* $\wedge_0^* = \{0\}$ *such that each* $(\wedge_\beta^*, \wedge_\alpha)$-*block of the character table* P *of* $\mathfrak{X}$ *has a constant row sum. In this case, the constant row sum* $\sum_{j \in \wedge_\alpha} p_j(i)$ *for* $i \in \wedge_\beta^*$ *of the block* $(\wedge_\beta^*, \wedge_\alpha)$ *is the* (β, α)-*entry* $\tilde{p}_\alpha(\beta)$ *of the fusion table* $\tilde{P}$.

For a given symmetric scheme $\tilde{\mathfrak{X}}$, a nonsymmetric commutative fission scheme $\mathfrak{X}$ of $\tilde{\mathfrak{X}}$ is called symmetrizable fission scheme of $\tilde{\mathfrak{X}}$ if $\tilde{\mathfrak{X}}$ can be obtained from $\mathfrak{X}$ by fusing every non-self-paired class with its conjugate counterpart. For the calculation of the character tables for the symmetrizable fission schemes, the following lemma in [3, Theorem 2.3] is useful.

Lemma 1.3 *Let* $\tilde{\mathfrak{X}} = (X, \{\tilde{R}_\eta\}_{0 \le \eta \le \delta})$ *be a commutative association scheme with* $|X| = n$. *Let* $\mathfrak{X} = (X, \{R_i\}_{i \in I})$ *where* $I = \{0, 1, \dots, \delta\} \cup \{d\}$ *be a* $(\delta + 1)$-*class commutative fission scheme of* $\tilde{\mathfrak{X}}$ *such that* $\tilde{R}_\nu$ *and* $\tilde{E}_\mu$ *of* $\tilde{\mathfrak{X}}$ *split into pairs as* $\tilde{R}_\nu = R_\nu \cup R_d$ *and* $\tilde{E}_\mu = E_\mu \oplus E_d$ *in* $\mathfrak{X}$. *If* $R_\nu' = R_d$, *then the character table* $P = (p_j(i))$ *of* $\mathfrak{X}$ *is determined from* $\tilde{P} = (\tilde{p}_\eta(\xi))$ *as follows:*

$$\begin{cases} p_j(i) = \tilde{p}_j(i) \\ p_j(\mu) = p_j(d) = \tilde{p}_j(\mu) \\ p_\nu(i) = \tilde{p}_\nu(i) k_\nu \cdot \tilde{k}_\nu^{-1} \\ p_d(i) = \tilde{p}_\nu(i) k_d \cdot \tilde{k}_\nu^{-1}, \text{ for } i \in I - \{\mu, d\}, \quad j \in I - \{\nu, d\}, \text{ and} \end{cases}$$

$$p_\nu(\mu) = p_d(d) = \overline{p_\nu(d)} = \overline{p_d(\mu)} = \tfrac{1}{2}\{\tilde{p}_\nu(\mu) + (-n \cdot \tilde{k}_\nu \cdot \tilde{m}_\nu^{-1})^{1/2}\}.$$

Lemma 1.3 will be used repeatedly whenever we need to calculate fission tables for putative symmetrizable fission schemes for given symmetric association schemes. If a fission table is realized as the character table of an association scheme, we want to describe the scheme and classify it if it is possible. We now recall the notion of wreath product and some examples of association schemes that will be used in our discussion.

1.4. Wreath products

Let $A_0, A_1, \dots, A_d$ be the adjacency matrices of an association scheme $\mathfrak{X}$. The matrix $R(\mathfrak{X}) = \sum_{i=1}^d i A_i$ is called the relation table of $\mathfrak{X}$. The wreath product of two association schemes is defined in terms of the relation tables of the two schemes as follows. Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ with $|X| = n$, $\mathfrak{Y} = (Y, \{S_j\}_{0 \le j \le e})$ with $|Y| = m$ be two association

schemes. Let W be the $(d + e)$-class association scheme on $X \times Y$ whose relation table is given by

$$R(W) = I_m \otimes (R(\mathfrak{X}) + e(J_n - I_n)) + R(\mathfrak{Y}) \otimes J_n$$

where I_ℓ and J_ℓ denote the $\ell \times \ell$ identity matrix and the all-1 matrix, respectively, and $A \otimes B = (a_{ij}B)$, the tensor product of $A = (a_{ij})$ and B. The scheme W is called the wreath product of $\mathfrak{X}$ with $\mathfrak{Y}$, and denoted by $W = \mathfrak{X} \int \mathfrak{Y}$. (The notion of wreath product is due to [14, pp. 45–47].)

