Journal of Algebraic Combinatorics **6** (1997), 133–140

Buekenhout-Tits Unitals

G.L. EBERT* ebert@math.udel.edu
University of Delaware, Department of Mathematical Sciences, Newark, DE 19716-2553, USA

Received December 23, 1994; Revised May 2, 1996

Abstract. A Buekenhout-Tits unital is defined to be a unital in $PG(2, q^2)$ obtained by coning the Tits ovoid using Buekenhout's parabolic method. The full linear collineation group stabilizing this unital is computed, and related design questions are also addressed. While the answers to the design questions are very similar to those obtained for Buekenhout-Metz unitals, the group theoretic results are quite different.

Keywords: Buekenhout unital, Tits ovoid

1. Introduction

In [12] the even order Buekenhout-Metz unitals were studied in detail. In that paper it is remarked that the unital obtained by forming the ovoidal cone of a Tits ovoid using Buekenhout's parabolic method should not be considered a Buekenhout-Metz unital since the ovoid which is coned is not an elliptic quadric. Other authors (see [5] or [14], for instance) have included such unitals in the class of Buekenhout-Metz unitals. In this paper we compute the full linear collineation group stabilizing a Buekenhout-Tits unital, thereby obtaining a group that is significantly smaller than the group one would obtain if the "starting ovoid" were an elliptic quadric. This lends credence to the viewpoint that these unitals do not belong to the Buekenhout-Metz class. Related design questions for these unitals are also addressed.

2. Preliminary results

A *unital* is any $2-(n^3+1, n+1, 1)$ design. It is well known that unitals are found embedded in any square order desarguesian projective plane; namely, the absolute points and nonabsolute lines of an hermitian polarity of $PG(2, q^2)$ form a unital, called the *classical* or *hermitian* unital. In addition, unitals of order n which do not embed in any projective plane of order n^2 (desarguesian or not) have been constructed (see [6] and [15]), as have unitals which embed in more than one projective plane (see [6] and [13]). Moreover, it is known that unitals are embedded in every Hughes plane (see [17]), in every Δ-plane (see [4]), in every derived Hughes plane (see [1]), and in every square order Figueroa plane (see [10]).

*This research was partially supported by NSA grant MDA904–94–H–2033.

While it is still unknown which projective planes contain unitals, Buekenhout [8] showed that every (projective) translation plane of order q^2 with $GF(q)$ in its kernel contains a "parabolic" unital. Here *parabolic* means that the line at infinity meets the unital in exactly one point. In [16] Metz showed how to use Buekenhout's method to construct a nonclassical parabolic unital in the desarguesian plane $PG(2, q^2)$ for any prime power $q > 2$. In [8] Buekenhout also showed that every derivable translation plane of order q^2 with $GF(q)$ in its kernel contains a "hyperbolic" unital; that is, such a plane contains a unital meeting the line at infinity in $q+1$ points. However, recently Barwick [5] has shown that the only hyperbolic unital in $PG(2, q^2)$ that is obtainable from Buekenhout's method is the classical unital.

We now briefly discuss Buekenhout's parabolic method as applied to $PG(2, q^2)$. Let $\Sigma = PG(4, q)$ denote projective 4-space over the finite field $GF(q)$, and let $H \cong PG(3, q)$ be some fixed hyperplane of Σ. Let S be a regular spread of H. We then may model $\pi = PG(2, q^2)$ by taking the points of $\Sigma \backslash H$ as our affine points, the lines of S as our points at infinity, the planes of $\Sigma \backslash H$ which meet H in a line of S as our extended affine lines, and S as our line at infinity. Incidence is defined by inclusion (see [7], for instance).

