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The Enumeration of Fully Commutative Elements of Coxeter Groups

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Abstract. A Coxeter group element w is fully commutative if any reduced expression for w can be obtained from any other via the interchange of commuting generators. For example, in the symmetric group of degree n, the number of fully commutative elements is the nth Catalan number. The Coxeter groups with finitely many fully commutative elements can be arranged into seven infinite families A_n, B_n, D_n, E_n, F_n, H_n and $I_2(m)$. For each family, we provide explicit generating functions for the number of fully commutative elements and the number of fully commutative involutions; in each case, the generating function is algebraic.

Keywords: Coxeter group, reduced word, braid relation

0. Introduction

A Coxeter group element w is said to be fully commutative if any reduced word for w can be obtained from any other via the interchange of commuting generators. (More explicit definitions will be given in Section 1 below.)

For example, in the symmetric group of degree n, the fully commutative elements are the permutations with no decreasing subsequence of length 3, and they index a basis for the Temperley-Lieb algebra. The number of such permutations is the nth Catalan number.

In [9], we classified the irreducible Coxeter groups with finitely many fully commutative elements. The result is seven infinite families of such groups; namely, A_n, B_n, D_n, E_n, F_n, H_n and $I_2(m)$. An equivalent classification was obtained independently by Graham [7], and in the simply-laced case by Fan [4]. In this paper, we consider the problem of enumerating the fully commutative elements of these groups. The main result (Theorem 2.6) is that for six of the seven infinite families (we omit the trivial dihedral family $I_2(m)$), the generating function for the number of fully commutative elements can be expressed in terms of three simpler generating functions for certain formal languages over an alphabet with at most four letters. The languages in question vary from family to family, but have a uniform description. The resulting generating function one obtains for each family is algebraic, although in some cases quite complicated (see (3.7) and (3.11)).

In a general Coxeter group, the fully commutative elements index a basis for a natural quotient of the corresponding Iwahori-Hecke algebra [7]. (See also [4] for the simply-laced

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case.) For A_n, this quotient is the Temperley-Lieb algebra. Recently, Fan [5] has shown that for types A, B, D, E and (in a sketched proof) F, this quotient is generically semisimple, and gives recurrences for the dimensions of the irreducible representations. (For type H, the question of semisimplicity remains open.) This provides another possible approach to computing the number of fully commutative elements in these cases; namely, as the sum of the squares of the dimensions of these representations. Interestingly, Fan also shows that the sum of these dimensions is the number of fully commutative involutions.

With the above motivation in mind, in Section 4 we consider the problem of enumerating fully commutative involutions. In this case, we show (Theorem 4.3) that for the six nontrivial families, the generating function can be expressed in terms of the generating functions for the palindromic members of the formal languages that occur in the unrestricted case. Again, each generating function is algebraic, and in some cases, the explicit form is quite complicated (see (4.8) and (4.10)).

In Section 5, we provide asymptotic formulas for both the number of fully commutative elements and the number of fully commutative involutions in each family. In the Appendix we provide tables of these numbers up through rank 12.

1. Full commutativity

Throughout this paper, W shall denote a Coxeter group with (finite) generating set S, Coxeter graph Γ, and Coxeter matrix $M = [m(s,t)]_{s,t\in S}$. A standard reference is [8].

1.1. Words

For any alphabet A, we use A^* to denote the free monoid consisting of all finite-length words $\mathbf{a} = (a_1, \ldots, a_l)$ such that $a_i \in A$. The multiplication in A^* is concatenation, and on occasion will be denoted '$\cdot$'. Thus $(a,b)(b,a) = (a,b)\cdot(b,a) = (a,b,b,a)$. A subsequence of **a** obtained by selecting terms from a set of consecutive positions is said to be a *subword* or *factor* of **a**.

For each $w \in W$, we define $\mathcal{R}(w) \subset S^*$ to be the set of reduced expressions for w; i.e., the set of minimum-length words $\mathbf{s} = (s_1, \ldots, s_l) \in S^*$ such that $w = s_1 \cdots s_l$.

For each integer $m \geq 0$ and $s, t \in S$, we define

$$\langle s,t\rangle_m = \underbrace{(s,t,s,t,\ldots)}_{m},$$

and let $\approx$ denote the congruence on S^* generated by the so-called *braid relations*; namely,

$$\langle s,t\rangle_{m(s,t)} \approx \langle t,s\rangle_{m(s,t)}$$

for all $s, t \in S$ such that $m(s,t) < \infty$.

It is well-known that for each $w \in W$, $\mathcal{R}(w)$ consists of a single equivalence class relative to $\approx$. That is, any reduced word for w can be obtained from any other by means of a sequence of braid relations [2, Section IV.1.5].

1.2. *Commutativity classes*

Let $\sim$ denote the congruence on S^* generated by the interchange of commuting generators; i.e., $(s, t) \sim (t, s)$ for all $s, t \in S$ such that $m(s, t) = 2$. The equivalence classes of this congruence will be referred to as *commutativity classes*.

Given $\mathbf{s} = (s_1, \ldots, s_l) \in S^*$, the *heap* of $\mathbf{s}$ is the partial order of $\{1, 2, \ldots, l\}$ obtained from the transitive closure of the relations $i \prec j$ for all $i < j$ such that $s_i = s_j$ or $m(s_i, s_j) \geq 3$. It is easy to see that the isomorphism class of the heap is an invariant of the commutativity class of $\mathbf{s}$. In fact, although it is not needed here, it can be shown that $\mathbf{s} \sim \mathbf{t} = (t_1, \ldots, t_l)$ if and only if there is an isomorphism $i \mapsto i'$ of the corresponding heap orderings with $s_i = t_{i'}$ (for example, see Proposition 1.2 of [9]).

In [9], we defined $w \in W$ to be *fully commutative* if $\mathcal{R}(w)$ consists of a single commutativity class; i.e., any reduced word for w can be obtained from any other solely by use of the braid relations that correspond to commuting generators. It is not hard to show that this is equivalent to the property that for all $s, t \in S$ such that $m(s, t) \geq 3$, no member of $\mathcal{R}(w)$ has $\langle s, t\rangle_m$ as a subword, where $m = m(s, t)$.

It will be convenient to let W^{FC} denote the set of fully commutative members of W.

As mentioned in the introduction, the irreducible FC-finite Coxeter groups (i.e., Coxeter groups with finitely many fully commutative elements) occur in seven infinite families denoted A_n, B_n, D_n, E_n, F_n, H_n and $I_2(m)$. The Coxeter graphs of these groups are displayed in figure 1. It is interesting to note that there are no "exceptional" groups.

For the dihedral groups, the situation is quite simple. Only the longest element of $I_2(m)$ fails to be fully commutative, leaving a total of $2m - 1$ such elements.

Henceforth, we will be concerned only with the groups in the remaining six families.

1.3. *Restriction*

For any word $\mathbf{s} \in S^*$ and any $J \subset S$, let us define $\mathbf{s}|_J$ to be the restriction of $\mathbf{s}$ to J; i.e., the subsequence formed by the terms of $\mathbf{s}$ that belong to J. Since the interchange of adjacent commuting generators in $\mathbf{s}$ has either the same effect or no effect in $\mathbf{s}|_J$, it follows that for any commutativity class C, the restriction of C to J is well-defined.

A family $\mathcal{F}$ of subsets of S is *complete* if for all $s \in S$ there exists $J \in \mathcal{F}$ such that $s \in J$, and for all $s, t \in S$ such that $m(s, t) \geq 3$ there exists $J \in \mathcal{F}$ such that $s, t \in J$.

Proposition 1.1 *If $\mathcal{F}$ is a complete family of subsets of S, then for all $\mathbf{s}, \mathbf{s}' \in S^*$, we have $\mathbf{s} \sim \mathbf{s}'$ if and only if $\mathbf{s}|_J \sim \mathbf{s}'|_J$ for all $J \in \mathcal{F}$.*

Proof: The necessity of the stated conditions is clear. For sufficiency, suppose that s is the first term of $\mathbf{s}$. Since $s \in J$ for some $J \in \mathcal{F}$, there must also be at least one occurrence of s in $\mathbf{s}'$. We claim that any term t that precedes the first s in $\mathbf{s}'$ must commute with s. If not, then we would have $\mathbf{s}|_{\{s,t\}} \not\sim \mathbf{s}'|_{\{s,t\}}$, contradicting the fact that $\mathbf{s}|_J \sim \mathbf{s}'|_J$ for some J containing $\{s, t\}$. Thus we can replace $\mathbf{s}'$ with some $\mathbf{s}'' \sim \mathbf{s}'$ whose first term is s. If we delete the initial s from $\mathbf{s}$ and $\mathbf{s}''$, we obtain words that satisfy the same restriction conditions as $\mathbf{s}$ and $\mathbf{s}'$. Hence $\mathbf{s} \sim \mathbf{s}''$ follows by induction with respect to length. $\square$

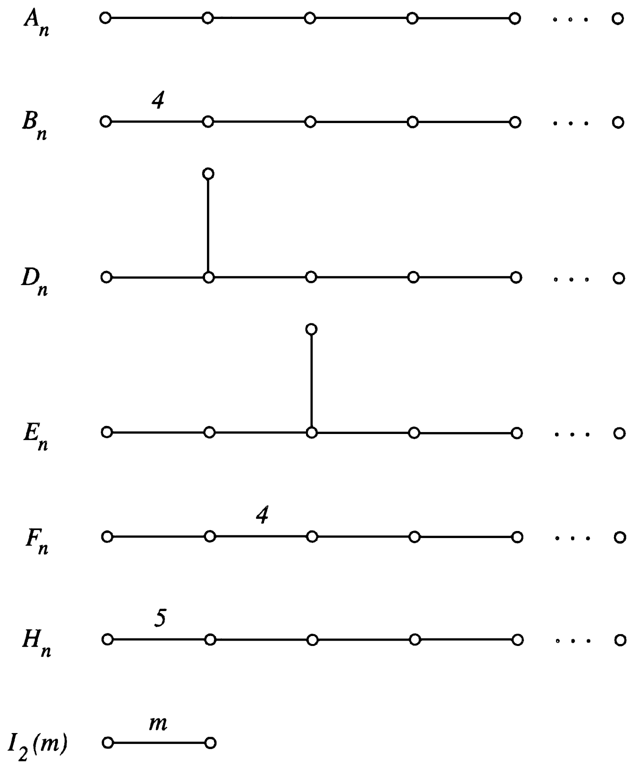

Figure 1. The irreducible FC-finite Coxeter groups.