1.5. $\mathfrak{X}(K_n)$

The complete graph on n vertices is denoted by K_n and the (trivial) association scheme given by the relation table $J_n - I_n$ is denoted by $\mathfrak{X}(K_n)$. The character table of the scheme $\mathfrak{X}(K_n)$ is given by $P(\mathfrak{X}(K_n)) = \begin{pmatrix} 1 & n-1 \\ 1 & -1 \end{pmatrix} \begin{matrix} m_0 = 1 \\ m_1 = n - 1 \end{matrix}$.

1.6. Group schemes $\mathfrak{X}(G)$

For a given finite group G with its conjugacy classes $C_0 = \{1\}, C_1, \ldots, C_d$, the scheme defined by $(a, b) \in R_i$ iff $ba^{-1} \in C_i, i = 0, 1, \ldots, d$, is called the group scheme of G and denoted by $\mathfrak{X}(G) = (G, \{R_i\}_{0 \leq i \leq d})$. Suppose $f_0 = 1, f_1, f_2, \ldots, f_d$ are degrees, of distinct irreducible characters of G. Then the multiplicities m_i of $\mathfrak{X}(G)$ are given by f_i^2 and $k_i = |C_i|, i = 0, 1, \ldots, d$. In this case, the character table $P(\mathfrak{X}(G))$ is given by $F^{-1} \cdot T \cdot K$, where T is the group character table of G, $F^{-1} = \text{diag}[1, f_1^{-1}, \ldots, f_d^{-1}]$, and $K = \text{diag}[1, k_1, \ldots, k_d]$.

1.7. Cyclotomic schemes $\mathfrak{X}(d, GF(q))$

Let q be a prime power number, and d be a positive integer that divides $q - 1$. Let α be a generator of the multiplicative group $GF(q)^*$, and H be the cyclic subgroup generated by α^d. Thus $[GF(q)^* : H] = d$, and its cosets are $\{H\alpha^i \mid i = 0, 1, \ldots, d - 1\}$. Define $R_0 = \{(x, x) \mid x \in GF(q)\}$, and $R_i = \{(x, y) \mid x - y \in H\alpha^{i-1}\}$, for $i = 1, 2, \ldots, d$. Then $\mathfrak{X}(d, GF(q)) = (GF(q), \{R_i\}_{0 \leq i \leq d})$ forms an association scheme and is called the cyclotomic scheme of class d over $GF(q)$.

The character table of $\mathfrak{X}(d, GF(q))$ can be derived from the group character table for the elementary abelian group $GF(q)$ by a fusion. Let T be the character table of the group $GF(q)$ whose rows are indexed by the set of additive characters and the columns are indexed by the elements of $GF(q)$. Take the coset decomposition by the subgroup $H = \langle \alpha^d \rangle$ as a partition of $GF(q) - \{0\}$. From the table T, first we combine (add) the corresponding columns in each part according to the partition $\{\{0\}, H, H\alpha, \ldots, H\alpha^{d-1}\}$, and then from the resulting $q \times (d + 1)$ table, delete all the duplicated rows leaving $d + 1$ distinct rows. (There are exactly $d + 1$ distinct rows!) The character table P of $\mathfrak{X}(d, GF(q))$ is the $(d + 1) \times (d + 1)$ fusion table of T. The rows of P can be rearranged in such a way that

$$P = \begin{pmatrix} 1 & k & \ldots & k \\ \vdots & & P_0 & \\ 1 & & & \end{pmatrix}$$

where $k = \frac{q-1}{d}$, $P_0 = \sum_{i=0}^{d-1} \phi_i C^i$ with

$$C = \begin{pmatrix} 0 & 1 & & & 0 \\ & \ddots & \ddots & & \\ & & 0 & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

