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Subspace Arrangements of Type B_n and D_n*

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Abstract. Let $\mathcal{D}_{n,k}$ be the family of linear subspaces of $\mathbb{R}^n$ given by all equations of the form

$$\epsilon_1 x_{i_1} = \epsilon_2 x_{i_2} = \cdots = \epsilon_k x_{i_k},$$

for $1 \le i_1 < \cdots < i_k \le n$ and $(\epsilon_1, \ldots, \epsilon_k) \in \{+1, -1\}^k$. Also let $\mathcal{B}_{n,k,h}$ be $\mathcal{D}_{n,k}$ enlarged by the subspaces

$$x_{j_1} = x_{j_2} = \cdots = x_{j_h} = 0,$$

for $1 \le j_1 < \cdots < j_h \le n$. The special cases $\mathcal{B}_{n,2,1}$ and $\mathcal{D}_{n,2}$ are well known as the reflection hyperplane arrangements corresponding to the Coxeter groups of type B_n and D_n, respectively.

In this paper we study combinatorial and topological properties of the intersection lattices of these subspace arrangements. Expressions for their Möbius functions and characteristic polynomials are derived. Lexicographic shellability is established in the case of $\mathcal{B}_{n,k,h}$, $1 \le h < k$, which allows computation of the homology of its intersection lattice and the cohomology groups of the manifold $M_{n,k,h} = \mathbb{R}^n \setminus \bigcup \mathcal{B}_{n,k,h}$. For instance, it is shown that $H^d(M_{n,k,k-1})$ is torsion-free and is nonzero if and only if $d = t(k-2)$ for some t, $0 \le t \le \lfloor n/k \rfloor$. Torsion-free cohomology follows also for the complement in $\mathbb{C}^n$ of the complexification $\mathcal{B}^{\mathbb{C}}_{n,k,h}$, $1 \le h < k$.

Keywords: cohomology, characteristic polynomial, Coxeter subspace arrangement, homotopy, homology, lexicographic shellability, signed graph

1. Introduction

A *subspace arrangement* is a finite set

$$\mathcal{A} = \{K_1, \ldots, K_h\}$$

of linear subspaces K_i in real Euclidean space $\mathbb{R}^n$. We assume that there are no containments $K_i \subseteq K_j$, $i \neq j$. An extensive theory exists for the case of real and complex *hyperplane arrangements* (i.e., codim $K_i = 1$), see Orlik and Terao [10]. Work on subspace arrangements of more general type has begun only in the last few years. See Björner [1] for an overview of this development.

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The *k-equal arrangement*, $\mathcal{A}_{n,k}$, consists of all subspaces of the form

$$\{\mathbf{x} \in \mathbb{R}^n : x_{i_1} = x_{i_2} = \cdots = x_{i_k}\},$$

for k-subsets $1 \leq i_1 < \cdots < i_k \leq n$. Here the x_i are the coordinate functions in $\mathbb{R}^n$. This arrangement has been thoroughly investigated in several recent papers. It first appears in the work of Björner and Lovász [2] and Björner et al. [3], motivated by its relevance for a certain problem in computational complexity. Obtaining good expressions for the Möbius function of the intersection lattice $L(\mathcal{A}_{n,k})$ was of crucial importance in that work. Later, Björner and Welker [5] computed the homotopy type of $L(\mathcal{A}_{n,k})$ and then, via the Goresky-MacPherson theorem [7], the cohomology of the complement $M_{n,k} = \mathbb{R}^n \setminus \bigcup \mathcal{A}_{n,k}$. Then Björner and Wachs [4] found a new approach to these computations via lexicographic shellability. Finally Sundaram and Wachs [12] and Sundaram and Welker [13] determined the representations arising from the S_n action on $L(\mathcal{A}_{n,k})$ and on $H^*(M_{n,k})$.

One way to think of $\mathcal{A}_{n,k}$ is as the orbit $W_{A_{n-1}}(K_k)$ of the single subspace

$$K_k = \{\mathbf{x} \in \mathbb{R}^n : x_1 = x_2 = \cdots = x_k\},$$

under the standard action of the reflection group $W_{A_{n-1}} \cong S_n$ on $\mathbb{R}^n$ (permuting coordinates). Now consider the *k-equal subspace arrangement of type* D_n, $\mathcal{D}_{n,k}$, consisting of all subspaces of the form

$$\{\mathbf{x} \in \mathbb{R}^n : \epsilon_1 x_{i_1} = \epsilon_2 x_{i_2} = \cdots = \epsilon_k x_{i_k}\},$$

for $1 \leq i_1 < \cdots < i_k \leq n$ and $(\epsilon_1, \ldots, \epsilon_k) \in \{+1, -1\}^k$. Adding to $\mathcal{D}_{n,k}$ the subspaces

$$\{\mathbf{x} \in \mathbb{R}^n : x_{j_1} = x_{j_2} = \cdots = x_{j_h} = 0\},$$

for $1 \leq j_1 < \cdots < j_h \leq n$ results in the *k, h-equal subspace arrangement of type* B_n. Denote by W_{B_n} and W_{D_n} the reflection groups of types B_n and D_n with their standard action on $\mathbb{R}^n$, see Humphreys [8]. Then $\mathcal{B}_{n,k,h}$ is the orbit union $W_{B_n}(K_k) \cup W_{B_n}(K'_h)$ where

$$K'_h = \{\mathbf{x} \in \mathbb{R}^n : x_1 = x_2 = \cdots = x_h = 0\},$$

$1 \leq h < k \leq n$, while $\mathcal{D}_{n,k}$ is the orbit $W_{D_n}(K_k) = W_{B_n}(K_k)$, $1 < k < n$. Thus it is geometrically motivated to view $\mathcal{B}_{n,k,h}$ and $\mathcal{D}_{n,k}$ as the type B_n and D_n analogs of $\mathcal{A}_{n,k}$, respectively, especially since for $k = 2$ and $h = 1$ all these subspace arrangements specialize to the hyperplane arrangements of the respective reflection groups.

After a review of definitions and some preliminary material in Section 2, we begin in Section 3 with the combinatorial study of the intersection lattices $L(\mathcal{B}_{n,k,h})$ and $L(\mathcal{D}_{n,k})$. These are both isomorphic to lattices of signed graphs, as can be seen from work of Zaslavsky [15, 16]. Generating functions for the Möbius functions and characteristic polynomials of such lattices are determined in a setting which is more general than what the motivating geometric examples would demand. In particular, we derive an explicit expression for the characteristic polynomials of $L(\mathcal{B}_{n,k,h})$ and $L(\mathcal{D}_{n,k})$ using a lattice point counting method due to Blass and Sagan [6]. This is the only purely combinatorial technique we know of to get such a result. It follows that these polynomials factor partially over the nonnegative integers, $\mathbb{Z}^+$. It is interesting to compare this with the hyperplane case where the corresponding polynomials factor completely over $\mathbb{Z}^+$.

In Section 4 we prove lexicographic shellability of the intersection lattices $L(\mathcal{B}_{n,k,h})$. This makes possible the computation of the homotopy type (which turns out to be a wedge of spheres) and the homology groups of these lattices. The homology of $L(\mathcal{A}_{n,k})$ was computed via a certain recursive procedure in [5] and later using lexicographic shellability in [4]. We remark that we were able to adapt the recursive procedure to $L(\mathcal{B}_{n,k,h})$ only in the case $h = 1$, whereas lexicographic shelling works in general. However, we have so far been unable to apply either approach to $L(\mathcal{D}_{n,k})$, so the homology computations for that lattice remain to be done.

The results from Section 4 are used in Section 5 together with the Goresky-MacPherson formula to compute the cohomology of the complement $\mathbb{R}^n \backslash \bigcup \mathcal{B}_{n,k,h}$. These cohomology groups are torsion-free, and for the pure arrangement $\mathcal{B}_{n,k,k-1}$ there is nonzero cohomology only in dimensions that are multiples of $k - 2$. The paper ends with some comments on related results and open questions.

2. Preliminaries

We will review some notions related to subspace arrangements and also establish notation. Associated with any subspace arrangement $\mathcal{A} = \{K_1, \ldots, K_h\}$ is its *intersection lattice*, $L = L(\mathcal{A})$, which consists of all intersections of subspaces in $\mathcal{A}$ ordered by reverse inclusion. This lattice has unique minimal element $\hat{0} = \mathbb{R}^n$ and unique maximal element $\hat{1} = \bigcap_{i=1}^h K_i$. Any partially ordered set with $\hat{0}$ and $\hat{1}$ is called *bounded*. (Any terminology from the theory of lattices and posets that we do not define can be found explained in Stanley's book [11].)