2. The generic case

Choose a distinguished generator $s_1 \in S$, and let $W = W_1, W_2, W_3, \ldots$ denote the infinite sequence of Coxeter groups in which W_i is obtained from W_{i-1} by adding a new generator s_i such that $m(s_i, s_{i-1}) = 3$ and s_i commutes with all other generators of W_{i-1}. In the language of [9], $\{s_2, \ldots, s_n\}$ is said to form a simple branch in the graph of W_n. For $n \geq 1$, let $S_n = S \cup \{s_2, \ldots, s_n\}$ denote the generating set for W_n, and let Γ_n denote the corresponding Coxeter graph (see figure 2). It will be convenient also to let S_0 and Γ_0 denote the corresponding data for the Coxeter group W_0 obtained when s_1 is deleted from S. Thus $S_n = S_0 \cup \{s_1, \ldots, s_n\}$ for all $n \geq 0$.

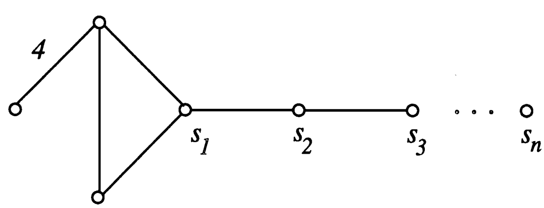

Figure 2. A simple branch.

2.1. *Spines, branches, and centers*

For any $w \in W_n^{\mathrm{FC}}$, we define the *spine* of w, denoted $\sigma(w)$, to be the pair (l, A), where l denotes the number of occurrences of s_1 in some (equivalently, every) reduced word for w, and A is the subset of $\{1, \dots, l-1\}$ defined by the property that $k \in A$ iff there is no occurrence of s_2 between the kth and $(k+1)$th occurrences of s_1 in some (equivalently, every) reduced word for w. We refer to l as the *length* of the spine.

Continuing the hypothesis that w is fully commutative, for $J \subseteq S_n$ we let $w|_J$ denote the commutativity class of $\mathbf{s}|_J$ for any reduced word $\mathbf{s} \in \mathcal{R}(w)$. (It follows from the discussion in Section 1.3 that this commutativity class is well-defined.) In particular, for each $w \in W_n^{\mathrm{FC}}$, we associate the pair

$$(w|_{S_n - S_0},\ w|_{S_1}).$$

We refer to $w|_{S_n - S_0}$ and $w|_{S_1}$ as the *branch* and *central* portions of w, respectively.

For example, consider the Coxeter group F_7. We label its generators $\{u, t, s_1, \dots, s_5\}$ in the order they appear in figure 1, so that $\{s_2, \dots, s_5\}$ is a simple branch. The heap of a typical fully commutative member of F_7 is displayed in figure 3. Its spine is $(5, \{1, 4\})$, and the heaps of its central and branch portions are displayed in figure 4.

Define $\mathcal{B}_n$ (the "branch set") to be the set of all commutativity classes B over the alphabet $S_n - S_0 = \{s_1, \dots, s_n\}$ such that

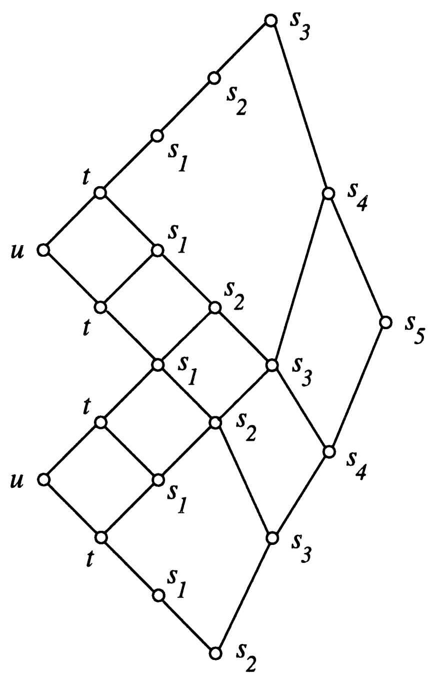

Figure 3. An F_7-heap.

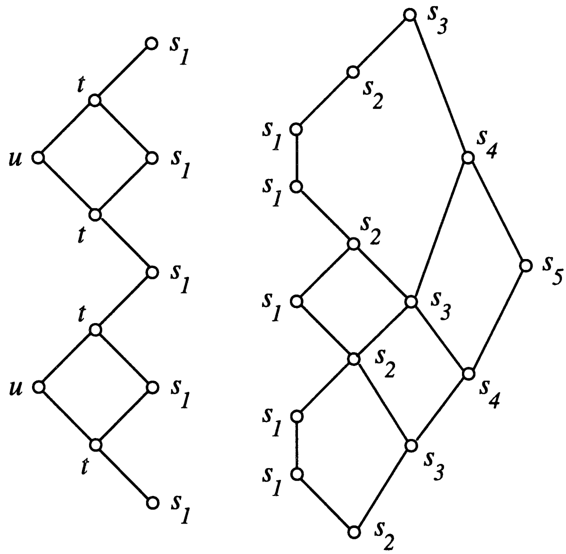

Figure 4. Center and branch.

(B1) If (s_i, s_i) is a subword of some member of B, then $i = 1$.
(B2) If (s_i, s_j, s_i) is a subword of some member of B, then $i = 1$.

Furthermore, given a spine $\sigma = (l, A)$, we define $\mathcal{B}_n(\sigma)$ to be the set of commutativity classes $B \in \mathcal{B}_n$ such that there are l occurrences of s_1 in every member of B, and

(B3) The kth and $(k+1)$th occurrences of s_1 occur consecutively in some member of B if and only if $k \in A$.

We claim (see Lemma 2.1) that $\mathcal{B}_n(\sigma)$ contains the branch portions of every fully commutative $w \in W_n$ with spine σ. Note also that $\mathcal{B}_n$ depends only on n, not W.

Similarly, let us define $\mathcal{C} = \mathcal{C}_W$ (the "central set") to be the set of commutativity classes C over the alphabet $S_1 = S$ such that

(C1) For all $s \in S_1$, no member of C has (s, s) as a subword.
(C2) If $\langle s, t\rangle_m$ is a subword of some member of C, where $m = m(s, t) \geq 3$, then s_1 occurs at least twice in this subword. (In particular, $s_1 = s$ or $s_1 = t$.)

In addition, we say that $C \in \mathcal{C}_W$ is *compatible* with the spine $\sigma = (l, A)$ if every member of C has l occurrences of s_1, and

(C3) If $\langle s, t\rangle_m$ is a subword of some member of C, where $m = m(s, t) \geq 3$, then this subword includes the kth and $(k+1)$th occurrences of s_1 for some $k \notin A$.

Let $\mathcal{C}(\sigma) = \mathcal{C}_W(\sigma)$ denote the set of σ-compatible members of $\mathcal{C}$. We claim (again, see Lemma 2.1) that $\mathcal{C}(\sigma)$ contains the central portions of every $w \in W_n^{\mathrm{FC}}$ with spine σ. Note also that $\mathcal{C}(\sigma)$ depends only on $W = W_1$ (more precisely, on the Coxeter graph Γ), not the length of the branch attached to it.

Lemma 2.1 *The mapping $w \mapsto (w|_{S_n - S_0}, w|_{S_1})$ defines a bijection*

$$W_n^{\mathrm{FC}} \longrightarrow \bigcup_{\sigma} \mathcal{B}_n(\sigma) \times \mathcal{C}_W(\sigma).$$

Proof: For all non-commuting pairs $s, t \in S_n$, we have $\{s, t\} \subseteq S_1$ or $\{s, t\} \subseteq S_n - S_0$, so by Proposition 1.1, the commutativity class of any $w \in W_n^{\mathrm{FC}}$ (and hence w itself) is uniquely determined by $w|_{S_n - S_0}$ and $w|_{S_1}$. Thus the map is injective.

Now choose an arbitrary fully commutative $w \in W_n$ with spine $\sigma = (l, A)$, and set $B = w|_{S_n - S_0}$, $C = w|_{S_1}$. The defining properties of the spine immediately imply the validity of (B3). Since consecutive occurrences of any $s \in S_n$ do not arise in any $\mathbf{s} \in \mathcal{R}(w)$, it follows that for all $k \geq 1$, the kth and $(k+1)$th occurrences of s in $\mathbf{s}$ must be separated by some $t \in S_n$ such that $m(s, t) \geq 3$. For $s = s_2, s_3, \ldots, s_n$, the only possibilities for t are in $S_n - S_0$; hence (B1) holds. For $s \in S_0$, the only possibilities for t are in S_1, so (C1) could fail only if $s = s_1$ and for some k, the only elements separating the kth and $(k+1)$th occurrences of s_1 in $\mathbf{s}$ that do not commute with s_1 are one or more occurrences of s_2. In that case, we could choose a reduced word for w so that the subword running from the kth to the $(k+1)$th occurrences of s_1 forms a reduced word for a fully commutative element of the parabolic subgroup isomorphic to A_n generated by $\{s_1, \ldots, s_n\}$. However, it is easy to show (e.g., Lemma 4.2 of [9]) that every reduced word for a fully commutative member of A_n has at most one occurrence of each "end node" generator. Thus (C1) holds.

Concerning (B2), (C2) and (C3), suppose that (s_i, s_j, s_i) occurs as a subword of some member of the commutativity class B. If $i > 1$, then every $s \in S_n$ that does not commute with s_i belongs to $S_n - S_0$. Hence, some reduced word for w must also contain the subword (s_i, s_j, s_i), contradicting the fact that w is fully commutative. Thus (B2) holds. Similarly, if we suppose that $\langle s, t\rangle_m$ occurs as a subword of some member of C, where $m = m(s, t) \geq 3$ and $s, t \in S_1$, then again we contradict the hypothesis that w is fully commutative unless $s = s_1$ or $t = s_1$, since s_1 is the only member of S_1 that may not commute with some member of $S_n - S_1$. In either case, since $\langle s, t\rangle_m$ cannot be a subword of any $\mathbf{s} \in \mathcal{R}(w)$, it follows that s_1 occurs at least twice in $\langle s, t\rangle_m$ (proving (C2)), and between two such occurrences of s_1, say the kth and $(k+1)$th, there must be an occurrence of s_2 in $\mathbf{s}$. By definition, this means $k \notin A$, so (C3) holds. Thus $B \in \mathcal{B}_n(\sigma)$ and $C \in \mathcal{C}_W(\sigma)$.