Here the number ϕ_i's are known as Gaussian periods and given by

$$\phi_i = \sum_{\beta \in H\alpha^{i-1}} \chi(\beta)$$

for a fixed nontrivial additive character χ. Finally, we note that the cyclotomic scheme $\mathfrak{X}(d, GF(q))$ is symmetric if and only if $\frac{q-1}{d}$ is even or q is a power of 2.

1.8. Strongly regular graphs

Let $\Gamma = (V, E)$ be a strongly regular graph (with vertex set V and edge set E) having the parameters (n, k, λ, μ). Let $\mathfrak{X}(\Gamma) = (V, \{R_0, R_1, R_2\})$ be the symmetric association scheme on V where relations are defined by $R_0 = \{(x, x) \mid x \in V\}$, $R_1 = \{(x, y) \mid (x, y) \in E\}$, and $R_2 = \{(x, y) \in V \times V \mid x \neq y, (x, y) \notin E\}$. Then $\lambda = p_{11}^1$, $\mu = p_{11}^2$, $k = k_1 = |\{y \mid (x, y) \in R_1\}|$, and the character table P of $\mathfrak{X}(\Gamma)$ is given by

$$P = \begin{pmatrix} 1 & k & n-k-1 \\ 1 & r & t \\ 1 & s & u \end{pmatrix} \begin{matrix} 1 \\ m \\ n-m-1 \end{matrix}$$

where $r = \frac{1}{2}(\lambda - \mu + \sqrt{D})$, $s = \frac{1}{2}(\lambda - \mu - \sqrt{D})$, $t = -r - 1$, $u = -s - 1$, and $m = \frac{1}{2}[n - 1 - \{2k + (n-1)(\lambda - \mu)\} \cdot D^{-\frac{1}{2}}]$ with $D = (\lambda - \mu)^2 + 4(k - \mu)$. Conversely, for any given two-class symmetric association scheme $\mathfrak{X} = (X, \{R_0, R_1, R_2\})$, the associated relation graphs $\Gamma(\mathfrak{X}) = (X, R_1)$ and $\bar{\Gamma}(\mathfrak{X}) = (X, R_2)$ are strongly regular graphs having parameters $(|X|, k_1, p_{11}^1, p_{11}^2)$ and $(|X|, k_2, p_{22}^2, p_{22}^1)$ respectively.

2. Fission schemes of completely multipartite strongly regular graphs

2.1. Completely multipartite strongly regular graphs

We now consider a special class of strongly regular graphs each of which has the parameters $(n, k, 2k - n, k)$ for some n and k with $n \leq 2k$. For such graphs $r = 0$, $s = k - n$, and $m = n(n - k - 1)(n - k)^{-1}$. Moreover, such a graph Γ is a completely multipartite graph with the size of each part $f = n - k$, and its complement $\bar{\Gamma}$ is the union of $g = \frac{n}{n-k}$ copies of the complete graph K_f. Thus, the graph Γ is also described by the parameters $(fg, f(g - 1), f(g - 2), f(g - 1))$, and in the notion of wreath product in 1.4, $\mathfrak{X}(\Gamma) = \mathfrak{X}(K_f) \int \mathfrak{X}(K_g)$.

