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Hilbert Polynomials in Combinatorics

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Abstract. We prove that several polynomials naturally arising in combinatorics are Hilbert polynomials of standard graded commutative k-algebras.

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1. Introduction

The purpose of this paper is to investigate which polynomials naturally arising in combinatorics are Hilbert polynomials of standard graded (commutative) k-algebras. Our motivation comes from the fact (first proved by R. Stanley [34]) that the order polynomial of a partially ordered set is a Hilbert polynomial. Since Stanley informally told me of this result I have been wondering whether it was an isolated one or an instance of a more general phenomenon. Several works of Stanley (see, e.g., [31, 32], and the references cited there) show that many sequences arising in combinatorics are Hilbert functions, but Stanley never explicitly considered Hilbert polynomials.

In this paper we begin such a systematic investigation. Our results show that several polynomials arising in combinatorics are Hilbert polynomials, and in many (but not all) cases we find general reasons for this. The techniques that we use are based on combinatorial characterizations of Hilbert functions and polynomials obtained by Macaulay in 1927 [24]. Though the characterization of Hilbert functions is very well-known and has been extensively used since then, the one for Hilbert polynomials is not, and is our main tool. Most of our results are non-constructive. More precisely, we often prove that a given combinatorial polynomial is Hilbert but we are unable to construct (in a natural way) a standard graded k-algebra having the given Hilbert polynomial.

The organization of the paper is as follows. In the next section we collect several definitions, notation, and results that will be used in the rest of this work. In Section 3 we develop a general theory of Hilbert polynomials. More precisely, using Macaulay's result, and other techniques, we present several operations on polynomials that preserve the Hilbert property, as well as results that give sufficient conditions on the coefficients of a polynomial (when expanded in terms of several different bases) that insure that the polynomial is

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Hilbert. We also introduce a new concept, which is naturally suggested by one of our results, which gives a measure of "how far" a polynomial is from being Hilbert. In Section 4 we apply the general theory developed in Section 3 to polynomials arising in enumerative and algebraic combinatorics. In particular, we prove that the σ and τ-polynomials of a graph, the zeta polynomial of a partially ordered set, the R-polynomial of two generic elements in a Coxeter system, the Kazhdan-Lusztig polynomials and the descent generating function of a finite Coxeter system, various generalizations of the Eulerian polynomials related to Stirling (multi)-permutations, Stirling polynomials, and several polynomials obtained by specializing certain symmetric functions, are all Hilbert polynomials (up to a shift by 1 in some cases). Finally, in Section 5, we present several conjectures arising from the present work together with the evidence that we have in their support, and we indicate directions and open problems for further research.

2. Notation, definitions, and preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the ring of integers, and $\mathbf{Q}$ be the field of rational numbers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \ldots, a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). Given $n, m \in \mathbf{P}$, $n \leq m$, we let $[n, m] \stackrel{\text{def}}{=} [m] \backslash [n-1]$. The cardinality of a set A will be denoted by $|A|$. Given a polynomial $P(x)$, and $i \in \mathbf{Z}$, we will denote by $[x^i](P(x))$ the coefficient of x^i in $P(x)$. For $a \in \mathbf{R}$ we let $\lfloor a \rfloor$ (respectively, $\lceil a \rceil$) denote the largest integer $\leq a$ (respectively, smallest integer $\geq a$).

Given a ring R and a variable x we denote by $R[[x]]$ the ring of formal power series in x with coefficients in R. For $i \in \mathbf{P}$ we let $(x)_i \stackrel{\text{def}}{=} x(x-1)\cdots(x-i+1)$, $\langle x \rangle_i \stackrel{\text{def}}{=} x(x+1)\cdots(x+i-1)$, $\binom{x}{i} \stackrel{\text{def}}{=} \frac{(x)_i}{i!}$, and $(\binom{x}{i}) \stackrel{\text{def}}{=} \frac{\langle x \rangle_i}{i!}$. We also let $(x)_0 \stackrel{\text{def}}{=} \langle x \rangle_0 \stackrel{\text{def}}{=} \binom{x}{0} \stackrel{\text{def}}{=} (\binom{x}{0}) \stackrel{\text{def}}{=} 1$. Note that, for $d \in \mathbf{P}$, $\{(x)_i\}_{i=0,\ldots,d}$, $\{\langle x \rangle_i\}_{i=0,\ldots,d}$, $\{\binom{x}{i}\}_{i=0,\ldots,d}$, $\{(\binom{x}{i})\}_{i=0,\ldots,d}$ and $\{\binom{x+d-i}{d}\}_{i=0,\ldots,d}$ are all bases of the real vector space V_d of real polynomials of degree $\leq d$. We call the first four the bases of *lower factorials, upper factorials, binomial coefficients,* and *twisted binomial coefficients*, respectively.

A sequence $\{a_0, a_1, \ldots, a_d\}$ (of real numbers) is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $i = 1, \ldots, d-1$. It is *unimodal* if there exists an index $0 \leq j \leq d$ such that $a_i \leq a_{i+1}$ for $i = 0, \ldots, j-1$ and $a_i \geq a_{i+1}$ for $i = j, \ldots, d-1$. It has no *internal zeros* if there are not three indices $0 \leq i < j < k \leq d$ such that $a_i, a_k \neq 0$ and $a_j = 0$. It is *symmetric* if $a_i = a_{d-i}$ for $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$. A polynomial $\sum_{i=0}^{d} a_i x^i$ is *log-concave* (respectively, *unimodal*, with no *internal zeros, symmetric*) if the sequence $\{a_0, a_1, \ldots, a_d\}$ has the corresponding property. It is well known that if $\sum_{i=0}^{d} a_i x^i$ is a polynomial with nonnegative coefficients and with only real zeros, then the sequence $\{a_0, a_1, \ldots, a_d\}$ is log-concave and unimodal, with no internal zeros (see, e.g., [8], or [14], Theorem B, p. 270).

We follow [33] for enumerative combinatorics notation and terminology. In particular, we denote by $S(n, k)$ (respectively, $c(n, k)$) the *Stirling numbers of the second kind* (respectively, *signless Stirling numbers of the first kind*) for $n, k \in \mathbf{N}$, and we follow Chapter 3 of [33] for notation and terminology related to the theory of partially ordered sets.

We follow [25], Chapter I, for notation and terminology related to partitions and symmetric functions. In particular, we denote by $\mathcal{P}$ the set of all (integer) partitions, and by Λ the ring of symmetric functions. Also, given $\lambda \in \mathcal{P}$, we denote by λ' its conjugate, and by s_λ (respectively e_λ, h_λ, p_λ, m_λ) the *Schur* (respectively *elementary*, *complete homogeneous*, *power sum*, *monomial*) symmetric function associated to λ. We will usually identify a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ with its diagram $\{(i, j) \in \mathbf{P} \times \mathbf{P} : 1 \le i \le r,\ 1 \le j \le \lambda_i\}$.

We follow [31] for notation and terminology concerning graded algebras and Hilbert functions. In particular, by a *graded k-algebra* (k being a field, fixed once and for all) we mean a commutative, associative ring R, with identity, containing a copy of the field k (so that R is a vector space over k) together with a collection of k-subspaces $\{R_i\}_{i \in \mathbf{N}}$ such that:

(i) $R = \bigoplus_{i \ge 0} R_i$ (as a k-vector space);
(ii) $R_0 = k$;
(iii) $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbf{N}$;
(iv) R is finitely generated as a k-algebra.

Note that this implies that each R_i is a finite dimensional vector space over k. The *Hilbert series* of R is the formal power series

$$P(R; x) \stackrel{\text{def}}{=} \sum_{i \ge 0} \dim_k(R_i) x^i.$$

The following fundamental result is well-known, and a proof of it can be found, e.g., in [3], Theorem 11.1, or in [31], Theorem 8.

Theorem 2.1 *Let R be a graded k-algebra as above. Then*

$$P(R; x) = \frac{h(R; x)}{\prod_{i=1}^{r} (1 - x^{k_i})},$$

in $\mathbf{Z}[[x]]$, where $h(R; x) \in \mathbf{Z}[x]$ and $k_1, \ldots, k_r$ are the degrees of a homogeneous generating set of R (as a k-algebra).

We call

$$H(R; i) \stackrel{\text{def}}{=} \dim_k(R_i)$$

the *Hilbert function* of R. We say that a k-algebra R as above is *standard* if it can be finitely generated (as a k-algebra) by elements of R_1. From now on we will always assume that all our graded k-algebras are standard. If R is a standard graded k-algebra then we can take $k_1 = \cdots = k_r = 1$ in Theorem 2.1 and this, by well known results from the theory of rational generating functions (see, e.g., [33], Proposition 4.2.2(iii)), implies the following fundamental result which was first proved by Hilbert (in a more general setting, see, e.g., [32], Corollary 9, [3], Corollary 11.2, or [12], Theorem 4.1.3).

Theorem 2.2 *Let R be a standard graded k-algebra. Then there exists a polynomial $P_R(x) \in \mathbf{Q}[x]$ and $N \in \mathbf{P}$ such that $H(R; i) = P_R(i)$ for all $i \ge N$.*

The polynomial $P_R(x)$ uniquely defined by the previous theorem is called the *Hilbert polynomial* of R. Note that if $P(R; x) \in \mathbf{Z}[x]$ then $P_R(x) = 0$.

Given $n, i \in \mathbf{P}$ it is not hard to show (see, e.g., [2], or [12], Lemma 4.2.6, p. 158) that there exist unique integers $a_i > a_{i-1} > \cdots > a_j \geq j$ (for some $j \in [i]$) such that

$$n = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}. \tag{1}$$

We then define

$$n^{\langle i \rangle} \stackrel{\text{def}}{=} \binom{a_i + 1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j + 1}{j+1},$$

and we also set $0^{\langle i \rangle} \stackrel{\text{def}}{=} 0$. We call (1) the *$i$-binomial expansion* of n. We say that a sequence $\{h_0, h_1, h_2, \ldots\}$ of nonnegative integers is an *O-sequence* if the following two conditions are satisfied:

(i) $h_0 = 1$;
(ii) $h_{i+1} \leq (h_i)^{\langle i \rangle}$ for all $i \in \mathbf{P}$.

We say that a finite sequence $\{h_0, h_1, \ldots, h_d\}$ is an O-sequence if $\{h_0, h_1, \ldots, h_d, 0, 0, \ldots\}$ is an O-sequence. An O-sequence is sometimes also called an *M-vector* (see, e.g., [32]) or an *M-sequence* (see, e.g., [6], where a different, but equivalent, definition is given). Note that an O-sequence $\{h_i\}_{i \in \mathbf{N}}$ has no internal zeros (since if $h_i = 0$ then $h_{i+1} \leq (h_i)^{\langle i \rangle} = 0^{\langle i \rangle} = 0$). However, an O-sequence is not necessarily unimodal (take, e.g., (1, 4, 3, 4)). We say that a formal power series $\sum_{i \geq 0} h_i x^i \in \mathbf{N}[[x]]$ is an *O-series* if $\{h_i\}_{i \in \mathbf{N}}$ is an O-sequence.

Let $x_1, \ldots, x_d$ be a set of independent variables. Recall (see, e.g., [12], Definition 4.2.1, p. 155, or [30], Section 2, p. 59) that a (non-empty) set $\mathcal{M}$ of monomials in $x_1, \ldots, x_d$ is said to be an *order ideal of monomials* if $p \in \mathcal{M}$ and q divides p implies $q \in \mathcal{M}$. In other words, if $x_1^{a_1} \ldots x_d^{a_d} \in \mathcal{M}$ and $b_i \in [0, a_i]$ for $i \in [d]$ then $x_1^{b_1} \ldots x_d^{b_d} \in \mathcal{M}$. In particular, since $\mathcal{M} \neq \emptyset$, $1 = x_1^0 \ldots x_d^0 \in \mathcal{M}$. For $i \in \mathbf{P}$ we let $\mathcal{M}_i \stackrel{\text{def}}{=} \{p \in \mathcal{M} : \deg(p) = i\}$ (where $\deg(x_1^{a_1} \ldots x_d^{a_d}) \stackrel{\text{def}}{=} a_1 + \cdots + a_d$), so $\mathcal{M}_0 = \{1\}$. The link between O-series, order ideals of monomials, and Hilbert series of standard graded k-algebras is given by the following fundamental and well known result which is due to Macaulay [24]. We refer the reader to [24, 13], or [12], Theorem 4.2.10, p. 160, for a proof (see also [30], Section 2).