To establish coordinates we let (x, y_1, y_2, z_1, z_2) denote homogeneous coordinates for Σ, where $x = 0$ is the equation of the hyperplane H at infinity, and we let (x, y, z) denote homogeneous coordinates for π, where $x = 0$ is the equation of the line at infinity for π. By picking $\epsilon \in GF(q^2) \backslash GF(q)$ and treating $\{1, \epsilon\}$ as an ordered basis for $GF(q^2)$ over $GF(q)$, we may establish the identification that $y = y_1 + y_2\epsilon$ and $z = z_1 + z_2\epsilon$. Now choose a 3-dimensional ovoid O which meets H in a single point P, and let Q be any point other than P on the unique spread line of S containing P. Buekenhout showed in [8] that the cone over O with vertex Q corresponds to a parabolic unital of $\pi = PG(2, q^2)$, using the above model for π. The argument given by Metz in [16] to show that for any $q > 2$ it is possible to choose O so the resulting unital is nonclassical uses only elliptic quadrics as candidates for O. Hence when we refer to a *Buekenhout-Metz unital*, we mean a unital embedded in $PG(2, q^2)$ obtained via Buekenhout's parabolic method in the special case when O is an elliptic quadric. We include the classical unital in this category, as it may be obtained in this fashion.

Of course, when q is an odd prime power, the only ovoids contained in $PG(3, q)$ are elliptic quadrics (see [3], for instance). However, when $q > 2$ is an odd power of 2, it is known that ovoids exist in $PG(3, q)$ which are not quadrics (see [18]). As we shall see in the next section, the automorphism group of a nonclassical unital in $PG(2, q^2)$ obtained via Buekenhout's parabolic method depends heavily on whether the ovoid being coned is a quadric or not.

3. The Buekenhout-Tits unital

For the remainder of this paper, let $q = 2^e$ for some odd integer $e > 1$. Let σ be the automorphism of $GF(q)$ defined by $\sigma : x \to x^{2^{(e+1)/2}}$. Thus $\sigma^2 : x \to x^2$ for all $x \in GF(q)$. Using left normalized row vectors to uniquely represent points of $PG(3, q)$, the Tits ovoid [18] may be coordinatized as $\bar{O} = \{(0, 0, 0, 1)\} \cup \{(1, s, t, s^{\sigma+2} + t^\sigma + st) : s, t \in GF(q)\}$. It is well known that every nontrivial planar section of $\bar{O}$ is a nonconic oval, and the unique tangent plane to $\bar{O}$ at $(1, s, t, s^{\sigma+2} + t^\sigma + st)$ is $[s^{\sigma+2} + t^\sigma + st, t, s, 1]$. Throughout this paper

$[\cdots]$ will denote the ordered coefficients of a linear equation representing a hyperplane in the appropriate projective space. Clearly, the tangent plane to $\bar{O}$ at $(0, 0, 0, 1)$ is $[1, 0, 0, 0]$.

Using the coordinates for $\Sigma = PG(4, q)$ described in the previous section, we embed $\bar{O}$ in Σ by taking $z_1 = 0$ as the hyperplane containing $\bar{O}$. Letting O denote the embedded Tits ovoid and left normalizing point coordinates for uniqueness as above, we have $P = O \cap H = (0, 0, 0, 0, 1)$. Without loss of generality, we may assume one of the lines in our regular spread of H is $\langle(0, 0, 0, 0, 1), (0, 0, 0, 1, 0)\rangle$, and hence we may choose $Q = (0, 0, 0, 1, 0)$ as the vertex of our cone. Thus the cone C over O with vertex Q is given by

$$\begin{aligned} C = \{(0, 0, 0, 0, 1)\} &\cup \{(0, 0, 0, 1, \lambda); \lambda \in GF(q)\} \\ &\cup \{(1, s, t, r, s^{\sigma+2} + t^{\sigma} + st) : r, s, t \in GF(q)\}. \end{aligned}$$

Modeling $\pi = PG(2, q^2)$ as in the previous section, C corresponds to the parabolic unital

$$U = \{(0, 0, 1)\} \cup \{(1, s + t\epsilon, r + (s^{\sigma+2} + t^{\sigma} + st)\epsilon) : r, s, t \in GF(q)\}$$

embedded in π. We call U a *Buekenhout-Tits unital*. To make our computations simpler, we pick $\epsilon \in GF(q^2)\backslash GF(q)$ so that $\epsilon^q = 1 + \epsilon$ and $\epsilon^2 = \delta + \epsilon$ for some $1 \neq \delta \in GF(q)$ with the trace of δ over $GF(2)$ equal to 1 (see [12], for instance).