2.2. Three-class fission schemes of $\mathfrak{X}(\Gamma)$

In [13, Lemma 5.3], it is shown that if $\mathfrak{X}(\Gamma)$ for the strongly regular graph Γ of $(fg, f(g - 1), f(g - 2), f(g - 1))$, f, $g \geq 2$ is realized as the symmetrization of a three-class association scheme $\mathfrak{X}$ then the character table of $\mathfrak{X}$ must be one of the following three:

$$(i) \quad P_\rho = \begin{bmatrix} 1 & \frac{1}{2}f(g-1) & \frac{1}{2}f(g-1) & f-1 \\ 1 & \rho & \bar{\rho} & -1 \\ 1 & \bar{\rho} & \rho & -1 \\ 1 & -\frac{1}{2}f & -\frac{1}{2}f & f-1 \end{bmatrix} \begin{matrix} 1 \\ \frac{1}{2}g(f-1) \\ \frac{1}{2}g(f-1) \\ g-1 \end{matrix}$$

$$(ii) \quad P_\sigma = \begin{bmatrix} 1 & \frac{1}{2}f(g-1) & \frac{1}{2}f(g-1) & f-1 \\ 1 & 0 & 0 & -1 \\ 1 & \sigma & \bar{\sigma} & f-1 \\ 1 & \bar{\sigma} & \sigma & f-1 \end{bmatrix} \begin{matrix} 1 \\ g(f-1) \\ \frac{1}{2}(g-1) \\ \frac{1}{2}(g-1) \end{matrix}$$

$$(iii) \quad P_\tau = \begin{bmatrix} 1 & f(g-1) & \frac{1}{2}(f-1) & \frac{1}{2}(f-1) \\ 1 & 0 & \tau & \bar{\tau} \\ 1 & 0 & \bar{\tau} & \tau \\ 1 & -f & \frac{1}{2}(f-1) & \frac{1}{2}(f-1) \end{bmatrix} \begin{matrix} 1 \\ \frac{1}{2}g(f-1) \\ \frac{1}{2}g(f-1) \\ g-1 \end{matrix}$$

where $\rho = \frac{1}{2}\{-(g - 1)/(f - 1)\}^{\frac{1}{2}}$, $\sigma = \frac{1}{2}f\{-1 + (-g)^{\frac{1}{2}}\}$ and $\tau = \frac{1}{2}\{-1 + (-f)^{\frac{1}{2}}\}$. Furthermore, it is shown that each of the above tables becomes feasible if and only if

(i) f and g are even and $g - 1$ is divisible by $f - 1$, for P_ρ;
(ii) either f is even and g is odd, or f is odd and $g \equiv 3 \pmod 4$, for P_σ;
(iii) $f \equiv 3 \pmod 4$ for P_τ.

As an easy consequence of this, we have the following observation:

Lemma 2.3 *Let Γ be a strongly regular graph having the parameters $(fg, f(g-1), f(g-2), f(g - 1))$. Suppose $\mathfrak{X}(\Gamma)$ has two distinct symmetrizable fission schemes of class 3. Then*

(1) *Both f and g are congruent to 3 modulo 4.*
(2) *The character tables of the two three-class fission schemes must be of type P_σ and P_τ in 2.2.*
(3) *$\mathfrak{X}(\Gamma)$ admits a symmetrizable fission scheme of class 4 having the character table*

$$P = \begin{bmatrix} 1 & \frac{1}{2}f(g-1) & \frac{1}{2}f(g-1) & \frac{1}{2}(f-1) & \frac{1}{2}(f-1) \\ 1 & 0 & 0 & \tau & \bar{\tau} \\ 1 & 0 & 0 & \bar{\tau} & \tau \\ 1 & \sigma & \bar{\sigma} & \frac{1}{2}(f-1) & \frac{1}{2}(f-1) \\ 1 & \bar{\sigma} & \sigma & \frac{1}{2}(f-1) & \frac{1}{2}(f-1) \end{bmatrix} \begin{matrix} 1 \\ \frac{1}{2}g(f-1) \\ \frac{1}{2}g(f-1) \\ \frac{1}{2}(g-1) \\ \frac{1}{2}(g-1) \end{matrix}$$

where $\sigma = \frac{f}{2}(-1 + \sqrt{-g})$, $\tau = \frac{1}{2}(-1 + \sqrt{-f})$.

