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On the Number of Group-Weighted Matchings

JEFF KAHN*
Dept. of Mathematics and RUTCOR, Rutgers University, New Brunswick, NJ 08903

ROY MESHULAM
Dept. of Mathematics, Technion, Haifa 32000, Israel

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Abstract. Let G be a bipartite graph with a bicoloration $\{A, B\}$, $|A| = |B|$, and let $w : E(G) \to \mathbf{K}$ where $\mathbf{K}$ is a finite abelian group with k elements. For a subset $S \subseteq E(G)$ let $w(S) = \prod_{e \in S} w(e)$. A perfect matching $M \subseteq E(G)$ is a w-*matching* if $w(M) = 1$.

It is shown that if $\deg(a) \geq d$ for all $a \in A$, then either G has no w-matchings, or G has at least $(d - k + 1)!$ w-matchings.

Keywords: bipartite matching, digraph, finite abelian group, group algebra, Olson's Theorem

1. Introduction

Let G be a bipartite graph with a bicoloration $\{A, B\}$, $|A| = |B|$. Let $E(G) \subseteq A \times B$ denote the edge set of G, and let $m(G)$ denote the number of perfect matchings of G.

Let $\mathbf{K}$ be a (multiplicative) finite abelian group $|\mathbf{K}| = k$, and let $w : E(G) \to \mathbf{K}$ be a weight assignment on the edges of G. For $S \subseteq E(G)$ let $w(S) = \prod_{e \in S} w(e)$.

A perfect matching M of G is a w-*matching* if $w(M) = 1$. We shall be interested in $m(G, w)$, the number of w-matchings of G.

M. Hall (see exercise 7.15 in [9]) showed that if $m(G) \geq 1$ and if $\deg(a) \geq d$ for all $a \in A$, then $m(G) \geq d!$.

Hall's result is the case $k = 1$ of the following Theorem. The case $k = 2$ was proved in [1].

Theorem 1.1 *Let $w : E(G) \to \mathbf{K}$. If $m(G, w) \geq 1$ and $\deg(a) \geq d$ for all $a \in A$, then $m(G, w) \geq (d - k + 1)!$.*

Theorem 1.1 is tight when $\mathbf{K} = \mathbf{C}_k$, the cyclic group of order k. More generally, for a finite abelian group $\mathbf{K}$, let $s = s(\mathbf{K})$ denote the maximal s for which there exists a sequence $x_1, \ldots, x_s \in \mathbf{K}$ such that $\prod_{i \in I} x_i \neq 1$ for all $\emptyset \neq I \subseteq [s] = \{1, \ldots, s\}$. The problem of determining $s(\mathbf{K})$ was suggested by Davenport (see [11]) and addressed by a number of authors [3, 5, 10, 11].

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Let $d \geq s = s(\mathbf{K})$ and let $G = K_{d,d}$ denote the complete bipartite graph on $[d] \times [d]$. Let $w : E(K_{d,d}) \to \mathbf{K}$ be given by $w(i, j) = 1$ if $i = j$ or $i \geq s + 1$, and $w(i, j) = x_i$ otherwise, where $x_1, \ldots, x_s$ are as above. Then $m(G, w) = (d - s)!$.

This construction suggests the following conjecture (which contains Theorem 1.1 since it's easy to see that $s(\mathbf{K}) \leq k - 1$).

Conjecture 1.2 ([1]) *Let $w : E(G) \to \mathbf{K}$. If $m(G, w) \geq 1$ and $\deg(a) \geq d$ for all $a \in A$, then $m(G, w) \geq (d - s(\mathbf{K}))!$.*

In Section 2 we utilize the complex group algebra of $\mathbf{K}$ to prove Theorem 1.1. In Section 3 we discuss a possible approach to Conjecture 1.2 when $\mathbf{K}$ is a p-group (the nice point here is the reduction to Conjecture 3.2 via Claim 3.3), and observe a connection with a conjecture of Griggs and Walker [8].

2. Proof of Theorem 1.1

The main ingredient of the proof of Theorem 1.1 is the following result on group-weighted digraphs.

Theorem 2.1 *Let $D = (V, E)$ be a simple digraph and let $\varphi : E \to \mathbf{K}$. If $\deg^+(v) \geq k$ for all $v \in V$, then there exist vertex-disjoint directed cycles $C_1, \ldots, C_l$ such that $\prod_{i=1}^{l} \prod_{e \in C_i} \varphi(e) = 1$.*

Proof: It is, of course, enough to prove the Theorem when $\deg^+(v) = k$ for all v. Let $\mathbb{C}[\mathbf{K}]$ denote the complex group algebra of $\mathbf{K}$, $M_{n\times k}(\mathbb{C}[\mathbf{K}])$ the $n \times k$ matrices over $\mathbb{C}[\mathbf{K}]$ and $M_n(\mathbb{C}[\mathbf{K}]) = M_{n\times n}(\mathbb{C}[\mathbf{K}])$. (For group algebras see e.g., [12].) For a matrix $Q = (q_{ij}) \in M_n(\mathbb{C}[\mathbf{K}])$ and $x \in \mathbb{C}[\mathbf{K}]$, let $S(Q, x) = \{\sigma \in S_n : \prod_{i=1}^{n} q_{i\sigma(i)} = x\}$ (where S_n is the symmetric group).