Finally, it remains to be shown that the map is surjective. For this, let $\sigma = (l, A)$ be a spine, and choose commutativity classes $B \in \mathcal{B}_n(\sigma)$ and $C \in \mathcal{C}_W(\sigma)$. Select representatives $\mathbf{s}_B \in (S_n - S_0)^*$ and $\mathbf{s}_C \in S_1^*$ for B and C. Since $S_1 \cap (S_n - S_0) = \{s_1\}$ is a singleton, and this singleton appears the same number of times in $\mathbf{s}_B$ and $\mathbf{s}_C$ (namely, l times), it follows that there is a word $\mathbf{s} \in S_n^*$ whose restrictions to $S_n - S_0$ and S_1 are $\mathbf{s}_B$ and $\mathbf{s}_C$, respectively. We claim that $\mathbf{s}$ is a reduced word for some $w \in W_n^{\mathrm{FC}}$, and hence $w \mapsto (B, C)$.

To prove the claim, first consider the possibility that for some $s \in S_n$, (s, s) occurs as a subword of some member of the commutativity class of $\mathbf{s}$. In that case, depending on whether $s \in S_1$, the same would be true of either B or C, contradicting (B1) or (C1). Next consider the possibility that $\langle s, t\rangle_m$ occurs as a subword of some word $\mathbf{s}'$ in the commutativity class of $\mathbf{s}$, where $m = m(s,t) \geq 3$. We must have either $s, t \in S_n - S_0$ or $s, t \in S_1$, and hence the same subword appears in some member of B or C, respectively. In the former case, (B2) requires that $s = s_1$ and $m = 3$. However the restriction of $\mathbf{s}'$ to S_1 would then have consecutive occurrences of s_1, contradicting (C1). In the latter case, (C2) and (C3) require that $s_1 = s$ or $s_1 = t$, and that the subword $\langle s, t\rangle_m$ includes the kth and $(k+1)$th occurrences of s_1 in $\mathbf{s}'$ for some $k \notin A$. It follows that s_2 does not occur between these two instances of s_1 in $\mathbf{s}'$, and thus they appear consecutively in the restriction of $\mathbf{s}'$ to $S_n - S_0$, contradicting (B3). Hence the claim follows. $\square$

The above lemma splits the enumeration of the fully commutative parts of the Coxeter groups $W_0, W_1, W_2, \ldots$ into two subproblems. The first subproblem, which is universal for all Coxeter groups, is to determine the number of branch commutativity classes with spine σ; i.e., the cardinality of $\mathcal{B}_n(\sigma)$ for all integers $n \geq 0$ and all σ. The second subproblem, which needs only to be done once for each series W_n, is to determine the number of central commutativity classes with spine σ; i.e., the cardinality of $\mathcal{C}_W(\sigma)$.

2.2. *Spinal analysis*

The possible spines that arise in the FC-finite Coxeter groups are severely limited. To make this claim more precise, suppose that $W = W_1, W_2, \ldots$ is one of the six nontrivial families of FC-finite Coxeter groups (i.e., A, B, D, E, F, or H). The Coxeter graph of W can then be chosen from one of the six in figure 5. For convenience, we have used s as the label for the distinguished generator previously denoted s_1.

Lemma 2.2 *If $C \in \mathcal{C}_W$ is compatible with the spine $\sigma = (l, A)$ and W is one of the Coxeter groups in figure* 5, *then $A \subseteq \{1, l-1\}$.*

Proof: Let $\mathbf{s} \in S^*$ be a representative of C, and towards a contradiction, let us suppose that A includes some k such that $1 < k < l-1$. Note that it follows that the kth and $(k+1)$th occurrences of s in $\mathbf{s}$ are neither the first nor the last such occurrences.

For the H-graph, property (C1) implies that the occurrences of s and t alternate in $\mathbf{s}$. Hence, the kth and $(k+1)$th occurrences of s appear in the middle of a subword of the form

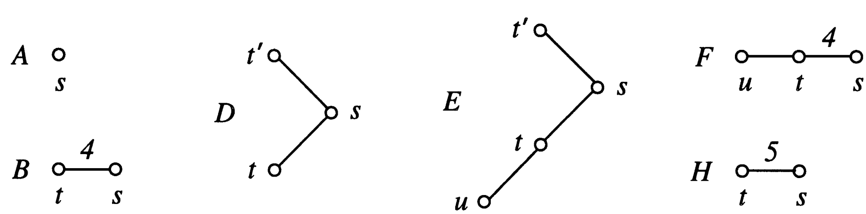

Figure 5.

(s, t, s, t, s, t, s). In particular, these two occurrences of s participate in a subword of $\mathbf{s}$ of the form (t, s, t, s, t), contradicting (C3).

For the F-graph, property (C1) implies that any two occurrences of s must be separated by at least one t. On the other hand, the subword between two occurrences of s must be a reduced word for some fully commutative member of the subgroup generated by $\{t, u\}$ (property (C2)), so the occurrences of s and t must alternate, and in the restriction of $\mathbf{s}$ to $\{s, t\}$, the kth and $(k+1)$th occurrences of s appear in the middle of a subword of the form (s, t, s, t, s, t, s). By (C3), these two occurrences of s cannot participate in an occurrence of (t, s, t, s) or (s, t, s, t) in $\mathbf{s}$. Hence, the two occurrences of t surrounding the kth (respectively, $(k+1)$th) occurrence of s must be separated by an occurrence of u. However in that case, (u, t, u) is a subword of some member of the commutativity class of $\mathbf{s}$, contradicting (C2).

For the E-graph, at least one of t and t' must appear between any two occurrences of s (otherwise (C1) is violated), and *both* t and t' must appear between the kth and $(k+1)$th occurrences of s, by (C3). On the other hand, property (C3) also implies that the subword (strictly) between the $(k-1)$th and $(k+2)$th occurrences of s in $\mathbf{s}$ must be a reduced word for some fully commutative member of W, a Coxeter group isomorphic to A_4. In particular, this implies that t' can appear at most once, and t at most twice, in this subword. Since we have already accounted for at least four occurrences of t and t', we have a contradiction.

This completes the proof, since the remaining three graphs are subgraphs of the preceding ones. $\square$

2.3. *Branch enumeration*

The previous lemma shows that for the FC-finite Coxeter groups, we need to solve the branch enumeration problem (i.e., determine the cardinality of $\mathcal{B}_n(\sigma)$) only for the spines $\sigma = (l, A)$ such that $A \subseteq \{1, l-1\}$. For this, we first introduce the notation

$$B_{n,l} := \binom{2n-1}{n+l-1} - \binom{2n-1}{n+l+1} = \frac{2l+1}{n+l+1}\binom{2n}{n+l}$$

for the number of $(n+l, n-l)$-*ballot sequences*. That is, $B_{n,l}$ is the number of orderings of votes for two candidates so that the winning candidate never trails the losing candidate, with the final tally being $n+l$ votes to $n-l$ votes (for example, see [3, Section 1.8]). This quantity is also the number of standard Young tableaux of shape $(n+l, n-l)$.

Let $\chi(P) = 1$ if P is true, and 0 otherwise.

Lemma 2.3 *For integers* $n, l \geq 0$, *we have*

$$|\mathcal{B}_n(l, \emptyset)| = B_{n,l},$$
$$|\mathcal{B}_n(l, \{1\})| = |\mathcal{B}_n(l, \{l-1\})| = B_{n,l-2} + B_{n,l-1} - \binom{n}{l-2} \quad (l \geq 2),$$
$$|\mathcal{B}_n(l, \{1, l-1\})| = B_{n+1,l-3} - 2\binom{n+1}{l-3} + \chi(l \leq n+4) \quad (l \geq 3).$$

Proof: For $i = 0, 1, 2$, let $B_{n,l}^{(i)}$ denote the cardinality of $\mathcal{B}_n(\sigma)$ for $\sigma = (l, \emptyset)$, $(l, \{1\})$ and $(l, \{1, l-1\})$, respectively. In the case $\sigma = (l, \emptyset)$, the defining properties (B1) and (B3) for membership of B in $\mathcal{B}_n(\sigma)$ can be replaced with

(B1′) For $1 \leq i \leq n$, no member of B has (s_i, s_i) as a subword.

It follows that for $1 \leq k < l$, the kth and $(k+1)$th occurrence of s_1 in any member of B must be separated by exactly one s_2, and the total number of occurrences of s_2 must be $l-1$, l, or $l+1$, according to whether the first and last occurrences of s_1 are preceded (respectively, followed) by an s_2. Furthermore, the restriction of B to $\{s_2, \ldots, s_n\}$ is a commutativity class with no subwords of the form (s_i, s_i) or (s_i, s_j, s_i) except possibly (s_2, s_3, s_2). By shifting indices ($i + 1 \to i$), we thus obtain any one of the members of $\mathcal{B}_{n-1}(l', \emptyset)$, where l' denotes the number of occurrences of s_2. Accounting for the four possible ways that s_1 and s_2 can be interlaced (or two, if $l = 0$), we obtain the recurrence

$$B_{n,l}^{(0)} = \begin{cases} B_{n-1,l-1}^{(0)} + 2B_{n-1,l}^{(0)} + B_{n-1,l+1}^{(0)} & \text{if } l \geq 1, \\ B_{n-1,0}^{(0)} + B_{n-1,1}^{(0)} & \text{if } l = 0. \end{cases}$$

On the other hand, it is easy to show that $B_{n,l}$ satisfies the same recurrence and initial conditions, so $B_{n,l}^{(0)} = B_{n,l}$. (In fact, one can obtain a bijection with ballot sequences by noting that the terms of the recurrence correspond to specifying the last two votes.)