Let $G = \{\theta \in PGL(3, q^2) : \theta(U) = U\}$ denote the linear collineation group of π leaving U invariant. As we shall soon see, G must fix the special point $P_\infty = (0, 0, 1)$ of U. Note that $l_\infty = [1, 0, 0]$ is the unique tangent line to U at P_∞. Of course, a simple counting argument shows that every point of U is incident with a unique tangent line, and every point of $\pi \backslash U$ is incident with exactly $q + 1$ tangent lines to U. Our proofs will frequently involve shifting our viewpoint from the unital U in $PG(2, q^2)$ to the ovoid $\bar{O}$ in $PG(3, q)$, to the cone C in $PG(4, q)$, and so forth.

Lemma 1 *Let $P_s = (1, s, s^{\sigma+2}\epsilon)$ for some $s \in GF(q)$. Then $P_s \in U$ and the unique tangent line to U at P_s is $l_s = [s^2 + s^{\sigma+2}\epsilon, s, 1]$.*

Proof: Clearly, $P_s \in U$ from the definition of U, and hence P_s is incident with a unique tangent line to U. Also P_s is incident with l_s from a computation of the inner product. Suppose $(1, \bar{s} + \bar{t}\epsilon, \bar{r} + (\bar{s}^{\sigma+2} + \bar{t}^{\sigma} + \bar{s}\bar{t})\epsilon)$ is another point of $U \cap l_s$, where $\bar{r}, \bar{s}, \bar{t} \in GF(q)$. Then $s^2 + s^{\sigma+2}\epsilon + s\bar{s} + s\bar{t}\epsilon + \bar{r} + (\bar{s}^{\sigma+2} + \bar{t}^{\sigma} + \bar{s}\bar{t})\epsilon = 0$ and hence

(i) $s^2 + s\bar{s} + \bar{r} = 0$
(ii) $s^{\sigma+2} + s\bar{t} + \bar{s}^{\sigma+2} + \bar{t}^{\sigma} + \bar{s}\bar{t} = 0.$

Returning momentarily to the representation $\bar{O} = \{(0, 0, 0, 1)\} \cup \{(1, a, b, a^{\sigma+2} + b^{\sigma} + ab) : a, b \in GF(q)\}$ for the Tits ovoid $\bar{O}$ in $PG(3, q)$, we know that $(1, s, 0, s^{\sigma+2}) \in \bar{O}$ and the unique tangent plane to $\bar{O}$ at this point is $[s^{\sigma+2}, 0, s, 1]$. Since $(1, \bar{s}, \bar{t}, \bar{s}^{\sigma+2} + \bar{t}^{\sigma} + \bar{s}\bar{t})$ is also a point of $\bar{O}$ incident with the plane $[s^{\sigma+2}, 0, s, 1]$ by (ii), we necessarily have $\bar{s} = s$ and $\bar{t} = 0$. From (i) we then obtain $\bar{r} = 0$ and hence l_s is tangent to U at P_s. □

Lemma 2 *The unique tangent line to U at the point $P_{rst} = (1, s+t\epsilon, r+(s^{\sigma+2}+t^{\sigma}+st)\epsilon)$ of U is the line $l_{rst} = [s^2 + t^2\delta + st + r + (s^{\sigma+2} + t^{\sigma} + st)\epsilon, s + t + t\epsilon, 1]$.*

Proof: Follows from Lemma 1, using the collineation induced by the matrix

$$M = \begin{bmatrix} 1 & t\epsilon & r + ts + t^{\sigma}\epsilon \\ 0 & 1 & t + t\epsilon \\ 0 & 0 & 1 \end{bmatrix}$$

acting on row vectors, which maps l_s to l_{rst}. □

If R is a point of $\pi \backslash U$, the $q + 1$ points of U incident with the $q + 1$ tangent lines to U passing through R will be called the *feet* of R. We now characterize when the feet of R form a collinear set, and use this characterization to help determine the group G previously defined.