Proof: It follows directly from 2.2. Notice that neither P_ρ and P_σ, nor P_ρ and P_τ combination can occur together.   □

In the above lemma, if $\mathfrak{X}(\Gamma)$ has a symmetrizable fission scheme of class 4, then each of the associated strongly regular graphs Γ and $\bar{\Gamma}$ of $\mathfrak{X}(\Gamma)$ is decomposed into two regular subdigraphs in such a way that two sets of the directed edges form a paired relations in the resulting fission scheme. An important key observation is that such a fission scheme exists only if both $\mathfrak{X}(K_f)$ and $\mathfrak{X}(K_g)$ are fissionable into pairs of nonsymmetric two-class schemes simultaneously. When both f and g are prime power integers that are congruent to 3 modulo 4, we have desired fission schemes $\mathfrak{X}(2, GF(f))$ and $\mathfrak{X}(2, GF(g))$ for $\mathfrak{X}(K_f)$ and $\mathfrak{X}(K_g)$, respectively. Thus we now have the following theorem.

Theorem 2.4 *Let $\mathfrak{X}(\Gamma)$ be the symmetric association scheme which comes from the strongly regular graph having the parameters $(fg, f(g-1), f(g-2), f(g-1))$ for $f, g \geq 3$. Then we have the following:*

(1) *If $f \equiv 3 \pmod 4$ and f is a prime power, then P_τ is realized as the character table of*
$$W_\tau = \mathfrak{X}(2, GF(f)) \int \mathfrak{X}(K_g).$$

(2) *If $g \equiv 3 \pmod 4$ and g is a prime power, then P_σ is realized as the character table of*
$$W_\sigma = \mathfrak{X}(K_f) \int \mathfrak{X}(2, GF(g)).$$

(3) *If f and g both are prime powers and congruent to 3 modulo 4, then P in (3) of Lemma 2.3 is realized as the character table of $W = \mathfrak{X}(2, GF(f)) \int \mathfrak{X}(2, GF(g))$.*

Proof: It suffices to show that the character tables of the schemes W_τ, W_σ and W are P_τ, P_σ, and P, respectively, for given f and g. We note that the adjacency matrices of the wreath product of two schemes, over the sets of cardinalities f and g with their adjacency matrices F's and G's, respectively, are the matrices $I_g \otimes F$'s and $G \otimes J_f$'s. It is shown that the eigenvalues of $I_g \otimes F$ are the same as those of F but the multiplicities of the eigenvalues for $I_g \otimes F$ are g times those of the same eigenvalues for F. Also the spectrum of $G \otimes J_f$ consists of 0 with multiplicity $g(f-1)$ and f times the eigenvalues of G with the same multiplicities. Therefore, it is straightforward to construct the character tables of W_τ, W_σ, and W by computing the spectra of their adjacency matrices directly from those of $\mathfrak{X}(K_f)$, $\mathfrak{X}(K_g)$, $\mathfrak{X}(2, GF(f))$, and $\mathfrak{X}(2, GF(g))$. $\square$

2.5. Parameters

The parameters of the schemes W_τ, W_σ, and W can be obtained by a straightforward computation. For instance, the intersection matrices of the scheme $W = \mathfrak{X}(2, GF(f)) \int \mathfrak{X}(2, GF(g))$ are given by B_1 and B_3 below.

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4}f(g-3) & \frac{1}{4}f(g+1) & 0 & 0 \\ \frac{1}{2}f(g-1) & \frac{1}{4}f(g-3) & \frac{1}{4}f(g-3) & \frac{1}{2}f(g-1) & \frac{1}{2}f(g-1) \\ 0 & \frac{1}{2}(f-1) & 0 & 0 & 0 \\ 0 & \frac{1}{2}(f-1) & 0 & 0 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2}(f-1) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(f-1) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4}(f-3) & \frac{1}{4}(f+1) \\ \frac{1}{2}(f-1) & 0 & 0 & \frac{1}{4}(f-3) & \frac{1}{4}(f-3) \end{bmatrix}.$$

