Journal of Algebraic Combinatorics **7** (1998), 53–75

The Structure of Nonthin Irreducible *T*-modules of Endpoint 1: Ladder Bases and Classical Parameters

S. HOBART
Department of Mathematics, University of Wyoming, Laramie, WY 82071

T. ITO
Department of Computational Science, Kanazawa University, Kakuna-niachi, Kanazawa 920-11 Japan

Received June 29, 1995; Revised October 22, 1996

Abstract. Building on the work of Terwilliger, we find the structure of nonthin irreducible T-modules of endpoint 1 for P- and Q-polynomial association schemes with classical parameters. The isomorphism class of such a given module is determined by the intersection numbers of the scheme and one additional parameter which must be an eigenvalue for the first subconstituent graph. We show that these modules always have what we call a ladder basis, and find the structure explicitly for the bilinear, Hermitean, and alternating forms schemes.

Keywords: Association scheme, Terwilliger algebra

1. Introduction

The study of Terwilliger algebras for association schemes was begun by Terwilliger in [4], where they are called subconstituent algebras. These noncommutative algebras are generated by the Bose-Mesner algebra of the scheme, together with matrices containing local information about the structure with respect to a fixed vertex. It is expected that these algebras will contribute significantly to the classification of P- and Q-polynomial schemes.

The irreducible modules for thin P- and Q-polynomial schemes were thoroughly investigated by Terwilliger in [4]. Roughly speaking, he shows that such modules inherit the P- and Q-polynomial property and have structures described by Askey–Wilson polynomials related to those of the scheme. He also relates thinness to the combinatorial structure of the scheme.

Little is known about nonthin irreducible modules, and their structures seem to be much more complicated. For one particular family of schemes, the Doob schemes, all irreducible modules were found by Tanabe [3]. However for the classical forms schemes (bilinear, alternating, Hermitean, and quadratic forms), which for diameter ≥ 6 are the other known examples of nonthin P- and Q-polynomial association schemes [4], the irreducible modules have not yet been determined.

Some basic theory for the case of endpoint 1, is found in Terwilliger's unpublished lecture notes [5]. In particular, he shows that the isomorphism classes of modules are determined by the eigenvalues of the subgraph on the first subconstituent, and that they are only a little larger than thin modules, in the sense that they intersect the distance i subspaces in dimension 2 for $2 \leq i \leq d - 1$ and dimension 1 for $i = 1$ and $i = d$.

In this paper, we consider only schemes with classical parameters; for these schemes, we describe the nonthin irreducible modules of endpoint 1. We show that for classical parameters, there exists a particularly nice basis for any nonthin irreducible module of endpoint 1, which we call a *ladder basis*. Such bases were first shown to exist for modules of the Doob schemes in [3]; part of the motivation for our work was a question of Terwilliger as to whether such bases exist in general. In fact, we show something stronger: for classical parameters, half of the elements of a ladder basis are multiples of elements of the basis given in [5] (the *Terwilliger basis*). We also find the matrix for the action of the adjacency matrix A_1 on the module, with respect to both the Terwilliger and ladder bases.

All the known examples of P- and Q-polynomial schemes with diameter ≥ 6 which are not thin have classical parameters. They are also self-dual, and so have what we may call dual classical parameters for the Q-polynomial, or dual, structure. Since our methods are algebraic rather than combinatorial, the theorems have dual versions (describing A_1^* instead of A_1) for schemes with dual classical parameters. However, we do not explore this further in this paper.

Terwilliger algebras contain a lot of information about the scheme, possibly enough to reconstruct the scheme if it is P- and Q-polynomial. However, the determination of all nonthin modules is extremely difficult in general. The advantage of investigating irreducible modules of endpoint 1 is that they are small enough to be manageable, but still can be expected to reflect the nature of the local structure in the scheme. We hope that the investigation of irreducible modules of small endpoint, together with combinatorial methods, will be sufficient to finish the classification of P- and Q-polynomial schemes.

The organization of the paper is as follows. Section 2 gives definitions and a summary of previous results from [4] and [5]. In Section 3, we define the term ladder basis, and give necessary and sufficient conditions for existence. Section 4 shows that if the scheme has classical parameters, any nonthin irreducible T-module of endpoint 1 for the scheme has a ladder basis. In Section 5, we explicitly find the action of A_1 on the module, with respect to both the Terwilliger and ladder bases. The final section is devoted to examples: we find the eigenvalues of the subgraph on the first subconstituent for the bilinear, alternating, and Hermitean forms schemes, and use these to determine the nonthin irreducible endpoint 1 modules.

2. Terwilliger algebras

In this section, we review the definition and basic results for Terwilliger algebras from [4] and [5]. The books [1] and [2] are basic references for P- and Q-polynomial association schemes.

2.1. Definitions and notation

Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a commutative association scheme with d classes. As usual, let A_i be the adjacency matrix for relation R_i, $\mathcal{A}$ the linear span of $\{A_0, A_1, \ldots, A_d\}$ over $\mathbf{C}$, that is, the Bose-Mesner algebra, and $\{E_i\}_{i=0}^d$ the set of primitive idempotents of $\mathcal{A}$. E_i is the projection onto the ith common eigenspace of the adjacency matrices.

Fix $x \in X$. Let $\Gamma_i(x)$ be the ith subconstituent of $\mathcal{X}$, so $\Gamma_i(x) = \{y \in X \mid (x, y) \in R_i\}$. Define $E_i^* = E_i^*(x) \in \mathrm{Mat}_X(\mathbf{C})$ to be the $|X| \times |X|$ diagonal matrix with (y, y) entry 1 if $(x, y) \in R_i$ and 0 otherwise. The *Terwilliger algebra* $T = T(x)$ is the subalgebra of $\mathrm{Mat}_X(\mathbf{C})$ generated by $\{A_0, \ldots, A_d, E_0^*, \ldots, E_d^*\}$.

Let $V = \mathbf{C}^{|X|}$ be the unitary space over $\mathbf{C}$ with an orthonormal basis which we identify with X, and inner product $\langle\ ,\ \rangle$. V is a module for T, called the *standard module*. For $0 \le i \le d$, V has a subspace $V_i^* = V_i^*(x)$ with basis $\Gamma_i(x)$; E_i^* is the orthogonal projection onto V_i^*.

T is semi-simple, so V decomposes into an orthogonal direct sum of irreducible T-modules. In this paper, each T-module will be considered as a submodule of V; we can do this since V is faithful.

An irreducible T-module W is *thin* if $\dim(E_i^* W) \le 1$ for all i. The *endpoint* of W is $\min\{i : E_i^* W \neq 0\}$ (note that this is called the *dual endpoint* in [4]). There is a unique irreducible T-module of endpoint 0, called the *trivial module*; it is thin and has basis $\{E_i^*\mathbf{1} : 0 \le i \le d\}$, where $\mathbf{1}$ is the vector of all 1's.

Throughout the paper, we will assume $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ is a P- and Q-polynomial scheme of diameter $d \ge 3$. As usual, we denote the intersection numbers by a_i, b_i, and c_i, where $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$, $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$, and $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ for $y \in \Gamma_i(x)$. We also let $k = |\Gamma_1(x)|$, the valency of the graph. Let $\{\theta_i\}$ be the eigenvalues and $\{\theta_i^*\}$ the dual eigenvalues of $\mathcal{X}$, so $A_1 = \sum \theta_i E_i$ and $E_1 = \frac{1}{|X|} \sum \theta_i^* A_i$. It is well known that there exists a nonzero constant Q such that [1]

$$(\theta_{i+3} - \theta_{i+2}) - \rho(\theta_{i+2} - \theta_{i+1}) + (\theta_{i+1} - \theta_i) = 0 \qquad (0 \le i \le d-3)$$
$$(\theta_{i+3}^* - \theta_{i+2}^*) - \rho(\theta_{i+2}^* - \theta_{i+1}^*) + (\theta_{i+1}^* - \theta_i^*) = 0 \qquad (0 \le i \le d-3).$$

where $\rho = q + q^{-1}$.

It is convenient to define θ_i and θ_i^* for all integers i by the above recurrence. Note that θ_i (and similarly θ_i^*) are distinct for $0 \le i \le d$, but they may not be so in general.