Assume now that $V = [n]$ and associate with D the matrix $Q = (q_{ij}) \in M_n(\mathbb{C}[\mathbf{K}])$ given by $q_{ii} = 1$ for all $1 \leq i \leq n$, $q_{ij} = \varphi(i, j)$ if $(i, j) \in E$, and $q_{ij} = 0$ otherwise.

Claim 2.2 *There exists a matrix $C = (c_{ij}) \in M_n(\mathbb{C})$ with $c_{ii} = 1$ for all $1 \leq i \leq n$, such that the matrix $R = (r_{ij}) \in M_n(\mathbb{C}[\mathbf{K}])$ given by $r_{ij} = c_{ij} q_{ij}$ satisfies $\det R = 0$ in $\mathbb{C}[\mathbf{K}]$.*

Proof: Let $\chi_1, \ldots, \chi_k$ denote the complex characters of $\mathbf{K}$. For $1 \leq i \leq n$ let $N^+(i) = \{j : (i, j) \in E\}$.

With any $n \times k$ matrix $H = (h_{jl}) \in M_{n\times k}(\mathbb{C})$ and $1 \leq i \leq n$, we associate a $k \times k$ matrix H_i indexed by $N^+(i) \times [k]$ and given by $H_i(j, l) = \chi_l(q_{ij}) h_{jl}$ for $j \in N^+(i)$, $1 \leq l \leq k$.

We may clearly choose an $H \in M_{n\times k}(\mathbb{C})$ such that rank $H_i = k$ for all $1 \leq i \leq n$. The non-singularity of H_i implies that there exists a vector $(c_{ij} : j \in N^+(i))$ such that for all $1 \leq l \leq k$

$$h_{il} = -\sum_{j\in N^+(i)} c_{ij} H_i(j, l) = -\sum_{j\in N^+(i)} c_{ij} \chi_l(q_{ij}) h_{jl}. \tag{1}$$

Now define $c_{ii} = 1$ and $c_{ij} = 0$ if $i \neq j$ and $(i, j) \notin E$, and let $R = (r_{ij}) = (c_{ij}q_{ij}) \in M_n(\mathbb{C}[\mathbf{K}])$.

Then (1) implies that for all $1 \leq l \leq k$, $0 \neq (h_{1l}, \ldots, h_{nl})^T$ is a nullvector of the matrix $\chi_l(R) = (\chi_l(r_{ij})) \in M_n(\mathbb{C})$. It follows that $\chi_l(\det R) = \det(\chi_l(R)) = 0$ for all $1 \leq l \leq k$, hence $\det R = 0$ in $\mathbb{C}[\mathbf{K}]$. □

Claim 2.2 implies that

$$0 = \det R = \sum_{x \in \mathbf{K}} \left(\sum_{\sigma \in S(Q,x)} \mathrm{Sg}(\sigma) \prod_{i=1}^{n} c_{i\sigma(i)} \right) x.$$

Since the identity permutation id belongs to $S(Q, 1)$ and $\prod_{i=1}^{n} c_{ii} = 1$, it follows that there exists $\mathrm{id} \neq \sigma \in S(Q, 1)$. Then $E_0 := \{(i, \sigma(i)) : i \neq \sigma(i)\}$ is a non-empty union of vertex-disjoint directed cycles, say $E_0 = \bigcup_{i=1}^{l} C_i$, and

$$\prod_{i=1}^{l} \prod_{e \in C_i} \varphi(e) = \prod q_{i\sigma(i)} = 1. \qquad \square$$

Theorem 1.1 now follows from Theorem 2.1 as in [1]: Let G be a bipartite graph on $\{A, B\}$, $|A| = |B| = n$ and $w : E(G) \to \mathbf{K}$. For $a \in A$ let $U_G(a, w)$ denote the set of all edges incident with a which participate in w-matchings of G, and $|U_G(a, w)| = u_G(a, w)$.

The following result clearly implies Theorem 1.1 by induction on d.