Example 2.6   Let Γ and Φ be the strongly regular graphs having the parameters $(21, 14, 7, 14)$ with $f = 7, g = 3$ and $(21, 18, 15, 18)$ with $f = 3, g = 7$, respectively. Then the fission schemes of $\mathfrak{X}(\Gamma)$ and $\mathfrak{X}(\Phi)$ are described by

$$W(\Gamma) = \mathfrak{X}(2, GF(7)) f \mathfrak{X}(Z_3)$$
$$W(\Phi) = \mathfrak{X}(Z_3) f \mathfrak{X}(2, GF(7)), \text{ respectively.}$$

Their relation tables are

$$R(W(\Gamma)) = I_3 \otimes (S + 2J_7') + T \otimes J_7$$
$$R(W(\Phi)) = I_7 \otimes (T + 2J_3') + S \otimes J_3$$

where $J_n' = J_n - I_n$,

$$T = R(\mathfrak{X}(Z_3)) = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$S = R(\mathfrak{X}(2, GF(7))) = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 2 & 2 \\ 2 & 0 & 1 & 1 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}.$$

Also $\mathfrak{X}(\Gamma)$ has two intermediate 3-class fission schemes $\mathfrak{X}(2, GF(7)) f \mathfrak{X}(K_3)$ and $\mathfrak{X}(K_7) f \mathfrak{X}(Z_3)$, and $\mathfrak{X}(\Phi)$ has two fission schemes $\mathfrak{X}(Z_3) f \mathfrak{X}(K_7)$ and $\mathfrak{X}(K_3) f \mathfrak{X}(2, GF(7))$ as well. These are fusion schemes of $W(\Gamma)$ and $W(\Phi)$, respectively. (See also (5.6) of [13] where one intermediate 3-class fission scheme from each of $\mathfrak{X}(\Gamma)$ and $\mathfrak{X}(\Phi)$ are described in detail.)

3.   Symmetrizable fission schemes of Paley graphs

The relation graphs of cyclotomic schemes $\mathfrak{X}(2, GF(q))$ for $q \equiv 1(\bmod 4)$ are known as Paley graphs, the strongly regular graphs having the parameters $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$. In [13, (7.1)] we have shown that all strongly regular graphs having the parameters $(n, \frac{1}{2}(n-1), \frac{1}{4}(n-5), \frac{1}{4}(n-1))$ for any $n \geq 5$ are not realized as the symmetrization of any three-class schemes. However, for each prime power $q \equiv 5(\bmod 8)$, it is easy to see that $\mathfrak{X}(2, GF(q))$ is the symmetrization of four-class nonsymmetric scheme $\mathfrak{X}(4, GF(q))$.

3.1.   Four-class symmetrizable fission tables

Let $\tilde{\mathfrak{X}}$ be the symmetric scheme of class two associated with the strongly regular graph Γ having the parameters (n, k, λ, μ). (Then the character table $\tilde{P}$ of $\tilde{\mathfrak{X}}$ is given by 1.8.)

Suppose $\tilde{\mathfrak{X}}$ admits a four-class symmetrizable fission scheme $\mathfrak{X}$. Then a feasible fission table P for $\mathfrak{X}$ may be described, up to a permutation of rows and columns, by

$$
P = \begin{pmatrix}
1 & \frac{1}{2}k & \frac{1}{2}k & \frac{1}{2}(n-k-1) & \frac{1}{2}(n-k-1) \\
1 & \rho & \bar{\rho} & \tau & \bar{\tau} \\
1 & \bar{\rho} & \rho & \bar{\tau} & \tau \\
1 & \sigma & \bar{\sigma} & \omega & \bar{\omega} \\
1 & \bar{\sigma} & \sigma & \bar{\omega} & \omega
\end{pmatrix}
\begin{matrix}
1 \\
\frac{1}{2}m \\
\frac{1}{2}m \\
\frac{1}{2}(n-m-1) \\
\frac{1}{2}(n-m-1)
\end{matrix}
$$

where either the pair ρ and ω or the pair τ and σ are nonreal. The reason is that when both of the nontrivial relations of $\tilde{\mathfrak{X}}$ split into paired relations, each of the two primitive idempotents must split into a pair of paired idempotents. From the orthogonality relations of the character table, we have a set of equations on ρ, τ, σ, and ω in terms of n, k, r, and s, although these are not enough to determine ρ, τ, σ, and ω explicitly. For the case when $\tilde{\mathfrak{X}} = \mathfrak{X}(2, GF(q))$, we have one free parameter to determine due to the pseudocyclic property of cyclotomic schemes.