By word reversal, the cases corresponding to $\sigma = (l, \{1\})$ and $\sigma = (l, \{l-1\})$ are clearly equivalent, so we restrict our attention to the former. Properties (B1) and (B3) imply that the restriction of any B in $\mathcal{B}_n(\sigma)$ to $\{s_1, s_2\}$ must then take the form

$$(*, s_1, s_1, s_2, s_1, s_2, s_1, \ldots, s_2, s_1, *),$$

where each '$*$' represents an optional occurrence of s_2. We declare the left side of B to be *open* if the above restriction has the form $(s_2, s_1, s_1, s_2, \ldots)$, and there is no s_3 separating the first two occurrences of s_2. Otherwise, the left side is *closed*.

Case I. The left side is open. In this case, if we restrict B to $\{s_2, \ldots, s_n\}$ (and shift indices), we obtain any one of the members of $\mathcal{B}_{n-1}(l', \{1\})$, where $l' = l$ or $l - 1$, according to whether there is an occurrence of s_2 following the last s_1. (If $l = 2$, then there is no choice: $l' = l = 2$ is the only possibility.)

Case II. The left side is closed. In this case, if we delete the first occurrence of s_1 from B, we obtain any one of the commutativity classes in $\mathcal{B}_n(l-1, \emptyset)$.

The above analysis yields the recurrence

$$B_{n,l}^{(1)} = \begin{cases} B_{n-1,l-1}^{(1)} + B_{n-1,l}^{(1)} + B_{n,l-1}^{(0)} & \text{if } l \geq 3, \\ B_{n-1,2}^{(1)} + B_{n,1}^{(0)} & \text{if } l = 2. \end{cases}$$

It is easy to verify that the claimed formula for $B^{(1)}_{n,l}$ satisfies the same recurrence and the proper initial conditions.

For $\sigma = (l, \{1, l-1\})$, the restriction of any B in $\mathcal{B}_n(\sigma)$ to $\{s_1, s_2\}$ takes the form

$$(*, s_1, s_1, s_2, s_1, s_2, s_1, \ldots, s_2, s_1, s_1, *),$$

where again each '$*$' represents an optional occurrence of s_2. In the special case $l = 3$, this becomes $(*, s_1, s_1, s_1, *)$; by deleting one of the occurrences of s_1, we obtain any one of the commutativity classes in $\mathcal{B}_n(2, \{1\})$.

Assuming $l \geq 4$, we now have not only the possibility that the left side of B is open (as in the case $\sigma = (l, \{1\})$), but the right side may be open as well, *mutatis mutandis*.

Case I. The left and right sides of B are both open. In this case, if we restrict B to $\{s_2, \ldots, s_n\}$ (and shift indices), we obtain any one of the members of $\mathcal{B}_{n-1}(l-1, \{1, l-2\})$.

Case II. Exactly one of the left or right sides of B is open. Assuming it is the left side that is open, if we restrict B to $\{s_2, \ldots, s_n\}$ (and shift indices), we obtain any one of the members of $\mathcal{B}_{n-1}(l', \{1\})$, where $l' = l-1$ or $l-2$, according to whether there is an occurrence of s_2 following the last s_1.

Case III. The left and right sides of B are both closed. In this case, if we delete the first and last s_1 from B, we obtain any one of the members of $\mathcal{B}_n(l-2, \emptyset)$.

The above analysis yields $B^{(2)}_{n,3} = B^{(1)}_{n,2}$ and the recurrence

$$B^{(2)}_{n,l} = B^{(2)}_{n-1,l-1} + 2\big(B^{(1)}_{n-1,l-1} + B^{(1)}_{n-1,l-2}\big) + B^{(0)}_{n,l-2}$$

for $l \geq 4$. Once again, it is routine to verify that the claimed formula for $B^{(2)}_{n,l}$ satisfies the same recurrence and initial conditions. $\square$

Remark 2.4 The union of $\mathcal{B}_n(l, \emptyset)$ for all $l \geq 0$ is the set of commutativity classes corresponding to the fully commutative members of the Coxeter group B_n whose reduced words do not contain the subword (s_1, s_2, s_1). In the language of [10], these are the "fully commutative top elements" of B_n; in the language of [4], these are the "commutative elements" of the Weyl group C_n.

Let $R(x)$ denote the generating series for the Catalan numbers. That is,

$$R(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 0} B_{n,0}x^n = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}x^n.$$

Note that $xR(x)^2 = R(x) - 1$. The following is a standard application of the Lagrange inversion formula (cf., Exercise 1.2.1 of [6]). We include below a combinatorial proof.

Lemma 2.5 *We have* $\sum_{n\geq 0} B_{n,l}x^n = x^l R(x)^{2l+1} = R(x)(R(x)-1)^l$.

Proof: A ballot sequence in which A defeats B by $2l$ votes can be factored uniquely into $2l+1$ parts by cutting the sequence after the last moment when candidate B trails by i votes, $i = 0, 1, \ldots, 2l-1$. The first part consists of a ballot sequence for a tie vote, and all remaining parts begin with a vote for A, followed by a ballot sequence for a tie. After deleting the $2l$ votes for A at the beginnings of these parts, we obtain an ordered $(2l+1)$-tuple of ballot sequences for ties, for which the generating series is $R(x)^{2l+1}$. $\square$

2.4. The generic generating function

To enumerate the fully commutative elements of the family $W = W_1, W_2, \ldots$, all that remains is the "central" enumeration problem; i.e., determining the cardinalities of $\mathcal{C}_W(\sigma)$ for all spines σ of the form described in Lemma 2.2. Setting aside the details of this problem until Section 3, let us define

$$c_{l,0} = |\mathcal{C}_W(l, \emptyset)|, \quad c_{l,1} = |\mathcal{C}_W(l, \{1\})| = |\mathcal{C}_W(l, \{l-1\})|, \quad c_{l,2} = |\mathcal{C}_W(l, \{1, l-1\})|,$$

and let $C_i(x)$ $(i = 0, 1, 2)$ denote the generating series defined by

$$C_0(x) = \sum_{l\geq 0} c_{l,0}x^l, \quad C_1(x) = c_{2,1} + 2\sum_{l\geq 3} c_{l,1}x^{l-2}, \quad C_2(x) = \sum_{l\geq 3} c_{l,2}x^{l-4}.$$

Although these quantities depend on W, we prefer to leave this dependence implicit.

Theorem 2.6 *If W is one of the six Coxeter groups displayed in figure* 5, *we have*

$$\sum_{n\geq 0} \left|W_n^{\mathrm{FC}}\right| x^n = R(x)C_0(R(x)-1) + R(x)^2C_1(R(x)-1) + R(x)^3C_2(R(x)-1)$$
$$- \frac{1}{1-x}C_1\left(\frac{x}{1-x}\right) - \frac{2}{(1-x)^2}C_2\left(\frac{x}{1-x}\right) + \frac{1}{1-x}C_2(x).$$

Proof: Successive applications of Lemmas 2.1, 2.2, and 2.3 yield

$$\begin{aligned}
\left|W_n^{\mathrm{FC}}\right| &= \sum_{\sigma} |\mathcal{B}_n(\sigma)| \cdot |\mathcal{C}_W(\sigma)| \\
&= \sum_{l\geq 0} c_{l,0}B_{n,l}^{(0)} + c_{2,1}B_{n,2}^{(1)} + 2\sum_{l\geq 3} c_{l,1}B_{n,l}^{(1)} + \sum_{l\geq 3} c_{l,2}B_{n,l}^{(2)} \\
&= \sum_{l\geq 0} c_{l,0}B_{n,l} + c_{2,1}(B_{n,0} + B_{n,1} - 1) \\
&\quad + 2\sum_{l\geq 3} c_{l,1}\left(B_{n,l-2} + B_{n,l-1} - \binom{n}{l-2}\right) \\
&\quad + \sum_{l\geq 3} c_{l,2}\left(B_{n+1,l-3} - 2\binom{n+1}{l-3} + \chi(l \leq n+4)\right).
\end{aligned} \tag{2.1}$$

Using Lemma 2.5 to simplify the corresponding generating function,[1] we obtain

$$\begin{aligned}
&\sum_{n\geq 0} \left|W_n^{\mathrm{FC}}\right| x^n \\
&\quad = \sum_{l\geq 0} c_{l,0} R(x)(R(x)-1)^l + c_{2,1}\left(R(x) + R(x)(R(x)-1) - \frac{1}{1-x}\right) \\
&\qquad + 2\sum_{l\geq 3} c_{l,1}\left(R(x)(R(x)-1)^{l-2} + R(x)(R(x)-1)^{l-1} - \frac{x^{l-2}}{(1-x)^{l-1}}\right) \\
&\qquad + \sum_{l\geq 3} c_{l,2}\left(x^{-1}R(x)(R(x)-1)^{l-3} - 2\frac{x^{l-4}}{(1-x)^{l-2}} + \frac{x^{l-4}}{1-x}\right).
\end{aligned}$$

Bearing in mind that $R(x)^2 = x^{-1}(R(x)-1)$, it is routine to verify that this agrees with the claimed expression. □

Remark 2.7 As we shall see in the next section, for each series W_n the generating functions $C_i(x)$ are rational, so the above result implies that the generating series for $|W_n^{\mathrm{FC}}|$ belongs to the algebraic function field $\mathbf{Q}(R(x)) = \mathbf{Q}(\sqrt{1-4x})$.

3. Enumerating the central parts

In this section, we determine the cardinalities of the central sets $\mathcal{C}(\sigma) = \mathcal{C}_W(\sigma)$ for each of the six Coxeter groups W displayed in figure 5. (The reader may wish to review the labeling of the generators in these cases, and recall that the distinguished generator s_1 has been given the alias s.) We subsequently apply Theorem 2.6, obtaining the generating function for the number of fully commutative elements in W_n.

3.1. The A-series

In this case, s is a singleton generator, so there is only one commutativity class of each length. It follows easily from the defining properties that the only central commutativity classes are those of (s) and $(\,)$ (the empty word). These are compatible only with the spines $\sigma = (1, \emptyset)$ and $(0, \emptyset)$, respectively. Thus we have

$$C_0(x) = 1 + x, \quad C_1(x) = C_2(x) = 0,$$

and Theorem 2.6 implies

$$\sum_{n\geq 0} \left|A_n^{\mathrm{FC}}\right| x^n = R(x)^2 = x^{-1}(R(x)-1).$$

Extracting the coefficient of x^n, we obtain

$$\left|A_n^{\mathrm{FC}}\right| = \frac{1}{n+2}\binom{2n+2}{n+1}, \tag{3.1}$$

a result first proved in [1, Section 2].