As in [2], we say that $\mathcal{X}$ has *classical parameters* (d, q, α, β) if $\mathcal{X}$ has diameter d and intersection numbers

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad (0 \le i \le d)$$
$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \qquad (0 \le i \le d),$$

where $\left[\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right]_q = 1 + q + q^2 + \cdots + q^{j-1}$ is the usual q-binomial coefficient. In this case, the eigenvalues and dual eigenvalues of $\mathcal{X}$ satisfy

$$\theta_i = q^{-i} b_i - \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad (0 \le i \le d)$$
$$\theta_i^* = \theta_0^* + (\theta_1^* - \theta_0^*)\begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \qquad (0 \le i \le d).$$

Fix $x \in X$, and write $T = T(x)$, $E_i^* = E_i^*(x)$, and $V_i^* = V_i^*(x)$. Let

$$\begin{aligned} F &= \sum_{i=0}^{d} E_i^* A_1 E_i^* && \text{(the \textit{flat operator})} \\ R &= \sum_{i=0}^{d-1} E_{i+1}^* A_1 E_i^* && \text{(the \textit{raise operator})} \\ L &= \sum_{i=1}^{d} E_{i-1}^* A_1 E_i^* && \text{(the \textit{lower operator}).} \end{aligned}$$

Note that $F, L, R \in T$, and $A_1 = F + L + R$. F is a symmetric 0, 1 matrix, and R and L are 0, 1 matrices such that $R^t = L$.

We will use extensively the following relations on these operators, which were given in [4] in a slightly different form.

Proposition 2.1 ([4], Lemmas 5.5 and 5.6)

$$(g_i^- FL^2 + LFL + g_i^+ L^2 F - \gamma L^2)E_i^* = 0 \quad (2 \le i \le d) \tag{2.1}$$

$$(g_i^- R^2 F + RFR + g_i^+ FR^2 - \gamma R^2)E_{i-2}^* = 0 \quad (2 \le i \le d) \tag{2.2}$$

$$\begin{aligned} (e_i^- RL^2 + (\rho+2)LRL + e_i^+ L^2 R + LF^2 - \rho FLF \\ + F^2 L - \gamma(LF + FL) - \delta L)E_i^* = 0 \quad (1 \le i \le d) \end{aligned} \tag{2.3}$$

$$\begin{aligned} (e_i^- R^2 L + (\rho+2)RLR + e_i^+ LR^2 + F^2 R - \rho FRF \\ + RF^2 - \gamma(FR + RF) - \delta R)E_{i-1}^* = 0 \quad (1 \le i \le d) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \rho &= q + q^{-1} = \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} && (0 \le i \le d-3) \\ \gamma &= \theta_i - \rho\theta_{i+1} + \theta_{i+2} && (0 \le i \le d-2) \\ \delta &= \theta_i^2 - \rho\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) && (0 \le i \le d-1), \end{aligned}$$

which are constants independent of i, *and*

$$\begin{aligned} g_i^- &= \frac{\theta_{i-2}^* - \theta_{i-3}^*}{\theta_{i-2}^* - \theta_i^*} && (2 \le i \le d) \\ g_i^+ &= \frac{\theta_i^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i-2}^*} && (2 \le i \le d) \\ e_i^- &= \frac{\theta_{i-3}^* - \theta_{i-1}^*}{\theta_i^* - \theta_{i-1}^*} && (1 \le i \le d) \\ e_i^+ &= \frac{\theta_i^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i-1}^*} && (1 \le i \le d). \end{aligned}$$

If $\mathcal{X}$ has classical parameters (d, q, α, β), the above are given by

$$\rho = q + q^{-1}$$
$$\gamma = q^{d-1}\alpha + \beta - 1 + \frac{\alpha - \beta + 1}{q}$$
$$\delta = \frac{1}{q}\left\{\begin{bmatrix} d+1 \\ 1 \end{bmatrix}\begin{bmatrix} d-1 \\ 1 \end{bmatrix}\alpha^2 - \left(\begin{bmatrix} d+1 \\ 1 \end{bmatrix} + \begin{bmatrix} d-1 \\ 1 \end{bmatrix}\right)(\beta - 1)\alpha \right.$$
$$\left. + (q+1)^2 q^{d-1}\beta + (\beta - 1)^2 \right\}$$
$$g_i^- = \frac{-q^2}{q+1}$$
$$g_i^+ = \frac{-1}{q(q+1)}$$
$$e_i^- = -q(q+1)$$
$$e_i^+ = \frac{-(q+1)}{q^2}.$$

2.2. *Irreducible modules of endpoint 1*

We now summarize the relevant results of [5] and [7] about irreducible modules of endpoint 1.

Let U_1^* be the subspace of V_1^* which is orthogonal to $\mathbf{1}$. Note that U_1^* is the subspace of V_1^* which is orthogonal to the trivial module, and hence $E_1^* T E_1^*$ acts on U_1^*.

Theorem 2.2 ([5]) *Let W be an irreducible T-module of endpoint* 1. *Then $E_1^* W$ is a one-dimensional subspace of U_1^*. In particular, any nonzero $v \in E_1^* W$ is an eigenvector of $E_1^* A_1 E_1^*$, and $W = Tv$. Conversely, let $v \in U_1^*$ be an eigenvector of $E_1^* A_1 E_1^*$. Then Tv is an irreducible T-module of endpoint* 1.

Theorem 2.3 ([5]) *Let v, $v' \in U_1^*$ be eigenvectors for $E_1^* A_1 E_1^*$ with corresponding eigenvalues λ, λ'. Then Tv and Tv' are isomorphic as T-modules if and only if $\lambda = \lambda'$.*

Let $W = Tv$ be an irreducible T-module of endpoint 1, where v is an eigenvector of $E_1^* A_1 E_1^*$ acting on U_1^*. Define

$$v_i^+ = E_i^* A_{i-1} v$$
$$v_i^- = E_i^* A_{i+1} v.$$

Theorem 2.4 ([5]) *The set of vectors* $\{v = v_1^+, v_2^+, v_2^-, \ldots, v_{d-1}^+, v_{d-1}^-, v_d^+\}$ spans W. *It is a basis for W if W is not thin. In particular, if W is not thin*, $\dim E_i^* W = 2$ *for* $2 \le i \le d-1$, *and* $\dim E_i^* W = 1$ *for* $i = 1, d$.

Theorem 2.5 ([7]) *Suppose W is a nonthin irreducible T-module of endpoint* 1. *Then the following maps are nonsingular.*

$$R|_{E_i^*W} : E_i^*W \to E_{i+1}^*W \qquad (2 \le i \le d-2)$$
$$L|_{E_i^*W} : E_i^*W \to E_{i-1}^*W \qquad (3 \le i \le d-1)$$

The basis $\{v = v_1^+, v_2^+, v_2^-, \dots, v_{d-1}^+, v_{d-1}^-, v_d^+\}$ will be called the *Terwilliger basis* for W.

The following lemma gives some easy consequences of the definition of these vectors. Note that v_0^-, v_0^+, v_d^-, and v_{d+1}^+ are all the zero vector.

Lemma 2.6 ([5]) *For* $1 \le i \le d$,

$$\begin{aligned} Rv_i^+ &= c_i v_{i+1}^+ && (2.5)\\ Lv_i^- &= b_i v_{i-1}^- && (2.6)\\ E_i^* A_i v &= -v_i^+ - v_i^- && (2.7)\\ Fv_i^+ &= Rv_{i-1}^- + (a_{i-1} + c_{i-1} - c_i)v_i^+ - c_i v_i^- && (2.8)\\ Lv_i^+ &= Fv_{i-1}^- + b_{i-1}v_{i-1}^+ + (c_i - a_{i-1} - c_{i-1})v_{i-1}^-. && (2.9) \end{aligned}$$

For classical parameters, the following theorem tells exactly which modules are thin.

Theorem 2.7 ([5]) *Suppose $\mathcal{X}$ has classical parameters (d, q, α, β). Let $v \in U_1^*$ be an eigenvector for $E_1^* A_1 E_1^*$ with eigenvalue λ. The irreducible module Tv of endpoint* 1 *is thin if and only if*

$$\lambda \in \left\{-1, -q-1, \beta - \alpha - 1, \alpha q \begin{bmatrix} d-1 \\ 1 \end{bmatrix} - 1\right\}.$$

3. Ladder bases

In this section, we describe a particularly nice sort of basis for a nonthin irreducible module of endpoint 1, which we will call a ladder basis. We will then give criteria for the existence of such a basis.