Theorem 2.3 *Let $\{h_i\}_{i \in \mathbf{N}}$ be a sequence of nonnegative integers. Then the following are equivalent*:

(i) $\sum_{i \geq 0} h_i x^i$ *is an O-series*;
(ii) *there exists a standard graded k-algebra R such that $P(R; x) = \sum_{i \geq 0} h_i x^i$, in $\mathbf{Z}[[x]]$*;
(iii) *there exists an order ideal of monomials $\mathcal{M}$ such that $h_i = |\mathcal{M}_i|$ for all $i \in \mathbf{N}$.*

The preceding result allows us, among other things, to prove that certain natural operations in the ring of formal power series preserve the property of being an O-series. The following result is known, but for lack of an adequate reference we give a proof of it here.

Proposition 2.4 *Let $\sum_{i\geq 0} a_i x^i$ and $\sum_{i\geq 0} b_i x^i$ be two O-series, and $j \in \mathbf{P}$. Then the following are also O-series*:

(i) $(\sum_{i\geq 0} a_i x^i)(\sum_{i\geq 0} b_i x^i)$;

(ii) $\sum_{i\geq 0} a_i x^i + \sum_{i\geq 0} b_i x^i - 1$;

(iii) $\sum_{i=0}^{j} a_i x^i$;

(iv) $\sum_{i\geq 0} a_i b_i x^i$;

(v) $\sum_{i\geq 0} a_{ji} x^i$.

Proof: (iii) is immediate from the definition of an O-series. The other statements all follow from corresponding constructions in the theory of graded algebras and Theorem 2.3. More precisely, let $R = \bigoplus_{i\geq 0} R_i$ and $S = \bigoplus_{i\geq 0} S_i$ be two standard graded k-algebras such that $P(R; x) = \sum_{i\geq 0} a_i x^i$ and $P(S; x) = \sum_{i\geq 0} b_i x^i$. Then $R \oplus S$, $R \otimes_k S$, $R * S$ (where $*$ denotes the Segre product, i.e., $R * S \stackrel{\text{def}}{=} \bigoplus_{i\geq 0}(R_i \otimes_k S_i)$) and $R^{(j)}$ (where $R^{(j)}$ denotes the jth Veronese subalgebra of R, i.e., $R^{(j)} \stackrel{\text{def}}{=} \oplus_{i\geq 0} R_{ij}$) are again standard graded k-algebras and $P(R \oplus S; x) = P(R; x) + P(S; x) - 1$, $P(R \otimes_k S; x) = P(R; x)\, P(S; x)$, $P(R * S; x) = \sum_{i\geq 0} a_i b_i x^i$ and $P(R^{(j)}; x) = \sum_{i\geq 0} a_{ji} x^i$ which, by Theorem 2.3, proves (i), (ii), (iv), and (v). □

Note that it is also possible to prove the preceding result by using the equivalence of parts (i) and (iii) in Theorem 2.3, thus avoiding commutative algebra.

Throughout this work, we say that a sequence $\{h_i\}_{i\in\mathbf{N}}$ (respectively, a polynomial $H(x)$) is a *Hilbert function* (respectively, a *Hilbert polynomial*) if there exists a standard graded k-algebra R such that $h_i = H(R; i)$ for all $i \in \mathbf{N}$ (respectively, $H(x) = P_R(x)$). We say that a finite sequence $\{h_0, h_1, \ldots, h_d\}$ is a Hilbert function if the sequence $\{h_0, h_1, \ldots, h_d, 0, 0, \ldots\}$ is a Hilbert function.

Just as Theorem 2.3 provides a numerical characterization of Hilbert functions, there is a numerical characterization of Hilbert polynomials, also due to Macaulay.

Theorem 2.5 *Let $P(x) \in \mathbf{Q}[x]$ be such that $P(\mathbf{Z}) \subseteq \mathbf{Z}$, and let $m_0, \ldots, m_d$ be the unique integers such that*

$$P(x) = \sum_{i=0}^{d}\left[\left(\!\!\binom{x}{i+1}\!\!\right) - \left(\!\!\binom{x-m_i}{i+1}\!\!\right)\right], \tag{2}$$

(*where* $d = \deg(P(x))$). *Then $P(x)$ is a Hilbert polynomial if and only if* $m_0 \geq m_1 \geq \cdots \geq m_d \geq 0$.

The existence and uniqueness of the integers $m_0, \ldots, m_d$ is an elementary statement, and the "if" part of the above theorem is easy to show. A proof of the "only if" part of Theorem 2.5 is given, e.g., in [24], p. 536, [20], Corollary 5.7, p. 47, and [28], Theorem 2.1, (see also [12], Exercise 4.2.15, p. 165).

Because of the previous result, given a polynomial $P(x) \in \mathbf{Q}[x]$ such that $P(\mathbf{Z}) \subseteq \mathbf{Z}$, we call the integers $m_0, \ldots, m_d$ uniquely determined by (2) the *Macaulay parameters* of $P(x)$, and we write $M(P) = (m_0, \ldots, m_d)$.

By a *simplicial complex* we mean a collection of sets Δ with the property that if $A \in \Delta$ and $B \subseteq A$ then $B \in \Delta$. We call the elements of Δ the *faces* of Δ. For $S \in \Delta$, the *dimension* of S is $|S|-1$. The dimension of Δ is $\dim(\Delta) \stackrel{\text{def}}{=} \max\{|A|-1 : A \in \Delta\}$. Given a simplicial complex Δ of dimension $d-1$ we let $f_{i-1}(\Delta) \stackrel{\text{def}}{=} |\{A \in \Delta : |A| = i\}|$, for $i = 0, \dots, d$, and call $\mathbf{f}(\Delta) \stackrel{\text{def}}{=} (f_0(\Delta), f_1(\Delta), \dots, f_{d-1}(\Delta))$ the *f-vector* of Δ. We then define the *h-vector* of Δ, $h(\Delta) \stackrel{\text{def}}{=} (h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta))$, by letting

$$\sum_{i=0}^{d} h_i(\Delta) x^{d-i} \stackrel{\text{def}}{=} \sum_{i=0}^{d} f_{i-1}(\Delta)(x-1)^{d-i}. \tag{3}$$

Clearly, knowledge of the f-vector of Δ is equivalent to the knowledge of its h-vector. Note that, by Theorem 2.3, if $\{f_0, \dots, f_{d-1}\}$ is the f-vector of a simplicial complex then $\{1, f_0, \dots, f_{d-1}\}$ is a Hilbert function.

Recall (see, e.g., [31], Definition 1.1, p. 62, or [12], Definition 5.1.2, p. 201) that we may associate a standard graded k-algebra to any finite simplicial complex Δ as follows. Let $\{v_1, \dots, v_n\}$ be the set of 0-dimensional faces of Δ, $x_1, \dots, x_n$ be indeterminates, and I_Δ be the ideal of $k[x_1, \dots, x_n]$ generated by $\{x_{i_1} \cdots x_{i_r} : 1 \le i_1 < \cdots < i_r \le n$ and $\{v_{i_1}, \dots, v_{i_r}\} \notin \Delta\}$. Then it is well known, and easy to see, that $R_\Delta \stackrel{\text{def}}{=} k[x_1, \dots, x_n]/I_\Delta$ is a standard graded k-algebra, called the *Stanley-Reisner* (or *face*) ring of Δ. This ring has been extensively studied and we refer the reader to [31], Chapter 2, and [12], Chapter 5, for its fundamental properties.

3. A general theory

Despite the fact that Theorems 2.3 and 2.5 completely characterize Hilbert functions and polynomials, it is, in practice, a difficult task to decide if a given polynomial is Hilbert using just these theorems. For example, the reader can check (preferably with the aid of a computer) that $M(x^5) = [73125997584400089301233649866440583794687734855985916 3646, 382429072075856811032080944 27, 276560696326610, 23520860, 6900, 120]$ and that (according to Maple) the first entry of this sequence is two times a prime number! Thus, the computation of the Macaulay parameters of a polynomial is in general not an easy task, neither computationally nor theoretically. Our purpose in this section is to use Theorems 2.3 and 2.5 to deduce other results on Hilbert polynomials that are easier to apply, even though they do not characterize these objects completely. In particular, we wish to obtain conditions on the coefficients of a polynomial with respect to the bases defined in the previous section that insure that it is a Hilbert polynomial.

We begin with the following result which expresses the relationship between the Macaulay parameters of a polynomial and its coefficients with respect to the basis of twisted binomial coefficients.

Proposition 3.1 *Let $P(x) \in \mathbf{Q}[x]$ be such that $P(\mathbf{Z}) \subseteq \mathbf{Z}$, and let*

$$P(x) = \sum_{i=0}^{d} \left[\left(\!\!\binom{x}{i+1}\!\!\right) - \left(\!\!\binom{x-m_i}{i+1}\!\!\right) \right] = \sum_{i=0}^{d} c_i \left(\!\!\binom{x}{i}\!\!\right), \tag{4}$$

where $d = \deg(P(x))$. *Then*

$$c_i = \sum_{j=0}^{d-i} (-1)^j \binom{m_{i+j}}{j+1}, \tag{5}$$

for $i = 0, \dots, d$.

Proof: It is easy to see that

$$\left(\!\!\binom{x-m}{i+1}\!\!\right) = \sum_{j=0}^{i+1} (-1)^{i+1-j} \binom{m}{i+1-j} \left(\!\!\binom{x}{j}\!\!\right) \tag{6}$$

for all $m, i \in \mathbf{N}$. Therefore

$$\left(\!\!\binom{x}{i+1}\!\!\right) - \left(\!\!\binom{x-m}{i+1}\!\!\right) = \sum_{j=0}^{i} (-1)^{i-j} \binom{m}{i+1-j} \left(\!\!\binom{x}{j}\!\!\right). \tag{7}$$

Summing (7) (with $m = m_i$) for $i = 0, \dots, d$ and comparing with (4) yields (5), as desired. □

Note that the previous result makes it easy to compute the coefficients of a polynomial with respect to the basis of twisted binomial coefficients from its Macaulay parameters (as implicitly noted also in [24], p. 537), but not conversely (even though the relations (5) are, of course, invertible). Hence, even a reasonably detailed knowledge of the coefficients $\{c_0, \dots, c_d\}$ in (4) will not make it easy to decide if the polynomial is Hilbert. However, the relations (5) do have the following interesting consequence.

Theorem 3.2 *For* $i \in \mathbf{N}$ *there exist* $\Phi_i \in \mathbf{Q}[x_0, \dots, x_i]$ *such that*:

(i) $\deg(\Phi_i) = 2^i$;

(ii) *if* $P(x) = \sum_{i=0}^{d} c_i \left(\!\binom{x}{i}\!\right) \in \mathbf{Q}[x]$ *is such that* $P(\mathbf{Z}) \subseteq \mathbf{Z}$ *and* $M(P(x)) = (m_0, \dots, m_d)$, *then* $m_{d-i} = \Phi_i(c_d, \dots, c_{d-i})$ *for* $i = 0, \dots, d$;

(iii) *the leading monomial of* Φ_i *is* $2\left(\frac{x_0}{2}\right)^{2^i}$.