Let $\mathcal{A}$ be a hyperplane arrangement and let $\mathcal{A}'$ be a subspace arrangement. Then we will say that *$\mathcal{A}'$ is embedded in $\mathcal{A}$* if each $K_i \in \mathcal{A}'$ is an element of $L(\mathcal{A})$. The subspace arrangements with which we will be concerned are embedded in the reflection hyperplane arrangements $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{D}_n$ defined as follows:

$$\begin{aligned}
\mathcal{A}_n &= \{x_i = x_j : 1 \leq i < j \leq n\},\\
\mathcal{B}_n &= \{x_i = \pm x_j : 1 \leq i < j \leq n\} \cup \{x_i = 0 : 1 \leq i \leq n\},\\
\mathcal{D}_n &= \{x_i = \pm x_j : 1 \leq i < j \leq n\}.
\end{aligned}$$

Note that $\mathcal{A}_n \subseteq \mathcal{D}_n \subseteq \mathcal{B}_n$.

It will be useful to have a combinatorial description of the intersection lattices $L(\mathcal{A}_n)$, $L(\mathcal{B}_n)$ and $L(\mathcal{D}_n)$. This is provided by Zaslavsky's theory of signed graphs [15, 16]. A *signed graph*, G, has vertex set $V(G) = [n]$ where $[n] = \{1, 2, \ldots, n\}$. The edges $E(G)$ of G can be of three types:

- a positive edge between vertices i and j, denoted ij^+,
- a negative edge between vertices i and j, denoted ij^-,
- a half edge with only one endpoint i, denoted i^h.

Note that both edges ij^+ and ij^- can be present. The idea is that the edges ij^+, ij^- and i^h correspond to the hyperplanes $x_i = x_j$, $x_i = -x_j$ and $x_i = 0$, respectively, in $\mathcal{B}_n$. So associated with any arrangement $\mathcal{A} \subseteq \mathcal{B}_n$ we have the *associated signed graph* $G_{\mathcal{A}}$ where an edge is in $G_{\mathcal{A}}$ if and only if the corresponding hyperplane is in $\mathcal{A}$.

We can now characterize $L(\mathcal{A})$ where $\mathcal{A} = \mathcal{A}_n$, $\mathcal{B}_n$ or $\mathcal{D}_n$ using these graphs. (With a little more work, one can characterize $L(\mathcal{A})$ for any $\mathcal{A} \subseteq \mathcal{B}_n$.) Given $V, W \subseteq [n]$ with $V \cap W = \emptyset$ (and we permit one of V or W to be empty) we let $K^b_{V,W}$ denote the *complete balanced graph* consisting of all positive edges between vertices of V, all positive edges between vertices of W, and all negative edges from a vertex of V to a vertex of W. It is called *balanced* because multiplying the signs around any cycle gives a positive sign. Also let K^u_V be the *complete unbalanced graph*, i.e., the one that has all edges of both signs between vertices in V. We also include all half edges on V in K^u_V in the case where $\mathcal{A} = \mathcal{B}_n$. It is *unbalanced* because there exist cycles with negative edge product. (A half edge is considered a negative cycle.) By *component* we mean a connected component in the usual sense of graph theory.

Theorem 2.1 (Zaslavsky [16]) *Let $\mathcal{A} = \mathcal{A}_n$, $\mathcal{B}_n$ or $\mathcal{D}_n$. The lattice $L(\mathcal{A})$ is isomorphic to the lattice of subgraphs $G \subseteq G_{\mathcal{A}}$ such that*
1. *every component of G is complete balanced or complete unbalanced, and*
2. *there is at most one unbalanced component.*

The graphs are partially ordered by inclusion of their edge sets.

The isomorphism is obtained by sending each graph G to the subspace $\bigcap_{e \in G} H_e$ where H_e is the hyperplane corresponding to the edge e. Because of this isomorphism, we will often talk about these intersection lattices as if they were lattices of graphs.

We can now combinatorially describe the subspace arrangements defined in Section 1. Call a component of a graph *trivial* or a *singleton* if it consists of a single vertex. Note that according to this definition, a single vertex i together with the half edge i^h is nontrivial. Now the k-equal subspace arrangement of type A_n, $\mathcal{A}_{n,k}$, consists of all graphs in $L(\mathcal{A}_n)$ having exactly one nontrivial component K and satisfying $|V(K)| = k$, where $|\cdot|$ denotes cardinality. Note that we must have $k \geq 2$ and that this component must be a complete positively signed (hence balanced) graph. Also, $\mathcal{A}_{n,2} = \mathcal{A}_n$. The k,h-equal subspace arrangement of type B_n, $\mathcal{B}_{n,k,h}$, consists of all graphs in $L(\mathcal{B}_n)$ having a unique nontrivial component K and satisfying $|V(K)| = k$ if K is balanced or $|V(K)| = h$ if it is unbalanced. Note that we can assume $h < k$, since if $h \geq k$ then there are containments among the subspaces. We also have the specialization $\mathcal{B}_{n,2,1} = \mathcal{B}_n$. In fact a natural one-parameter geometric analog of the type A_n k-equal arrangement is the pure arrangement $\mathcal{B}_{n,k,k-1}$. Finally the k-equal subspace arrangement of type D_n, $\mathcal{D}_{n,k}$, consists of all graphs in $L(\mathcal{D}_n)$ having a unique nontrivial component K which is balanced and satisfies $|V(K)| = k$. Note that nothing is to be gained by having a second parameter h in the D_n case since, with the obvious definition, $\mathcal{D}_{n,k,h} = \mathcal{B}_{n,k,h}$ for $2 \leq h < k$.

Let $\Pi_{n,k}$ and $\Pi_{n,k,h}$ be the induced posets of all graphs from $L(\mathcal{A}_n)$ and $L(\mathcal{B}_n)$, respectively, whose nontrivial components have at least k vertices if balanced and at least h vertices if unbalanced. In this definition we make no assumption on k and h other than $k \geq 2$ and $h \geq 1$. From Theorem 2.1 we immediately obtain the following.

Corollary 2.2 *For $1 \leq h \leq k$ we have the following lattice isomorphisms.*
1. $L(\mathcal{A}_{n,k}) \cong \Pi_{n,k}$,
2. $L(\mathcal{B}_{n,k,h}) \cong \Pi_{n,k,h}$ *if* $h < k$,
3. $L(\mathcal{D}_{n,k}) \cong \Pi_{n,k,k}$.

Note that the posets $\Pi_{n,k,h}$ also exist for $h > k$, but they are not even lattices in that case and so are not of interest to us.

Let $\mathcal{A}$ be any subspace arrangement and consider the Möbius function $\mu(X) = \mu(\hat{0}, X)$ where $X \in L = L(\mathcal{A})$. We also let $\mu(L) = \mu(\hat{0}, \hat{1})$. The *characteristic polynomial of L* (or of $\mathcal{A}$) is

$$\chi(L, t) = \sum_{X \in L} \mu(X) t^{\dim X}. \tag{1}$$

In order to give a combinatorial proof of one of the results in Section 3, we will use the following theorem. In it, $\mathbb{Z}$ represents the integers.

Theorem 2.3 (Blass and Sagan [6]) *Let $\mathcal{A}$ be a subspace arrangement embedded in $\mathcal{B}_n$ and let $t = 2s + 1$ be a positive odd integer. Consider the cube*

$$Q_t = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : -s \leq x_i \leq s\}.$$

Then

$$\chi(\mathcal{A}, t) = |Q_t \backslash \mathcal{A}|$$

i.e., the characteristic polynomial of $\mathcal{A}$ evaluated at an odd integer t is the number of lattice points left in the cube of side t once the subspaces of $\mathcal{A}$ are removed.

If L is any partially ordered set then the pair $x, y \in L$ determines a *closed interval* $[x, y] = \{z \in L : x \leq z \leq y\}$. The corresponding *order complex*, $\Delta(x, y)$, is the abstract simplicial complex of all sets $\{x_1, \dots, x_{l-1}\}$ coming from chains $x = x_0 < x_1 < \cdots < x_l = y$ in $[x, y]$. We also write $\Delta(L)$ for $\Delta(\hat{0}, \hat{1})$. We will say that L has a certain topological property if $\Delta(L)$ does.

A property that we will be very concerned with is shellability. A *cover* in a poset P is an edge $x \to y$ of P's Hasse diagram, i.e., a pair $x, y \in P$ such that $x < y$ and there is no $z \in P$ satisfying $x < z < y$. Let $C(P)$ denote the set of covers of P. If $\mathcal{L}$ is a totally ordered set then a function $\lambda : C(P) \to \mathcal{L}$ is called a *labeling with label set $\mathcal{L}$*. This induces a labeling of every maximal chain

$$C : x = x_0 \to x_1 \to \cdots \to x_l = y$$

in $[x, y]$ where

$$\lambda(C) = \lambda(x_0 \to x_1), \ldots, \lambda(x_{l-1} \to x_l).$$

The parameter l is called *the length of* C and denoted $l(C)$. Define the lexicographic order on maximal chains by saying that $C < C'$ if $\lambda(C) < \lambda(C')$ in lexicographic order. Note that C and C', and thus $\lambda(C)$ and $\lambda(C')$, can have different lengths. We say that C is *increasing* (respectively, *decreasing*) if $\lambda(C)$ is a strictly increasing (respectively, weakly decreasing) sequence. Note the difference between the strict and weak cases. A *lexicographic shelling* or *EL-labeling* of a poset P is a labeling such that

S1. every interval $[x, y]$ has a unique increasing maximal chain C, and
S2. $C < C'$ for any other maximal chain C' in $[x, y]$.