3.2. *The B-series*

In this case, we have $S = \{s, t\}$, and the defining properties imply that the central commutativity classes are singletons in which the occurrences of s and t alternate. It follows that $c_{l,0}$ is simply the number of alternating $\{s, t\}$-words in which s occurs l times; namely, 4 (if $l > 0$) or 2 (if $l = 0$). Also, the only alternating $\{s, t\}$-word that is compatible with a spine (l, A) with $A \neq \emptyset$ is (s, t, s), which is compatible with $(2, \{1\})$. Thus we have

$$C_0(x) = 2 + \frac{4x}{1-x}, \quad C_1(x) = 1, \quad C_2(x) = 0.$$

After some simplifications, Theorem 2.6 yields

$$\sum_{n\geq 0}\left|B_{n+1}^{\mathrm{FC}}\right|x^n = x^{-1}((1-4x)^{-1/2}-1) + x^{-1}(R(x)-1) - \frac{1}{1-x}.$$

Extracting the coefficient of x^{n-1}, we obtain

$$\left|B_n^{\mathrm{FC}}\right| = \frac{n+2}{n+1}\binom{2n}{n} - 1, \tag{3.2}$$

a result first proved in [10, Section 5].

3.3. *The D-series*

In this case, a set of representatives for the central commutativity classes consist of the subwords of $(s, t, s, t', s, t, s, t', \ldots)$, together with (t, t'), (s, t, t'), (t, t', s), and (s, t, t', s). Of these, only (s, t, t', s) is compatible with a spine (l, A) with $A \neq \emptyset$; the remainder are compatible only with $(l, \emptyset)$ for some l. Among the subwords of $(s, t, s, t', s, t, s, t', \ldots)$, the number with l occurrences of s is 8 (if $l \geq 2$), 7 (if $l = 1$), or 3 (if $l = 0$). Thus we have

$$C_0(x) = (1 + 2x + x^2) + \left(3 + 7x + \frac{8x^2}{1-x}\right), \quad C_1(x) = 1, \quad C_2(x) = 0,$$

and after some simplifications, Theorem 2.6 implies

$$\sum_{n\geq 0}\left|D_{n+2}^{\mathrm{FC}}\right|x^n = \frac{1}{2x^2}((1-4x)^{-1/2}-1-2x) + x^{-2}(R(x)-1-x) - \frac{1}{1-x}.$$

Extracting the coefficient of x^{n-2}, we obtain

$$\left|D_n^{\mathrm{FC}}\right| = \frac{n+3}{2n+2}\binom{2n}{n} - 1, \tag{3.3}$$

a result obtained previously in [4] and [10, Section 10].

3.4. The H-series

As in the B-series, the central commutativity classes are the singletons formed by each of the alternating $\{s, t\}$-words. In particular, the value of $C_0(x)$ is identical to its B-series version. The words that are compatible with spines of the form $(l, \{1\})$ are those that begin with s (and have at least two occurrences of s), and (t, s, t, s); thus $c_{2,1} = 3$ and $c_{l,1} = 2$ for $l \geq 3$. The words compatible with spines of the form $\{1, l-1\}$ are those that both begin and end with s and have at least four occurrences of s; i.e., $c_{3,2} = 0$ and $c_{l,2} = 1$ for $l \geq 4$. Thus we have

$$C_0(x) = 2 + \frac{4x}{1-x}, \quad C_1(x) = 3 + \frac{4x}{1-x}, \quad C_2(x) = \frac{1}{1-x}.$$

After some simplifications, Theorem 2.6 yields

$$\sum_{n\geq 0} \left|H_{n+1}^{\mathrm{FC}}\right| x^n = x^{-2}((1-4x)^{-1/2} - 1 - 2x) - \frac{8}{1-2x} + \frac{4-3x}{(1-x)^2}.$$

Extracting the coefficient of x^{n-1}, we obtain

$$\left|H_n^{\mathrm{FC}}\right| = \binom{2n+2}{n+1} - 2^{n+2} + n + 3. \tag{3.4}$$

3.5. The F-series

In this case, we can select a canonical representative $\mathbf{s} \in S^*$ from each central commutativity class by insisting that whenever s and u are adjacent in $\mathbf{s}$, u precedes s. Any such word has a unique factorization $\mathbf{s} = \mathbf{s}_0\mathbf{s}_1 \cdots \mathbf{s}_l$ with $\mathbf{s}_0 \in \{t, u\}^*$ and $\mathbf{s}_1, \ldots, \mathbf{s}_l$ each being words consisting of an initial s followed by a $\{t, u\}$-word. In fact, given our conventions, we must have $\mathbf{s}_0 \in \{(), (t), (u), (t, u), (u, t)\}$ and $\mathbf{s}_i \in \{(s), (s, t), (s, t, u)\}$ for $1 \leq i \leq l$, with $\mathbf{s}_i = (s)$ allowed only if $i = l$. We also cannot have (s, t, u) preceded by (u), (t, u), or (s, t, u); otherwise, some member of the commutativity class of $\mathbf{s}$ contains the forbidden subword (u, t, u). Conversely, any word meeting these specifications is the canonical representative of some central commutativity class. The language formed by these words therefore consists of

$$\begin{aligned} &\{(), (t), (u, t), (u, s, t), (t, u, s, t)\} \cdot \{(s, t, u, s, t), (s, t)\}^* \\ &\quad \cdot \{(), (s, t, u)\} \cdot \{(), (s)\}, \end{aligned} \tag{3.5}$$

together with the exceptional cases $\{(u), (t,u), (u,s), (t,u,s)\}$. Hence

$$C_0(x) = (2+2x) + \frac{(3+2x)(1+x)^2}{1-x-x^2} = \frac{(1+x)(5+3x)}{1-x-x^2}.$$

Turning now to $C_1(x)$, note that the central commutativity classes that are compatible with a spine of the form $(l, \{1\})$ are those for which the first two occurrences of s do not participate in an occurrence of the subwords (s,t,s,t), or (t,s,t,s). If s occurs three or more times, this requires $(\,)$ to be the first factor in (3.5), followed by an occurrence of (s,t,u,s,t). Hence, the canonical representatives compatible with $(l, \{1\})$ consist of

$$(s,t,u,s,t) \cdot \{(s,t,u,s,t), (s,t)\}^* \cdot \{(\,), (s,t,u)\} \cdot \{(\,), (s)\} \tag{3.6}$$

and four additional cases with $l = 2$: $\{(s,t,s), (u,s,t,s), (s,t,s,u), (t,u,s,t,s)\}$. It follows that $c_{2,1} = 5$, and therefore

$$C_1(x) = c_{2,1} + 2\sum_{l\geq 3} c_{l,1}x^{l-2} = 3 + 2\frac{(1+x)^2}{1-x-x^2} = 1 + \frac{4+2x}{1-x-x^2}.$$

To determine $C_2(x)$, note first that (s,t,u,s,t,s) is the unique canonical representative compatible with the spine $(3, \{1,2\})$. For the spines $(l, \{1, l-1\})$ with $l \geq 4$, compatibility requires (s) to be the last factor in (3.6), and it must be preceded by (s,t,u,s,t). Hence

$$C_2(x) = \sum_{l\geq 3} c_{l,2}x^{l-4} = x^{-1} + \frac{x}{1-x-x^2}.$$

After simplifying the generating function provided by Theorem 2.6, we obtain

$$\begin{aligned}\sum_{n\geq 0} \left|F_{n+2}^{\mathrm{FC}}\right| x^n = {}& \frac{10-5(1+x)(R(x)-1)}{1-4x-x^2} + x^{-1}(R(x)-1) \\ & - \frac{6-4x}{1-3x+x^2} + \frac{1+x}{1-x-x^2} - \frac{1}{1-x}.\end{aligned} \tag{3.7}$$

While it is unlikely that there is a simple closed formula for $|F_n^{\mathrm{FC}}|$, it is interesting to note that the Fibonacci numbers f_n satisfy

$$\sum_{n\geq 0} f_n x^n = \frac{1}{1-x-x^2}, \quad \sum_{n\geq 0} f_{2n} x^n = \frac{1-x}{1-3x+x^2}, \quad \sum_{n\geq 0} f_{3n} x^n = \frac{1-x}{1-4x-x^2},$$

so when the coefficient of x^{n-2} is extracted in (3.7), we obtain

$$\begin{aligned}\left|F_n^{\mathrm{FC}}\right| = {}& 5f_{3n-4} - 5\sum_{k=2}^{n-1} \frac{f_{3k-5}}{n-k+1}\binom{2n-2k}{n-k} + \frac{1}{n}\binom{2n-2}{n-1} \\ & - 2f_{2n-2} - 2f_{2n-4} + f_{n-1} - 1.\end{aligned}$$

3.6. The E-series

We claim that there is a unique member of each central commutativity class (in fact, *any* commutativity class in S^*) with the property that (s, u), (t', u), and (t', t) do not occur as subwords. To see this, note first that the set of left members of these pairs is disjoint from the set of right members. Secondly, these pairs are precisely the set of commuting generators of W. Hence, for any pair of words that differ by the interchange of two adjacent commuting generators, one member of the pair can be viewed as a "reduction" of the other, in the sense that the set of positions where u and t occur are farther to the left. Furthermore, since the set of instances of the forbidden pairs in any given word are pairwise disjoint, it follows by induction that any sequence of reductions eventually terminates with the same word, proving the claim.

Let L denote the formal language over the alphabet S formed by the canonical representatives (in the sense defined above) of the central commutativity classes. Given any formal language K over S, we will write $K(x)$ for the generating function obtained by assigning the weight x^l to each $\mathbf{s} \in K$ for which s occurs l times. Note that by this convention, we have $C_0(x) = L(x)$.

Any word $\mathbf{s} \in S^*$ has a unique factorization $\mathbf{s} = \mathbf{s}_0\mathbf{s}_1 \cdots \mathbf{s}_l$ with $\mathbf{s}_0 \in \{t, t', u\}^*$ and $\mathbf{s}_1, \ldots, \mathbf{s}_l$ each being words consisting of an initial s followed by a $\{t, t', u\}$-word. For membership in L, every subword of $\mathbf{s}$ not containing s must be a member of

$$E := \{(\,), (t), (u), (t, u), (u, t), (t'), (t, t'), (u, t'), (t, u, t'), (u, t, t')\},$$

the set of canonical representative for the fully commutative members of the subgroup generated by $\{t, t', u\}$. When s is prepended to these words, only six remain canonical:

$$\mathbf{a}_1 = (s),\ \mathbf{a}_2 = (s, t),\ \mathbf{a}_3 = (s, t, u),\ \mathbf{a}_4 = (s, t'),\ \mathbf{a}_5 = (s, t, t'),\ \mathbf{a}_6 = (s, t, u, t').$$

Thus we have $L \subset E \cdot \{\mathbf{a}_1, \ldots, \mathbf{a}_6\}^*$.