Proof: We define $\Phi_i \in \mathbf{Q}[x_0, \dots, x_i]$ inductively as follows,

$$\Phi_0 \stackrel{\text{def}}{=} x_0, \tag{8}$$

$$\Phi_i \stackrel{\text{def}}{=} x_i - \sum_{j=1}^{i} (-1)^j \binom{\Phi_{i-j}}{j+1}, \tag{9}$$

if $i \geq 1$. Then (i), (ii), and (iii) follow easily by induction on $i \in \mathbf{P}$. In fact, by our induction hypotheses, $\deg(\binom{\Phi_{i-j}}{j+1}) = (j+1)2^{i-j}$ for $j = 1, \dots, i$ and hence, by (9), $\deg(\Phi_i) =$

$\deg(\binom{\Phi_{i-1}}{2}) = 2^i$ and the leading monomial of Φ_i is $\frac{1}{2}(2(\frac{x_0}{2})^{2^{i-1}})^2 = 2(\frac{x_0}{2})^{2^i}$. Similarly, we deduce from (5) and our induction hypotheses that

$$\begin{aligned} m_{d-i} &= c_{d-i} - \sum_{j=1}^{i}(-1)^j \binom{m_{d-i+j}}{j+1} \\ &= c_{d-i} - \sum_{j=1}^{i}(-1)^j \binom{\Phi_{i-j}(c_d, \dots, c_{d-i+j})}{j+1} \\ &= \Phi_i(c_d, \dots, c_{d-i}), \end{aligned}$$

by (9), as desired. □

The following easy consequence of Proposition 3.1 will be useful later on.

Lemma 3.3 *Let $P(x) = \sum_{i=0}^{d} c_i ((\binom{x}{i}))$ be a Hilbert polynomial such that $c_d = 1$. Then $c_{d-1} \geq 1$.*

Proof: From (5) we deduce that $m_d = c_d$, and $m_{d-1} = c_{d-1} + \binom{c_d}{2}$, and the thesis follows from Theorem 2.5. □

Note that a Hilbert polynomial satisfying the hypotheses of Lemma 3.3 does not necessarily have all its coefficients nonnegative when expanded in terms of the basis of the twisted binomial coefficients. For example, $M(((\binom{x}{2})) + 4((\binom{x}{1})) - 2((\binom{x}{0}))) = (4, 4, 1)$.

It is of course easy to compute the Macaulay parameters of polynomials of small degree, and the following computational result will be convenient later on.

Proposition 3.4 *Let $a, b, c \in \mathbf{Z}$. Then $ax^2 + bx + c$ is a Hilbert polynomial if and only if $0 \leq 2a \leq b + 2a^2 - 2a \leq c + \binom{b+2a^2-2a}{2} - \binom{2a}{3}$.*

Proof: One computes that

$$\sum_{i=0}^{2}\left[\left(\binom{x}{i+1}\right) - \left(\binom{x-m_i}{i+1}\right)\right] = x^2\binom{m_2}{2} + \left(m_1 + m_2 - \frac{m_2^2}{2}\right)x \\ + m_0 - \binom{m_1}{2} + \binom{m_2}{3},$$

and the thesis follows from Theorem 2.5. □

The next result gives some fundamental operations on polynomials that preserve the property of being a Hilbert polynomial. Some of these are known, but for lack of an adequate reference we give a complete proof here.

Theorem 3.5 *Let $A(x), B(x) \in \mathbf{Q}[x]$ be two Hilbert polynomials, $k \in \mathbf{P}$, $m \in \mathbf{N}$, and $\{h_0, \dots, h_r\}$ be a Hilbert function. Then the following are Hilbert polynomials:*

(i) $A(x) + B(x)$;
(ii) $A(x)B(x)$;
(iii) $A(kx + m)$;
(iv) $A(x) - A(x - 1)$;
(v) $k\,A(x) + m$;
(vi) $\sum_{i=0}^{r} h_i A(x - i)$.

Proof: By hypothesis there exist $H_1, H_2 : \mathbf{N} \to \mathbf{N}$ such that $\{H_1(n)\}_{n\in\mathbf{N}}$ and $\{H_2(n)\}_{n\in\mathbf{N}}$ are O-sequences, and $H_1(n) = A(n)$, $H_2(n) = B(n)$ if $n \geq n_0$ (for some $n_0 \in \mathbf{N}$). Hence, by (ii) of Proposition 2.4, $\{1, H_1(1) + H_2(1), H_1(2) + H_2(2), \ldots\}$ is an O-sequence and $H_1(n) + H_2(n) = A(n) + B(n)$ for $n \geq n_0$ and this shows that $A(x) + B(x)$ is a Hilbert polynomial. In an exactly analogous way (using (iv) and (v) of Proposition 2.4) one proves (ii), and (iii) for $m = 0$.

To prove (iv) note that by Theorem 2.5 and our hypotheses we have that

$$A(x) = \sum_{i=0}^{d}\left[\left(\!\!\binom{x}{i+1}\!\!\right) - \left(\!\!\binom{x-m_i}{i+1}\!\!\right)\right] \tag{10}$$

where $m_0 \geq m_1 \geq \cdots \geq m_d \geq 0$, and $d \overset{\text{def}}{=} \deg(A(x))$. Therefore

$$\begin{aligned} A(x) - A(x-1) &= \sum_{i=0}^{d}\left[\left(\!\!\binom{x}{i+1}\!\!\right) - \left(\!\!\binom{x-1}{i+1}\!\!\right) - \left(\!\!\binom{x-m_i}{i+1}\!\!\right)\right. \\ &\qquad\qquad \left. + \left(\!\!\binom{x-1-m_i}{i+1}\!\!\right)\right] \\ &= \sum_{i=0}^{d-1}\left[\left(\!\!\binom{x}{i+1}\!\!\right) - \left(\!\!\binom{x-m_{i+1}}{i+1}\!\!\right)\right] \end{aligned}$$

and (iv) follows from Theorem 2.5. Also, (10) implies that

$$A(x+1) = \sum_{i=0}^{d}\left[\left(\!\!\binom{x+1}{i+1}\!\!\right) - \left(\!\!\binom{x+1-m_i}{i+1}\!\!\right)\right]. \tag{11}$$

Now note that,

$$\left(\!\!\binom{x+1}{i+1}\!\!\right) - \left(\!\!\binom{x+1-m}{i+1}\!\!\right) = \sum_{j=0}^{i}\left[\left(\!\!\binom{x}{j+1}\!\!\right) - \left(\!\!\binom{x-m}{j+1}\!\!\right)\right]$$

and hence

$$M\left(\left(\!\!\binom{x+1}{i+1}\!\!\right) - \left(\!\!\binom{x+1-m}{i+1}\!\!\right)\right) = (\underbrace{m, m, \ldots, m}_{i+1}) \tag{12}$$

for all $m, i \in \mathbf{N}$. Therefore, by Theorem 2.5, every summand on the RHS of (11) is a Hilbert polynomial and this, by (i), implies that $A(x+1)$ is a Hilbert polynomial. Hence $A(x+m)$ is a Hilbert polynomial and this concludes the proof of (iii) since we have already observed that (iii) holds if $m = 0$.

Now note that it follows easily from (10) that $M(A(x)+m) = (m+m_0, m_1, \ldots, m_d)$. Hence, by Theorem 2.5, $A(x)+m$ is a Hilbert polynomial and (v) follows from (i).

To prove (vi) let R and S be two standard graded k-algebras such that $P(R; x) = \sum_{n\geq 0} H_1(n)x^n$ and $P(S; x) = \sum_{n=0}^{r} h_n x^n$. Then

$$P(R \otimes_k S; x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} h_i H_1(n-i)\right)x^n$$

(where $h_i \stackrel{\text{def}}{=} 0$ if $i > r$), and hence

$$H(R \otimes_k S; n) = \sum_{i=0}^{r} h_i\, A(n-i)$$

if $n \geq n_0 + r$, and (vi) follows. □

Note that while parts (i), (ii), (iii) (for $m = 0$), and (vi) of Theorem 3.5 have a clear algebraic and geometric interpretation, we have been unable to find any algebraic or geometric explanation for parts (iii) (when $m > 0$), and (iv).

It is natural to ask whether there are other operations on polynomials which preserve the property of being a Hilbert polynomial. One operation to consider, in view of part (iv) of Theorem 3.5, is the anti-difference of a polynomial $A(x)$ (i.e., the unique polynomial ∇A such that $(\nabla A)(n) = \sum_{j=0}^{n} A(j)$ for all $n \in \mathbf{N}$). This, however, fails to preserve the property of being a Hilbert polynomial. For example, $A(x) = 3x$ is a Hilbert polynomial (since $M(3x) = (3, 3)$) but

$$(\nabla A)(x) = 3\left(\binom{x}{2}\right)$$

is not a Hilbert polynomial (since $M(3(\binom{x}{2})) = (2, 3, 3)$). However, it is not hard to compute the Macaulay parameters of a polynomial $A(x)$ if the Macaulay parameters of $A(x) - A(x-1)$ are known, and hence to obtain a necessary and sufficient condition on a Hilbert polynomial so that its anti-difference is again a Hilbert polynomial.

Proposition 3.6 *Let $A(x) \in \mathbf{Q}[x]$ be such that $A(\mathbf{Z}) \subseteq \mathbf{Z}$ and suppose that $M(A(x) - A(x-1)) = (m_0, \ldots, m_d)$. Then*

$$M(A(x)) = (m_{-1}, m_0, \ldots, m_d)$$

where $m_{-1} \stackrel{\text{def}}{=} \sum_{i=0}^{d}(-1)^i\binom{m_i+1}{i+2} + A(-1)$.

Proof: Let $B(x) \stackrel{\text{def}}{=} A(x) - A(x-1)$, for brevity. Then we have from our hypothesis and the definition of $(\nabla B)(x)$ that, for all $n \in \mathbf{N}$,

$$\begin{aligned}(\nabla B)(n) &= \sum_{j=0}^{n} B(j)\\ &= \sum_{i=0}^{d}\sum_{j=0}^{n}\left[\left(\!\!\binom{j}{i+1}\!\!\right) - \left(\!\!\binom{j-m_i}{i+1}\!\!\right)\right]\\ &= \sum_{i=0}^{d}\left[\left(\!\!\binom{n}{i+2}\!\!\right) - \left(\!\!\binom{n-m_i}{i+2}\!\!\right) + \left(\!\!\binom{-m_i-1}{i+2}\!\!\right)\right]\\ &= \sum_{i=0}^{d+1}\left[\left(\!\!\binom{n}{i+1}\!\!\right) - \left(\!\!\binom{n-m'_i}{i+1}\!\!\right)\right]\end{aligned}$$

where $m'_0 = \sum_{i=0}^{d}((\binom{-m_i-1}{i+2}))$, and $m'_i = m_{i-1}$, for $i = 1, \ldots, d+1$. Therefore $M(\nabla B) = (m'_0, m_0, \ldots, m_d)$ and hence $M((\nabla B)(x) + A(-1)) = (A(-1) + m'_0, m_0, \ldots, m_d)$, as desired. □

Corollary 3.7 *Let $A(x) \in \mathbf{Q}[x]$ be a Hilbert polynomial of degree d with Macaulay parameters $(m_0, \ldots, m_d)$, and $B(x) \in \mathbf{Q}[x]$ be such that $B(x) - B(x-1) = A(x)$. Then $B(x)$ is a Hilbert polynomial if and only if*

$$m_0 \leq \sum_{i=0}^{d}(-1)^i \binom{m_i+1}{i+2} + B(-1).$$

We now wish to study which polynomials of the bases defined in the previous section are Hilbert polynomials.