Throughout Sections 3 to 5, we assume that W is a nonthin irreducible T-module of endpoint 1, and v is a nonzero vector in E_1^*W. Then v is an eigenvector for $E_1^* A_1 E_1^*$, and we denote the corresponding eigenvalue by λ. By Theorem 2.2, $W = Tv$.

3.1. Criteria for existence

A *ladder basis* for W is an orthogonal basis $\{w_1^-, w_2^+, w_2^-, \dots, w_{d-1}^+, w_{d-1}^-, w_d^+\}$ which satisfies the following.

(i) $E_i^*W = \text{span}\{w_i^+, w_i^-\}$.
(ii) w_i^+, w_i^- are eigenvectors for F.

(iii) For $2 \le i \le d-1$,

$$Rw_i^+ \in \operatorname{span}\{w_{i+1}^+\}$$
$$Lw_i^- \in \operatorname{span}\{w_{i-1}^-\}.$$

If the eigenvalues of F on E_i^*W are distinct, then the corresponding eigenvectors are automatically orthogonal. In this case, the following lemma shows that it is enough to check one of the properties of (iii).

Lemma 3.1 *Suppose i is a fixed integer with $2 \le i \le d-2$, and w_i^+, w_i^-, w_{i+1}^+, w_{i+1}^- are mutually orthogonal vectors such that $E_i^*W = \operatorname{span}\{w_i^+, w_i^-\}$ and $E_{i+1}^*W = \operatorname{span}\{w_{i+1}^+, w_{i+1}^-\}$.*

Then $Lw_{i+1}^- \in \operatorname{span}\{w_i^-\}$ if and only if $Rw_i^+ \in \operatorname{span}\{w_{i+1}^+\}$.

Proof: Since $\langle w_i^+, w_i^-\rangle = 0 = \langle w_{i+1}^+, w_{i+1}^-\rangle$, it follows that $Lw_{i+1}^- \in \operatorname{span}\{w_i^-\}$ if and only if $\langle Lw_{i+1}^-, w_i^+\rangle = 0$. But $\langle Lw_{i+1}^-, w_i^+\rangle = \langle w_{i+1}^-, Rw_i^+\rangle$, so this occurs if and only if $Rw_i^+ \in \operatorname{span}\{w_{i+1}^+\}$. □

We can now use this and the relations on the operators F, L, and R to show that we only need to check a few eigenvectors to see if a ladder basis exists.

Proposition 3.2 *The following are equivalent.*

(i) *W has a ladder basis.*

(ii) *There exist eigenvectors w_2^+ and w_3^+ for F such that $w_2^+ \in E_2^*W$, $w_3^+ \in E_3^*W$, and $Rw_2^+ \in \operatorname{span}\{w_3^+\}$.*

(iii) *There exist eigenvectors w_2^- and w_3^- for F such that $w_2^- \in E_2^*W$, $w_3^- \in E_3^*W$, and $Lw_3^- \in \operatorname{span}\{w_2^-\}$.*

Proof:

(ii ⇒ iii): Since F is symmetric, there exist eigenvectors w_2^- and w_3^- such that $E_2^*W = \operatorname{span}\{w_2^+, w_2^-\}$, $\langle w_2^+, w_2^-\rangle = 0$, $E_3^*W = \operatorname{span}\{w_3^+, w_3^-\}$ and $\langle w_3^+, w_3^-\rangle = 0$. By Lemma 3.1, $Lw_3^- \in \operatorname{span}\{w_2^-\}$.

(iii ⇒ ii): Similar.

(i ⇒ ii): Clear.

(ii ⇒ i): Define w_i^+ inductively by $w_i^+ = Rw_{i-1}^+$, $4 \le i \le d-1$, and let w_d^+ be any nonzero vector of E_d^*W. By Theorem 2.5, $w_i^+ \ne 0$ for $i < d$.

Suppose w_{i-1}^+ and w_{i-2}^+ are eigenvectors for F, with eigenvalues λ_{i-1}^+ and λ_{i-2}^+, respectively. Applying the operator of (2.2) to w_{i-2}^+, we find

$$g_i^-\lambda_{i-2}^+ w_i^+ + \lambda_{i-1}^+ w_i^+ + g_i^+ F w_i^+ - \gamma w_i^+ = 0.$$

Since g_i^+ is nonzero, $Fw_i^+ \in \operatorname{span}\{w_i^+\}$, and w_i^+ is an eigenvector for F. Since $\dim E_d^*W = 1$, w_d^+ is an eigenvector for F and $Rw_{d-1}^+ \in \operatorname{span}\{w_d^+\}$. So the set $\{w_2^+, w_3^+, \ldots, w_d^+\}$ is half of a ladder basis. Now choose w_i^- an eigenvector for F so that $E_i^*W = \text{span}\{w_i^+, w_i^-\}$ and $\langle w_i^+, w_i^-\rangle = 0$, and we are finished by Lemma 3.1. $\square$

3.2. Eigenvalues of F

Suppose $\{w_1^-, w_2^+, w_2^-, \ldots, w_{d-1}^+, w_{d-1}^-, w_d^+\}$ is a ladder basis for W, and that $\{\lambda_i^+\}$ and $\{\lambda_i^-\}$ are the corresponding eigenvalues, so $Fw_i^+ = \lambda_i^+ w_i^+$, and $Fw_i^- = \lambda_i^- w_i^-$. Note that $\lambda_1^- = \lambda$. For convenience, we will define

$$\lambda_1^+ = \frac{\gamma - \lambda_2^+ - g_3^+\lambda_3^+}{g_3^-},$$

where γ, g_3^+, and g_3^- are as in Proposition 2.1.

Theorem 3.3 *The eigenvalues* λ_i^+ *and* λ_i^- $(3 \le i \le d-1)$ *of* F *are given in terms of* λ_1^+, λ_2^+, λ_1^-, *and* λ_2^- *as follows*:

$$\lambda_i = \frac{-(\gamma(\theta_i^* - \theta_2^*)(\theta_i^* - \theta_1^*) - \lambda_2(\theta_3^* - \theta_2^*)(\theta_i^* - \theta_1^*) + \lambda_1(\theta_1^* - \theta_0^*)(\theta_i^* - \theta_2^*))}{(\theta_{i+1}^* - \theta_i^*)(\theta_i^* - \theta_{i-1}^*)} \tag{3.1}$$

where either

$$(+) \quad \lambda_i = \lambda_i^+ \quad (2 \le i \le d-1),$$

or

$$(-) \quad \lambda_i = \lambda_i^- \quad (2 \le i \le d-1).$$

If $g_d^+ \neq 0$, *then this formula applies for* λ_d^+ *also.*

Proof of Theorem 3.3: Note that in any case we can find λ_d^+ using $\text{tr}(A_1|_W) = \sum \lambda_i^+ + \sum \lambda_i^- = \theta_1 + 2\sum_{i=2}^{d-1}\theta_i + \theta_d$. $\square$

Proof: The equation

$$g_i^-\lambda_{i-2} + \lambda_{i-1} + g_i^+\lambda_i - \gamma = 0 \quad (3 \le i \le d) \tag{3.2}$$

holds in either case. To see this for $(+)$, $i \ge 4$, apply (2.2) to w_{i-2}^+; for $i = 3$, it holds by definition of λ_1^+. For $(-)$, apply (2.1) to w_i^-. If $g_i^+ \neq 0$ (which holds at least for $2 \le i \le d-1$), this determines λ_i recursively.

It is easy to check that (3.1) is a solution to this equation. However, we will give below a method for solving the recursion.