Theorem 3 *Let R be any point of $\pi \backslash U$. Then the feet of R are collinear if and only if $R \in l_\infty$.*

Proof: The fact that the feet of any point on l_∞, other than P_∞, must be collinear follows from the geometry of an ovoidal cone embedded in $PG(4, q)$ as described above (see [11] for the general case).

Conversely, suppose $R \notin l_\infty$. Then $R = (1, y, z)$ for some $y, z \in GF(q^2)$. Expressing $y = y_1 + y_2\epsilon$ and $z = z_1 + z_2\epsilon$ uniquely for $y_1, y_2, z_1, z_2 \in GF(q)$, the $q + 1$ tangent lines incident with R are easily seen to be the lines l_{rst} where $r, s, t \in GF(q)$ satisfy

$$\begin{cases} s^2 + t^2\delta + st + r + y_1 s + y_1 t + y_2\delta t + z_1 = 0 \\ s^{\sigma+2} + t^{\sigma} + st + y_2 s + y_1 t + z_2 + 0. \end{cases}$$

The corresponding feet are

$$\begin{aligned} F = \{&(1, s + t\epsilon, s^2 + t^2\delta + st + y_1 s + y_1 t + y_2\delta t + z_1 + (s^{\sigma+2} + t^{\sigma} + st)\epsilon) : \\ &s^{\sigma+2} + t^{\sigma} + st = y_2 s + y_1 t + z_2\}. \end{aligned}$$

If these feet were incident with a line of the form $[A, B, 1]$, then by expressing $A = a_1 + a_2\epsilon$ and $B = b_1 + b_2\epsilon$ for $a_1, a_2, b_1, b_2 \in GF(q)$, we obtain

(i) $s^2 + \delta t^2 + st + (y_1 + b_1)s + (y_1 + y_2\delta + b_2\delta)t + z_1 + a_1 = 0$
(ii) $s^{\sigma+2} + t^{\sigma} + st = b_2 s + (b_1 + b_2)t + a_2$.

Viewing the ordered pair (s, t) as a point in the desarguesian affine plane $AG(2, q)$ of order q, equation (ii) represents the $q + 1$ points (s, t) on some (affine) planar section of the Tits ovoid, while equation (i) represents the points (s, t) of an affine conic. Since the $q + 1$ ordered pairs (s, t) corresponding to the feet F must satisfy both (i) and (ii), we arrive at an obvious contradiction.

Similarly, if the feet F lie on a line of the form $[A, 1, 0]$, the corresponding ordered pairs (s, t) satisfy the equation $a_1 + a_2\epsilon + s + t\epsilon = 0$, and hence $s = a_1, t = a_2$. This contradicts the fact that we must have $q+1$ choices for (s, t). Therefore, all cases considered, the feet of a point $R \notin l_\infty$ do not form a collinear set. $\square$

Theorem 4 *Let G denote the linear collineation group of π leaving U invariant. Then G is an abelian group of order q^2 consisting of those collineations induced by the matrices*

$$\left\{ \begin{bmatrix} 1 & u\epsilon & v+u^\sigma\epsilon \\ 0 & 1 & u+u\epsilon \\ 0 & 0 & 1 \end{bmatrix} : u, v \in GF(q) \right\}.$$

Proof: Straight forward computations show that the q^2 linear collineations induced by the above matrices leave U invariant. Moreover, these collineations clearly form an abelian group.

Conversely, any element of G must fix the point P_∞ and hence the line l_∞ by Theorem 3. Since nonsingular matrices that are scalar multiples of one another induce the same collineation, we may therefore assume that every element of G is induced by a matrix of the form

$$M = \begin{bmatrix} 1 & a & b \\ 0 & e & c \\ 0 & 0 & f \end{bmatrix},$$

for some choice of $a, b, c, e, f \in GF(q^2)$ with $ef \neq 0$. We now determine the conditions imposed on the entries of any such matrix M. Once again we uniquely express $a = a_1 + a_2\epsilon$ for $a_1, a_2 \in GF(q)$, and so on for b, c, e, f.