Theorem 3.8 *Let $d \in \mathbf{P}$, $a_0, \ldots, a_d \in \mathbf{P}$, and $i \in [0, d]$. Then:*

(i) *x^d is a Hilbert polynomial if and only if $d \geq 3$;*

(ii) *$\langle x\rangle_d$ is a Hilbert polynomial if and only if $d \geq 3$;*

(iii) *$a_0(x+a_1)\cdots(x+a_d)$ is a Hilbert polynomial;*

(iv) *$\binom{x}{d}$ is not a Hilbert polynomial;*

(v) *$((\binom{x}{d}))$ is not a Hilbert polynomial;*

(vi) *$\binom{x+d-i}{d}$ is a Hilbert polynomial if and only if $i = 0$.*

Proof: A straightforward computation using Theorem 2.5 shows that x and x^2 are not Hilbert polynomials, while x^3, x^4, and x^5 are. So (i) follows from part (ii) of Theorem 3.5. Also, it is easily verified, using Theorem 2.5, that $\langle x\rangle_2$ is not a Hilbert polynomial, while $\langle x\rangle_3$ is. But, by Proposition 3.4, r and $x+r$ are Hilbert polynomials whenever $r \geq 1$, so

(ii) and (iii) follow from part (ii) of Theorem 3.5. Furthermore, we have from (6) that

$$\binom{x+d-i}{d} = \sum_{j=d-i+1}^{d} (-1)^{d-j} \binom{i-1}{d-j} \left(\!\!\binom{x}{j}\!\!\right),$$

if $i \in [d]$, so (iv), (v), and the "only if" part of (vi) follow from Lemma 3.3. On the other hand, if $i = 0$ then

$$\binom{x+d}{d} = \left(\!\!\binom{x+1}{d}\!\!\right),$$

which is a Hilbert polynomial since $\{((\binom{n+1}{d}))\}_{n\in\mathbf{N}}$ is the Hilbert function of $k[x_0, \ldots, x_d]$ (in fact, it follows from (12) that the Macaulay parameters of $\binom{x+d}{d}$ are $(\overbrace{1, 1, \ldots, 1}^{d+1})$). $\square$

The reader will notice that the basis of lower factorials $\{(x)_i\}_{i\in\mathbf{N}}$ is missing from Theorem 3.8. It can be easily checked that $(x)_2$, and $(x)_3$ are not Hilbert polynomials, and we conjecture (see Section 5) that $(x)_d$ is always a Hilbert polynomial if $d \geq 4$.

Theorem 3.8 has several interesting consequences.

Corollary 3.9 *Let $P(x) \in \mathbf{Q}[x]$ and suppose that $P(x-1) \in \mathbf{N}[x]$. Then $P(x)$ is a Hilbert polynomial.*

Proof: By hypothesis we can write $P(x) = \sum_{i=0}^{d} a_i (x+1)^i$ where $a_0, \ldots, a_d \in \mathbf{N}$, so the thesis follows from part (i) of Theorem 3.5 and part (iii) of Theorem 3.8. $\square$

Corollary 3.10 *Let $P(x) = \sum_{i=0}^{d} a_i x^i = \sum_{i=0}^{d} \gamma_i \langle x \rangle_i \in \mathbf{Q}[x]$. Suppose that at least one of the following conditions is satisfied*:

(i) $a_0, \ldots, a_d \in \mathbf{N}$ *and* $a_1, a_2 \geq 3$;

(ii) $\gamma_0, \ldots, \gamma_d \in \mathbf{N}$ *and* $\gamma_1 \geq 3$, $\gamma_2 \geq 2$.

Then $P(x)$ is a Hilbert polynomial.

Proof: Assume first that (i) holds. It follows from part (i) of Theorem 3.5 and part (i) of Theorem 3.8 that $a_i x^i$ is a Hilbert polynomial for $0 \leq i \leq d$, $i \neq 1, 2$ (since $a_i \in \mathbf{N}$). On the other hand, it follows from Proposition 3.4 that $a_1 x$ and $a_2 x^2$ are also Hilbert polynomials (since $a_1, a_2 \geq 3$), so the thesis follows from part (i) of Theorem 3.5.

Similarly, if (ii) holds then it follows from part (i) of Theorem 3.5 and part (ii) of Theorem 3.8 that $\gamma_i \langle x \rangle_i$ is a Hilbert polynomial for $0 \leq i \leq d$, $i \neq 1, 2$. On the other hand, it follows from Proposition 3.4 that $\gamma_1 \langle x \rangle_1$ and $\gamma_2 \langle x \rangle_2$ are also Hilbert polynomials (since $\gamma_1 \geq 3$, $\gamma_2 \geq 2$), so the thesis follows from part (i) of Theorem 3.5. $\square$

Another interesting consequence of Theorem 3.8 is the following:

Theorem 3.11 *Let $P(x) = \sum_{i=0}^{d} c_i ((\binom{x}{i})) = \sum_{i=0}^{d} w_i \binom{x+d-i}{d} = \sum_{i=0}^{d} a_i x^i$ be such that $P(\mathbf{Z}) \subseteq \mathbf{Z}$. Suppose that any one of the following conditions is satisfied*:

(i) $c_0 \geq c_1 \geq \cdots \geq c_d \geq 0$;
(ii) $w_0 \geq w_1 \geq \cdots \geq w_d \geq 0$;
(iii) $a_0, \ldots, a_d \in \mathbf{N}$, $d \geq 3$, *and* $a_0 \leq a_1 \leq a_2 \leq a_3$.
Then $P(x)$ *is a Hilbert polynomial.*

Proof: Since $P(\mathbf{Z}) \subseteq \mathbf{Z}$ we conclude easily (considering $P(0), P(-1), \ldots, P(-d+1)$) that $c_0, \ldots, c_d \in \mathbf{Z}$, and $w_0, \ldots, w_d \in \mathbf{Z}$.

Assume now that (i) holds. Then there exist $\beta_0, \ldots, \beta_d \in \mathbf{N}$ such that $c_i = \beta_i + \beta_{i+1} + \cdots + \beta_d$ for $i = 0, \ldots, d$. Hence

$$P(x) = \sum_{i=0}^{d} c_i \left(\!\!\binom{x}{i}\!\!\right) = \sum_{i=0}^{d}\sum_{j=i}^{d} \beta_j \left(\!\!\binom{x}{i}\!\!\right) = \sum_{j=0}^{d} \beta_j \left(\sum_{i=0}^{j} \left(\!\!\binom{x}{i}\!\!\right)\right)$$
$$= \sum_{j=0}^{d} \beta_j \left(\!\!\binom{x+1}{j}\!\!\right),$$

and the thesis follows from Theorems 3.5 and 3.8.

Similarly, if (ii) holds then there exist $b_0, \ldots, b_d \in \mathbf{N}$ such that $w_i = b_i + b_{i+1} + \cdots + b_d$ for $i = 0, \ldots, d$. Hence

$$P(x) = \sum_{i=0}^{d} w_i \binom{x+d-i}{d} = \sum_{i=0}^{d}\sum_{j=i}^{d} b_j \binom{x+d-i}{d}$$
$$= \sum_{j=0}^{d} b_j \left(\sum_{i=0}^{j} \binom{x+d-i}{d}\right). \tag{13}$$

Now note that

$$\sum_{i=0}^{j} \binom{x+d-i}{d} = \sum_{i=0}^{d} \left[\left(\!\!\binom{x+1-i}{d+1}\!\!\right) - \left(\!\!\binom{x-i}{d+1}\!\!\right)\right]$$
$$= \left(\!\!\binom{x+1}{d+1}\!\!\right) - \left(\!\!\binom{x-j}{d+1}\!\!\right).$$

Hence $\sum_{i=0}^{j}\binom{x+d-i}{d}$ is a Hilbert polynomial by (12) and the thesis follows from (13), and Theorem 3.5.

Finally, assume that (iii) holds. It is easily verified (using Theorem 2.5) that $x^2 + x^3$, $x + x^2 + x^3$, and $1 + x + x^2 + x^3$ are Hilbert polynomials. But

$$P(x) = a_0(1 + x + x^2 + x^3) + (a_1 - a_0)(x + x^2 + x^3) + (a_2 - a_1)(x^2 + x^3)$$
$$+ (a_3 - a_2)x^3 + \sum_{i=4}^{d} a_i x^i,$$

so the thesis follows from our hypotheses and Theorems 3.5 and 3.8. □

For the basis $\{\binom{x}{i}\}_{i=0,\dots,d}$ we have been unable to prove any results similar to Theorem 3.11. The main reason is that, in general, neither $\sum_{i=0}^{j}\binom{x}{i}$ nor $\sum_{i=j}^{d}\binom{x}{i}$ are Hilbert polynomials. For example, $M(\binom{x}{0}+\binom{x}{1}+\binom{x}{2}) = (1,0,1)$ and $M(\binom{x}{2}+\binom{x}{3}) = (1,1,-1,1)$, (in fact, computations suggest that $\sum_{i=0}^{j}\binom{x}{i}$ is never a Hilbert polynomial if $j \geq 2$ and that $\sum_{i=j}^{d}\binom{x}{i}$ is never a Hilbert polynomial if $d \geq 3$ and $0 \leq j \leq d$). So, in some sense, the basis $\{\binom{x}{i}\}_{i=0,\dots,d}$ is, among the six bases considered, the "farthest" from Hilbert polynomials.

Given the close connection existing between Hilbert functions and polynomials it is natural to wonder if there are other ways to produce Hilbert polynomials from a Hilbert function.

Theorem 3.12 *Let $\{h_0,\dots,h_d\}$ be a Hilbert function. Then the following are Hilbert polynomials*:

(i) $\sum_{i=0}^{d} h_i\binom{x+d-i}{d}$;

(ii) $\sum_{i=0}^{d} h_i\langle x\rangle_i$.

Proof: (i) is an immediate consequence of part (vi) of Theorems 3.5 and 3.8 (just take $A(x) = \binom{x+d}{d}$).

To prove (ii) note that if $h_1 \geq 3$ and $h_2 \geq 2$ then the result follows from part (ii) of Corollary 3.10. If $h_2 \leq 1$ then, by our hypothesis, $h_3 \leq h_2^{\langle 2\rangle} \leq 1$, and therefore $h_4 \leq h_3^{\langle 3\rangle} \leq 1$, etc., so we conclude that $h_2 = \cdots = h_r = 1$ and $h_{r+1} = \cdots = h_d = 0$ for some $r \in [2,d]$. But, by Proposition 3.4, $1 + h_1\langle x\rangle_1$ and $1 + h_1\langle x\rangle_1 + \langle x\rangle_2$ are Hilbert polynomials for any $h_1 \in \mathbf{P}$. Hence, by part (i) of Theorem 3.5 and part (ii) of Theorem 3.8 we conclude that $1 + h_1\langle x\rangle_1 + \langle x\rangle_2 + \cdots + \langle x\rangle_r$ is a Hilbert polynomial for any $r \geq 2$. If $h_2 \geq 2$ but $h_1 \leq 2$, then by our hypothesis we conclude that $2 \leq h_2 \leq h_1^{\langle 1\rangle} = \binom{h_1+1}{2} \leq \binom{3}{2} = 3$ and hence that $h_1 = 2$ and $2 \leq h_2 \leq 3$. But, by Proposition 3.4, $1 + 2\langle x\rangle_1 + 2\langle x\rangle_2$ and $1 + 2\langle x\rangle_1 + 3\langle x\rangle_2$ are both Hilbert polynomials, so the result follows also in this case from part (i) of Theorem 3.5 and part (ii) of Theorem 3.8. $\square$

Note that the preceding result fails for the bases $\{x^i\}_{i=0,\dots,d}$, $\{(x)_i\}_{i=0,\dots,d}$ and $\{\binom{x}{i}\}_{i=0,\dots,d}$. For example, $1 + x + x^2$, $1 + (x)_1 + (x)_2$, and $1 + \binom{x}{1} + \binom{x}{2}$ are not Hilbert polynomials.

Since, by Theorem 2.3, if $(f_0,\dots,f_{d-1}) \in \mathbf{N}^d$ is the f-vector of some simplicial complex then $(1, f_0,\dots,f_{d-1})$ is a Hilbert function, it is natural to investigate the analogue of Theorem 3.12 for f-vectors of simplicial complexes.

Theorem 3.13 *Let $(f_0,\dots,f_{d-1}) \in \mathbf{N}^d$ $(d \in \mathbf{P})$ be the f-vector of some simplicial complex. Then the following are Hilbert polynomials*:

(i) $\sum_{i=0}^{d-1} f_i\binom{x}{i}$;

(ii) $\sum_{i=0}^{d-1} f_i\langle x\rangle_i$;

(iii) $\sum_{i=0}^{d-1} f_i x^i$.