For each $\mathbf{e} \in E$, let $L_{\mathbf{e}}$ denote the set of $\mathbf{s} \in L$ for which the initial factor $\mathbf{s}_0$ is $\mathbf{e}$. If $\mathbf{s}_0 = (\,)$, then either $\mathbf{s} = (\,)$, $\mathbf{s} = (s)$, or deletion of the initial s in $\mathbf{s}$ yields a member of $L_{\mathbf{e}}$ for some $\mathbf{e} \in \{(t), (t, u), (t'), (t, t'), (t, u, t')\}$, and conversely. In terms of generating functions, we have

$$L_{(\,)}(x) = 1 + x + x(L_{(t)}(x) + L_{(t,u)}(x) + L_{(t')}(x) + L_{(t,t')}(x) + L_{(t,u,t')}(x)).$$

Similarly, deletion of s from the second position defines a bijection from $L_{(u)} - \{(u), (u, s)\}$ to $L_{(u,t)} \cup L_{(u,t')} \cup L_{(u,t,t')}$, so we have

$$L_{(u)}(x) = 1 + x + x(L_{(u,t)}(x) + L_{(u,t')}(x) + L_{(u,t,t')}(x)).$$

Combining these two decompositions, we obtain

$$\begin{aligned} L(x) &= \sum_{\mathbf{e}\in E} L_{\mathbf{e}}(x) \\ &= L_{(\,)}(x) + x^{-1}(L_{(\,)}(x) - 1 - x) + L_{(u)}(x) + x^{-1}(L_{(u)}(x) - 1 - x) \\ &= x^{-1}(1 + x)(L_{(\,)}(x) + L_{(u)}(x) - 2). \end{aligned} \tag{3.8}$$

Now consider the language $K = L \cap \{\mathbf{a}_2, \ldots, \mathbf{a}_6\}^*$, and the refinements K_i $(2 \le i \le 6)$ consisting of those nonvoid members of K whose initial factor is $\mathbf{a}_i$. Since the result of appending $\mathbf{a}_1 = (s)$ to any $\mathbf{s} \in L$ remains in L if and only if $\mathbf{s}$ does not already end in s, it follows that $L_{()} = K\{(), \mathbf{a}_1\}$. Similarly, we have

$$(s, t) \cdot L_{(u)} = K_3\{(), \mathbf{a}_1\},$$

so (3.8) can be rewritten in the form

$$L(x) = x^{-1}(1+x)^2K(x) + x^{-2}(1+x)^2K_3(x) - 2x^{-1}(1+x). \tag{3.9}$$

For $2 \le i \le 6$, the commutativity classes of

$$\mathbf{a}_2\mathbf{a}_3, \quad \mathbf{a}_3\mathbf{a}_4\mathbf{a}_3, \quad \mathbf{a}_i\mathbf{a}_i, \quad \mathbf{a}_i\mathbf{a}_5\ (i \ne 3), \quad \mathbf{a}_5\mathbf{a}_i, \quad \mathbf{a}_i\mathbf{a}_6, \quad \mathbf{a}_6\mathbf{a}_i\ (i \ne 2),$$

each have representatives in which one or more of the subwords (t, s, t), (t', s, t'), (u, t, u) and (t, u, t) appear, and hence cannot be central. Conversely, as a subset of $\{\mathbf{a}_2, \ldots, \mathbf{a}_6\}^*$, membership in K is characterized by avoidance of the subwords listed above. It follows that $K_5 = \{\mathbf{a}_5\}$, $K_6 = \{\mathbf{a}_6\} \cup \mathbf{a}_6K_2$, and

$$\begin{aligned}
K_2 &= \{\mathbf{a}_2\} \cup \mathbf{a}_2K_4,\\
K_3 &= \{\mathbf{a}_3, \mathbf{a}_3\mathbf{a}_5, \mathbf{a}_3\mathbf{a}_4\} \cup \{\mathbf{a}_3, \mathbf{a}_3\mathbf{a}_4\}K_2,\\
K_4 &= \{\mathbf{a}_4\} \cup \mathbf{a}_4K_2 \cup \mathbf{a}_4K_3.
\end{aligned}$$

Solving this recursive description of the languages K_i (essentially a computation in the ring of formal power series in noncommuting variables $\mathbf{a}_2, \ldots, \mathbf{a}_6$), we obtain

$$\begin{aligned}
K_2 &= \{\mathbf{a}_2, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_5\} \cup \{\mathbf{a}_2\mathbf{a}_4, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_4\}K_2^+,\\
K_3 &= \{\mathbf{a}_3\mathbf{a}_5\} \cup \{\mathbf{a}_3, \mathbf{a}_3\mathbf{a}_4\}K_2^+,\\
K_4 &= \{\mathbf{a}_4\mathbf{a}_3\mathbf{a}_5\} \cup \{\mathbf{a}_4, \mathbf{a}_4\mathbf{a}_3, \mathbf{a}_4\mathbf{a}_3\mathbf{a}_4\}K_2^+,\\
K_6 &= \mathbf{a}_6K_2^+,
\end{aligned}$$

where $K_2^+ = \{()\} \cup K_2 = \{\mathbf{a}_2\mathbf{a}_4, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_4\}^* \cdot \{(), \mathbf{a}_2, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_5\}$. Thus

$$\begin{aligned}
K_3(x) &= x^2 + (x+x^2)(1+x+x^4)(1-x^2-x^3-x^4)^{-1} = \frac{x(1+2x-x^2)}{1-x-x^3},\\
K(x) &= 1 + \sum_{i=2}^{6} K_i(x) = \frac{1+5x+6x^2+3x^3}{1-x^2-x^3-x^4},
\end{aligned}$$

and hence (3.9) implies

$$C_0(x) = \frac{(1+x)(10+7x+4x^2)}{1-x-x^3}.$$

The central commutativity classes compatible with spines of the form $(l, \{1\})$ are those for which the first two occurrences of s do not participate in an occurrence of the subwords (s, t, s) or (s, t', s). These correspond to the members of L for which the first occurrence of one of the factors $\mathbf{a}_i$ is either $\mathbf{a}_5$ or $\mathbf{a}_6$, followed by at least one more occurrence of $\mathbf{a}_1, \ldots, \mathbf{a}_6$. If $\mathbf{a}_5$ is the first factor, the possibilities are limited to $\{(\,), (u), (t, u)\}\mathbf{a}_5\mathbf{a}_1$, since $\mathbf{a}_5$ can be followed only by $\mathbf{a}_1$. If the first factor is $\mathbf{a}_6$, then the choices consist of the members of $K_6\{(\,), \mathbf{a}_1\}$ other than $\mathbf{a}_6$, since no nonvoid member of E can precede $\mathbf{a}_6$. Hence, the language of canonical representatives compatible with the spines $(l, \{1\})$ is

$$\{(\,), (u), (t, u)\}\mathbf{a}_5\mathbf{a}_1 \cup K_6\{(\,), \mathbf{a}_1\} - \{\mathbf{a}_6\}.$$

In particular, (s, t, t', s), (u, s, t, t', s), (t, u, s, t, t', s), (s, t, u, t', s), and (s, t, u, t', s, t) are the members compatible with the spine $(2, \{1\})$, so $c_{2,1} = 5$. Hence, using the decomposition of K_6 determined above, we obtain

$$C_1(x) = -5 + 2x^{-2}(-x + 3x^2 + (1 + x)K_6(x)) = -1 + \frac{6 - 2x + 2x^2}{1 - x - x^3}.$$

The canonical representatives of the central commutativity classes compatible with spines of the form $(l, \{1, l - 1\})$ must have a factorization in which there are at least three occurrences of the words $\mathbf{a}_i$, the first and penultimate of these being $\mathbf{a}_5$ or $\mathbf{a}_6$. Since $\mathbf{a}_6$ cannot be preceded by any of the factors $\mathbf{a}_i$, $\mathbf{a}_5$ must be the penultimate factor. Since $\mathbf{a}_5$ can be followed only by $\mathbf{a}_1$, the first factor must therefore be $\mathbf{a}_6$, there is no non-void member of E preceding $\mathbf{a}_6$, and the last factor must be $\mathbf{a}_1$. From the above decompositions of K_6 and K_2^+, it follows that the language formed by the members of L that start with $\mathbf{a}_6$ and terminate with $\mathbf{a}_5\mathbf{a}_1$ is

$$\mathbf{a}_6 \cdot \{\mathbf{a}_2\mathbf{a}_4, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_4\}^* \cdot \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_5\mathbf{a}_1, \tag{3.10}$$

and therefore

$$C_2(x) = \frac{x^2}{1 - x^2 - x^3 - x^4} = \frac{x^2}{(1 + x)(1 - x - x^3)}.$$

Combining our expressions for $C_i(x)$ $(i = 0, 1, 2)$, the generating function provided by Theorem 2.6 can be simplified to the form

$$\sum_{n\geq 0} \left|E^{\mathrm{FC}}_{n+3}\right| x^n = \frac{16 - 52x + 45x^2 - x^{-1}(R(x) - 1)}{1 - 7x + 14x^2 - 9x^3} - \frac{6 - 14x + 12x^2}{1 - 4x + 5x^2 - 3x^3} + \frac{1 - x^3 - x^4}{(1 - x^2)(1 - x - x^3)}. \tag{3.11}$$

4. Fully commutative involutions

We will say that a commutativity class C is *palindromic* if it includes the reverse of some (equivalently, all) of its members. A fully commutative $w \in W$ is an involution if and only if $\mathcal{R}(w)$ is palindromic.

In the following, we will adopt the convention that if X is a set of commutativity classes, then $\bar{X}$ denotes the set of palindromic members of X. Similarly, $\bar{W}$ and $\bar{W}^{\mathrm{FC}}$ shall denote the set of involutions in W and W^{FC}, respectively.