3.2. Pseudocyclic fission tables

A commutative scheme is said to be pseudocyclic if its multiplicities coincide with each other. Notice that in a pseudocyclic scheme all the valencies are also the same. In particular if $\tilde{\mathfrak{X}}$ is pseudocyclic two-class symmetric scheme, then in the notations used in 1.8, its character table $\tilde{P}$ will be given by $k = m = \frac{1}{2}(n-1)$ and $s = -r-1 = \frac{1}{2}(-1-\sqrt{n})$ while $r = \frac{1}{2}(-1+\sqrt{n})$. Furthermore, if $\mathfrak{X}$ is a four-class symmetrizable fission scheme of $\tilde{\mathfrak{X}}$, then the character table of $\mathfrak{X}$ will be given by P in 3.1 with the identities $k = m = \frac{1}{2}(n-1)$, $\omega = \rho, \sigma = \tau, \rho + \bar{\rho} = \frac{1}{2}(-1+\sqrt{n})$, $\tau + \bar{\tau} = \frac{1}{2}(-1-\sqrt{n})$, and $\rho\bar{\rho} + \tau\bar{\tau} = \frac{1}{8}(3n+1)$. Thus, in particular, we have the following.

Lemma 3.3 *Suppose P is the character table of any four-class symmetrizable fission scheme of $\tilde{\mathfrak{X}} = \mathfrak{X}(2, GF(q))$ for $q \equiv 5(mod\ 8)$. Then P is given by*

$$
P = \begin{pmatrix}
1 & f & f & f & f \\
1 & \rho & \bar{\rho} & \tau & \bar{\tau} \\
1 & \bar{\rho} & \rho & \bar{\tau} & \tau \\
1 & \bar{\tau} & \tau & \rho & \bar{\rho} \\
1 & \tau & \bar{\tau} & \bar{\rho} & \rho
\end{pmatrix}
\begin{matrix}
1 \\ f \\ f \\ f \\ f
\end{matrix}
\quad where \quad
\begin{cases}
f = \frac{1}{4}(q-1) \\
\rho + \bar{\rho} = \frac{1}{2}(-1+\sqrt{q}) \\
\tau + \bar{\tau} = \frac{1}{2}(-1-\sqrt{q}) \\
\rho\bar{\rho} + \tau\bar{\tau} = \frac{1}{8}(3q+1).
\end{cases}
\quad (*)
$$

Proof: Straightforward.

The following particular solution (1) or (2) for the system (*) yields the character tables of $\mathfrak{X}(4, GF(q))$ depending on the quadratic residue of given q:

$$
(1) \quad \begin{cases}
\rho = \frac{1}{4}\left[-1 + \sqrt{q} + \left\{-2q - \frac{1}{2}(q-1)\sqrt{q}\right\}^{\frac{1}{2}}\right] \\
\tau = \frac{1}{4}\left[-1 - \sqrt{q} + \left\{-2q + \frac{1}{2}(q-1)\sqrt{q}\right\}^{\frac{1}{2}}\right]
\end{cases}
$$

$$
(2) \quad \begin{cases}
\rho = \frac{1}{4}\left[-1 + \sqrt{q} + \left\{-2q + \frac{1}{2}(q-1)\sqrt{q}\right\}^{\frac{1}{2}}\right] \\
\tau = \frac{1}{4}\left[-1 - \sqrt{q} + \left\{-2q - \frac{1}{2}(q-1)\sqrt{q}\right\}^{\frac{1}{2}}\right]
\end{cases}
$$