Substitute the values for g_i^- and g_i^+ into (3.2); this results in the equation

$$((\theta^*_{i+1}-\theta^*_i)\lambda_i-(\theta^*_{i-1}-\theta^*_{i-2})\lambda_{i-1})-((\theta^*_i-\theta^*_{i-1})\lambda_{i-1}-(\theta^*_{i-2}-\theta^*_{i-3})\lambda_{i-2})$$
$$+(\theta^*_i-\theta^*_{i-2})\gamma=0. \tag{3.3}$$

Summing (3.3) for $i=3$ to j,

$$(\theta^*_{j+1}-\theta^*_j)\lambda_j-(\theta^*_{j-1}-\theta^*_{j-2})\lambda_{j-1}-(\theta^*_3-\theta^*_2)\lambda_2+(\theta^*_1-\theta^*_0)\lambda_1$$
$$+\gamma(\theta^*_j+\theta^*_{j-1}-\theta^*_2-\theta^*_1)=0, \tag{3.4}$$

which holds for $2\le j\le d$.

Define μ_j by

$$\lambda_j=\frac{\mu_j}{(\theta^*_{j+1}-\theta^*_j)(\theta^*_j-\theta^*_{j-1})}$$

and let

$$\sigma=-(\theta^*_3-\theta^*_2)\lambda_2+(\theta^*_1-\theta^*_0)\lambda_1-\gamma(\theta^*_2+\theta^*_1),$$

so (3.4) becomes

$$\mu_j-\mu_{j-1}+\gamma\left(\theta_j^{*2}-\theta_{j-1}^{*2}\right)+\sigma(\theta^*_j-\theta^*_{j-1})=0. \tag{3.5}$$

Summing (3.5) from 2 to i results in

$$\mu_i-\mu_1+\gamma\left(\theta_i^{*2}-\theta_1^{*2}\right)+\sigma(\theta^*_i-\theta^*_1)=0,$$

from which the theorem follows. □

4. The Terwilliger basis

Recall that the Terwilliger basis $\{v=v_1^+,v_2^+,v_2^-,\ldots,v_{d-1}^+,v_{d-1}^-,v_d^+\}$ of W is defined by $v_i^+=E_i^*A_{i-1}v$, $v_i^-=E_i^*A_{i+1}v$. Occasionally it is useful to let $v_1^-=E_1^*A_2v$ and $v_d^-=0$. It follows from (2.7) that $v_1^-=(-\lambda-1)v$.

In this section, we find explicitly the action of A_1 on v_i^+ and v_i^-, and show that an association scheme with classical parameters must have a ladder basis.

4.1. Notation

We will use the following notation for entries of $A_1|_W$.

$$\begin{aligned}
Rv_i^- &= r_i^+ v_{i+1}^+ + r_i^- v_{i+1}^- && (2 \le i \le d-2)\\
Rv_{d-1}^- &= r_{d-1}^+ v_d^+ \\
Lv_i^+ &= l_i^+ v_{i-1}^+ + l_i^- v_{i-1}^- && (3 \le i \le d)\\
Lv_2^+ &= l_2^+ v_1^+ \\
Fv_i^+ &= f_i^{++} v_i^+ + f_i^{-+} v_i^- && (2 \le i \le d-1)\\
Fv_i^- &= f_i^{+-} v_i^+ + f_i^{--} v_i^- && (2 \le i \le d-1)\\
Fv_d^+ &= f_d^{++} v_d^+
\end{aligned}$$

Note that $Fv_1^+ = \lambda v_1^+$, and from (2.6), $Lv_2^- = b_2 v_1^- = (-\lambda - 1)b_2 v_1^+$. We know Rv_i^+ and Lv_i^- from Lemma 2.6.

Thus, with respect to the Terwilliger basis, $A_1|_W$ has matrix

$$\begin{pmatrix}
\lambda & l_2^+ & -(\lambda+1)b_2 & & & & & & & \\
1 & f_2^{++} & f_2^{+-} & l_3^+ & 0 & & & & & \\
0 & f_2^{-+} & f_2^{--} & l_3^- & b_3 & & & & & \\
 & c_2 & r_2^+ & f_3^{++} & f_3^{+-} & l_4^+ & 0 & & & \\
 & 0 & r_2^- & f_3^{-+} & f_3^{--} & l_4^- & b_4 & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & & & & f_{d-1}^{++} & f_{d-1}^{+-} & l_d^+ \\
 & & & & & & & f_{d-1}^{-+} & f_{d-1}^{--} & l_d^- \\
 & & & & & & & c_{d-1} & r_{d-1}^+ & f_d^{++}
\end{pmatrix}.$$

Lemma 4.1 *For* $2 \le i \le d-1$,

$$f_i^{+-} = l_{i+1}^+ - b_i \tag{4.1}$$

$$f_i^{--} = l_{i+1}^- + a_i + c_i - c_{i+1} \tag{4.2}$$

$$f_i^{++} = r_{i-1}^+ + a_{i-1} + c_{i-1} - c_i \tag{4.3}$$

$$f_i^{-+} = r_{i-1}^- - c_i, \quad \text{where } r_1^- = 0. \tag{4.4}$$

Proof: This follows directly from (2.8) and (2.9). □

The remainder of this section will be restricted to the case of classical parameters. The methods here clearly give formulae in the general P- and Q-polynomial case for the entries of $A_1|_W$, however they are quite complicated. We expect to consider the general case using a different approach in a subsequent paper.

4.2. *Showing a ladder basis exists*

Theorem 4.2 *Suppose $\mathcal{X}$ has classical parameters. Then $f_i^{+-} = 0$, and hence v_i^- is an eigenvector for F, $2 \le i \le d-1$.*

Proof: We first show that f_2^{+-} and f_3^{+-} are equal to 0.

Most of the calculations use Eqs. (2.1) to (2.4) on the operators F, L, and R.

From (2.9) with $i = 2$ and the fact that $v_1^- = -(\lambda+1)v_1^+$, we find

$$l_2^+ = (\lambda+1)(a_1 - c_2 - \lambda + 1) + b_1. \tag{4.5}$$

Since $\mathcal{X}$ is P-polynomial, we can write

$$A_2 = \frac{1}{c_2}\left(A_1^2 - kI - a_1 A_1\right)$$

and this together with (2.7) results in

$$f_2^{++} = a_1 - c_2 - \lambda \tag{4.6}$$
$$f_2^{-+} = -c_2. \tag{4.7}$$

Now apply (2.4) to v_1^+, and consider coefficients of v_2^+. This results in the equation

$$(\rho+2)l_2^+ + e_2^+ c_2 l_3^+ + f_2^{++2} + f_2^{-+} f_2^{+-} - \rho\lambda f_2^{++} + \lambda^2 - \gamma(f_2^{++} + \lambda) - \delta = 0$$

which we may consider as being linear in the unknowns f_2^{+-} and l_3^+; everything else is known as a function of q, d, α, and β. Solving simultaneously with the equation $f_2^{+-} - l_3^+ + b_2 = 0$ from (4.1), we find

$$f_2^{+-} = 0 \tag{4.8}$$
$$l_3^+ = b_2. \tag{4.9}$$

Similarly, apply (2.4) to v_1^+ and consider the coefficients of v_2^- to get a linear equation in l_3^- and f_2^{--}

$$e_2^+ c_2 l_3^- + f_2^{++} f_2^{-+} + f_2^{-+} f_2^{--} - \rho\lambda f_2^{-+} - \gamma f_2^{-+} = 0.$$

Solving simultaneously with $f_2^{--} - l_3^- - a_2 - c_2 + c_3 = 0$ from (4.2), we find

$$f_2^{--} = q(\lambda - a_1) + a_2 \tag{4.10}$$
$$l_3^- = q(\lambda - a_1) + c_3 - c_2. \tag{4.11}$$

Apply (2.2) to v_1^+, and consider the coefficient of v_3^+ resulting in the equation

$$g_3^- c_2 \lambda + f_2^{++} c_2 + f_2^{-+} r_2^+ + g_3^+ c_2 f_3^{++} - \gamma c_2 = 0$$

which may be solved simultaneously with

$$f_3^{++} - r_2^+ + c_3 - a_2 - c_2 = 0,$$

and the coefficient of v_3^- resulting in the equation

$$f_2^{-+} r_2^- + g_3^+ c_2 f_3^{-+} = 0$$

which may be solved simultaneously with

$$f_3^{-+} - r_2^- + c_3 = 0.$$

The results are

$$\begin{aligned} f_3^{++} &= q(a_1 - \lambda) + \frac{k - b_3}{q^2 + q + 1} - c_3 \\ &= -(\lambda + 1)q + a_2 + c_2 - c_3 && (4.12) \\ r_2^+ &= q(a_1 - \lambda) - k + b_2 + \frac{k - b_3}{q^2 + q + 1} = -(\lambda + 1)q && (4.13) \\ f_3^{-+} &= \frac{-q^2 - q}{q^2 + q + 1} c_3 && (4.14) \\ r_2^- &= \frac{c_3}{q^2 + q + 1}. && (4.15) \end{aligned}$$