Theorem 2.2 *If G has a w-matching then there exists an $a \in A$ such that $u_G(a, w) \geq \deg_G(a) - k + 1$.*

Proof: Let $M = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ be a w-matching of G. With no loss of generality we may assume that $w(a_i, b_i) = 1$ for all i. (Otherwise for each i and $e \ni a_i$, multiply $w(e)$ by $w^{-1}(a_i, b_i)$.)

Construct a directed graph D on $\{1, \ldots, n\}$ by taking $(i, j) \in E(D)$ iff $(a_i, b_j) \in E(G) \backslash U_G(a_i, w)$, and let $\varphi : E(D) \to \mathbf{K}$ be defined by $\varphi(i, j) = w(a_i, b_j)$. Suppose for a contradiction that the assertion of the theorem is false, so that $\deg^+(v) \geq k$ for all $v \in V(D)$. It then follows from Theorem 2.1 that D contains vertex-disjoint cycles $C_1, \ldots, C_l$ with $\prod_{i=1}^{l} \prod_{e \in C_i} \varphi(e) = 1$. Let $V_0 = \bigcup_{i=1}^{l} V(C_i)$ and define a permutation σ on V_0 by $\sigma(v_1) = v_2$ if $(v_1, v_2) \in \bigcup_{i=1}^{l} E(C_i)$. Then the perfect matching

$$M' = \{(a_i, b_i) : i \notin V_0\} \cup \{(a_i, b_{\sigma(i)}) : i \in V_0\}$$

is a w-matching. So, since $(a_i, b_{\sigma(i)}) \notin U_G(a_i, w)$ for any $i \in V_0$, we have the desired contradiction. □

3. An approach to the *p*-group case

Let $\mathbf{K} = \mathbf{C}_{p^{e_1}} \times \cdots \times \mathbf{C}_{p^{e_t}}$ be an abelian p-group, and let $\mathbb{Z}_p[\mathbf{K}]$ denote the group algebra of $\mathbf{K}$ over $\mathbb{Z}_p$. Let $I_p(\mathbf{K}) = \{\sum_{x \in \mathbf{K}} a_x x \in \mathbb{Z}_p[\mathbf{K}] : \sum_{x \in \mathbf{K}} a_x = 0\}$, the *augmentation ideal* of $\mathbb{Z}_p[\mathbf{K}]$. It was shown by Olson [11], and independently Kruyswijk (see [2]), that $I_p(\mathbf{K})$ is nilpotent of degree $\sum_{i=1}^{t}(p^{e_i}-1)+1$, and that this implies $s(\mathbf{K})=\sum_{i=1}^{t}(p^{e_i}-1)$.

For $S \subseteq \mathbb{Z}_p[\mathbf{K}]$ let $M_n(S)$ denote the set of $n \times n$ matrices with entries in S. For $l \leq n-1$ let $U_{\mathbf{K}}(n,l)$ denote the set of matrices $Q=(q_{ij}) \in M_n(\mathbf{K} \cup \{0\})$ such that for each $i \in [n]$, $q_{ii}=1$ and $Q(i) := \{j \neq i : q_{ij} \neq 0\}$ has cardinality l.

By the proof of Theorem 1.1, Conjecture 1.2 for the p-group $\mathbf{K}$ will follow from the following analogue of Claim 2.2.

Conjecture 3.1 *For any $Q=(q_{ij}) \in U_{\mathbf{K}}(n, s(\mathbf{K})+1)$ there exists a matrix $C=(c_{ij}) \in M_n(\mathbb{Z}_p)$ with $c_{ii}=1$ for $1 \leq i \leq n$, such that $R=(r_{ij})=(c_{ij}q_{ij}) \in M_n(\mathbb{Z}_p[\mathbf{K}])$ satisfies* $\det R = 0$ *in* $\mathbb{Z}_p[\mathbf{K}]$.

We next formulate another, perhaps more natural, conjecture which implies Conjecture 3.1.

Let $\mathcal{A}=(A_1,\ldots,A_n)$ be an ordered family of subsets of $[n]$ such that $i \notin A_i$ for all $1 \leq i \leq n$. Let $W_p(\mathcal{A})$ denote the affine space of all matrices $C=(c_{ij}) \in M_n(\mathbb{Z}_p)$ such that $c_{ii}=1$, and $c_{ij}=0$ whenever $i \neq j$ and $j \notin A_i$.

Conjecture 3.2 *If $|A_i| \geq l$ for all $1 \leq i \leq n$, then $W_p(\mathcal{A})$ contains a matrix of rank at most $n-l$.*

To show that Conjecture 3.2 implies Conjecture 3.1 we need

Claim 3.3 *Let $Q=(q_{ij}) \in M_n(\mathbf{K})$ and $C=(c_{ij}) \in M_n(\mathbb{Z}_p)$. If* $\operatorname{rank} C \leq n-s(\mathbf{K})-1$, *then $R=(r_{ij})=(c_{ij}q_{ij}) \in M_n(\mathbb{Z}_p[\mathbf{K}])$ satisfies* $\det R = 0$.