Proof: Let Δ be a simplicial complex such that $\mathbf{f}(\Delta) = (f_0,\dots,f_{d-1})$, and let R_Δ be the Stanley-Reisner ring of Δ. It is then well known (see, e.g., [31], Theorem 1.4, p. 63, or

[12], Theorem 5.1.7, p. 204), and also easy to see, that the Hilbert function of R_Δ is given by

$$H(R_\Delta; n) = \begin{cases} 1, & \text{if } n = 0, \\ \sum_{i=0}^{d-1} f_i \binom{n-1}{i}, & \text{if } n \in \mathbf{P}, \end{cases} \tag{14}$$

and (i) follows from part (iii) of Theorem 3.5.

To prove (ii) and (iii) note that if $f_2 \geq 3$ then necessarily $f_1 \geq 3$ and (ii) and (iii) follow from Corollary 3.10. If $f_2 \leq 2$ then $\dim(\Delta) \leq 2$ and it is easy to check, using Proposition 3.4, that $\sum_{i=1}^{2} f_i x^i$ and $\sum_{i=0}^{2} f_i \langle x \rangle_i$ are always Hilbert polynomials in this case. □

Note that there exists a complete numerical characterization, similar to Theorem 2.3, of the sequences that are the f-vector of some simplicial complex (see, e.g., [12], Section 5.1, p. 201, or [31], Theorem 2.1, p. 64). Therefore, one could state Theorem 3.13 without any reference to simplicial complexes.

We conclude our general discussion on Hilbert polynomials by introducing a concept which measures “how far” a polynomial is from being Hilbert. The crucial result for this definition is the following.

Theorem 3.14 *Let $P(x) \in \mathbf{Z}[x]$ be a polynomial with positive leading term. Then there exists $M \in \mathbf{N}$ such that $P(x+i)$ is a Hilbert polynomial for any $i \geq M$.*

Proof: Let $P(x) = \sum_{j=0}^{d} a_j x^j$ where $a_j \in \mathbf{Z}$ and $a_d \in \mathbf{P}$. Then

$$\begin{aligned} P(x+i) &= \sum_{j=0}^{d} a_j (x+i)^j \\ &= \sum_{j=0}^{d} a_j \sum_{k=0}^{j} \binom{j}{k} x^k i^{j-k} \\ &= \sum_{k=0}^{d} \left(\sum_{j=k}^{d} a_j \binom{j}{k} i^{j-k} \right) x^k. \end{aligned}$$

Hence the coefficient of x^k in $P(x+i)$ is a polynomial in i of degree $d-k$ and positive leading term, for $k = 0, \ldots, d$. Therefore there exists $N \in \mathbf{N}$ such that $P(x+i) \in \mathbf{N}[x]$ if $i > N$. The thesis follows from Corollary 3.9. □

The preceding theorem suggests, and allows us to make, the following definition. Given a polynomial $P(x) \in \mathbf{Z}[x]$ with positive leading term we let

$$H\{P\} \stackrel{\text{def}}{=} \max\{i \in \mathbf{N} : P(x+i) \ \text{ is not a Hilbert polynomial}\} + 1 \tag{15}$$

(where $\max\{\emptyset\} \stackrel{\text{def}}{=} -1$). We call $H\{P\}$ the *Hilbert index* of $P(x)$. Hence $P(x+i)$ is a Hilbert polynomial for all $i \geq H\{P\}$, and $P(x)$ is a Hilbert polynomial if $H\{P\} = 0$.

Part (iii) of Theorem 3.5 (with $k = m = 1$) enables us to give the following useful characterization of the Hilbert index of a polynomial.

Corollary 3.15 *Let $P(x) \in \mathbf{Z}[x]$ be a polynomial with positive leading term and $i \in \mathbf{N}$. Then $P(x+i)$ is a Hilbert polynomial if and only if $i \geq H\{P\}$.*

Since $\binom{x}{d}$ and $(\binom{x}{d})$ are not Hilbert polynomials it is natural to ask, in light of the concept just introduced, what their Hilbert index is. As a matter of fact, we have already answered this question in Theorem 3.8, essentially, but we record the result here.

Proposition 3.16 *Let $d \in \mathbf{P}$. Then $H\{\binom{x}{d}\} = d$ and $H\{(\binom{x}{d})\} = 1$.*

Proof: Part (vi) of Theorem 3.8 shows that $(\binom{x}{d}) = \binom{x+d-1}{d}$ is not a Hilbert polynomial while $(\binom{x+1}{d}) = \binom{x+d}{d}$ is, hence by Corollary 3.15 $H\{\binom{x}{d}\} = d$, and $H\{(\binom{x}{d})\} = 1$. □

Note that Theorem 3.2 has the following interesting consequence.

Theorem 3.17 *Let $P(x) \in \mathbf{Z}[x]$ be a polynomial with positive leading term. Then there exists $M \in \mathbf{N}$ such that $iP(x)$ is a Hilbert polynomial for any $i \geq M$.*

Proof: Let $P(x) = \sum_{j=0}^{d} c_j (\binom{x}{j})$ where $c_d > 0$. Then we have from Theorem 3.2 that

$$M(iP(x)) = (\Phi_d(ic_d, \ldots, ic_0), \Phi_{d-1}(ic_d, \ldots, ic_1), \ldots, \Phi_1(ic_d, ic_{d-1}), \Phi_0(ic_d)) \tag{16}$$

for all $i \in \mathbf{Z}$. But, again by Theorem 3.2, $\Phi_j(ic_d, \ldots, ic_{d-j}) - \Phi_{j-1}(ic_d, \ldots, ic_{d-j+1})$ is a polynomial in i of degree 2^j and positive leading term, for $j = 1, \ldots, d$. Therefore there exists $N \in \mathbf{N}$ such that

$$\Phi_d(ic_d, \ldots, ic_0) \geq \Phi_{d-1}(ic_d, \ldots, ic_1) \geq \cdots \geq \Phi_1(ic_d, ic_{d-1}) \geq \Phi_0(ic_d)$$

for all $i \geq N$, and the result follows from (16), Theorem 2.5, and our hypothesis since $\Phi_0(ic_d) = ic_d \geq 0$ if $i \geq 0$. □

Therefore, one could define a second "Hilbert index" in analogy with (15). We leave the investigation of this "Hilbert index" to the interested reader. In particular, it would be interesting to know if the analogue of Corollary 3.15 holds for it.

4. Applications to combinatorics

In this section we apply the general results obtained in the previous section to prove that several polynomials arising in enumerative and algebraic combinatorics are actually Hilbert polynomials.

We begin by considering several polynomials associated to graph colorings. Let $G = (V, E)$ be a graph (without loops and multiple edges). A map $\varphi\colon V \to \mathbf{P}$ is said to be a *coloring* of G if $\varphi(x) \neq \varphi(y)$ for all $x, y \in V$ such that $(x, y) \in E$. Given $n \in \mathbf{P}$ we denote by $P_G(n)$ the number of colorings $\varphi\colon V \to \mathbf{P}$ such that $\varphi(V) \subseteq [n]$. It is then well known (see, e.g., [26], or [14], Section 4.1, p. 179) that there exists a polynomial $\chi(G; x) \in \mathbf{Z}[x]$, of degree $|V|$, such that $\chi(G; n) = P_G(n)$ for all $n \in \mathbf{P}$. This polynomial is called the *chromatic polynomial* of G and has been extensively studied (see, e.g., [27], for a survey). Since $\chi(G; x)$ is a polynomial one may write

$$\chi(G; x) = \sum_{i=0}^{|V|} a_i (x)_i = \sum_{i=0}^{|V|} (-1)^{|V|-i} c_i \langle x \rangle_i .$$

Then the polynomials $\sigma(G; x) \stackrel{\text{def}}{=} \sum_{i=0}^{|V|} a_i x^i$ and $\tau(G; x) \stackrel{\text{def}}{=} \sum_{i=0}^{|V|} c_i x^i$ are called the *σ-polynomial* and the *τ-polynomial* of G, respectively. Despite the fact that knowledge of one of these three polynomials implies knowledge of the other two it is often the case that $\sigma(G; x)$ and $\tau(G; x)$ are more convenient to handle then $\chi(G; x)$ itself. For this reason $\sigma(G; x)$ and $\tau(G; x)$ have also been studied, and we refer the reader to [9, 10], and the references cited therein, for more information on these two polynomials.

Theorem 4.1 *Let $G = (V, E)$ be a graph on p vertices, with $p \geq 3$. Then the following are Hilbert polynomials*:

(i) $\sigma(G; x)$;

(ii) $\tau(G; x)$;

(iii) $(-1)^p \chi(G; -(x + 1))$.

Proof: It is easy to verify directly (using Theorem 2.5 and some patience) that the theorem holds if $p = 3$.

We first prove (i) by induction on $p \geq 3$. Assume that $p \geq 4$. If $G = K_p$ (the complete graph on p vertices) then $\sigma(G; x) = x^p$ and (i) holds by Theorem 3.8. If $G \neq K_p$ then it follows from Theorem 1 of [26] that

$$\chi(G; x) = \chi(K_p; x) + \sum_{j=1}^{\binom{p}{2}-|E|} \chi(G_j; x),$$

and (therefore) that

$$\sigma(G; x) = \sigma(K_p; x) + \sum_{j=1}^{\binom{p}{2}-|E|} \sigma(G_j; x),$$

where each G_j has $p - 1$ vertices, and (i) follows from our induction hypothesis and Theorem 3.5.

Similarly, we prove (ii) by induction on $p \geq 3$. If $G = N_p$ (the empty graph on p vertices) then it is easy to see (see, e.g., [33], p. 209, or [9], p. 748) that

$$\tau(N_p; x) = \sum_{i=1}^{p} S(p, i)x^i = \sigma(N_p; x)$$

and the result follows from (i). If $G \neq N_p$ then there follows from repeated application of Proposition 5.1 of [9] that

$$\tau(G; x) = \tau(N_p; x) + \sum_{j=1}^{|E|} \tau(G_j; x)$$

where each G_j has $p - 1$ vertices, and (ii) follows from our induction hypothesis and Theorem 3.5.

Finally, note that (iii) follows immediately from Corollary 3.9 and the well known fact (see, e.g., [26]) that $(-1)^p \chi(G; -x) \in \mathbf{N}[x]$. $\square$

Note that the above proof shows that $(-1)^p \chi(G; -(x + 1))$ is a Hilbert polynomial for any $p \geq 1$.

Regarding the chromatic polynomial itself we have the following result (see also Conjecture 5.3) whose proof is analogous to that of part (i) of Theorem 4.1 and is therefore omitted.

Proposition 4.2 *The following statements are equivalent*:
(i) $\chi(G; x)$ *is a Hilbert polynomial for all graphs G with at least* 4 *vertices*;
(ii) $(x)_p$ *is a Hilbert polynomial for all* $p \geq 4$.

Another connection between chromatic polynomials and Hilbert functions appears in [4].