This notion of EL-shellability, introduced in [4], extends the standard one for graded posets. The fundamental result about shellings that we will need is as follows.

Theorem 2.4 (Björner and Wachs [4]) *Suppose the bounded poset P admits a lexicographic shelling. Then P has the homotopy type of a wedge of spheres and thus its integral homology groups are free. For each $d \geq -1$ the number of d-spheres, and hence also the dth reduced Betti number, equals the number of decreasing maximal chains of P having length $d+2$.*

A lexicographic shelling in fact produces additional topological information that we will not need. Namely, the decreasing maximal chains determine a basis for the homology and cohomology of P [4].

3. Möbius functions and characteristic polynomials

We will now investigate the combinatorics of the posets $\Pi_{n,k,h}$. More generally, suppose $T, V \subseteq \mathbb{Z}^+$ with $1 \in T$ and let $\Pi_{n,T,V}$ denote the subposet of all graphs in $L(\mathcal{B}_n)$ with components K such that $|V(K)| \in T$ if K is balanced and $|V(K)| \in V$ if K is unbalanced. Note that $\Pi_{n,k,h} = \Pi_{n,T,V}$ if $T = \{1, k, k+1, \ldots\}$ and $V = \{h, h+1, \ldots\}$. Similarly define $\Pi_{n,T}$ as a subposet of $L(\mathcal{A}_n)$. Although these posets need not be lattices, they still contain the minimal element $\hat{0}$ (the empty graph), so we can still talk about their Möbius functions. By convention, we let $\mu(P) = 0$ if P is a poset without a unique maximal element.

If we have a subspace $X \in L(\mathcal{B}_n)$ then $\dim X = b(G)$, where $b(G)$ is the number of balanced blocks in the corresponding graph G. Thus it is consistent with definition (1) to let

$$\chi(P, t) = \sum_{G \in P} \mu(G) t^{b(G)}.$$

for any subposet $P \subseteq L(\mathcal{B}_n)$ with $\hat{0} \in P$. Since determining μ, χ and the corresponding generating functions is no more complicated for arbitrary T and V, we will do everything in this generality.

It will be convenient to introduce another subposet of $L(\mathcal{B}_n)$. Let $\Pi^b_{n,T}$ contain the graphs all of whose components K are balanced and satisfy $|V(K)| \in T$. Also let $\Pi^b_{n,k}$ be the special case where $T = \{1, k, k+1, \ldots\}$. Our first proposition gives a recursion for the characteristic polynomial of this poset.

Proposition 3.1 *We have the recurrence relation*

$$\chi\left(\Pi^b_{n,T}, t\right) = t \sum_{m=1}^{n} 2^{m-1} \binom{n-1}{m-1} \mu(\Pi_{m,T}) \chi\left(\Pi^b_{n-m,T}, t\right).$$

Proof: If $G \in \Pi^b_{n,T}$, then let K be the component of G containing the vertex n and let $m = |K|$. There are $2^{m-1}\binom{n-1}{m-1}$ choices for K since it is a balanced complete bipartite graph. This explains the first factor in the sum.

Now fix K and consider the subposet

$$P = \left\{G \in \Pi^b_{n,T} \colon K \text{ is a component of } G\right\}.$$

The lower ideal generated by P is isomorphic to the product $\Pi^b_{n-m,T} \times \Pi_{m,T}$, with P being the cross-section $\Pi^b_{n-m,T} \times K$. Thus the contribution of P to $\chi(\Pi^b_{n,T}, t)$ is $\chi(\Pi^b_{n-m,T}, t) \cdot \mu(\Pi_{m,T})t$. This completes the proof. □

Let $P = (P_n)_{n\geq 0}$ be a family of subposets $P_n \subseteq L(\mathcal{B}_n)$ with $\hat{0} \in P_n$. Note that when $n = 0$ we have $L(\mathcal{B}_0) = \{\hat{0}\}$ and $\chi(L(\mathcal{B}_0), t) = 1$. We consider the following generating functions

$$\begin{aligned} M(P, x) &= \sum_{n\geq 1} \mu(P_n) \frac{x^n}{n!} \\ F(P, x, t) &= \sum_{n\geq 0} \chi(P_n, t) \frac{x^n}{n!} \\ p(P, x) &= F(P, x, 1) \end{aligned}$$

Note that if P_n has a unique maximal element not equal to $\hat{0}$ for all sufficiently large n, then $p(P, x)$ is a polynomial in x. This will be the case for the lattices of our type B_n and D_n subspace arrangements. In particular, we will need the truncated exponential function

$$p_k(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!}.$$

Also note that if $\hat{1} \in P_n$ for all n then $F(P, x, 0) = M(P, x)$. Finally we will need the following result about the k-equal arrangement $\mathcal{A}_{n,k}$.

Theorem 3.2 (Björner and Lovász [2]) *We have the functional equations*

$$\exp M(\Pi_T, x) = p(\Pi_T, x),$$
$$F(\Pi_T, x, t) = p(\Pi_T, x)^t$$

and also

$$p(\Pi_k, x) = p_k(x).$$

Combining the two previous results, we obtain the following formulae. It is interesting to compare the expression for $\chi(\Pi^b_{n,2}, t)$ with the well known

$$\chi(\Pi_{n,2,1}) = (t-1)(t-3)\cdots(t-2n+1).$$

Theorem 3.3 *We have the functional equation*

$$F\left(\Pi^b_T, x, t\right) = p(\Pi_T, 2x)^{t/2}.$$

In particular

$$F\left(\Pi^b_k, x, t\right) = p_k(2x)^{t/2} = \left[1 + 2x + \cdots + \frac{(2x)^{k-1}}{(k-1)!}\right]^{t/2},$$

and

$$\chi\left(\Pi^b_{n,2}, t\right) = t(t-2)(t-4)\cdots(t-2n+2).$$

Proof: Multiplying the equation in Proposition 3.1 by $x^{n-1}/(n-1)!$ and summing, we obtain

$$F_x\left(\Pi^b_T, x, t\right) = tM_x(\Pi_T, 2x)F\left(\Pi^b_T, x, t\right)$$

where the subscript x denotes the derivative with respect to that variable. This differential equation is easily solved by separating variables and then simplified by using Theorem 3.2:

$$F\left(\Pi^b_T, x, t\right) = \exp\left(\frac{t}{2}M(\Pi_T, 2x)\right) = p(\Pi_T, 2x)^{t/2}.$$

The first special case follows from Theorem 3.2 again. The second is obtained from the first by extracting the coefficient of $x^n/n!$ using the binomial theorem. □

Although $\chi(\Pi^b_{n,k}, t)$ factors over the integers for $k = 2$, it does not do so in general. However, it does factor partially. To see this, it is convenient to expand this polynomial in the basis of *double falling factorials*

$$\{t\}_n = \chi\left(\Pi^b_{n,2}, t\right) = t(t-2)(t-4)\cdots(t-2n+2).$$

We will also need a certain refinement of the Stirling numbers of the second kind, namely let

$$S_k(n, j) = \text{the number of partitions of } [n] \text{ into } j \text{ subsets, each subset of size } \leq k.$$

Corollary 3.4 *We have the expansion*

$$\chi\left(\Pi^b_{n,k}, t\right) = \sum_{j=1}^{n} 2^{n-j} S_{k-1}(n, j)\{t\}_j,$$

and the divisibility relation

$$\{t\}_{\lceil n/(k-1)\rceil} \mid \chi\left(\Pi^b_{n,k}, t\right)$$

where $\lceil \cdot \rceil$ *is the round-up function.*

Proof: The second expression follows from the first because $S_{k-1}(n, j) = 0$ for $n > (k-1)j$.

To obtain the expression for χ, write $p_k(x) = 1 + \bar{p}_k(x)$ and use Theorem 3.3:

$$\begin{aligned} F\left(\Pi^b_k, x\right) &= (1 + \bar{p}_k(2x))^{t/2} \\ &= \sum_{j \geq 0} \binom{t/2}{j} \bar{p}_k(2x)^j \\ &= \sum_{j \geq 0} \{t\}_j 2^{-j} \bar{p}_k(2x)^j / j! \end{aligned}$$

Now take the coefficient of $x^n/n!$ on both sides, using the fact that this coefficient in $\bar{p}_k(2x)^j/j!$ is just $2^n S_{k-1}(n, j)$. □

We can now follow the same path for $\Pi_{n,T,V}$ that we did for $\Pi^b_{n,T}$.