4.1. *The generic generating function*

Consider the enumeration of fully commutative involutions in a series of Coxeter groups $W = W_1, W_2, \ldots$ of the type considered in Section 2. It is clear that $w \in W_n^{\mathrm{FC}}$ is an involution if and only if its branch and central portions are palindromic. Thus by Lemma 2.1, determining the cardinality of $\bar{W}_n^{\mathrm{FC}}$ can be split into two subproblems: enumerating $\bar{\mathcal{B}}_n(\sigma)$ (the palindromic branch classes) and $\bar{\mathcal{C}}_W(\sigma)$ (the palindromic central classes).

For integers $n, l \geq 0$, we define $\bar{B}_{n,l} = \binom{n}{k}$, where $k = \lceil \frac{n+l}{2} \rceil$.

Lemma 4.1 *We have*

$$\begin{aligned} |\bar{\mathcal{B}}_n(l, \emptyset)| &= \bar{B}_{n,l}, \\ |\bar{\mathcal{B}}_n(l, \{1\})| &= \bar{B}_{n+1,0} - 1 & &\text{(if } l = 2\text{; or 0, if } l > 2), \\ |\bar{\mathcal{B}}_n(l, \{1, l-1\})| &= \bar{B}_{n+1,l-3} - \chi(l \leq n+4) & &\text{(if } l \geq 3). \end{aligned}$$

Proof: Following the proof of Lemma 2.3, for $i = 0, 1, 2$, let $\bar{B}_{n,l}^{(i)}$ denote the cardinality of $\bar{\mathcal{B}}_n(\sigma)$ for $\sigma = (l, \emptyset)$, $(l, \{1\})$ and $(l, \{1, l-1\})$, respectively. Recall that the occurrences of s_1 and s_2 must be interlaced in any representative of $B \in \mathcal{B}_n(l, \emptyset)$, and that when we restrict B to $\{s_2, \ldots, s_n\}$ (and shift indices), we obtain a member of $\mathcal{B}_{n-1}(l', \emptyset)$, where l' denotes the number of occurrences of s_2. To be palindromic, it is therefore necessary and sufficient that the $\{s_2, \ldots, s_n\}$-restriction of B is palindromic, and that $l' = l + 1$ or $l - 1$ (or 0, if $l = 0$). This yields the recurrence

$$\bar{B}_{n,l}^{(0)} = \begin{cases} \bar{B}_{n-1,l+1}^{(0)} + \bar{B}_{n-1,l-1}^{(0)} & \text{if } l \geq 1, \\ \bar{B}_{n-1,1}^{(0)} + \bar{B}_{n-1,0}^{(0)} & \text{if } l = 0. \end{cases}$$

It is easy to verify that $\bar{B}_{n,l}$ satisfies the same recurrence and initial conditions.

For spines of the form $\sigma = (l, \{1\})$, it is clear that there can be no palindromic classes unless $l = 2$, since for $l > 2$, there must be an occurrence of s_2 between the last two occurrences of s_1, but not for the first two. Assuming $l = 2$, the bijection provided in the proof of Lemma 2.3 preserves palindromicity, and thus proves the recurrence

$$\bar{B}_{n,2}^{(1)} = \bar{B}_{n-1,2}^{(1)} + \bar{B}_{n,1}^{(0)}.$$

It is routine to check that the claimed formula for $\bar{B}^{(1)}_{n,2}$ satisfies the same recurrence and initial conditions.

For $\sigma = (l, \{1, l-1\})$, the left and right sides of any palindromic $B \in \mathcal{B}_n(\sigma)$ must be both open or both closed, in the sense defined in the proof of Lemma 2.3. Furthermore, a branch class with this property is palindromic if and only if its restriction to $\{s_2, \ldots, s_n\}$ is palindromic, so the bijection provided in Lemma 2.3 for this case yields

$$\bar{B}^{(2)}_{n,l} = \bar{B}^{(2)}_{n-1,l-1} + \bar{B}^{(0)}_{n,l-2} \quad (l \geq 4)$$

and $\bar{B}^{(2)}_{n,3} = \bar{B}^{(1)}_{n,2}$. Once again, it is routine to check that the claimed formula for $\bar{B}^{(2)}_{n,l}$ satisfies the same recurrence and initial conditions. □

Lemma 4.2 *We have* $\sum_{n\geq 0} \bar{B}_{n,l} x^n = \frac{1+xR(x^2)}{\sqrt{1-4x^2}} x^l R(x^2)^l$.

Proof: We have $\sum_{n\geq 0} \bar{B}_{n,l} x^n = F_{l,0}(x) + F_{l,1}(x)$, where $F_{l,j}(x) = \sum_{n+l=j \bmod 2} \bar{B}_{n,l} x^n$. We can interpret $F_{l,j}(x)$ as the generating function for sequences of votes in an election in which A defeats B by $l+j$ votes. Such sequences can be uniquely factored by cutting the sequence after the last moment when B trails A by i votes, $i = 0, 1, \ldots, l+j-1$. The first factor consists of an arbitrary sequence for a tie vote, which has generating function $1/\sqrt{1-4x^2}$, and the remaining $l+j$ factors each consist of a vote for A, followed by a "ballot sequence" for a tie vote (cf., Section 2.3), which has generating function $xR(x^2)$. □

Turning now to the palindromic central commutativity classes, let us define

$$\bar{c}_{l,0} = |\bar{\mathcal{C}}_W(l, \emptyset)|, \quad \bar{c}_{2,1} = |\bar{\mathcal{C}}_W(2, \{1\})|, \quad \bar{c}_{l,2} = |\bar{\mathcal{C}}_W(l, \{1, l-1\})|,$$

and associated generating functions

$$\bar{C}_0(x) = \sum_{l\geq 0} \bar{c}_{l,0} x^l, \quad \bar{C}_{12}(x) = \bar{c}_{2,1} x^{-1} + \sum_{l\geq 3} \bar{c}_{l,2} x^{l-4}.$$

Theorem 4.3 *If W is one of the Coxeter groups displayed in figure* 5, *then*

$$\sum_{n\geq 0} \left|\bar{W}^{\mathrm{FC}}_n\right| x^n = \frac{1+xR(x^2)}{\sqrt{1-4x^2}} (\bar{C}_0(xR(x^2)) + R(x^2)\bar{C}_{12}(xR(x^2))) - \frac{1}{1-x}\bar{C}_{12}(x).$$

Proof: As noted previously, $w \in W^{\mathrm{FC}}$ is an involution if and only if the central and branch portions of w are palindromic. Successive applications of Lemmas 2.1, 2.2, and 4.1 therefore yield

$$\begin{aligned} \left|\bar{W}^{\mathrm{FC}}_n\right| &= \sum_{\sigma} |\bar{\mathcal{B}}_n(\sigma)| \cdot |\bar{\mathcal{C}}_W(\sigma)| = \sum_{l\geq 0} \bar{c}_{l,0} \bar{B}^{(0)}_{n,l} + \bar{c}_{2,1} \bar{B}^{(1)}_{n,2} + \sum_{l\geq 3} \bar{c}_{l,2} \bar{B}^{(2)}_{n,l} \\ &= \sum_{l\geq 0} \bar{c}_{l,0} \bar{B}_{n,l} + \bar{c}_{2,1}(\bar{B}_{n+1,0} - 1) + \sum_{l\geq 3} \bar{c}_{l,2}(\bar{B}_{n+1,l-3} - \chi(l \leq n+4)). \end{aligned} \tag{4.1}$$

The corresponding generating function thus takes the form

$$\sum_{n\geq 0} \left|\bar{W}_n^{\mathrm{FC}}\right| x^n = \frac{1 + xR(x^2)}{\sqrt{1-4x^2}} \sum_{l\geq 0} \bar{c}_{l,0} x^l R(x^2)^l + \bar{c}_{2,1} x^{-1} \left(\frac{1 + xR(x^2)}{\sqrt{1-4x^2}} - \frac{1}{1-x} \right)$$
$$+ \sum_{l\geq 3} \bar{c}_{l,2} \left(\frac{1 + xR(x^2)}{\sqrt{1-4x^2}} x^{l-4} R(x^2)^{l-3} - \frac{x^{l-4}}{1-x} \right),$$

using Lemma 4.2. □

As we shall see below, both $\bar{C}_0(x)$ and $\bar{C}_{12}(x)$ are rational, so the generating series for $|\bar{W}_n^{\mathrm{FC}}|$ belongs to the algebraic function field $\mathbf{Q}(x, R(x^2)) = \mathbf{Q}(x, \sqrt{1-4x^2})$.

4.2. *The A-series*

In this case, we have $\bar{C}_0(x) = 1 + x$ and $\bar{C}_{12}(x) = 0$, since there are only two central commutativity classes (namely, those of $()$ and (s)), and both are palindromic. Hence

$$\sum_{n\geq 0} \left|\bar{A}_n^{\mathrm{FC}}\right| x^n = \frac{(1 + xR(x^2))^2}{\sqrt{1-4x^2}} = x^{-1} \left(\frac{1 + xR(x^2)}{\sqrt{1-4x^2}} - 1 \right).$$

Either by extracting the coefficient of x^n, or more directly from (4.1), we obtain

$$\left|\bar{A}_n^{\mathrm{FC}}\right| = \bar{B}_{n+1,0} = \binom{n+1}{\lceil (n+1)/2 \rceil}. \tag{4.2}$$

4.3. *The B-series*

In this case, the central commutativity classes are singletons in which the occurrences of s and t alternate. For each $l \geq 0$, there are two such words that are palindromic and have l occurrences of s. Among these, (s, t, s) is the only one that is compatible with a spine (l, A) with $A \neq \emptyset$. Hence $\bar{C}_0(x) = 2/(1-x)$, $\bar{C}_{12}(x) = x^{-1}$, and Theorem 4.3 implies

$$\sum_{n\geq 0} \left|\bar{B}_{n+1}^{\mathrm{FC}}\right| x^n = x^{-1} \left(\frac{1 + xR(x^2)}{\sqrt{1-4x^2}} - 1 \right) + \frac{2}{1-2x} - \frac{1}{1-x}.$$

Extracting the coefficient of x^{n-1}, we obtain

$$\left|\bar{B}_n^{\mathrm{FC}}\right| = 2^n + \binom{n}{\lceil n/2 \rceil} - 1. \tag{4.3}$$