Since $(0, 0, 1)$ and $(1, 0, 1)$ are points of U, the points $(0, 0, 1)M = (1, a, b)$ and $(1, 0, 1)M = (1, a, b+f)$ must also be points of U. This forces f to be an element of $GF(q)$ and thus $f_2 = 0$. Next let $P_{rst} = (1, s+t\epsilon, r+(s^{\sigma+2}+t^\sigma+st)\epsilon)$ denote any point of $U \backslash \{P_\infty\}$, where $r, s, t \in GF(q)$ are arbitrarily chosen. Then $P_{rst}M \in U$ implies that

$$\begin{aligned}(\#) \qquad & (a_1+e_1s+e_2\delta t)(a_2+e_2s+(e_1+e_2)t) \\ & + \left(a_1^\sigma + e_1^\sigma s^\sigma + e_2^\sigma \delta^\sigma t^\sigma\right)\left(a_1^2+e_1^2s^2+e_2^2\delta^2t^2\right) + a_2^\sigma + e_2^\sigma s^\sigma + (e_1+e_2)^\sigma t^\sigma \\ & = b_2 + c_2 s + (c_1+c_2)t + f_1(s^{\sigma+2}+t^\sigma+st)\end{aligned}$$

must hold for all choices of s and t in $GF(q)$. Letting $s = 0 = t$, one obtains

$$(*) \qquad a_1a_2 + a_1^{\sigma+2} + a_2^\sigma = b_2.$$

Letting $t = 0$ in (#) and using $(*)$, one sees that

$$\left(e_1^{\sigma+2}+f_1\right)s^{\sigma+2} + \left(e_2^\sigma + a_1^\sigma e_1^\sigma\right)s^\sigma + \left(e_1e_2 + a_1^\sigma e_1^2\right)s^2 + (a_2e_1 + a_1e_2 + c_2)s = 0$$

must hold for all $s \in GF(q)$. Treating the left-hand side of this equation as a polynomial in s, the degree is at most $2^{(e+1)/2}+2$, which is strictly less than $q = 2^e$ since $e \geq 3$. Thus, in order to have q roots, the polynomial must be identically zero. This forces

$$(**)\qquad \begin{cases} f_1 = e_1^{\sigma+2} \\ e_2^{\sigma} = a_1^2 e_1^{\sigma} \\ e_1 e_2 = a_1^{\sigma} e_1^2 \\ a_2 e_1 + a_1 e_2 = c_2 \end{cases}.$$

Similarly, letting $s = 0$ in (#) and using $(*)$, another polynomial degree argument (in t) shows that

$$(***)\qquad \begin{cases} e_2^{\sigma+2}\delta^{\sigma+2} = 0 \\ f_1 = e_1^{\sigma} + e_2^{\sigma} + a_1^2 e_2^{\sigma}\delta^{\sigma} \\ a_1^{\sigma} e_2^2 \delta^2 = (e_1 + e_2) e_2 \delta \\ c_1 + c_2 = a_2 e_2 \delta + a_1(e_1 + e_2) \end{cases}$$

Since $\delta \neq 0$ from its definition, one immediately obtains $e_2 = 0$ from $(***)$ and thus $e_1 \neq 0$ as $e \neq 0$. Solving $(**)$ and $(***)$ simultaneously, it is quickly seen that the matrix M must be of the form indicated in the theorem. □

Corollary *Let G be the linear collineation group of π stabilizing U. Then G fixes P_∞, has q orbits of size q^2 on $U\backslash\{P_\infty\}$, has q orbits of size q on $l_\infty\backslash\{P_\infty\}$, and has $q^2 - q$ orbits of size q^2 on $\pi\backslash(U \cup l_\infty)$.*