Finally, apply (2.4) to v_2^+ and consider the coefficient of v_3^+ to obtain

$$\begin{aligned} 0 = {} & e_3^- l_2^+ c_2 + (\rho + 2)\big(l_3^+ c_2^2 + l_3^- c_2 r_2^+\big) + e_3^+ c_2 c_3 l_4^+ + c_2 f_3^{++2} + c_2 f_3^{-+} f_3^{+-} \\ & - \rho(f_2^{++} f_3^{++} c_2 + f_3^{++} f_2^{-+} r_2^+ + f_3^{+-} f_2^{-+} r_2^-) + c_2\big(f_2^{++2} + f_2^{-+} f_2^{+-}\big) \\ & + r_2^+ f_2^{-+} (f_2^{++} + f_2^{--}) - \gamma (c_2 f_3^{++} + c_2 f_2^{++} + f_2^{-+} r_2^+) - \delta c_2. \end{aligned}$$

Solving this simultaneously with the equation

$$f_3^{+-} - l_4^+ + b_3 = 0,$$

we find

$$\begin{aligned} f_3^{+-} &= 0 && (4.16) \\ l_4^+ &= b_3. && (4.17) \end{aligned}$$

Now we can show $f_i^{+-} = 0$ for all i.

Apply (2.1) to v_i^-, $4 \le i \le d-1$, and consider the coefficients of v_{i-2}^+. The resulting equation is

$$g_i^- b_i b_{i-1} f_{i-2}^{+-} + b_i f_{i-1}^{+-} l_{i-1}^+ + g_i^+ f_i^{+-} l_i^+ l_{i-1}^+ = 0 \quad (4 \le i \le d-1).$$

Note also that by (4.1), $l_i^+ = f_{i-1}^{+-} + b_{i-1}$. Now inductively, starting with $f_2^{+-} = f_3^{+-} = 0$ and hence $l_3^+ = b_2$, $l_4^+ = b_3$, we can show that $f_i^{+-} = 0$ (and hence $l_{i+1}^+ = b_i$). □

Corollary 4.3 *If $\mathcal{X}$ has classical parameters, then W has a ladder basis.*

Proof: By Theorem 4.2, v_2^- and v_3^- are eigenvectors with $Lv_3^- \in \text{span}\{v_2^-\}$. Now by Proposition 3.2, W has a ladder basis. □

5. The entries of the matrix

In this section, we give our computational results about the entries of the matrix for $A_1|_W$ with respect to both the Terwilliger and ladder bases.

5.1. *The matrix with respect to the Terwilliger basis*

Theorem 5.1 *Suppose $\mathcal{X}$ has classical parameters. Then the entries of $A_1|_W$ with respect to the Terwilliger basis, using the notation of Section* 4.1, *are given by*

$$l_2^+ = (\lambda+1)(a_1 - c_2 - \lambda + 1) + b_1 \tag{5.1}$$

$$f_i^{++} = q^{i-2}(a_1 - \lambda) + \frac{\left[{i-2 \atop 1}\right]}{\left[{i \atop 1}\right]}(k - b_i) - c_i \qquad (2 \le i \le d) \tag{5.2}$$

$$f_i^{--} = q^{i-1}(\lambda - a_1) + a_i \qquad (2 \le i \le d-1) \tag{5.3}$$

$$f_i^{+-} = 0 \qquad (2 \le i \le d-1) \tag{5.4}$$

$$f_i^{-+} = \left(\frac{\left[{i-2 \atop 1}\right]}{\left[{i \atop 1}\right]} - 1\right)c_i \qquad (2 \le i \le d-1) \tag{5.5}$$

$$l_i^+ = b_{i-1} \qquad (3 \le i \le d) \tag{5.6}$$

$$l_i^- = q^{i-2}(\lambda - a_1) + c_i - c_{i-1} \qquad (3 \le i \le d) \tag{5.7}$$

$$r_i^+ = q^{i-1}(a_1 - \lambda) - k + b_i + \frac{\left[{i-1 \atop 1}\right]}{\left[{i+1 \atop 1}\right]}(k - b_{i+1}) \quad (2 \le i \le d-1) \tag{5.8}$$

$$r_i^- = \frac{\left[{i-1 \atop 1}\right]}{\left[{i+1 \atop 1}\right]} c_{i+1} \qquad (2 \le i \le d-2). \tag{5.9}$$

Proof: We will make extensive use of the intermediate calculations in the proof of Theorem 4.2.

The formula (5.1) is just (4.5).

To find (5.2) and (5.3), we apply Theorem 3.3; we can do this since W has a ladder basis, and $E_i^* W$ has eigenvalues f_i^{++} and f_i^{--}. Here, $\lambda_1^- = \lambda$, and $\lambda_2^- = q(\lambda - a_1) + a_2$, hence

$$\begin{aligned} f_i^{--} &= \lambda_i^- \\ &= -\frac{(\theta_i^* - \theta_2^*)(\theta_i^* - \theta_1^*)\gamma}{(\theta_{i+1}^* - \theta_i^*)(\theta_i^* - \theta_{i-1}^*)} + \frac{(\theta_3^* - \theta_2^*)(\theta_i^* - \theta_1^*)}{(\theta_{i+1}^* - \theta_i^*)(\theta_i^* - \theta_{i-1}^*)}(q(\lambda - a_1) + a_2) \\ &\quad - \frac{(\theta_1^* - \theta_0^*)(\theta_i^* - \theta_2^*)}{(\theta_{i+1}^* - \theta_i^*)(\theta_i^* - \theta_{i-1}^*)}\lambda \\ &= -\frac{(q^i - q)(q^i - q^2)}{q^2(q-1)^2}\gamma + \frac{q^i(q^i - q)}{q^3(q-1)}(q(\lambda - a_1) + a_2) - \frac{q^i(q^i - q^2)}{q^2(q-1)}\lambda \end{aligned}$$

and using the values of γ, a_1, and a_2 for classical parameters, we can check that this equals (5.3). Using (4.2), we also get (5.7).

$f_i^{++} = \lambda_i^+$ can be calculated similarly. From (4.6), $f_2^{++} = \lambda_2^+ = a_1 - c_2 - \lambda$ and from (4.12), $f_3^{++} = \lambda_3^+ = -(\lambda + 1)q + a_2 + c_2 - c_3$, so $\lambda_1^+ = (\gamma - \lambda_2^+ - g_3^+ \lambda_3^+)/g_3^- = -(\lambda + 1 + q)/q$. Now (5.2) follows from Theorem 3.3.

Using (4.3), we also get (5.8) for $i \le d-2$; the case $i = d-1$ can be checked directly using (2.8).

To show (5.9), we use another recursive equation. Apply (2.2) to v_{i-1}^+, $3 \le i \le d-2$, and consider the coefficients of v_{i+1}^-. The resulting equation is

$$g_{i+1}^- f_{i-1}^{-+} r_{i-1}^- r_i^- + c_{i-1} f_i^{-+} r_i^- + g_{i+1}^+ c_i c_{i-1} f_{i+1}^{-+} = 0.$$

By (4.4), $f_i^{-+} = r_{i-1}^- - c_i$ for $i \ge 2$. Hence substituting this and the values of g_{i+1}^+ and g_{i+1}^- for classical parameters and solving for r_i^-, we have for $3 \le i \le d-1$

$$r_i^- = \frac{c_{i-1} c_i c_{i+1}}{q^3(r_{i-2}^- - c_{i-1})r_{i-1}^- + c_i c_{i-1} - q(q+1)(r_{i-1}^- - c_i)c_{i-1}}. \tag{5.10}$$

It is easily checked that (5.9) satisfies this equation. □

Corollary 5.2 *Suppose $\mathcal{X}$ has classical parameters. For* $2 \le i \le d-1$,

i. $f_i^{-+} \neq 0$

ii. $f_i^{++} - f_i^{--} \neq 0$.