Proposition 3.5 *We have the recurrence relation*

$$\begin{aligned} \chi(\Pi_{n,T,V}, t) &= t \sum_{m=1}^{n} 2^{m-1} \binom{n-1}{m-1} \mu(\Pi_{m,T}) \chi(\Pi_{n-m,T,V}, t) \\ &\quad + \sum_{m=1}^{n} \binom{n-1}{m-1} \mu(\Pi_{m,T,V}) \chi\left(\Pi^b_{n-m,T}, t\right). \end{aligned}$$

Proof: The first and second sums correspond to graphs $G \in \Pi_{n,T,V}$ where the component K containing the vertex n is balanced and unbalanced, respectively. Since the details are very much like those in the proof of Proposition 3.1, we omit them. □

Theorem 3.6 *We have the functional equations*

$$M(\Pi_{T,V},x) = p(\Pi_T,2x)^{-1/2}p(\Pi_{T,V},x),$$
$$F(\Pi_{T,V},x,t) = p(\Pi_T,2x)^{(t-1)/2}p(\Pi_{T,V},x).$$

In particular for $h \le k$

$$F(\Pi_{k,h},x,t) = p_k(2x)^{(t-1)/2}p_h(x).$$

Proof: As in the proof of Theorem 3.3, we multiply the equation in Proposition 3.5 by $x^{n-1}/(n-1)!$ and sum:

$$F_x(\Pi_{T,V},x,t) = tM_x(\Pi_T,2x)F(\Pi_{T,V},x,t) + M_x(\Pi_{T,V},x)F\big(\Pi_T^b,x,t\big).$$

Moving the first term from the right to the left side of this equation, we see that it is linear in $F(\Pi_{T,V},x,t)$ with integrating factor

$$\exp\left(\int -tM_x(\Pi_T,2x)\,dx\right) = \exp\left(-\frac{t}{2}M(\Pi_T,2x)\right) = p(\Pi_T,2x)^{-t/2}$$

by Theorem 3.2. Applying the integrating factor, we get

$$\begin{aligned}\frac{\partial}{\partial x}[F(\Pi_{T,V},x,t)p(\Pi_T,2x)^{-t/2}] &= M_x(\Pi_{T,V},x)F\big(\Pi_T^b,x,t\big)p(\Pi_T,2x)^{-t/2}\\ &= M_x(\Pi_{T,V},x)\end{aligned}$$

by Theorem 3.3. Integration gives

$$F(\Pi_{T,V},x,t) = p(\Pi_T,2x)^{t/2}M(\Pi_{T,V},x).$$

If $t = 1$ this specializes to

$$M(\Pi_{T,V},x) = p(\Pi_T,2x)^{-1/2}p(\Pi_{T,V},x)$$

which, when plugged back into the previous equation, also gives the formula for $F(\Pi_{T,V},x,t)$ in the statement of the theorem.

For the "in particular", the factor of $p_k(2x)^{(t-1)/2}$ is obtained from Theorem 3.2. Also note that for $h \le k$ the poset $\Pi_{n,k,h}$ has $\hat{0} = \hat{1}$ if and only if $n < h$. Thus

$$\chi(\Pi_{n,k,h},1) = \begin{cases} 1 & \text{if } n < h \\ 0 & \text{if } n \ge h \end{cases}.$$

which gives the second factor. □

We can specialize this theorem to get the well known generating functions for the characteristic polynomial in the case of the B_n and D_n hyperplane arrangements

Corollary 3.7 *We have the generating functions*

$$F(L(\mathcal{B}), x, t) = (1+2x)^{(t-1)/2} \qquad \text{for } L(\mathcal{B}_n) = \Pi_{n,2,1}$$
$$F(L(\mathcal{D}), x, t) = (1+2x)^{(t-1)/2}(1+x) \quad \text{for } L(\mathcal{D}_n) = \Pi_{n,2,2}.$$

We can also get a nice formula involving $\chi(\Pi_{n,k,h})$ itself.

Corollary 3.8 *For $1 \le h \le k$ we have the expansion*

$$\chi(\Pi_{n,k,h}, t) = \sum_{i=0}^{h-1} \binom{n}{i} \sum_{j=1}^{n-i} 2^{n-i-j} S_{k-1}(n-i, j)\{t-1\}_j,$$

and the divisibility relation

$$\{t-1\}_{\lceil (n-h+1)/(k-1)\rceil} \mid \chi(\Pi_{n,k,h}, t).$$

Proof: Again, the second relation follows easily from the first. We could derive the expression for χ from Theorem 3.6 with a demonstration similar to that of Corollary 3.4. Instead, we will give a combinatorial proof based on Theorem 2.3.

It suffices to show that a polynomial equation holds for positive odd $t = 2s+1$, so consider the cube Q_t. The arrangement $\mathcal{A}$ consists of all subspaces containing points with at least k coordinates equal in absolute value or at least h coordinates equal to zero. Thus $Q_t \backslash \mathcal{A}$ contains those points $\mathbf{x} = (x_1, \ldots, x_n)$ in Q_t with at most $k-1$ of the numbers $|x_1|, \ldots, |x_n|$ equal to any given value and at most $h-1$ zero coordinates.

Let $A_{i,j} \subseteq Q_t \backslash \mathcal{A}$ consist of all those points with exactly i zero coordinates and exactly j different nonzero coordinate absolute values. So it suffices to show that

$$|A_{i,j}| = \binom{n}{i} 2^{n-i-j} S_{k-1}(n-i, j)\{t-1\}_j.$$

First, there are $\binom{n}{i}$ ways to choose the zero coordinates in $\mathbf{x}$. Next, we can choose the absolute values of nonzero coordinates in $s(s-1)\cdots(s-j+1) = 2^{-j}\{t-1\}_j$ ways. Once these j values have been chosen, we can distribute them among the $n-i$ nonzero coordinates in $S_{k-1}(n-i, j)$ ways, since a value can be repeated at most $k-1$ times. Finally, there are 2^{n-i} ways to sign the nonzero coordinates after choosing their absolute values. □

4. Shellability, homotopy type and homology

In this section we will determine the homotopy type of $\Pi_{n,k,h}$ for $h < k$. Since these lattices turn out to be homotopic to wedges of spheres, their homology groups are free and we will derive a formula for the corresponding Betti numbers. We will do this by applying Theorem 2.4. First, however, we will need to cite some general results about shellings.

Consider posets P and P' with covers labeled by totally ordered sets L and L', respectively. A *cover isomorphism* is a function $f: P \to P'$ such that

1. f is an isomorphism of posets and thus induces a bijection $f: \mathcal{C}(P) \to \mathcal{C}(P')$, furthermore
2. f induces a well-defined order-preserving bijection $\hat{f}: L \to L'$ given as follows: If $l \in L$ labels a cover, c, and $f(c) = c'$ with label l' then $\hat{f}(l) = l'$.

The next result can be obtained immediately from the definitions.

Lemma 4.1 *Let $f : P \to P'$ be a cover isomorphism. Then the labeling L of P is a lexicographic shelling if and only if the labeling L' of P' is a lexicographic shelling.*

Now let P_1, P_2 be cover-labeled posets with labelings $\lambda_1: \mathcal{C}(P_1) \to L_1$ and $\lambda_2: \mathcal{C}(P_2) \to L_2$. The *product labeling* of the poset product $P = P_1 \times P_2$ by $L_1 \cup L_2$ is

$$\lambda((a, b) \to (c, b)) = \lambda_1(a \to c)$$
$$\lambda((a, b) \to (a, d)) = \lambda_2(b \to d).$$

A useful lemma concerning products is as follows.

Lemma 4.2 (Björner and Wachs [4]) *Let $P = P_1 \times P_2$ have the product labeling λ and fix a linear extension of $L_1 \cup L_2$. If*
1. *$L_1 \cap L_2 = \emptyset$ and*
2. *λ_1, λ_2 are lexicographic shellings of P_1, P_2, respectively,*

then λ is a lexicographic shelling of $P_1 \times P_2$.

Finally, we will need to recall the shelling of $\Pi_{n,k}$ given in [4]. Any graph G has an *associated set partition*

$$\pi(G) = B_1/B_2/\cdots/B_m$$

where each B_i is the vertex set of a component of G. We call B_i an *l-block* if $|B_i| = l$. We also use the notation $[n] = \{1, 2, \ldots, n\}$. Now define a labeling $\lambda : \mathcal{C}(\Pi_{n,k}) \to [2] \times [n]$, where the label set is given the lexicographic ordering, as given in the following table.

Symbol	Operation to obtain the cover $G \to H$	$\lambda(G \to H)$
M	Merge two nontrivial blocks B_i, B_j	$(1, \max B_i \cup B_j)$
C	Create a new k-block B	$(2, \max B)$
S	Merge singleton $\{a\}$ into a nontrivial block	$(2, a)$

Here M, C and S stand for "merge", "create" and "singleton", respectively.