We now consider Hilbert polynomials arising from the theory of finite partially ordered sets. Let P be a finite poset. Recall (see, e.g., [33], Section 3.11, p. 129) that the *zeta polynomial* of P is the unique polynomial $Z(P; x)$ such that $Z(P; n + 1)$ equals the number of multichains of P of length $n - 1$, for all $n \in \mathbf{P}$ (see, [33], Section 3.11, for further information about zeta polynomials), and that the *order polynomial* of P is the unique polynomial $\Omega(P; x)$ such that $\Omega(P; n)$ equals the number of order preserving maps $\omega : P \to [n]$, for all $n \in \mathbf{P}$ (see [33], Section 4.5, for further information about order polynomials). Given a finite labeled poset (P, ω) (i.e., P is a finite poset, and $\omega : P \to [p]$ is a bijection, where $p \stackrel{\text{def}}{=} |P|$) and a linear extension τ of P (i.e., an order preserving bijection $\tau : P \to [p]$) we let

$$d(\tau, \omega) \stackrel{\text{def}}{=} |\{i \in [p - 1] : \omega(\tau^{-1}(i)) > \omega(\tau^{-1}(i + 1))\}|,$$

and $w_i(P, \omega)$ be the number of linear extensions τ of P such that $d(\tau, \omega) = i - 1$, for $i = 1, \ldots, p$. The sequence $\{w_1(P, \omega), \ldots, w_p(P, \omega)\}$ is one of the fundamental enumerative invariants of the labeled poset (P, ω) and has been studied extensively (see, e.g., [29], and [8]). In particular, it is known (see, e.g., [29], Section 1.2, Definition 3.2, p. 8, and Proposition 8.3, p. 24) that if ω is a linear extension of P then the numbers $w_i(P; \omega)$ do not depend on ω. In this case we write $w_i(P)$ instead of $w_i(P; \omega)$.

Theorem 4.3 *Let P be a finite poset of size p. Then*:

(i) $Z(P; x+1)$ *is a Hilbert polynomial*;

(ii) $\Omega(P; x+1)$ *is a Hilbert polynomial*;

(iii) $(w_1(P), \ldots, w_p(P))$ *is a Hilbert function.*

Proof: It is well known (see, e.g., [33], Proposition 3.11.1, p. 129) that

$$Z(P; x+1) = \sum_{i=0}^{l} b_i \binom{x-1}{i} \tag{17}$$

where b_i is the number of chains of P of length i (i.e., totally ordered subsets of P of cardinality $i+1$), and l is the length of the longest chain of P. But the collection of all chains of P is clearly a simplicial complex (usually denoted $\Delta(P)$ and called the *order complex* of P, see, e.g., [33], p. 120) and its f-vector is $(b_0, b_1, \ldots, b_l)$. Hence (i) follows from (17) and (14). Also, it is well known (see, e.g., [33], Section 3.11, p. 130), and easy to see, that

$$\Omega(P; x) = Z(J(P); x) \tag{18}$$

(where $J(P)$ denotes the lattice of order ideals of P, see, e.g., [33], Section 3.4) and so (ii) follows from (i). To prove (iii) note that using (17) and (18) we conclude that

$$\begin{aligned} 1 + \sum_{n \geq 1} \sum_{i=0}^{p} f_i \binom{n-1}{i} x^n &= \sum_{n \geq 0} Z(J(P); n+1) x^n \\ &= \sum_{n \geq 0} \Omega(P; n+1) x^n \\ &= \frac{\sum_{i=1}^{p} w_i(P) x^{i-1}}{(1-x)^{p+1}} \end{aligned} \tag{19}$$

by a well-known result from the theory of P-partitions (see, e.g., [33], Theorem 4.5.14, p. 219), where f_i is the number of chains of $J(P)$ of length i (i.e., the number of i-dimensional faces of $\Delta(J(P))$). This implies, by (3) and the binomial theorem (see, e.g., [33], p. 16), that $(w_1(P), \ldots, w_p(P))$ is the h-vector of $\Delta(J(P))$. But it is well-known (see, e.g., [33], Section 3.4) that $J(P)$ is always a distributive lattice. This, in turn, implies that $\Delta(J(P))$ is shellable (see, e.g., [12], Theorem 5.1.12, p. 208, and [33], Section 3.3) and (iii) follows from the fact that h-vectors of shellable complexes are O-sequences (see, e.g., Theorem 5.1.15 of [12]). □

Part (ii) of Theorem 4.3 was first proved by Stanley [34] and, as mentioned in the Introduction, was the motivation and origin of the present work. The preceding result suggests the more general question of whether the order polynomial $\Omega(P, \omega; x + 1)$ of a labeled poset (P, ω) (see, e.g., [29], Section 13, p. 45, or [8], p. 1, for definitions) is a Hilbert polynomial. This is easily seen to be false. For example, if $P = [3]$ and $\omega(1) = 3$, $\omega(2) = 2, \omega(3) = 1$, then it is easy to see that $\Omega(P, \omega; x+1) = \binom{x+1}{3}$ which is not a Hilbert polynomial by part (vi) of Theorem 3.8. Nonetheless, we feel that there are general classes of labeled posets for which $\Omega(P, \omega; x + 1)$ is a Hilbert polynomial (see Conjecture 5.7 and the comments following it). Note that, by part (i) of Theorem 3.12 and well-known results on the order polynomial (see, e.g., [8], Theorem 5.7.1, p. 66) part (ii) is a consequence of part (iii) in Theorem 4.3. Therefore, (iii) also fails, in general, if ω is not a linear extension.

Taking appropriate posets P for which the zeta polynomial is known allows us to find explicit classes of Hilbert polynomials. We give one such example here.

Corollary 4.4 *Let $k, m \in \mathbf{P}$. Then $\frac{1}{m}\binom{(kx+1)m}{m-1}$ is a Hilbert polynomial.*

Proof: Let $P_{k,m}$ be the poset of all non-crossing, k-divisible, partitions of $[km]$, ordered by refinement (see, e.g., [33], Chapter 3, Ex. 68.a, p. 169, [8], Section 6.3, p. 73, or [16], for definitions). Then by a result of Edelman (see [16], Corollary 4.4, or [33], loc. cit.) we have that

$$Z(P_{k,m}; x + 1) = \frac{1}{m}\binom{(kx + 1)m}{m - 1},$$

and the thesis follows from Theorem 4.3. $\square$

Note that using Theorems 3.5 and 3.8 one can easily prove that $\binom{(kx+1)m}{m-1}$ is a Hilbert polynomial for all $k, m \in \mathbf{P}$. However, we have been unable to find a similar proof (i.e., avoiding Theorem 4.3) for $\frac{1}{m}\binom{(kx+1)m}{m-1}$.

For our next two applications we assume that the reader is familiar with the basic theory of Coxeter groups as presented, e.g., in Part II of [21]. In particular, given a Coxeter system (W, S) we denote by $l\colon W \to \mathbf{N}$ its length function, and by $\preceq$ the Bruhat order on W. Given $u, v \in W$ we denote by $R_{u,v}(x)$ (respectively, $P_{u,v}(x)$) the R-polynomial (respectively, Kazhdan-Lusztig polynomial) of u and v and we let

$$d(v) \stackrel{\text{def}}{=} |\{s \in S : l(vs) \leq l(v)\}|. \tag{20}$$

We refer the reader to [21], Sections 5.2, 5.9, 7.4, and 7.5 for the definitions of, and further information about, these concepts.

We need first the following simple observation.

Lemma 4.5 *Let $i, j \in \mathbf{N}$, $j \geq 2$. Then $x^i(x + 1)^j$ is a Hilbert polynomial.*

Proof: By Theorems 3.5 and 3.8 it is clearly enough to prove the result for $j = 2$. If $i \geq 3$ then $x^i(x + 1)^2$ is a Hilbert polynomial by Theorems 3.5 and 3.8. On the other hand, one can verify directly (using Theorem 2.5) that $(x+1)^2$, $x(x+1)^2$, and $x^2(x+1)^2$ are all Hilbert polynomials and the result follows. □

Theorem 4.6 *Let (W, S) be a Coxeter system and $u, v \in W$, $u \preceq v$, be such that $l(v) - l(u) \geq 3$. Then $R_{u,v}(x+1)$ is a Hilbert polynomial.*

Proof: If $l(v) - l(u) = 3$ then it is easy to see (see, e.g., [21], Section 7.5) that $R_{u,v}(x)$ equals either $(x-1)^3$ or $(x-1)^3 + (x-1)x$ and one can check that the result holds in this case. So assume that $l(v) - l(u) \geq 4$. It is then well known (see, e.g., [21], Section 7.5, p. 154, or [15], Theorem 1.3) that

$$R_{u,v}(x+1) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} a_i (x+1)^i x^{d-2i} \tag{21}$$

where $d \stackrel{\text{def}}{=} l(v) - l(u)$ and $a_i \in \mathbf{N}$ for $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$. If $i = 1$ then $d - 2i \geq 2$ and hence $(x+1)x^{d-2i}$ is a Hilbert polynomial by Theorems 3.5 and 3.8 and the fact that $(x+1)x^2$ is a Hilbert polynomial. If $i = 0$ then $d - 2i \geq 4$ and hence x^{d-2i} is a Hilbert polynomial by Theorem 3.8. If $i \geq 2$ then $(x+1)^i x^{d-2i}$ is a Hilbert polynomial by Lemma 4.5. Hence the result follows from (21) and Theorem 3.5. □

Given a finite Coxeter system (W, S) we let

$$d_i(W) \stackrel{\text{def}}{=} |\{v \in W : \ d(v) = i\}| \tag{22}$$

for $i \in \mathbf{N}$.

Theorem 4.7 *Let (W, S) be a finite Coxeter system and $u, v \in W$, $u \preceq v$. Then*:
(i) $\sum_{w \in W} x^{d(w)} = \sum_{i \geq 0} d_i(W) x^i$ *is a Hilbert polynomial*;
(ii) $P_{u,v}(x+1)$ *is a Hilbert polynomial*;
(iii) $\{d_0(W), d_1(W), \ldots, d_{|S|}(W)\}$ *is a Hilbert function.*

Proof: Let, for brevity, $P(W; x) \stackrel{\text{def}}{=} \sum_{w \in W} x^{d(w)}$. If $|S| \leq 2$ then either $P(W; x) = 1 + x$ (if $|S| = 1$) or W is a finite dihedral group, in which case $P(W; x) = 1 + (2p-2)x + x^2$ for some $p \geq 2$. In both cases (i) holds by Proposition 3.4. So assume that $|S| \geq 3$. Note that it follows immediately from the definition (22) that

$$d_0(W) = 1, \quad d_1(W) \geq |S|$$

for any (W, S) (since $d(s) = 1$ for all $s \in S$). Furthermore, it is known (see, e.g., [11], Theorem 2.4) that $P(W; x)$ is always a symmetric unimodal polynomial. Hence

$$d_2(W) \geq |S|$$

(since $\deg(P(W; x)) = |S| \geq 3$). Therefore $d_1(W) \geq 3$ and $d_2(W) \geq 3$ and (i) follows from part (i) of Corollary 3.10.

Since it is well known (see [1, 23]) that Kazhdan-Lusztig polynomials of finite Coxeter systems have nonnegative coefficients, (ii) follows immediately from Corollary 3.9.

To prove (iii) let $\Delta(W, S)$ be the Coxeter complex of (W, S) (see, e.g., [5], or [21], Section 1.15, for the definition and further information about the Coxeter complex). By a result of Björner (see [5], Theorems 1.6, 2.1, and Proposition 1.2, or [11], Theorem 2.3) the h-vector of $\Delta(W, S)$ is $(d_0(W), d_1(W), \ldots, d_{|S|}(W))$. But, again by a result of Björner ([5], Theorem 2.1), $\Delta(W, S)$ is a shellable simplicial complex and (iii) follows from Theorem 5.1.15 of [12]. □

Note that it is a long standing conjecture (see, [22], p. 166, and also [21], Section 7.9, p. 159) that Kazhdan-Lusztig polynomials have always nonnegative coefficients. It is therefore natural to conjecture that part (ii) of Theorem 4.7 holds also for infinite Coxeter systems. Given the close connection existing between the R-polynomials and the Kazhdan-Lusztig polynomials of W (see, e.g., [21], Section 7.10, p. 160, Eq. (20)) one can ask whether there is a direct proof of part (ii) of Theorem 4.7 from Theorem 4.6. If this could be found then it would probably yield a proof of our conjecture. Also note that, by the remarks following Theorem 3.12, part (i) is not a consequence of part (iii) in Theorem 4.7.