Proof: Since $c_i \neq 0$, it follows from (5.5) that $f_i^{-+} \neq 0$. F is symmetric, and hence $F|_{E_i^* W}$ is diagonalizable, and has eigenvalues f_i^{++} and f_i^{--} by Theorem 4.2. If $f_i^{++} = f_i^{--}$, this implies $f_i^{-+} = 0$, a contradiction. □

We can also write all entries in terms of the d, q, α, β, and λ.

Corollary 5.3 *If $\mathcal{X}$ has classical parameters (d, q, α, β), then*

$$
\begin{aligned}
l_2^+ &= -\lambda^2 + \left\{\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - q - 2\right)\alpha + \beta - q - 2\right\}\lambda - (q+1)\alpha + \begin{bmatrix} d \\ 1 \end{bmatrix}\beta - q - 1 \\
f_i^{++} &= \left(q^i\begin{bmatrix} d-i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\alpha + \begin{bmatrix} i-1 \\ 1 \end{bmatrix}(\beta - 1) - q^{i-2}(\lambda + q + 1) \\
f_i^{--} &= \left(q^i\begin{bmatrix} d-i \\ 1 \end{bmatrix} - \begin{bmatrix} i+1 \\ 1 \end{bmatrix}\right)\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\alpha + \begin{bmatrix} i-1 \\ 1 \end{bmatrix}(\beta - 1) + q^{i-1}\lambda \\
f_i^{+-} &= 0 \\
f_i^{-+} &= -q^{i-2}(q+1)\left(\alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix} + 1\right) \\
l_i^+ &= q^{i-1}\begin{bmatrix} d-i+1 \\ 1 \end{bmatrix}\left(\beta - \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \quad \text{for } i \geq 3 \\
l_i^- &= q^{i-2}\left\{\left(\begin{bmatrix} i-1 \\ 1 \end{bmatrix} - q^i\begin{bmatrix} d-i \\ 1 \end{bmatrix}\right)\alpha - \beta + \lambda + q + 1\right\} \\
r_i^+ &= -q^{i-1}(\lambda + 1) \\
r_i^- &= \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\left(1 + \begin{bmatrix} i \\ 1 \end{bmatrix}\alpha\right).
\end{aligned}
$$

5.2. *The matrix with respect to the ladder basis*

We will assume that $\mathcal{X}$ has classical parameters (d, q, α, β). By Corollary 5.2, $f_i^{++} \neq f_i^{--}$, and the eigenvectors of $F|_W$ are orthogonal. Let

$$
\begin{aligned}
w_i^- &= v_i^- && (1 \leq i \leq d-1) \\
w_i^+ &= \frac{c_i}{\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]}\left(\frac{f_i^{++} - f_i^{--}}{f_i^{-+}}v_i^+ + v_i^-\right) && (2 \leq i \leq d-1) \\
w_d^+ &= \frac{c_d}{\left[\begin{smallmatrix} d \\ 1 \end{smallmatrix}\right]}\left(\frac{\lambda + 1 - q\left[\begin{smallmatrix} d-1 \\ 1 \end{smallmatrix}\right]\alpha}{1 + \left[\begin{smallmatrix} d-1 \\ 1 \end{smallmatrix}\right]\alpha}\right)v_d^+.
\end{aligned}
$$

Note that

$$
\frac{f_i^{++} - f_i^{--}}{f_i^{-+}} = \frac{\lambda + 1 - q\left[\begin{smallmatrix} i-1 \\ 1 \end{smallmatrix}\right]\alpha}{1 + \left[\begin{smallmatrix} i-1 \\ 1 \end{smallmatrix}\right]\alpha},
$$

so (since $v_d^- = 0$), w_d^+ follows the same formula as the other vectors w_i^+. By Theorem 2.7, $w_d^+ \neq 0$.

It is easily checked that w_i^+ and w_i^- are eigenvectors for F, and that

$$Lw_i^- = b_i w_{i-1}^-.$$

By Lemma 3.1, $Rw_i^+ \in \operatorname{span}\{w_{i+1}^+\}$. The coefficient of v_{i+1}^- in Rw_i^+ is

$$\frac{r_i^- c_i}{\left[{i \atop 1}\right]} = \frac{\left[{i-1 \atop 1}\right] c_{i+1} c_i}{\left[{i+1 \atop 1}\right]\left[{i \atop 1}\right]},$$

hence

$$Rw_i^+ = \frac{\left[{i-1 \atop 1}\right]}{\left[{i \atop 1}\right]} c_i w_{i+1}^+.$$

Therefore $\{w_1^-, w_2^+, w_2^-, \ldots, w_{d-1}^+, w_{d-1}^-, w_d^+\}$ is a ladder basis for $\mathcal{X}$.

For this basis, we use the notation

$$\begin{aligned}
Lw_i^+ &= \sigma_i^+ w_{i-1}^+ + \sigma_i^- w_{i-1}^- \quad (3 \le i \le d)\\
Lw_2^+ &= \sigma_2^- w_1^-\\
Rw_i^- &= \tau_i^+ w_{i+1}^+ + \tau_i^- w_{i+1}^- \quad (1 \le i \le d-2)\\
Rw_{d-1}^- &= \tau_{d-1}^+ w_d^+.
\end{aligned}$$

So $A_1|_W$ has matrix with respect to this basis given by

$$\begin{pmatrix}
\lambda_1^- & \sigma_2^- & b_2 & & & & & & \\
\tau_1^+ & \lambda_2^+ & 0 & \sigma_3^+ & 0 & & & & \\
\tau_1^- & 0 & \lambda_2^- & \sigma_3^- & b_3 & & & & \\
 & \frac{c_2}{q+1} & \tau_2^+ & \lambda_3^+ & 0 & \sigma_4^+ & 0 & & \\
 & 0 & \tau_2^- & 0 & \lambda_3^- & \sigma_4^- & b_4 & & \\
 & & & & & \ddots & \ddots & \ddots & \\
 & & & & & & \lambda_{d-1}^+ & 0 & \sigma_d^+ \\
 & & & & & & 0 & \lambda_{d-1}^- & \sigma_d^- \\
 & & & & & & \frac{\left[{d-2 \atop 1}\right]}{\left[{d-1 \atop 1}\right]} c_{d-1} & \tau_{d-1}^+ & \lambda_d^+
\end{pmatrix}.$$

Now, using Theorem 5.1, and the equations

$$\begin{aligned}
v_i^+ &= \frac{f_i^{-+}}{f_i^{++} - f_i^{--}} \left(\frac{\left[{i \atop 1}\right]}{c_i} w_i^+ - w_i^- \right)\\
v_i^- &= w_i^-,
\end{aligned}$$

we can easily calculate the entries of this matrix.

Theorem 5.4 *With notation as above*,

$$\lambda_i^+ = \left(q^i \begin{bmatrix} d-i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\alpha + \begin{bmatrix} i-1 \\ 1 \end{bmatrix}(\beta - 1) - q^{i-2}(\lambda + q + 1) \qquad (2 \le i \le d)$$

$$\lambda_i^- = \left(q^i \begin{bmatrix} d-i \\ 1 \end{bmatrix} - \begin{bmatrix} i+1 \\ 1 \end{bmatrix}\right) \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\alpha + \begin{bmatrix} i-1 \\ 1 \end{bmatrix}(\beta - 1) + q^{i-1}\lambda \qquad (1 \le i \le d-1)$$

$$\sigma_i^+ = \frac{q^{i-1}\left[{d+1-i \atop 1}\right]\left(\beta - \left[{i-1 \atop 1}\right]\alpha\right)\left(\lambda + 1 - q\left[{i-1 \atop 1}\right]\alpha\right)}{\left(\lambda + 1 - q\left[{i-2 \atop 1}\right]\alpha\right)} \qquad (3 \le i \le d)$$

$$\sigma_i^- = \frac{q^{i-2}(q + \lambda + 1)(\lambda + 1 + \alpha - \beta)\left(\lambda + 1 - q\left[{d-1 \atop 1}\right]\alpha\right)}{\left(\lambda + 1 - q\left[{i-2 \atop 1}\right]\alpha\right)} \qquad (2 \le i \le d)$$

$$\tau_i^+ = \frac{q^{i-1}(\lambda + 1)}{q\left[{i \atop 1}\right]\alpha - (\lambda + 1)} \qquad (1 \le i \le d-1)$$

$$\tau_i^- = \frac{\left[{i \atop 1}\right]\left(\left[{i \atop 1}\right]\alpha + 1\right)\left(\lambda + 1 - q\left[{i-1 \atop 1}\right]\alpha\right)}{\lambda + 1 - q\left[{i \atop 1}\right]\alpha} \qquad (1 \le i \le d-2).$$

6. Examples

The irreducible modules of endpoint 1 are determined up to isomorphism by the eigenvalues of the graph on the first subconstituent $\Gamma_1(x)$, since $E_1^* A_1 E_1^*$ is the adjacency matrix of this graph. In this section, we consider three families of P- and Q-polynomial schemes which are not thin: the bilinear forms schemes, the Hermitean forms schemes, and the alternating forms schemes. For each, we find the eigenvalues of $\Gamma_1(x)$ and then use the results of Section 5 to find the entries of $A_1|_W$ for each nonthin irreducible module W of endpoint 1. We use [2] as a reference for these schemes.