Theorem 4.3 (Björner and Wachs [4]) *The labeling rules M, C and S give a lexicographic shelling of $\Pi_{n,k}$.*

For $\Pi_{n,k,h}$ there are additional covers involving unbalanced blocks. Let $h < k$ and define a labeling $\lambda : C(\Pi_{n,k,h}) \to [4] \times [n]$ by

Symbol	Operation to obtain the cover $G \to H$	$\lambda(G \to H)$
UM	Convert a nontrivial balanced block B to unbalanced, or merge it with the unbalanced block	$(1, \max B)$
UC	Create a new unbalanced h-block B	$(2, \max B)$
US	Merge $\{a\}$ into the unbalanced block	$(2, a)$
BM	Merge two nontrivial balanced blocks B_i, B_j	$(3, \max B_i \cup B_j)$
BC	Create a new balanced k-block B	$(4, \max B)$
BS	Merge $\{a\}$ into a nontrivial balanced block	$(4, a)$

In this table U and B stand for "unbalanced" and "balanced", respectively.

The main theorem of this section is the following

Theorem 4.4 *The labeling rules UM-BS give a lexicographic shelling of* $\Pi_{n,k,h}$ *for* $h < k$.

Proof: Let $[G, H]$ be an interval in $\Pi_{n,k,h}$ and consider a nontrivial component K of H. We also let K stand for the graph in $\Pi_{n,k,h}$ obtained from H by breaking every component with vertices in $[n]\backslash V(K)$ into singletons. Let G_K denote the graph obtained from G by the same operation. Then we have the poset isomorphism

$$[G, H] \equiv \prod_K [G_K, K]. \tag{2}$$

Furthermore, if $V(K) = \{v_1 < \cdots < v_m\}$ then the map $f : V(K) \to [m]$ given by $v_i \mapsto i$ induces a map $f : [G_K, K] \to [G'_K, \hat{1}]$ for some G'_K. Here $[G'_K, \hat{1}]$ is in $\Pi_{m,k}$ or $\Pi_{m,k,h}$ if K is balanced or unbalanced, respectively. If $\Pi_{m,k}$ is labeled by M-S and $\Pi_{m,k,h}$ is labeled by UM-BS, then f is a cover isomorphism and (2) gives rise to the product labeling. When $H \neq \hat{1}$, each factor of (2) is lexicographically shellable according to the cover isomorphism lemma combined with either Theorem 4.3 for the factors $\Pi_{m,k}$ or with induction on n for the factors $\Pi_{m,k,h}$. The lemma on poset products now applies to show that the given labeling satisfies conditions S1 and S2 on intervals $[G, H]$ for $H \neq \hat{1}$.

When $H = \hat{1}$, we must first identify an increasing chain C in $[G, \hat{1}]$. There will be three cases depending on the form of G.

Case I. G has an unbalanced component. Say

$$\pi(G) = B_0/B_1/\cdots/B_m$$

where B_0 corresponds to the unbalanced component, B_i, $1 \le i \le l$, correspond to nontrivial balanced components, and B_i, $l < i \le m$, correspond to trivial components.

Let

$$b_i = \max B_i \quad \text{for } 1 \le i \le m. \tag{3}$$

Without loss of generality we can list the B_i so that

$$b_1 < \cdots < b_l \quad \text{and} \quad b_{l+1} < \cdots < b_m. \tag{4}$$

Define a chain C by

$$C : G = G_0 \to G_1 \to \cdots \to G_m = \hat{1}$$

where G_j, $0 \le j \le m$, has an unbalanced component on the vertex set $\cup_{i \le j} B_i$ and balanced components on $B_{j+1}, \ldots, B_m$. Thus

$$\lambda(C) : (1, b_1) < \cdots < (1, b_l) < (2, b_{l+1}) < \cdots < (2, b_m) \tag{5}$$

which is increasing.

Case II. G has nontrivial component(s) all of which are balanced. Say

$$\pi(G) = B_1/\cdots/B_l/\cdots/B_m$$

with the same conventions as in Case I. Note that B_0 does not exist and $l \ge 1$. Define $C : G = G_0 \to G_1 \to \cdots \to G_m = \hat{1}$ where G_j is as before but only for $j \ge 1$. So the first cover in C is the conversion of B_1 from balanced to unbalanced. Thus $\lambda(C)$ is still given by (5) and is increasing.

Case III. G has only trivial components, i.e., $G = \hat{0}$. Let

$$C : \hat{0} = G_0 \to G_1 \to \cdots \to G_{n-h+1} = \hat{1}$$

where G_j, $1 \le j \le n - h + 1$ has an unbalanced component on $[h + j - 1]$ and $\{h + j\}$, $\ldots$, $\{n\}$ are singletons. Thus

$$\lambda(C) : (2, h) < (2, h + 1) < \cdots < (2, n)$$

which is increasing.

We must verify condition S1 in the definition of a shelling. Let C' be any chain different from C in $[G, \hat{1}]$. To show that C' is not increasing, it suffices to find an *inversion*, i.e., a pair of labels λ_1, λ_2 such that $\lambda_1 \ge \lambda_2$ but λ_1 comes before λ_2 in $\lambda(C')$. The last cover of C' must come from an application of rule UM or rule US and so must have a label (i, b) where $i = 1, 2$. If C' contains a cover coming from one of the balanced labeling rules, then it has a previous label (j, c) where $j = 3, 4$ and thus an inversion. So we can restrict ourselves to chains C' using only UM-US. We now consider each of the three cases separately.

Case I. Since G already has an unbalanced component, we can't use rule UC, leaving only UM and US possible. If any application of US comes before a UM, then some $(2, b)$ forms an inversion with some $(1, c)$. Otherwise $\lambda(C')$ is a non-identity permutation of $\lambda(C)$ as given in (5) and hence contains an inversion.

Case II. Since G contains only balanced components, the first cover $G_0 \to G_1'$ of C' must come from rules UC or UM. Let C'' be the portion of C' in the interval $[G_1', \hat{1}]$. If C'' has an inversion, then we are done. Otherwise, from Case I, C'' must be the unique increasing chain in $[G_1', \hat{1}]$. There are now two possibilities.

If $G_0 \to G_1'$ is a UC cover then it has label $(2, b)$ for some b. This implies that the first cover of C'' must be UM, since G_1' has nontrivial balanced component B_1, and so has label $(1, b_1)$. Thus these two labels form an inversion.

If $G_0 \to G_1'$ is a UM cover then it has label $(1, b_i)$ for some i, where the b_i are given by (3). Furthermore we must have $2 \leq i \leq l$ since $i = 1$ leads to $C' = C$. As before, this implies that the first cover of C'' is UM with label $(1, b_1)$. Thus this pair of labels forms an inversion by (4).

Case III. Since $G = \hat{0}$, the first cover of C' must be of type UC. Let G_1', C'' be as before and reason as in Case II to reduce to the situation where C'' is the unique increasing chain in $[G_1', \hat{1}]$. Now G_1' has a unique nontrivial component which is unbalanced on some vertex set B with $|B| = h$ but $B \neq [h]$. (The case $B = [h]$ leads to $C' = C$.) Thus there is a singleton component b in G_1' with $b < \max B$. Hence the label $\lambda(\hat{0} \to G_1') = (2, \max B)$ forms an inversion with the label $(2, b)$ in $\lambda(C'')$. This completes the verification that C' contains an inversion.

Finally we must verify condition S2 in the definition of a shelling, showing that $\lambda(C)$ is lexicographically least. It suffices to show the following: If $G_i \to G_{i+1}$ is any cover in C and H is any element of $\Pi_{n,k,h}$ covering G_i then

$$\lambda(G_i \to G_{i+1}) \geq \lambda(G_i \to H) \quad \text{implies} \quad H = G_{i+1}.$$

Case I. Consider $G_0 \to G_1$ with label $(1, b_1)$. So $\lambda(G_0 \to H) \leq (1, b_1)$ implies $\lambda(G_0 \to H) = (1, b)$ for some $b \leq b_1$. Thus $G_0 \to H$ is a UM cover and this in turn implies that $b = b_i$ for some i, $1 \leq i \leq l$. Now (4) forces $b = b_1$ and $H = G_1$ as desired. Similar considerations apply for the covers $G_i \to G_{i+1}$, $1 < i < l$.

Now consider $G_l \to G_{l+1}$ with label $(2, b_{l+1})$. So $\lambda(G_l \to H) \leq (2, b_{l+1})$ implies $G_l \to H$ is a cover coming from some unbalanced rule. But G_l has only one nontrivial component, K, which is unbalanced. Thus only rule US can apply and $\lambda(G_l \to H) = (2, b)$ for some vertex label $b \leq b_{l+1}$. But b_{l+1} is the smallest vertex label outside K, so $b = b_{l+1}$ and $H = G_{l+1}$. Similar considerations apply for the covers $G_i \to G_{i+1}$, $l < i < m$.