Proof: From the proof of Theorem 4, we know that G fixes P_∞. A trivial computation shows that G acts fixed-point-freely on $\pi\backslash l_\infty$. If $(0, 1, z)$ is any point of $l_\infty\backslash\{P_\infty\}$, the G-stablizer of this point consists of all collineations induced by matrices of the form

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $b \in GF(q)$ is arbitrary. Elementary counting finishes the proof. □

The above results support our decision not to include the Buekenhout-Tits unital U in the class of Buekenhout-Metz unitals. In [12] it is shown that any nonclassical Buekenhout-Metz unital $\bar{U}$ (obtained by coning an elliptic quadric) of even order q admits a nonabelian linear collineation group $\bar{G}$ of order $q^3(q-1)$. The point orbits of $\bar{G}$ acting on π are $\{P_\infty\}$, $\bar{U}\backslash\{P_\infty\}$, $l_\infty\backslash\{P_\infty\}$ and $\pi\backslash(\bar{U} \cup l_\infty)$. In [2] similar results are obtained for odd order Buekenhout-Metz unitals. These results differ significantly from those obtained in Theorem 4 and its corollary above.

4. Related structures

In [2] and [12] it is shown that all Buekenhout-Metz unitals (as defined above) are self-dual. This is also true for the Buekenhout-Tits unital U defined in the previous section. If $U^\perp$ denotes the dual design (i.e., the points of $U^\perp$ are the tangent lines to U and the blocks of $U^\perp$ are the points of $\pi\backslash U$), then the points of $U^\perp$ are

$$\begin{aligned}&\{[s^2+t^2\delta+st+r+(s^{\sigma+2}+t^\sigma+st)\epsilon, s+t+t\epsilon, 1]: r,s,t\in GF(q)\}\cup\{[1,0,0]\}\\ &\quad=\{[r+(s^{\sigma+2}+t^\sigma+st)\epsilon, s+t+t\epsilon, 1]: r,s,t\epsilon GF(q)\}\cup\{[1,0,0]\}\end{aligned}$$

by Lemma 2. If ψ denotes the semilinear collineation of π induced by the Frobenius field automorphism $x\to x^q$ followed by interchanging first and third coordinates, then

$$\begin{aligned}\psi(U)&=\{(r+s^{\sigma+2}+t^\sigma+st+(s^{\sigma+2}+t^\sigma+st)\epsilon, s+t+t\epsilon, 1):\\ &\qquad r,s,t\in GF(q)\}\cup\{(1,0,0)\}\\ &=\{(r+(s^{\sigma+2}+t^\sigma+st)\epsilon, s+t+t\epsilon, 1): r,s,t\in GF(q)\}\cup\{(1,0,0)\}\end{aligned}$$

since $(s+t\epsilon)^q=s+t(1+\epsilon)=s+t+t\epsilon$. Thus the point set for the design $U^\perp$ is identified with $\psi(U)$ by simply interchanging square and round brackets. As ψ clearly maps the blocks of U onto the blocks of $U^\perp$ with this same identification, we have proven the following result.

Theorem 5 *Buekenhout-Tits unitals are self-dual as designs.*

In [12, Theorem 3] it is observed that an even order Buekenhout-Metz unital cannot contain an oval. The same proof applies here.

Theorem 6 *Buekenhout-Tits unitals contain no ovals.*

Finally, in [2] and [12] it is shown that one can construct a 2-design from any Buekenhout-Metz unital by "projecting" along the blocks incident with P_∞. This 2-design has the parameters of a point residual of an inversive plane, and, moreover, can be completed to a miquelian inversive plane in a natural way. The analogous result holds for a Buekenhout-Tits unital U. In fact, it holds for any parabolic Buekenhout unital embedded in any translation plane (see [11]).