Proof: Let $B \in M_n(\mathbb{Z}_p)$ be a non-singular matrix such that the first $s(\mathbf{K})+1$ rows of BC are zero. Then with $BR=(t_{ij}) \in M_n(\mathbb{Z}_p[\mathbf{K}])$, it follows that $t_{ij} \in I_p(\mathbf{K})$ for all $1 \leq i \leq s(\mathbf{K})+1$, $1 \leq j \leq n$. Since $I_p(\mathbf{K})$ is nilpotent of degree $s(\mathbf{K})+1$ it follows that $\det BR = \sum_{\sigma \in S_n} \operatorname{Sg}\sigma \prod_{i=1}^{n} t_{i\sigma(i)} = 0$, hence $\det R = 0$. □

Conjecture 3.2 ⇒ Conjecture 3.1 Suppose $Q=(q_{ij}) \in U_{\mathbf{K}}(n, s(\mathbf{K})+1)$, and let $\mathcal{A}=(Q(1),\ldots,Q(n))$. By Conjecture 3.2 there exists a $C \in W_p(\mathcal{A})$ such that $\operatorname{rank} C \leq n-s(\mathbf{K})-1$. Let $Q'=(q'_{ij}) \in M_n(\mathbf{K})$ be given by $q'_{ij}=q_{ij}$ if $q_{ij} \in \mathbf{K}$ and q'_{ij} an arbitrary element of $\mathbf{K}$ otherwise. Clearly $R' := (c_{ij}q'_{ij}) = (c_{ij}q_{ij}) = R$, therefore by Claim 3.3 $\det R = \det R' = 0$. □

Remarks.

1. The cases $l=1, n-1$ of Conjecture 3.2 are trivial. We have proved the cases $l=2, n-2$ by a graph theoretical argument similier to Proposition 4.1 in [1].
2. Again let $\mathcal{A}=(A_1,\ldots,A_n)$ with $i \notin A_i$ for all i. It is easy to see that the existence of a matrix of rank at most $n-l$ in $W_p(\mathcal{A})$ is equivalent to the existence of vectors $v_i \in \mathbb{Z}_p^l$, $i \in [n]$, satisfying

   (a) $v_i \in \langle v_j : j \in A_i \rangle$ for all i, and
   (b) $\langle v_i : i \in [n] \rangle = \mathbb{Z}_p^l$ (where $\langle\cdot\rangle$ is linear span).

(The $n \times l$ matrix H whose rows are the v_i's will then satisfy $MH = 0$ for an appropriate $M \in W_p(\mathcal{A})$.) So Conjecture 3.2 is equivalent to the statement that such v_i exist whenever $|A_i| = l$ for all i.

For an l-uniform hypergraph $\mathcal{F} = \{F_1, \ldots, F_m\}$ on $[n]$, say $\mathcal{F}$ has *property* G_p if there exists a matrix $H \in M_{n\times l}(\mathbb{Z}_p)$ such that the $l \times l$ minors of H corresponding to the F_i's are all non-singular. If $\mathcal{A}$ as above is l-uniform and has property G_p with corresponding H, then the rows of H satisfy (a), (b), giving Conjecture 3.2 for $\mathcal{A}$. (For example, since property G_p clearly does hold for any fixed n and large enough p, the same follows for Conjecture 3.2.)

Of course, we cannot expect (and do not need) property G_p in general; still, sufficient conditions for the property are of interest. A conjecture of Griggs and Walker [8] says that for any n and $A \subset \mathbb{Z}_n$, the hypergraph $\{A + i : i \in \mathbb{Z}_n\}$ has property G_2. (This was motivated by, and would imply, a conjecture proposed in [4, 6].) For $|A| = 3$, the Griggs-Walker Conjecture was proved by Füredi et al. [7], who actually showed property G_2 for an arbitrary 3-uniform, 3-regular $\mathcal{F}$.

For general l, such a generalization does not hold, but we believe the following, less extreme relaxation may be correct.

Conjecture 3.4 *If $\mathcal{F}$ is l-uniform on $[n]$ and for each t and distinct $F_1, \ldots, F_t \in \mathcal{F}$,*

$$|F_1 \cap \cdots \cap F_t| \le \max\{k - t + 1, 0\},$$

and

$$|F_1 \cup \cdots \cup F_t| \ge \min\{k + t - 1, n\},$$

then $\mathcal{F}$ has property G_p.

This contains (via the Cauchy-Davenport Theorem) the Griggs-Walker conjecture when n is prime, and would presumably shed some light on the general case as well.

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