Remark 3.4 In the notion of 1.7 with $d = 4$, this implies that the Gaussian periods ϕ_i, $i = 0, 1, 2, 3$, can be expressed by ρ and τ as follows:

$$\begin{cases} \phi_0 = \bar{\phi}_2 = \sum_{\beta \in \langle \alpha^4 \rangle} \chi(\beta) = \rho \\ \phi_1 = \bar{\phi}_3 = \sum_{\beta \in \langle \alpha^4 \rangle \alpha} \chi(\beta) = \tau \end{cases}$$

where the pair ρ and τ are given by either the formula (1) or (2) depending on the given q. For instance, the character table of $\mathfrak{X}(4, GF(5))$ is given by (1) while that of $\mathfrak{X}(4, GF(13))$ is given by (2).

We close this section with the following two questions:

Question 1. Is there any other solution for the above system (*) that yields a feasible fission table P for given $\tilde{P}$, perhaps, for a large prime power $q \equiv 5 \pmod 8$?

For a given association scheme $\mathfrak{X}$, if all the (nontrivial) relation graphs (X, R_i) of $\mathfrak{X}$ are connected, then we say that $\mathfrak{X}$ is primitive. The cyclotomic schemes associated to Paley graphs are examples of primitive schemes.

Question 2. Are there any other two-class primitive schemes (strongly regular graphs) that admit symmetrizable fission schemes besides $\mathfrak{X}(2, GF(q))$ (Paley graphs) for $q \equiv 5 \pmod 8$?

References

1. E. Bannai, "Subschemes of some association schemes," *J. Algebra* **144** (1991), 167–188.
2. E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin/Cummings, Menlo Park, California, 1984.
3. E. Bannai and S.Y. Song, "Character tables of fission schemes and fusion schemes," *European J. of Combinatorics* **14** (1993), 385–396.
4. E. Bannai and A. Munemasa, "Davenport-Hasse theorem and cyclotomic association schemes," *Proceedings of Algebraic Combinatorics Conference*, Hirosaki University, Japan (1990).
5. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance Regular Graphs*, Springer-Verlag, Berlin, 1989.
6. I.A. Faradzev, "Cellular subrings of the symmetric square of a cellular ring of rank 3," *Investigations in Algebraic Theory of Combinatorial Objects*, I.A. Faradzev, A.A. Ivanov, M.H. Klin, and A.J. Woldar (Eds.), Kluwer Acad. Publ., Dordrecht, 1992.
7. I.A. Faradzev, A.V. Ivanov, and M.H. Klin, "Galois correspondence between permutation groups and cellular rings," *Graphs and Combinatorics* **6** (1990), 303–332.
8. I.A. Faradzev, M.H. Klin, and M.E. Muzichuk, "Cellular rings and groups of automorphisms of graphs," *Investigations in Algebraic Theory of Combinatorial Objects*, I.A. Faradzev, A.A. Ivanov, M.H. Klin, and A.J. Woldar (Eds.), Kluwer Acad. Publ., Dordrecht, 1992.
9. D.G. Higman, "Coherent configurations, Part I, Ordinary representation theory," *Geometriae Dedicata* **4** (1975), 1–32.
10. K.W. Johnson and J.D.H. Smith, "Characters of finite quasigroups III: Quotients and fusion," *Europ. J. Comb.* **19** (1989), 47–56.
11. M.H. Klin and R. Poschel, "The König problem, the isomorphism problem for cyclic groups and the method of Schur rings," *Algebraic Methods in Graph Theory*, North Holland, Amsterdam, pp. 405–434, 1981.
12. M.E. Muzichuk, "Subcells of the symmetric cells," *Algebraic Structures and Their Applications*, Kiev, 1988, 172–174, in Russian.
13. S.Y. Song, "Class 3 association schemes whose symmetrizations have two classes," *J. of Combin. Theory (A)* **70** (1995), 1–29.
14. B. Weisfeiler, *On Construction and Identification of Graphs*, Springer-Verlag, Berlin, 1976.