Theorem 7 *Let U be a Buekenhout-Tits unital. Then the points of $U\backslash\{P_\infty\}$ and the blocks of U not incident with P_∞ project upon a $2-(q^2,q+1,q)$ design whose "points" are $O\backslash\{P\}$ and whose "blocks" are the planar sections of O not incident with P. Clearly, this 2-design can be completed to a Suzuki-Tits inversive plane.*

5. Open problems

The determination of the linear collineation group stabilizing a Buekenhout-Tits unital U used the fact that the feet of a point $R\in\pi\backslash U$ are collinear if and only if $R\in l_\infty$. The same is true for nonclassical Buekenhout-Metz unitals. It seems like an interesting problem to determine the possible geometric configurations that may arise for the feet of a point $R \in \pi \setminus \bar{U}$, where $\bar{U}$ is any nonclassical unital obtained via Buekenhout's parabolic method. It should be noted here that the feet of such a point form an arc or a collinear set when $\bar{U}$ is an odd order Buekenhout-Metz unital that can be expressed as the union of conics (see [2]). Such a study might lead to another geometric characterization of nonclassical unitals embedded in $PG(2, q^2)$ arising from Buekenhout's method (see [14]). In addition, knowledge of the potential configurations for the feet might help resolve the questions of which projective planes contain unitals and which unitals can be embedded in $PG(2, q^2)$.

The notion of "projection" discussed in the last section also seems worthy of further investigation. For instance, if removal of a point and all the incident blocks from an abstract unital enables the resulting structure to be projected upon the point residual of an inversive plane, must the unital be embeddable in a translation plane? Must such a unital be obtainable from Buekenhout's parabolic method?

References

1. V. Abatangelo, B. Larato, and L.A. Rosati, "Unitals in planes derived from Hughes planes," *J. Comb., Inform. and System Sciences* **15** (1990), 151–155.
2. R.D. Baker and G.L. Ebert, "On Buekenhout-Metz unitals of odd order," *J. Combin. Theory, Ser. A* **60** (1992), 67–84.
3. A. Barlotti, "Un'estensione del teorema di Segre-Kustaanheimo," *Boll. Un. Mat. Ital.* **10** (1955), 498–504.
4. A. Barlotti and G. Lunardon, "Una classe di unitals nei Δ-piani," *Riv. Mat. Univ. Parma (4)* **5** (1979), 781–785.
5. S.G. Barwick, "A characterization of the classical unital," *Geom. Dedicata.* **52** (1994), 175–180.
6. A.E. Brouwer, "Some unitals on 28 points and their embeddings in projective planes of order 9," *Geometries and Groups*, Lecture Notes in Mathematics, Springer-Verlag, New York/Berlin, 1981, Vol. 893, pp. 183–188.
7. R.H. Bruck and R.C. Bose, "The construction of translation planes from projective spaces," *J. Algebra* **1** (1964), 85–102.
8. F. Buekenhout, "Existence of unitals in finite translation planes of order q^2 with a kernel of order q," *Geom. Dedicata* **5** (1976), 189–194.
9. P. Dembowski, *Finite Geometries*, Springer-Verlag, New York/Berlin, 1968.
10. M.J. deResmini and N. Hamilton, "Hyperovals and Unitals in Figueroa Planes," to appear in *Europ. J. Combinatorics*.
11. J. Dover, *Theory and Applications of Spreads of Geometric Spaces*, Ph.D. Thesis, University of Delaware, 1996.
12. G.L. Ebert, "On Buekenhout-Metz unitals of even order," *Europ. J. Combinatorics* **13** (1992), 109–117.
13. K. Gruning, "A class of unitals of order q which can be embedded in two different planes of order q^2," *J. Geom.* **29** (1987), 61–77.
14. C. Lefèvre-Percsy, "Characterization of Buekenhout-Metz unitals," *Arch. Math.* **36** (1981), 565–568.
15. H. Lüneburg, "Some remarks concerning the Ree groups of type (G_2)," *J. Algebra* **3** (1966), 256–259.
16. R. Metz, "On a class of unitals," *Geom. Dedicata* **8** (1979), 125–126.
17. L.A. Rosati, "Disegni unitari nei piani di Hughes," *Geom. Dedicata* **27** (1988), 295–299.
18. J. Tits, "Ovoides et groupes de Suzuki," *Arch. Math.* **13** (1962), 187–198.