If (W, S) is a Coxeter system of type A_n then it is well known that $xP(W; x) = A_{n+1}(x)$ and $d_k(W) = A(n+1, k+1)$ for $n \in \mathbf{P}$ and $k \in \mathbf{N}$, where $A_n(x)$ and $A(n, k)$ denote the nth *Eulerian polynomial* and the (n, k)th *Eulerian number*, respectively. Though these objects have been widely studied (see, e.g., [17], [14], Section 6.5, and the references cited therein) the preceding result seems to be new even in this special case.

Corollary 4.8 *Let $n \in \mathbf{P}$. Then $\frac{1}{x}A_n(x)$ is a Hilbert polynomial and $\{A(n, k)\}_{k=1,\ldots,n}$ is a Hilbert function.*

We don't know any combinatorial proof (i.e., avoiding commutative algebra) of the fact that $\{A(n, k)\}_{k=1,\ldots,n}$ is an O-sequence. Note that $A_n(x)$ is not in general a Hilbert polynomial, for example $A_2(x) = \langle x\rangle_2$.

Some of the combinatorial sequences that have been most studied in enumerative combinatorics are those of the Stirling numbers. Strange as this may seem, we have not found in the literature any result relating to the question of whether these sequences are O-sequences. We answer a more general question in the affirmative here. For $n, k, r \in \mathbf{P}$ let $S_r(n, k)$ be the number of set partitions of $[n]$ into k blocks, each of size $\geq r$. The numbers $S_r(n, k)$ are usually called the *r-associated Stirling numbers of the second kind* (see, e.g., [14], Chapter V, Ex. 7, p. 221, for further information about these numbers).

Theorem 4.9 *Let $r, n \in \mathbf{P}$. Then*:

(i) $\{c(n, n-k)\}_{k=0,\ldots,n-1}$ *is a Hilbert function*;

(ii) $\{S_r(n, k)\}_{k=1,\ldots,\lfloor \frac{n}{r} \rfloor}$ *is a Hilbert function.*

Proof: It is clear from the definitions that $1 + ix$ is an O-series for all $i \geq 1$. But it is well-known (see, e.g., [33], Proposition 1.3.4, p. 19) that

$$\sum_{k=0}^{n-1} c(n, n-k)x^k = \prod_{i=1}^{n-1}(1 + ix)$$

so (i) follows from part (i) of Proposition 2.4.

To prove (ii) let $V = \{S \subseteq [n-1] : \; n - r \geq |S| \geq r\}$ and $\Delta \stackrel{\text{def}}{=} \{F \subseteq V : \; S \cap T = \emptyset$ for all $S, T \in F$ such that $S \neq T$, and $\sum_{S \in F} |S| \leq n - r\}$. It is then clear that Δ is a simplicial complex on vertex set V. Also,

$$f_{k-1}(\Delta) = S_r(n, k+1)$$

for all $k = 0, \ldots, \lfloor \frac{n}{r} \rfloor - 1$ (since if $\{S_1, \ldots, S_k\} \in \Delta$ then $\{S_1, \ldots, S_k, [n]\backslash(\bigcup_{i=1}^{k} S_i)\}$ is a partition of $[n]$ into $k + 1$ blocks, each of size $\geq r$, and this is a bijection). Hence $\{S_r(n, k)\}_{k=2,\ldots,\lfloor \frac{n}{r} \rfloor}$ is the f-vector of a simplicial complex and therefore $\{S_r(n, k)\}_{k=1,\ldots,\lfloor \frac{n}{r} \rfloor}$, by Theorem 2.3, is an O-sequence. □

We now prove that a rather general class of polynomials arising from the enumeration of Stirling permutations are always Hilbert polynomials. Fix $k \in \mathbf{P}$ and $m_1, m_2, \ldots \in \mathbf{P}$. Recall (see, e.g., [8], Section 6.6) that a permutation $a_1 a_2 \cdots a_{m_1+\cdots+m_k}$ of the multiset $M_k \stackrel{\text{def}}{=} \{1^{m_1}, 2^{m_2}, \ldots, k^{m_k}\}$ is called a *Stirling permutation* if $1 \leq u < v < w \leq m_1 + \cdots + m_k$ and $a_u = a_w$ imply $a_v \geq a_u$. Stirling permutations have been first introduced and studied in [19] in the case $m_1 = \cdots = m_k = 2$, and later in [18] (in the case $m_1 = \cdots = m_k$) and [8] (in the general case). We denote by Q_k the set of all Stirling permutations of M_k. So, for example, $Q_k = S_k$ if $m_1 = \cdots = m_k = 1$. Given a permutation $\pi = a_1 a_2 \cdots a_{m_1+\cdots+m_k}$ of M_k a *descent* of π is an index $j \in [m_1 + \cdots + m_k - 1]$ such that $a_j > a_{j+1}$. For $0 \leq i \leq |M_k| - 1$ we let $B_{k,i}$ be the number of Stirling permutations of M_k with exactly i descents. So, if $m_1 = \cdots = m_k = 1$, then $B_{k,i}$ is just the Eulerian number $A(k, i+1)$. There are (at least) two important generating functions associated with the numbers $B_{k,i}$, namely

$$B_k(x) \stackrel{\text{def}}{=} \sum_{i=0}^{|M_k|-1} B_{k,i} x^i, \tag{23}$$

and

$$f_k(x) \stackrel{\text{def}}{=} \sum_{i=1}^{|M_k|} B_{k,i-1} \binom{x + |M_k| - i}{|M_k|} \tag{24}$$

(see, [8], Section 6.6, for further information on these two polynomials). As noted in [8], p. 78, (24) is usually the better behaved of these two generating functions. This turns out to be true also from our present point of view. In fact, we will prove that (23) is always a Hilbert polynomial while $\{f_k(n+1)\}_{n \in \mathbf{N}}$ is always a Hilbert function.

Theorem 4.10 *Let $k \in \mathbf{P}$ and $m_1, \ldots, m_k \in \mathbf{P}$. Then*:

(i) $\{f_k(n+1)\}_{n\in\mathbf{N}}$ *is a Hilbert function*;

(ii) $B_k(x)$ *is a Hilbert polynomial.*

In particular, $f_k(x+1)$ *is a Hilbert polynomial.*

Proof: We first prove (i) by induction on $k \in \mathbf{P}$. If $k = 1$ then $B_1(x) = 1$ and hence $f_1(x) = \binom{x+m_1-1}{m_1} = \left(\binom{x}{m_1}\right)$ and the result follows since $\{\left(\binom{n+1}{m_1}\right)\}_{n\in\mathbf{N}}$ is the Hilbert function of $k[x_0, \ldots, x_{m_1}]$. So let $k \in \mathbf{P}$ and assume that $\sum_{n\geq 0} f_k(n+1)x^n$ is an O-series. From Proposition 6.6.1 (see also the proof of Theorem 6.6.2) of [8] we deduce that

$$\sum_{n\geq 0}(n+1) f_k(n+1)x^n = \sum_{n\geq 0}\sum_{i=1}^{|M_{k+1}|} B_{k+1,i-1}\binom{n+1+|M_k|+1-i}{|M_k|+1}x^n$$
$$= \frac{B_{k+1}(x)}{(1-x)^{|M_k|+2}}, \tag{25}$$

by (23) and the binomial theorem (see, e.g., [33], p. 16). Therefore, again by the binomial theorem, we conclude from (23), (24), and (25) that

$$\sum_{n\geq 0} f_{k+1}(n+1)x^n = \frac{B_{k+1}(x)}{(1-x)^{|M_{k+1}|+1}} = \frac{1}{(1-x)^{m_{k+1}-1}}\sum_{n\geq 0}(n+1) f_k(n+1)x^n. \tag{26}$$

Since $\sum_{n\geq 0} x^n$ and $\sum_{n\geq 0}(n+1)x^n$ are both O-series, (i) follows from our induction hypothesis and parts (i) and (iv) of Proposition 2.4.

To prove (ii) note that it follows from [8], Proposition 6.6.1 (and it can also be verified directly), that $B_2(x) = 1 + m_1x$, and

$$B_3(x) = 1 + (3m_1 + m_2)x + m_1(m_1 + m_2 - 1)x^2.$$

Therefore $B_1(x)$ and $B_2(x)$ are Hilbert polynomials, and $B_3(x)$ is a Hilbert polynomial if $m_1 \geq 2$ or if $m_2 \geq 3$ by part (i) of Corollary 3.10. But it can be verified directly, using Proposition 3.4, that $1 + 5x + 2x^2$ and $1 + 4x + x^2$ are also Hilbert polynomials and this proves (ii) if $k \leq 3$. If $k \geq 4$ then we see easily from our definitions that $B_{k,1} \geq 2^{k-1} \geq 8$ and $B_{k,2} \geq 3^{k-3} \geq 3$ (see also Proposition 6.6.1 of [8]) and (ii) again follows from part (i) of Corollary 3.10. □

Note that the proof of Theorem 4.10 actually yields an inductive procedure for constructing a ring R_k having $\{f_k(n+1)\}_{n\in\mathbf{N}}$ as Hilbert function by alternatively taking Segre products and tensor products with polynomial rings on 2 and $m_k - 1$ variables, respectively. It would be interesting to study these rings and the corresponding varieties, and to see whether they can be defined directly in some explicit way. Also, note that, as observed in [8], Section 6.6, p. 79, no combinatorial interpretation of the integers $f_k(n+1)$ is known except in the cases $m_1 = \cdots = m_k = 1$ and $m_1 = \cdots = m_k = 2$. While Theorem 4.10

does not provide such a combinatorial interpretation, it does provide an algebraic interpretation. Furthermore, if the rings R_k referred to above can be constructed explicitly, then a combinatorial interpretation of their Hilbert function would probably follow.

It is well-known (see, e.g., [19]), and also easy to see, that $S(n+k, n)$ is a polynomial function of n, for each $k \in \mathbf{N}$. An interesting consequence of Theorem 4.10 is the following.

Corollary 4.11 *Let $k \in \mathbf{N}$. Then $\{S(n+1+k, n+1)\}_{n\in\mathbf{N}}$ is a Hilbert function. In particular, $S(x+1+k, x+1)$ is a Hilbert polynomial.*

Proof: Taking $m_i = 2$ for all $i \in \mathbf{P}$ yields, by part (ii) of Proposition 6.6.4 of [8], that $f_k(n+1) = S(n+1+k, n+1)$ for all $n \in \mathbf{N}$, and the result follows from part (i) of Theorem 4.10. □

The polynomials $S(x+k, x)$ are usually called *Stirling polynomials* (see, e.g., [19], or [8], Section 6.6, p. 80).

Corollary 4.11 can, in turn, be generalized in another direction using the theory of symmetric functions. We need first the following simple observation.

Proposition 4.12 *Let $f \in \Lambda$. Then there exists a (necessarily unique) polynomial $\bar{f}(x) \in \mathbf{Q}[x]$ such that*

$$\bar{f}(n) = f(1, 2, \ldots, n, 0, 0, \ldots) \tag{27}$$

for all $n \in \mathbf{P}$.

Proof: It is well-known (see, e.g., [25], Chapter I, Section 2, Ex. 11, p. 23), and easy to see, that

$$S(n+k, n) = h_k(1, 2, \ldots, n, 0, 0, \ldots) \tag{28}$$

for all $n \in \mathbf{P}$ and $k \in \mathbf{N}$, and that, as noted before Corollary 4.11, $S(n+k, n)$ is a polynomial function of n for all $k \in \mathbf{N}$. By definition (see, e.g., [25], Chapter I, Section 2) we have that

$$h_\lambda(x_1, x_2, \ldots) = \prod_{i=1}^{l} h_{\lambda_i}(x_1, x_2, \ldots) \tag{29}$$

if $\lambda = (\lambda_1, \ldots, \lambda_l)$, hence the result holds for the complete homogeneous symmetric functions h_λ, $\lambda \in \mathcal{P}$. But every $f \in \Lambda$ can be expressed as a finite linear combination of h_λs, and the result follows. □

Thus Corollary 4.11 is asserting (by (28) and (27)) that $\{h_k(1, 2, \ldots, n+1)\}_{n\in\mathbf{N}}$ is a Hilbert function and $\bar{h}_k(x+1)$ is a Hilbert polynomial. This naturally suggests the problem of determining those symmetric functions $f \in \Lambda$ for which these properties hold.