Case II. The exact same reasoning as in Case I applies. One need only be careful about which of the two possibilities in the UM rule is being used.

Case III. Consider $G_0 \to G_1$ with label $(2, h)$. Since $G_0 = \hat{0}$, the cover $G_0 \to H$ must come from UC or BC. But $\lambda(G_0 \to H) \leq (2, h)$ rules out BC, so $\lambda(G_0 \to H) = (2, b)$ with $b \leq h$. Also $b = \max B$ where B is the vertex set of the unique nontrivial component of H and $|B| = h$. Hence $b = h$ and $H = G$. The covers $G_i \to G_{i+1}$, $i \geq 1$, are handled as in the second paragraph of Case I.

This completes the verification of condition S2 and the proof of the theorem. □

We now use Theorem 2.4 to calculate the Betti numbers of $\Pi_{n,k,h}$ by counting decreasing chains in our shelling. First, however, we need a lemma about Stirling numbers. Let $c(t, t')$ denote a signless Stirling number of the first kind, i.e., the number of permutations of $[t]$ that decompose into t' disjoint cycles.

Lemma 4.5 *We have*

$$\sum_{t'=0}^{t} 2^{t'} c(t, t - t') = (2t - 1)!!$$

Proof: This result follows easily by induction on t. However, we prefer to give a combinatorial proof. The sum counts all permutations of $[t]$ where the elements of each cycle have been colored red or blue and the smallest element in each cycle is always red. The double factorial counts all complete matchings of the set $\{1, 2, \ldots, t, -1, -2, \ldots, -t\}$.

Given a cycle $(c_1, c_2, \ldots, c_m)$ with minimum c_1 in a bicolored permutation we construct a matching on $\{c_1, \ldots, c_m, -c_1, \ldots, -c_m\}$ as follows. If $m = 1$ then match c_1 to $-c_1$. If $m > 1$ then match c_1 to one of $\pm c_2$ where the sign is chosen to be the same as c_1's (in this case positive) if c_1, c_2 are colored differently, or the opposite sign (in this case negative) if c_1, c_2 are colored the same. Now match whichever of $\pm c_2$ is unmatched to one of $\pm c_3$ using the same rule, and iterate this process (subscripts being taken modulo m). It is easy to see that this results in a matching and so applying the process to every cycle in a permutation gives a complete matching. It is also easy to construct an inverse for this procedure, proving that we have a bijection. □

Note that any maximal chain in $\Pi_{n,k,h}$ has at most one cover that comes from applying rule UC. Let $D^l_{n,k,h}$ denote the number of decreasing maximal chains in $\Pi_{n,k,h}$ of length l that have no UC cover. Thus we have $l = n - t(k - 2)$ where t is the number of covers of type BC. Note that $1 \leq t \leq \lfloor n/k \rfloor$. Similarly, let $\hat{D}^l_{n,k,h}$ denote the number of decreasing maximal chains in $\Pi_{n,k,h}$ of length l that have exactly one UC cover. In this case $l = n - t(k - 2) - (h - 1)$, where t again counts the BC covers and $0 \leq t \leq \lfloor (n - h)/k \rfloor$.

Theorem 4.6 *Suppose that* $1 \leq h < k$ *and* $k > 2$. *If* $l = n - t(k-2)$ *where* $1 \leq t \leq \lfloor n/k \rfloor$, *then*

$$D^l_{n,k,h} = 2^{n-t}(2t - 1)!! \sum_{0=i_0 \leq \cdots \leq i_t = n-tk} \prod_{j=0}^{t-1} \binom{n - jk - i_j - 1}{k - 1} (j + 1)^{i_{j+1} - i_j}. \quad (6)$$

If $l = n - t(k-2) - (h-1)$ *where* $0 \le t \le \lfloor (n-h)/k \rfloor$, *then*

$$\hat{D}^l_{n,k,h} = \begin{cases} \binom{n-1}{h-1} & \text{if } t = 0, \\ \sum_{m=kt}^{n-h} \binom{n}{m}\binom{n-m-1}{h-1} D^{m-t(k-2)}_{m,k,h} & \text{if } t \ge 1. \end{cases}$$

All other values of $D^l_{n,k,h}$ *and* $\hat{D}^l_{n,k,h}$ *are zero.*

Proof: Let us consider a chain C counted by $D^l_{n,k,h}$. Then $\lambda(C)$ can be broken into three consecutive parts.

First comes a sequence of all labels with first coordinate 4. This corresponds to a sequence of BC and BS covers which create k-blocks and merge singletons until t nontrivial blocks have been formed. Note that no singletons are left after this stage because if there were, then rule US would have to be used after conversion of a block to unbalanced by UM, and C would not be decreasing. Let

$$i_j = \text{ the number of singleton merges while there are } \le j \text{ nontrivial blocks.}$$

So $0 = i_0 \le \cdots \le i_t = n - tk$. When creating the $j+1$st nontrivial block we must always use the largest remaining singleton (to maintain a decreasing chain) together with $k-1$ of the other $n - jk - i_j - 1$ available singletons. The number of balanced blocks on k elements is 2^{k-1}. Thus

$$\text{number of choices for the } j+1\text{st BC cover } = 2^{k-1}\binom{n-jk-i_j-1}{k-1}.$$

Each singleton merge while there are $j+1$ nontrivial blocks present can be done in $2j+2$ ways for a total of $(2j+2)^{i_{j+1}-i_j}$ ways, until the creation of block $j+2$. Thus the total number of choices for this portion of the chain is

$$\sum_{i_j}\prod_j 2^{k-1}\binom{n-jk-i_j-1}{k-1}(2j+2)^{i_{j+1}-i_j}$$
$$= 2^{n-t}\sum_{i_j}\prod_j \binom{n-jk-i_j-1}{k-1}(j+1)^{i_{j+1}-i_j}. \tag{7}$$

Next comes a sequence of labels with first coordinate 3, i.e., applications of rule BM for merging nontrivial blocks. Suppose there are t' merges starting from some graph H with blocks $A_1/\cdots/A_t$ and $a_i = \max A_i$, $1 \le i \le t$. Consider the ordinary partition poset $\Pi_{t,2}$ on the set $\{a_1, \ldots, a_t\}$, labeling each cover by $\max B_i \cup B_j$ where B_i, B_j are the two blocks merged. It is known from [14] that this is a shelling of $\Pi_{t,2}$. Let Π_H be the subposet obtained as the union of all chains in $\Pi_{n,k,h}$ starting at H and only using BM covers. Then there is a function $f : \Pi_H \to \Pi_{t,2}$ given by taking each $H' \in \Pi_H$ and mapping it to the graph obtained by removing all vertices except the a_i and making all remaining edges positive.

This function is onto and label-preserving. The decreasing chains with t' merges starting at H are mapped onto the decreasing chains with t' merges starting at $\hat{0} \in \Pi_{t,2}$. Since $\Pi_{t,2}$ is shelled by the labeling, it contains $c(t, t-t')$ such chains. Furthermore, each such chain has $2^{t'}$ preimages. Thus by Lemma 4.5 the total contribution from this portion of the chain C is

$$\sum_{t'} 2^{t'} c(t, t-t') = (2t-1)!! \tag{8}$$

Finally, we have covers coming from the unbalanced rules. But to use any such rule, we must first create an unbalanced block which gives a label of the form $(1, b)$. Thus to maintain a decreasing chain, we can only use rule UM, first to create the block and then to merge the rest of the blocks with it. This can only be done in one way, namely in decreasing order of maxima. Hence $D^l_{n,k,h}$ is given by the product of (7) and (8) which agrees with formula (6).

To obtain the formula for $\hat{D}^l_{n,k,h}$ when $t \geq 1$, we follow the same argument as before with two changes. At the end of the first sequence of BC and BS covers it is no longer true that only nontrivial blocks are left. Suppose that the set of vertices in nontrivial blocks is $S \subseteq [n]$ where $|S| = m$, $kt \leq m \leq n-h$. Then there are $\binom{n}{m}$ choices for S and $D^{m-t(k-2)}_{m,k,h}$ ways to pick the deceasing chains once S is chosen. We must also modify the final sequence of covers to begin with an application of UC to create an unbalanced component, followed by some applications of US, and ending with some of UM. The UC cover can be chosen in $\binom{n-m-1}{h-1}$ ways since the largest remaining singleton must be put in the unbalanced h-block. But the US sequence and the UM sequence can each be done only in decreasing order of maxima. Putting together the various counts finishes the $t \geq 1$ case.