The only other known P- and Q-polynomial schemes of diameter ≥ 6 which are not thin are the Doob schemes and the quadratic forms schemes [4]. All irreducible modules for the Doob schemes were determined by Tanabe [3]. The case of the quadratic forms schemes appears to be much more difficult.

6.1. The bilinear forms schemes

Let X be the set of $d \times n$ matrices over $GF(q)$, $d \leq n$. Define matrices M and N to have relation i if $\mathrm{rk}(M - N) = i$. This gives a P- and Q-polynomial scheme, called the *bilinear forms scheme*, which has classical parameter (d, q, α, β) with $\alpha = q - 1$ and $\beta = q^n - 1$.

The automorphism group acts transitively on this scheme, therefore the structure of $T(x)$ is independent of x.

Assume $d \geq 3$.

Theorem 6.1 *The induced subgraph on* $\Gamma_1(x)$ *for the bilinear forms scheme has eigenvalues and multiplicities as shown below. We also show whether the corresponding module is the trivial module* (*of endpoint 0*), *or a thin or nonthin irreducible module of endpoint* 1.

Note that two of the eigenvalues coincide if $d = n$, *and that* -1 *does not occur as an eigenvalue if* $q = 2$.

eigenvalue	*multiplicity*	*corresponding module*
$q^d + q^n - q - 2$	*1*	*trivial*
$q^n - q - 1$	$\frac{q^d-q}{q-1}$	*thin*
$q^d - q - 1$	$\frac{q^n-q}{q-1}$	*thin*
-1	$\frac{(q^d-1)(q^n-1)(q-2)}{(q-1)^2}$	*thin*
$-q$	$\frac{(q^d-q)(q^n-q)}{(q-1)^2}$	*nonthin*

Proof: We may assume $x = 0$, the $d \times n$ zero matrix, so $\Gamma_1(x)$ has as vertices the set of all $d \times n$ matrices of rank 1. Any rank 1 matrix M can be written uniquely as $M = \mu u v^t$, where $\mu \in GF(q)^*$, and u and v are d and n-tuples respectively with first nonzero entry equal to 1.

Suppose $M = \mu u v^t$. Clearly M is adjacent to $\mu_1 u_1 v_1^t$ if $u = u_1$ or $v = v_1$. The number of such matrices is equal to a_1, hence this characterizes the matrices adjacent to M. (This could also be easily checked using the distance-transitivity of the automorphism group and suitable u and v.)

Given any such u and v, $\{\mu u v^t \mid \mu \in GF(q)^*\}$ forms a clique in $\Gamma_1(x)$, and the vertices of two such cliques are either all adjacent or all nonadjacent. We can therefore form a quotient graph on the $\frac{(q^d-1)(q^n-1)}{(q-1)^2}$ cliques of this form. The quotient graph has adjacency matrix

$$B = (J - I) \otimes I + I \otimes (J - I).$$

This matrix has eigenvalues $\frac{q^n+q^d-2q}{q-1}$, $\frac{q^n-1}{q-1} - 2$, $\frac{q^d-1}{q-1} - 2$, and -2 of multiplicity 1, $\frac{q^d-q}{q-1}$, $\frac{q^n-q}{q-1}$, and $\frac{(q^d-q)(q^n-q)}{(q-1)^2}$, respectively.

Now $\Gamma_1(x)$ has adjacency matrix $B \otimes J_{q-1} + I \otimes (J - I)$, and the result follows. □

Suppose W is the unique nonthin irreducible module. Using Theorems 5.1 and 5.4, we can find the matrix of $A_1|_W$ with respect to either the Terwilliger basis or the ladder basis of Section 5.2.

The eigenvalues of F on $E_i^* W$ are given by

$$\lambda_i^+ = f_i^{++} = \begin{bmatrix} i-1 \\ 1 \end{bmatrix}(q^d + q^n - q^i - 1) - \begin{bmatrix} i \\ 1 \end{bmatrix} q^{i-2}$$

$$\lambda_i^- = f_i^{--} = \begin{bmatrix} i-1 \\ 1 \end{bmatrix}(q^d + q^n - q^i - 1) - \begin{bmatrix} i \\ 1 \end{bmatrix} q^{i}$$

where $\lambda = \lambda_1^-$.

For the Terwilliger basis, the other entries of the matrix of $A_1|_W$ are

$$\begin{aligned}
l_2^+ &= \frac{(q^d - q^2 + q - 1)(q^n - q^2 + q - 1)}{q-1} + q^2 - q \\
f_i^{+-} &= 0 \\
f_i^{-+} &= -(q+1)q^{2i-3} \\
l_i^+ &= \frac{(q^n - q^{i-1})(q^d - q^{i-1})}{q-1} \\
l_i^- &= -q^{i-2}(q^n + q^d - q^i - q^{i-1} - 1) \\
r_i^+ &= q^{i-1}(q-1) \\
r_i^- &= q^i \begin{bmatrix} i-1 \\ 1 \end{bmatrix}.
\end{aligned}$$

For the ladder basis given in Section 5.2, the other entries of $A_1|_W$ are

$$\begin{aligned}
\sigma_i^+ &= \frac{(q^d - q^{i-1})(q^n - q^{i-1})(q^i - 1)}{(q-1)(q^{i-1}-1)} \\
\sigma_i^- &= \frac{-q^{i-1}(q^d - 1)(q^n - 1)}{(q^i - q)} \\
\tau_i^+ &= \frac{-q^{i-1}(q-1)}{q^{i+1}-1} \\
\tau_i^- &= \frac{q^i(q^i-1)^2}{(q-1)(q^{i+1}-1)}.
\end{aligned}$$

6.2. *The Hermitean forms schemes*

Let X be the set of $d \times d$ Hermitean matrices over $GF(r^2)$, where r is a prime power. Define matrices M and N to have relation i if $\mathrm{rk}(M-N) = i$. This gives a P- and Q-polynomial scheme, called the *Hermitean forms scheme*, which has classical paramters (d, q, α, β), with $q = -r$, $\alpha = -r-1$, and $\beta = -(-r)^d - 1$.

The automorphism group of this scheme acts transitively, therefore the structure of $T(x)$ is independent of the choice of x.

Assume $d \geq 3$.

Theorem 6.2 *The induced subgraph on $\Gamma_1(x)$ for the Hermitean forms scheme has eigenvalues and multiplicities as shown below. We also show whether the corresponding module is the trivial module (of endpoint 0), or a thin or nonthin irreducible module of endpoint 1.*

Note that if $r = 2$, -1 does not occur as an eigenvalue.

eigenvalue	*multiplicity*	*corresponding module*
$r-2$	1	*trivial*
$r-2$	$\frac{r^{2d}-r^2}{r^2-1}$	*nonthin*
-1	$\frac{(r^{2d}-1)(r-2)}{r^2-1}$	*thin*

Proof: It follows immediately from [6], Theorem 2.12, that $\Gamma_1(x)$ is a disjoint union of $\frac{r^{2d}-1}{r^2-1}$ cliques of size $r-1$. □

As in Section 6.1, we now give the entries of the matrix $A_1|_W$ for the Terwilliger basis and ladder basis, in terms of q and d. Recall that $q=-r$.