Theorem 4.13 *Let $\lambda \in \mathcal{P}$. Then the following are Hilbert functions*:
(i) $\{h_\lambda(1, 2, \ldots, n+1)\}_{n\in\mathbf{N}}$;
(ii) $\{p_\lambda(1, 2, \ldots, n+1)\}_{n\in\mathbf{N}}$.
In particular, *both $\bar{h}_\lambda(x+1)$ and $\bar{p}_\lambda(x+1)$ are Hilbert polynomials.*

Proof: (i) Follows immediately from (28), (29), Corollary 4.11, and Proposition 2.4.

To prove (ii) note that it follows from the definition (see, e.g., [25], Chapter I, Section 2) that

$$\sum_{n\geq 0} \bar{p}_k(n+1)x^n = \left(\sum_{n\geq 0} x^n\right)\left(\sum_{n\geq 0}(n+1)^k x^n\right) \tag{30}$$

for all $k \in \mathbf{P}$. Since $\sum_{n\geq 0} x^n$ and $\sum_{n\geq 0}(n+1)x^n$ are both Hilbert series this shows, by Proposition 2.4, that $\{\bar{p}_k(n+1)\}_{n\in\mathbf{N}}$ is a Hilbert function for all $k \in \mathbf{P}$. Since, by definition (see, e.g., [25], Chapter I, Section 2), $p_\lambda(x_1, x_2, \ldots) = \prod_{i=1}^{l} p_{\lambda_i}(x_1, x_2, \ldots)$ if $\lambda = (\lambda_1, \ldots, \lambda_l)$, (ii) follows from Proposition 2.4. □

Given a $\mathbf{Z}$-basis $\{b_\lambda\}_{\lambda\in\mathcal{P}}$ of Λ we say that $f \in \Lambda$ is *b-positive* if $f = \sum_{\lambda\in\mathcal{P}} a_\lambda b_\lambda$ implies that $a_\lambda \geq 0$ for all $\lambda \in \mathcal{P}$. Then Theorem 4.13 has the following immediate consequence.

Corollary 4.14 *Let $f \in \Lambda$ be h-positive. Then $\bar{f}(x+1)$ is a Hilbert polynomial.*

Another consequence of Theorem 4.13 is the following. We denote by $\mathcal{B}_k(x)$ the kth Bernoulli polynomial, for $k \in \mathbf{P}$ (see, e.g., [14], Chapter I, Section 14, p. 48, for the definition and further information about Bernoulli polynomials).

Corollary 4.15 *Let $k \in \mathbf{P}$. Then $\mathcal{B}_k(x+2) - \mathcal{B}_k(0)$ is a Hilbert polynomial.*

Proof: It is well known (see, e.g., [14], Section 3.9, p. 155) that $(k+1)\bar{p}_k(n+1) = \mathcal{B}_{k+1}(n+2) - \mathcal{B}_{k+1}(0)$ for all $k, n \in \mathbf{P}$. Hence

$$\mathcal{B}_{k+1}(x+2) - \mathcal{B}_{k+1}(0) = (k+1)\bar{p}_k(x+1) \tag{31}$$

and the result follows from Theorems 3.5 and 4.13 and the fact that $\mathcal{B}_1(x+2) - \mathcal{B}_1(0) = x+2$ is a Hilbert polynomial. □

Note that $\mathcal{B}_k(x+2)$ cannot be a Hilbert polynomial since, in general, $\mathcal{B}_k(\mathbf{N}) \not\subseteq \mathbf{Z}$, and that $\mathcal{B}_k(x+1) - \mathcal{B}_k(0)$ is not always a Hilbert polynomial (for example, $\mathcal{B}_2(x+1) - \mathcal{B}_2(0) = x^2 + x$).

5. Open problems

Despite the fact that Hilbert functions and polynomials are preserved by many natural operations on formal power series and polynomials, respectively (see Proposition 2.4 and

Theorem 3.5), there are many sequences and polynomials naturally arising in enumerative and algebraic combinatorics for which we have been unable to decide whether they are Hilbert. In this section we survey the most striking such cases, and we present some conjectures together with the evidence we have in their favor.

Our first conjecture is naturally suggested by Theorem 4.9.

Conjecture 5.1 *Let $n \in \mathbf{P}$. Then $\{S(n, n-k)\}_{k=0,\ldots,n-1}$ is a Hilbert function.*

We have verified this conjecture for $n \leq 24$. In addition to the numerical evidence, there is a heuristic reasoning that suggests the validity of Conjecture 5.1. A sequence of positive integers is a Hilbert function if it "does not grow too fast". Now, it is well-known (see, e.g., [14], Section 7.1, Theorem D, p. 271) that the sequence $\{S(n,k)\}_{k=1,\ldots,n}$ is log-concave and unimodal, hence the real content of Conjecture 5.1 is for the values of k that precede the mode of the sequence. But it is known (see, e.g., [33], Chapter 1, Exercise 18, p. 47) that the mode of $\{S(n,k)\}_{k=1,\ldots,n}$ is less than $\lfloor \frac{n}{2} \rfloor$. Hence one expects the sequence $\{S(n, n-k)\}_{k=0,\ldots,n-1}$ to grow "less rapidly" than $\{S(n,k)\}_{k=1,\ldots,n}$ and therefore we expect Conjecture 5.1 to be true since Theorem 4.9 holds.

Theorem 3.8 allows one to settle the question of whether a given polynomial is Hilbert pretty easily if its coefficients with respect to the basis $\{x^i\}_{i \in \mathbf{N}}$ are nonnegative and have a combinatorial interpretation. However, there are many polynomials for which this is not the case (especially polynomials that "count something" when evaluated at nonnegative integers) but that seem to be Hilbert. In this respect, we feel that the following is the most interesting and outstanding open problem arising from the present work.

Conjecture 5.2 *Let G be a graph on at least 4 vertices, and $\chi(G; x)$ be its chromatic polynomial. Then $\chi(G; x)$ is a Hilbert polynomial.*

We have verified the above conjecture for all graphs with at most 15 vertices. Two related conjectures are the following:

Conjecture 5.3 *Let $d \in \mathbf{P}$, $d \geq 4$. Then $(x)_d$ is a Hilbert polynomial.*

Conjecture 5.4 *Let $d \in \mathbf{P}$. Then $3d\binom{x}{d}$ is a Hilbert polynomial.*

We have verified these conjectures for $d \leq 15$. Note that, by Proposition 4.2, Conjectures 5.2 and 5.3 are equivalent, while by Theorem 3.5 Conjecture 5.4 implies Conjecture 5.3.

For what concerns the symmetric functions $f \in \Lambda$ such that $\bar{f}(x+1)$ is a Hilbert polynomial we have the following conjectures.

Conjecture 5.5 *Let $\lambda \in \mathcal{P}$. Then $\bar{s}_\lambda(x+1)$ is a Hilbert polynomial if and only if $|\lambda| \geq 3$.*

Conjecture 5.6 *Let $\lambda \in \mathcal{P}$. Then $\bar{m}_\lambda(x+1)$ is a Hilbert polynomial if and only if $|\lambda| \geq 3$.*

We have verified the above conjectures for $|\lambda| \leq 7$. Note that since any Schur symmetric function is m-positive (see, e.g., [25], Chapter I, Section 6), Conjecture 5.6 implies Conjecture 5.5 as well as Corollary 4.14.

While the specialization $f \mapsto f(1, 2, \ldots, n+1, 0, 0, \ldots)$, for $f \in \Lambda$, has been suggested by Corollary 4.11, there is one other specialization which is routinely used in the theory of symmetric functions (see, e.g., [25], Chapter I, Section 2, Ex. 1, and Section 3, Ex. 4) namely $f \mapsto f(\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots)$. Since it is easy to verify that $f(\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots)$ is a polynomial function of n for any $f \in \Lambda$ (see, e.g., [8], Proposition 6.2.1), it is natural to wonder for which symmetric functions $f \in \Lambda$ one has that $f(\underbrace{1, 1, \ldots, 1}_{n+1}, 0, 0, \ldots)$ is a Hilbert polynomial (as a function of n). The answer for the p_λs is of course trivial. For the Schur functions we believe that the following holds.

Conjecture 5.7 *Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}$ be such that $|\lambda| \geq 7$. Then $s_\lambda(\underbrace{1, 1, \ldots, 1}_{n+1}, 0, 0, \ldots)$ is a Hilbert polynomial if and only if $\lambda_1 + \lambda_2 \geq 4$.*

We have verified Conjecture 5.7 for $|\lambda| \leq 12$. It is worth noting that the "only if" part of Conjecture 5.7 also holds for $4 \leq |\lambda| \leq 6$ (but not the "if" part, take, e.g., $\lambda = (3, 3)$, $(2, 2, 2)$, $(2, 2, 1)$, or $(3, 1)$). What makes Conjecture 5.7 particularly tantalizing is that there is an explicit closed formula for $s_\lambda(\underbrace{1, \ldots, 1}_{n+1}, 0, 0, \ldots)$, namely

$$s_\lambda(\underbrace{1, \ldots, 1}_{n+1}, 0, 0, \ldots) = \prod_{(i,j)\in\lambda} \left(\frac{n + 1 + j - i}{h(i, j)} \right), \tag{32}$$

where $h(i, j)$ is the *hook length* of (i, j) in λ (see, e.g., [25], Chapter I, Section 1, Ex. 1, and Section 3, Ex. 4). This allows us to conclude in particular, by part (vi) of Theorem 3.8, that Conjecture 5.7 does hold if $l(\lambda) = 1$ (i.e., for complete homogeneous symmetric functions) or if $\lambda_1 = 1$ (i.e., for elementary symmetric functions). Note that $s_\lambda(\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots)$ is also equal to the order polynomial of a column strict labeled Ferrers poset of shape λ (see, e.g., [8], Section 5.2, for further information). The monomial symmetric functions seem to exhibit an extremely mysterious behavior and we have been unable to extract any general conjecture from the data that we have. For example, if $|\lambda| \leq 8$ and $l(\lambda) \geq 2$ then we have verified that $m_\lambda(\underbrace{1, \ldots, 1}_{n+1}, 0, 0, \ldots)$ is a Hilbert polynomial if and only if $\lambda \in \{(3, 2, 1), (4, 2, 1), (3, 2, 1, 1), (2, 2, 1, 1, 1, 1), (3, 2, 1, 1, 1), (3, 2, 2, 1), (4, 2, 1, 1), (4, 3, 1), (5, 2, 1)\}$.

The results in Section 3 also suggest several open problems. In particular, it would be interesting to answer the following questions, which are naturally suggested by Theorems 3.12 and 3.13.

Problem 5.8 *Let $\{h_0, \ldots, h_d\}$ be a Hilbert function. Is it true that then $\sum_{i=0}^{d} h_i (\binom{x}{i})$ is a Hilbert polynomial*?

Problem 5.9 *Let $(f_0, \ldots, f_d)$ be the f-vector of a simplicial complex. Is it true that then $\sum_{i=0}^{d} f_i (x)_i$, $\sum_{i=0}^{d} f_i \binom{x+d-i}{d}$, and $\sum_{i=0}^{d} f_i (\binom{x}{i})$ are Hilbert polynomials*?

Finally, there is a general "open problem" that arises naturally with almost any result presented in this work. Namely, whenever we prove that a certain polynomial (or sequence) is Hilbert it is natural to ask whether one can construct, in a natural way, a standard graded k-algebra having the given Hilbert polynomial or function. Besides giving a more illuminating proof of the original result, such a graded algebra would probably have interesting properties in its own right. We have not investigated this problem. However, we do believe that natural constructions of graded algebras exist that "explain" all parts of Theorems 3.5, 3.12, 3.13, and 4.7.

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