Finally we must consider what happens when $t = 0$. Since no balanced k-blocks are created we must start with a UC cover, which can be done in $\binom{n-1}{h-1}$ ways, and follow by merging in all the singletons in decreasing order. This completes the counting of the decreasing chains. □

If P is a poset, then let $\tilde{\beta}^d(P)$ be the reduced Betti number of the order complex $\Delta(P)$ in dimension d (the rank of reduced homology with integer coefficients). We will use the abbreviations $\tilde{\beta}^d(x, y)$ when P is an interval $[x, y]$ and $\tilde{\beta}^d_{n,k}$, $\tilde{\beta}^d_{n,k,h}$ when $P = \Pi_{n,k}, \Pi_{n,k,h}$, respectively. In order to give the formulae for $\tilde{\beta}^d_{n,k,h}$ it will be convenient to have the Kronecker delta, $\delta_{s,t}$, which equals 1 if $s = t$ and 0 otherwise. Combining Theorems 2.4 and 4.6 we immediately get the following result.

Theorem 4.7 *If $1 \leq h < k$ and $k > 2$ then $\Pi_{n,k,h}$ has the homotopy type of a wedge of spheres, so its integral homology groups are free. In the case $h \neq 1, k-1$ we have*

$$d = n-2-t(k-2) \text{ with } 1 \leq t \leq \lfloor n/k \rfloor \Rightarrow \tilde{\beta}^d_{n,k,h} = D^{d+2}_{n,k,h}$$
$$d = n-2-t(k-2)-(h-1) \text{ with } 0 \leq t \leq \lfloor (n-h)/k \rfloor \Rightarrow \tilde{\beta}^d_{n,k,h} = \hat{D}^{d+2}_{n,k,h}$$

In the case $h = 1, k-1$ we have

$$d = n - 2 - t(k-2) \text{ with } 1 - \delta_{h,1} \le t \le \lfloor (n+1-\delta_{h,1})/k \rfloor \Rightarrow \tilde{\beta}^d_{n,k,h} = D^{d+2}_{n,k,h} + \hat{D}^{d+2}_{n,k,h}.$$

Furthermore, these are exactly the cases when $\tilde{\beta}^d_{n,k,h} \neq 0$.

Note that when $h = k-1$ then the parameter t counts the *total* number of times a nontrivial block is formed, rather than just the number of formations of balanced k-blocks. It is interesting to compare this result to the one in [5] on the reduced Betti numbers in the type A_n case. We will also need this result in the next section when we compute the ranks of cohomology groups.

Theorem 4.8 (Björner and Welker [5]) *If $2 < k \le n$ then $\Pi_{n,k}$ has the homotopy type of a wedge of spheres, so its integral homology groups are free. Furthermore, $\tilde{\beta}^d_{n,k} \neq 0$ if and only if $d = n-3-t(k-2)$ for some t, $1 \le t \le \lfloor n/k \rfloor$ and in that case*

$$\tilde{\beta}^d_{n,k} = (t-1)! \sum_{0=i_0 \le \cdots \le i_t = n-tk} \prod_{j=0}^{t-1} \binom{n-jk-i_j-1}{k-1} (j+1)^{i_{j+1}-i_j}.$$

Although there does not seem to be a nice closed form expression for $\tilde{\beta}^d_{n,k,h}$, things simplify in high and low dimensions. The following corollary is an easy computation using Theorems 4.6 and 4.7 so its proof is omitted.

Corollary 4.9 *We have the following particular values for $\tilde{\beta}^d_{n,k,h}$ in high dimensions.*

$$h \neq 1, k-1 \Rightarrow \tilde{\beta}^{n-k}_{n,k,h} = 2^{n-1}\binom{n-1}{k-1},$$

$$h \neq k-1 \quad \Rightarrow \tilde{\beta}^{n-h-1}_{n,k,h} = \binom{n-1}{h-1},$$

$$h = k-1 \quad \Rightarrow \tilde{\beta}^{n-k}_{n,k,h} = 2^{n-1}\binom{n-1}{k-1} + \binom{n-1}{k-2}$$

We have the following particular values for $\tilde{\beta}^d_{n,k,h}$ in low dimensions.

$$h \neq 1 \quad \text{and} \quad \frac{n}{k} = q \in \mathbb{Z} \Rightarrow \tilde{\beta}^{2q-2}_{n,k,h} = 2^{n-q}(2q-1)!! \prod_{j=0}^{q-1} \binom{n-jk-1}{k-1}$$

$$h \neq k-1 \quad \text{and} \quad \frac{n-h}{k} = r \Rightarrow \tilde{\beta}^{2r-1}_{n,k,h} = \binom{n}{n-h} 2^{n-r}(2r-1)!! \prod_{j=0}^{r-1} \binom{n-jk-1}{k-1}$$

Furthermore, if $h \neq 1, k-1$ then for any $d = n-2-t(k-2)$, $1 \le t \le \lfloor n/k \rfloor$, we have the divisibility relation

$$2^{n-\lfloor n/k \rfloor}\binom{n-1}{k-1} \Big| \tilde{\beta}^d_{n,k,h}.$$

5. Cohomology of the complement

If $\mathcal{A}$ is a subspace arrangement in $\mathbb{R}^n$ then we let $M(\mathcal{A}) = \mathbb{R}^n \setminus \cup \mathcal{A}$ be the manifold obtained by removing the subspaces from $\mathbb{R}^n$. In particular, let $M_{n,k,h} = M(\mathcal{B}_{n,k,h})$. The fundamental result linking the homology of the lattice $L(\mathcal{A})$ and cohomology of the manifold $M(\mathcal{A})$ is as follows. Here $\tilde{H}(\hat{0}, x)$ denotes the reduced homology of the interval $[\hat{0}, x]$ viewed as an order complex.

Theorem 5.1 (Goresky-MacPherson [7]) *Let $\mathcal{A}$ be a subspace arrangement with intersection lattice L and manifold M. Then for all dimensions d*

$$\tilde{H}^d(M) \equiv \bigoplus_{x \in L \setminus \hat{0}} \tilde{H}_{\operatorname{codim} x - 2 - d}(\hat{0}, x).$$

Since the Goresky-MacPherson formula involves the homology of intervals in L we will have to investigate their structure in $\Pi_{n,k,h}$. They turn out to be poset products so the next result, whose proof can be found in Björner and Welker [5], will be useful.

Proposition 5.2 *Suppose that $x < x'$ in poset P, $y < y'$ in poset Q and consider the interval $[(x, y), (x', y')] \in P \times Q$.*
1. *If $\Delta(x, x')$ and $\Delta(y, y')$ are both homotopy equivalent to wedges of spheres, then so is $\Delta((x, y), (x', y'))$.*
2. *If $\tilde{H}_d(x, x')$ and $\tilde{H}_d(y, y')$ are free for all d then so is $\tilde{H}_d((x, y), (x', y'))$ and*

$$\tilde{H}_d((x, y), (x', y')) \equiv \bigoplus_{p+q=d-2} \tilde{H}_p(x, x') \otimes \tilde{H}_q(y, y').$$

Finally, we need some notation. Suppose $G \in \Pi_{n,k,h}$ has $\pi(G) = B_0/B_1/\cdots$ where B_0 is the block of the unbalanced component. We permit $B_0 = \emptyset$. Recall from the isomorphism of Corollary 2.2 that G corresponds to a certain subspace, that we will also denote by G. To make our formulas more compact, it will be convenient to assume that if $n = 0$ then $\Pi_{0,k,h}$ consists of a single element and

$$\tilde{\beta}^d_{0,k,h} = \begin{cases} 1 & \text{if } d = -2, \\ 0 & \text{else.} \end{cases}$$

Theorem 5.3 *Suppose $1 \le h < k$ and $k > 2$. Consider $G \in \Pi_{n,k,h}$ having an unbalanced component of size $a_0 \ge 0$ and nontrivial balanced components of size $a_1, \ldots, a_m \ge k$.*

1. codim $G = \sum_{i=0}^{m} a_i - m$.
2. $[\hat{0}, G] \cong \Pi_{a_0,k,h} \times \Pi_{a_1,k} \times \cdots \times \Pi_{a_m,k}$.
3. $[\hat{0}, G]$ *has the homotopy type of a wedge of spheres and so its homology groups are free.*
4. $\tilde{\beta}^d(\hat{0}, G) = \sum_{p_0+\cdots+p_m=d-2m} \tilde{\beta}^{p_0}_{a_0,k,h} \tilde{\beta}^{p_1}_{a_1,k} \cdots \tilde{\beta}^{p_m}_{a_m,k}$.
5. $\tilde{\beta}^d(\hat{0}, G)$ *is nonzero if and only if*

$$d = \sum_{i=0}^{m} a_i - m - 2 - t(k-2)$$

where

$$m + (1 - \delta_{h,1})(1 - \delta_{a_0,0}) \le t \le \left\lfloor \frac{a_0 + \delta_{h,k-1}(1 - \delta_{a_0,0})}{k} \right\rfloor + \sum_{i=1}^{m} \left\lfloor \frac{a_i}{k} \right\rfloor,$$

or when $a_0 \neq 0, h \neq 1, k-1$ *and*

$$d = \sum_{i=0}^{m} a_i - m - 2 - t(k-2) - (h-1)$$

where

$$m \le t \le \left\lfloor \frac{a_0 - h}{k} \right\rfloor + \sum_{i=1}^{m} \left\lfloor \frac{a_i}{k} \right\rfloor.$$