The eigenvalues of F on E_i^*W are

$$\lambda_i^+ = f_i^{++} = -\begin{bmatrix} i-1 \\ 1 \end{bmatrix} q^{i-1}(q+1) - \begin{bmatrix} i-2 \\ 1 \end{bmatrix}$$
$$\lambda_i^- = f_i^{--} = -\begin{bmatrix} i \\ 1 \end{bmatrix}(q^i + q^{i-1} + 1)$$

where $\lambda = \lambda_1^-$.

The other entries for the Terwilliger basis are

$$l_2^+ = \frac{-(q^{2d}-q^4)}{q-1} + q^2 - 1$$
$$f_i^{+-} = 0$$
$$f_i^{-+} = -(q+1)q^{2i-3}$$
$$l_i^+ = \frac{-(q^d-q^{i-1})(q^d+q^{i-1})}{q-1}$$
$$l_i^- = q^{i-2}(q^i+q^{i-1}-1)$$
$$r_i^+ = q^{i-1}(q+1)$$
$$r_i^- = q^i \begin{bmatrix} i-1 \\ 1 \end{bmatrix}.$$

The other entries for the ladder basis given in Section 5.2 are

$$\sigma_i^+ = \frac{-(q^i+1)(q^{2d}-q^{2i-2})}{(q-1)(q^{i-1}+1)}$$
$$\sigma_i^- = \frac{-q^{i-2}(q^{2d}-1)}{q^{i-1}+1}$$
$$\tau_i^+ = \frac{-q^{i-1}(q+1)}{q^{i+1}+1}$$
$$\tau_i^- = \frac{q^i(q^i-1)(q^i+1)}{(q-1)(q^{i+1}+1)}.$$

6.3. The alternating forms schemes

Let X be the set of $n \times n$ skew-symmetric matrices with 0 diagonal over $GF(r)$; note that such matrices have even rank. Two matrices M and N have relation i if $\mathrm{rk}(M-N) = 2i$. This defines the *alternating forms scheme*, which is a P- and Q-polynomial scheme with classical parameters $d = \lfloor n/2 \rfloor$, $q = r^2$, $\alpha = r^2 - 1$, and $\beta = r^m - 1$, where $m = 2d - 1$ if $n = 2d$, and $m = 2d + 1$ if $n = 2d + 1$.

Since the automorphism group is transitive, the structure of the Terwilliger algebra $T(x)$ for the scheme is independent of the choice of x.

Assume $d \geq 3$.

The following lemma about ranks follows from some easy matrix theory; or see [2], Lemmas 9.5.4 and 9.5.5.

Lemma 6.3 *Let M_1 and M_2 be skew symmetric matrices of rank m_1 and m_2, respectively.*
(i) *If* $\ker(M_1) + \ker(M_2) = GF(r)^n$, *then* $\ker(M_1 - M_2) = \ker(M_1) \cap \ker(M_2)$.
(ii) $\mathrm{rk}(M_1 - M_2) = m_1 + m_2$ *if and only if* $GF(r)^n = \ker(M_1) + \ker(M_2)$.

Theorem 6.4 *The induced subgraph on $\Gamma_1(x)$ for the alternating forms scheme has eigenvalues and multiplicities as shown below. We also show whether the corresponding module is the trivial module (of endpoint 0), or a thin or nonthin irreducible module of endpoint 1.*

Note that if $r = 2$, then -1 does not occur as an eigenvalue.

eigenvalue	*multiplicity*	*corresponding module*
$r^n + r^{n-1} - r^2 - 2$	1	*trivial*
$r^{n-1} - r^2 - 1$	$\frac{r^n - r}{r-1}$	*thin*
-1	$\frac{(r-2)(r^n-1)(r^{n-1}-1)}{(r^2-1)(r-1)}$	*thin*
$-r^2 + r - 1$	$\frac{(r^n-1)(r^{n-1}-r^2)}{(r^2-1)(r-1)}$	*nonthin*

Proof: Since the automorphism group is transitive, we can assume $x = 0$, the zero matrix. Thus the vertices of $\Gamma_1(x)$ are the matrices of X of rank 2.

If M and N are two distinct such matrices, $\mathrm{rk}(M - N) = 2$ or 4. By Lemma 6.3, $\mathrm{rk}(M-N) = 4$, and M and N are not adjacent, if and only if $GF(r)^n = \ker(M) + \ker(N)$. Thus M and N are adjacent if and only if $\dim(\ker(M) + \ker(N)) < n$. Given an $n-2$ dimensional subspace S, the set of matrices in X with kernel S forms a clique of size $r - 1$ in $\Gamma_1(x)$ and the vertices of two such cliques are either all adjacent or all nonadjacent.

As in the proof of Theorem 6.1, we now form the quotient graph on the $\left[{n \atop 2}\right]_r$ cliques, identifying each with the subspace which is the common kernel. This graph has as vertices the subspace of $GF(r)^n$ of dimension $n-2$, and two subspaces are adjacent if they intersect in a subspace of dimension $n - 3$. But this is isomorphic to the r-Johnson graph on the two-dimensional subspaces of $GF(r)^n$, and has as eigenvalues $\frac{r(r+1)(r^{n-2}-1)}{r-1}$, $\frac{r^2(r^{n-3}-1)}{r-1} - 1$, and $-r - 1$, with multiplicities 1, $\frac{r^n - r}{r-1}$, and $\frac{(r^n-1)(r^{n-1}-r^2)}{(r^2-1)(r-1)}$, respectively.

If the quotient graph has adjacency matrix B, then $\Gamma_1(x)$ has adjacency matrix $B \otimes J_{r-1} + I \otimes (J - I)$, and the result follows. $\square$

Below are the entries of $A_1|_W$ with respect to both bases. Recall that $[{n \atop 2}]_{r^2} = 1 + r^2 + \cdots + r^{2(i-1)}$ and $q = r^2$; we use this notation here to show the similarities with the previous two examples. Also, note that $\{2d, m\} = \{n, n-1\}$ for both even and odd n, and this allows us to treat both the even and odd cases together.

The eigenvalues of F on $E_i^* W$ are

$$\lambda_i^+ = f_i^{++} = \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_{r^2} (r^n + r^{n-1} - r^{2i}) - \begin{bmatrix} 2i-2 \\ 1 \end{bmatrix}_{r^2} - r^{2i-3}$$

$$\lambda_i^- = f_i^{--} = \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_{r^2} (r^n + r^{n-1} - r^{2i}) - \begin{bmatrix} 2i \\ 1 \end{bmatrix}_{r^2} + r^{2i-1}$$

where $\lambda = \lambda_1^-$.

The other entries for the Terwilliger basis are

$$l_2^+ = \frac{(r^n - r^4)(r^{n-1} - r^4)}{r^2 - 1} + r^{n+1} + r^n - r^5 - r^2$$

$$f_i^{+-} = 0$$

$$f_i^{-+} = -r^{4i-6}(r^2 + 1)$$

$$l_i^+ = \frac{(r^n - r^{2i-2})(r^{n-1} - r^{2i-2})}{r^2 - 1}$$

$$l_i^- = -r^{2i-4}(r^n + r^{n-1} - r^{2i} - r^{2i-2} - r)$$

$$r_i^+ = r^{2i-1}(r - 1)$$

$$r_i^- = r^{2i} \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_{r^2}.$$

The other entries for the ladder basis given in Section 5.2 are

$$\sigma_i^+ = \frac{(r^{2i-1} - 1)(r^n - r^{2i-2})(r^{n-1} - r^{2i-2})}{(r^2 - 1)(r^{2i-3} - 1)}$$

$$\sigma_i^- = \frac{-r^{2i-2}(r^{n-1} - 1)(r^{n-2} - 1)}{r^{2i-3} - 1}$$

$$\tau_i^+ = \frac{-r^{2i-2}(r - 1)}{r^{2i+1} - 1}$$

$$\tau_i^- = \frac{r^{2i}(r^{2i} - 1)(r^{2i-1} - 1)}{(r^2 - 1)(r^{2i+1} - 1)}.$$

Acknowledgments

Both authors were visitors at Kyushu University, Fukuoka, Japan during the time of this research, and they would like to thank the Department of Mathematics of that university for their hospitality. They would also like to thank Paul Terwilliger for making his lecture notes available, and for many helpful discussions.

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