Proof: The first two items follow from Zaslavsky's Theorem 2.1 characterizing $\Pi_{n,k,h}$. Number 3 comes from combining Theorems 4.7, 4.8, Proposition 5.2 and item 2. Similarly number 4 is an application of Proposition 5.2 to item 2. Note that when $a_0 = 0$ then $\tilde{\beta}^{p_0}_{a_0,k,h}$ only permits a term to be nonzero when $p_0 = -2$. In this case $p_1 + \cdots + p_m = d - 2(m-1)$ which is what is needed to correctly apply Proposition 5.2. Finally, the d values giving nonzero Betti numbers can be extracted by using item 4 and the bounds in Theorems 4.7 and 4.8. In particular, one must have $p_i = a_i - 3 - t_i(k-2)$ for $i > 0$ and $p_0 = a_0 - 2 - t_0(k-2)$ or $p_0 = a_0 - 2 - t_0(k-2) - (h-1)$ as appropriate and then the restrictions on the t_i will give the bounds on $t = \sum_{i=0}^{m} t_i$. □

Now let $\rho^d_{n,k,h}$, $\tilde{\rho}^d_{n,k,h}$ denote the ranks of $H^d(M_{n,k,h})$, $\tilde{H}^d(M_{n,k,h})$, respectively.

Theorem 5.4 *Suppose* $1 \le h < k$ *and* $k > 2$. *For each* $G \in \Pi_{n,k,h}$ *let* $a_0 \ge 0$ *be the size of its unbalanced component and let* $a_1, \ldots, a_m \ge k$ *be the sizes of its nontrivial balanced components.*

1. *The groups* $H^d(M_{n,k,h})$ *are free.*
2. $\tilde{\rho}^d_{n,k,h} = \sum_{G \in \Pi_{n,k,h} \setminus \hat{0}} \sum_{q_0+\cdots+q_m=d} \tilde{\beta}^{a_0-2-q_0}_{a_0,k,h} \tilde{\beta}^{a_1-3-q_1}_{a_1,k} \cdots \tilde{\beta}^{a_m-3-q_m}_{a_m,k}$.

3. $\rho^d_{n,k,h} \neq 0$ *if and only if*

$$d = t(k-2) \quad \textit{where} \quad 0 \le t \le \left\lfloor \frac{n}{k} \right\rfloor$$

or when $a_0 \neq 0, h \neq 1, k-1$ *and*

$$d = t(k-2) + (h-1) \quad \textit{where} \quad 0 \le t \le \left\lfloor \frac{n-h}{k} \right\rfloor.$$

Proof: The first two items follow from Theorem 5.1 together with numbers 3 and 4 of Theorem 5.3, respectively. To get the sum, we make the substitutions $p_0 = a_0 - 2 - q_0$ and $p_i = a_i - 3 - q_i$ for $i \ge 1$. The restrictions for the nonzero ranks are gotten by applying the bounds in Theorems 4.7 and 4.8 to the summation formula just proved. □

Again, the general form of $\tilde{\rho}^d_{n,k,h}$ is messy, but there is a nice low-dimensional case.

Corollary 5.5 *Suppose* $1 \le h < k$ *and* $k > 2$.

$$h \neq 1, k-1 \Rightarrow \tilde{\rho}^{k-2}_{n,k,h} = \sum_{i=k}^{n} 2^i \binom{n}{i}\binom{i-1}{k-1},$$

$$h = 1 \Rightarrow \tilde{\rho}^{k-2}_{n,k,1} = 2^n \sum_{i=k}^{n} \binom{n}{i}\binom{i-1}{k-1},$$

$$h = k-1 \Rightarrow \tilde{\rho}^{k-2}_{n,k,k-1} = \sum_{i=k}^{n} \binom{n}{i}\left[2^i\binom{i-1}{k-1} + \binom{i-1}{k-2}\right].$$

Proof: We will only do the case $h \neq 1, k-1$ as the others are similar. By Theorem 4.8, a factor in the sum for $\tilde{\rho}^d_{n,k,h}$ which corresponds to the ith balanced block will be nonzero only when $a_i - 3 - q_i = a_i - 3 - t_i(k-2)$. So we must have $q_i = t_i(k-2)$ for some $t_i \ge 1$, $1 \le i \le m$. Now $q_0 + \cdots q_m = d = k-2$ forces $q_0 = t_0(k-2)$ where $t_0 \ge 0$. Thus by Theorem 4.7 we see that G has exactly one nontrivial component which is unbalanced when $t_0 = 1$ and balanced when $t_1 = 1$. In the unbalanced case, if $a_0 = i$ then there are $\binom{n}{i}$ ways to choose the component and $\tilde{\beta}^{i-k}_{i,k,h} = 2^{i-1}\binom{i-1}{k-1}$ by Corollary 4.9. Thus the total contribution of these G is

$$\sum_{i=k}^{n} 2^{i-1}\binom{n}{i}\binom{i-1}{k-1}. \tag{9}$$

In the balanced case there are $2^{i-1}\binom{n}{i}$ ways to choose the component. Furthermore $\tilde{\beta}^{i-k-1}_{i,k} = \binom{i-1}{k-1}$ by a specialization of Theorem 4.8. Thus the balanced graphs give the same total contribution as the unbalanced ones, and doubling (9) gives our formula. □

There is another way to derive the formula for $\tilde{\rho}^{k-2}_{n,k,1}$ in this corollary. Since the arrangement $\mathcal{B}_{n,k,1}$ contains the coordinate hyperplanes its complement naturally decomposes into 2^n parts, one for each orthant. This way one easily sees that $M_{n,k,1}$ has the homotopy type of the disjoint union of 2^n copies of the complement of $\mathcal{A}_{n,k}$. Thus we derive that for all d

$$\rho^d_{n,k,1} = 2^n \rho^d_{n,k}$$

where $\rho^d_{n,k}$ is the rank of $H^d(M_{n,k})$. These ranks were computed in [5]. In particular, for $k > 2$ we get the previous corollary's formula for $\tilde{\rho}^{k-2}_{n,k,1}$.

6. Remarks

We end with some comments and questions raised by this work.

(1) In Section 3 we always assumed that $1 \in T$ when exploring the combinatorics of $\Pi_{n,T}$ and $\Pi_{n,T,V}$ since this was the case of interest to us. However, this restriction is not necessary. One can assume that T is arbitrary and add a $\hat{0}$ to the poset if $1 \notin T$. Linusson [9] has derived expressions for the generating functions that we considered in this generality for $\Pi_{n,T}$ and $\Pi_{n,T,V}$.

(2) We should explain the reasons for the restrictions that appear on k and h in our results from Sections 4 and 5. The inequality $k > 2$ is not really necessary. But when $k = 2$ the Betti number is nonzero only in dimension $n - 2$ and is given by a sum of many descending chain counts since the equations $n = d + 2 = n - t(k - 2) = n - t(k - 2) - (h - 1)$ put no restriction on t. The sum is, of course, much messier than the well-known value $\tilde{\beta}^{n-2}_{n,2,1} = (2n - 1)!!$. The reason for this is that shelling rules UM-BS are complicated precisely because they must take care of arrangements where the subspaces are not hyperplanes. If the subspaces do have codimension one then easier techniques are available.
On the other hand, the restriction $h < k$ is forced on us by the fact that UM-BS do not give a shelling when $h = k$, i.e., for $L(\mathcal{D}_{n,k})$. The problem is that the longest chains in $[\hat{0}, \hat{1}]$ no longer start with the creation of an unbalanced block, but instead must start by forming a balanced one. And one of these chains must be the unique increasing chain in order to obtain a lexicographic shelling. It seems that completely different techniques will have to be developed to handle this case. It would be interesting to either prove or disprove that the lattice is shellable when $h = k$.

(3) The topological results for $\mathcal{B}_{n,k,h}$ in Section 5 can be extended in the following ways. Using item 3 in Theorem 5.3 and the Ziegler-Živaljević Theorem [17] one can conclude that the singularity link $S^{n-1} \cap (\cup \mathcal{B}_{n,k,h})$ has the homotopy type of a wedge of spheres. Using Theorem 5.3 and the Goresky-MacPherson Theorem (Theorem 5.1) one can compute the cohomology groups, which are torsion-free, of the complement in $\mathbb{C}^n$ of the complexification $\mathcal{B}^{\mathbb{C}}_{n,k,h}$, for $1 \leq h < k$. The arguments are completely parallel to those carried out for $\mathcal{A}_{n,k}$ in [5] so they will be omitted here.

Note added in proof: The subspace arrangements $\mathcal{D}_{n,k}$ have been further investigated by E.M. Feichtner and D.N. Kozlov in their 1995 preprint "On subspace arrangements of type D." This paper contains various results on the homology of such arrangements, including a full characterization in the case $n \leq 2k$, as well as certain vanishing results in general.

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