4.4. The D-series

In this case, the palindromic central classes are represented by the odd-length subwords of $(s, t, s, t', s, t, s, t', \ldots)$ whose middle term is t or t', together with $()$, (s), (t, t'), and (s, t, t', s). In particular, leaving aside (s, t, t', s), there are exactly four such words with l occurrences of s for each even $l \geq 0$, so we have

$$\bar{C}_0(x) = x + x^2 + \frac{4}{1-x^2}.$$

Also $\bar{C}_{12}(x) = x^{-1}$, since (s, t, t', s) is the only representative compatible with a spine of the form (l, A) with $A \neq \emptyset$. After simplifying the expression in Theorem 4.3, we obtain

$$\sum_{n\geq 0} \left|\bar{D}^{\mathrm{FC}}_{n+2}\right| x^n = \frac{1+3x}{2x^3}\left(\frac{1}{\sqrt{1-4x^2}} - 1\right) + \frac{2}{1-2x} - \frac{x^{-1}}{1-x}.$$

Extracting the coefficient of x^{n-2} yields

$$\left|\bar{D}^{\mathrm{FC}}_n\right| = \begin{cases} 2^{n-1} + \dfrac{3}{2}\dbinom{n}{n/2} - 1 & \text{if } n \text{ is even,} \\ 2^{n-1} + \dfrac{1}{2}\dbinom{n+1}{(n+1)/2} - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{4.4}$$

4.5. The H-series

The palindromic central classes in this case are the same as those for the B-series; the only difference is that those corresponding to $\langle s, t\rangle_7$, $\langle s, t\rangle_9, \ldots$ are now compatible with spines of the form $(l, \{1, l-1\})$ for $l \geq 4$. Thus we have

$$\bar{C}_0(x) = \frac{2}{1-x}, \quad \bar{C}_{12}(x) = \frac{x^{-1}}{1-x}.$$

The generating function provided by Theorem 4.3 is therefore

$$\sum_{n\geq 0} \left|\bar{H}^{\mathrm{FC}}_{n+1}\right| x^n = \frac{4}{1-2x} - \frac{2-x}{(1-x)^2},$$

and hence

$$\left|\bar{H}^{\mathrm{FC}}_n\right| = 2^{n+1} - (n+1). \tag{4.5}$$

4.6. The F-series

Recall that in Section 3.5, we selected a set of canonical representatives for the central commutativity classes by forbidding the subword (s, u). If $\mathbf{s}$ is one such representative, let $\mathbf{s}^*$ denote the canonical representative obtained by reversing $\mathbf{s}$ and then reversing each offending (s, u)-subword.

If $\mathbf{s}$ is the canonical representative of a palindromic class (i.e., $\mathbf{s} = \mathbf{s}^*$), then either $\mathbf{s} \in \{(\,), (u)\}$, or else $\mathbf{s}$ has a unique factorization fitting one of the forms

$$\mathbf{a}(s)\mathbf{a}^*, \quad \mathbf{a}(u, s)\mathbf{a}^*, \quad \mathbf{a}(t)\mathbf{a}^*,$$

where $\mathbf{a}$ is itself a canonical representative for some central commutativity class. Conversely, any canonical representative ending in (s) can be uniquely factored into one of the two forms $\mathbf{a} \cdot (s)$ or $\mathbf{a} \cdot (u, s)$, and the corresponding word obtained by appending $\mathbf{a}^*$ remains central. Similarly, any canonical representative ending with (t) but not (u, t) or (u, s, t), when factored into the form $\mathbf{a} \cdot (t)$, remains central when $\mathbf{a}^*$ is appended.

Now from (3.5), the language of canonical representatives ending in (s) consists of the exceptional set $\{(u, s), (t, u, s)\}$, together with

$$\{(\,), (t), (u, t), (u, s, t), (t, u, s, t)\} \cdot \{(s, t, u, s, t), (s, t)\}^* \cdot \{(s), (s, t, u, s)\}, \tag{4.6}$$

and the language of representatives ending with (t) but not (u, t) or (u, s, t) is

$$\{(t)\} \cup \{(\,), (t), (u, t), (u, s, t), (t, u, s, t)\} \cdot \{(s, t, u, s, t), (s, t)\}^* \cdot (s, t). \tag{4.7}$$

Including the exceptional cases $(\,)$ and (s), this yields

$$\bar{C}_0(x) = 3 + 2x + \frac{(3 + 2x^2)(x + x^3)}{1 - x^2 - x^4} + \frac{x^2(3 + 2x^2)}{1 - x^2 - x^4} = 1 + \frac{2 + 5x + x^2 + 3x^3}{1 - x^2 - x^4}.$$

The unique palindromic classes compatible with the spines $(2, \{1\})$ and $(3, \{1, 2\})$ are represented by (s, t, s) and (s, t, u, s, t, s). For the spines $\sigma = (l, \{1, l - 1\})$ with $l \geq 4$, recall from Section 3.5 that a canonical representative compatible with σ must begin with (s, t, u, s, t) and end with (t, u, s, t, s). Selecting the portions of (4.6) and (4.7) that begin with (s, t, u, s, t) yields the languages

$$(s, t, u, s, t) \cdot \{(s, t, u, s, t), (s, t)\}^* \cdot \{(s), (s, t, u, s)\},$$
$$(s, t, u, s, t) \cdot \{(s, t, u, s, t), (s, t)\}^* \cdot (s, t),$$

so we have

$$\bar{C}_{12}(x) = 2x^{-1} + \frac{x + x^2 + x^3}{1 - x^2 - x^4}.$$

Simplification of the generating series provided by Theorem 4.3 yields

$$\sum_{n \geq 0} \left|\bar{F}^{\mathrm{FC}}_{n+2}\right| x^n = \frac{4 + 10x + 2x^2 + x^2(1 + 5x + 3x^2 - 5x^3)Q(x)}{1 - 4x^2 - x^4} + (1 + 3x)Q(x) - \frac{3 + 4x + 2x^2 + 3x^3}{1 - x^2 - x^4} + \frac{1}{1 - x}, \tag{4.8}$$

where $Q(x) = ((1 - 4x^2)^{-1/2} - 1)/2x^2$.

The coefficients can be expressed in terms of the Fibonacci numbers as follows:

$$|\bar{F}_{2n}^{\mathrm{FC}}| = f_{3n} + f_{3n-2} + \frac{1}{2}\sum_{k=1}^{n-1}(f_{3k-2} + f_{3k-4})\binom{2n-2k}{n-k} + \frac{1}{2}\binom{2n}{n} - f_{n+2} + 1,$$

$$|\bar{F}_{2n+1}^{\mathrm{FC}}| = 5f_{3n-1} + \frac{5}{2}\sum_{k=1}^{n-1} f_{3k-3}\binom{2n-2k}{n-k} + \frac{3}{2}\binom{2n}{n} - f_{n+2} - f_n + 1.$$

4.7. The E-series

In Section 3.6, we selected a canonical representative $\mathbf{s}$ for each central commutativity class. As in the previous section, we let $\mathbf{s}^*$ denote the canonical representative for the commutativity class of the reverse of $\mathbf{s}$.

If $\mathbf{s} \in S^*$ is a representative of any palindromic commutativity class, then the set of generators appearing an odd number of times in $\mathbf{s}$ must commute pairwise. Indeed, the "middle" occurrence of one generator would otherwise precede the "middle" occurrence of some other generator in every member of the commutativity class. Aside from the exceptional cases $(\,)$, (u), and (s) (which cannot be followed and preceded by the same member of S and remain central), it follows that every central palindromic class has a unique representative fitting one of the forms

$$\mathbf{a}^*(t)\mathbf{a}, \quad \mathbf{a}^*(t')\mathbf{a}, \quad \mathbf{a}^*(t,t')\mathbf{a}, \quad \mathbf{a}^*(u,t')\mathbf{a}, \quad \mathbf{a}^*(u,s)\mathbf{a}, \tag{4.9}$$

where $\mathbf{a}$ is the canonical representative of some central commutativity class. However, we cannot assert that the above representatives are themselves canonical; for example, if $\mathbf{a} = (s,t)$, then $\mathbf{a}^*(u,t')\mathbf{a}$ is a representative of a central palindromic class, but the canonical representative of this class is (t,u,s,t',s,t).

For the representatives whose middle factor is (t), (t'), (t,t'), or (u,t'), observe that s must be the first term of $\mathbf{a}$, assuming that $\mathbf{a}$ is nonvoid. Furthermore, if we prepend an initial s (or s,t, in the case of (u,t')), the resulting words $(s,t)\mathbf{a}$, $(s,t')\mathbf{a}$, $(s,t,t')\mathbf{a}$, and $(s,t,u,t')\mathbf{a}$ are (in the notation of Section 3.6) members of the formal languages $K_2\{(\,),(s)\}$, $K_4\{(\,),(s)\}$, $K_5\{(\,),(s)\}$, and $K_6\{(\,),(s)\}$, respectively. Conversely, any member of these languages arises in this fashion.

For a representative whose middle factor is (u,s), if we prepend (s,t,t') to $(u,s)\mathbf{a}$, we obtain a member of a central commutativity class whose canonical representative is $(s,t,u,t',s)\mathbf{a}$, and hence a member of $K_6\{(\,),(s)\}$. Conversely, any member of $K_6\{(\,),(s)\}$ other than $\mathbf{a}_6 = (s,t,u,t')$ arises this way.

Collecting the contributions of the five types of palindromic central classes, along with the exceptional cases $\{(\,),(u),(s)\}$, we obtain

$$\begin{aligned}\bar{C}_0(x) &= 2 + x + x^{-2}(1+x^2)(K_2(x^2) + K_4(x^2) + K_5(x^2)\\ &\quad + K_6(x^2)) + x^{-3}((1+x^2)K_6(x^2) - x^2) = \frac{6+3x+2x^2-x^3+3x^4+x^5}{1-x^2-x^6}.\end{aligned}$$

For the spine $\sigma = (2, \{1\})$, there is a unique σ-compatible central class that is palindromic; namely, the class of (s, t, t', s). For the spines σ of the form $(l, \{1, l-1\})$, recall from (3.10) that the canonical representatives of the σ-compatible classes all begin with $\mathbf{a}_6\mathbf{a}_2\mathbf{a}_4$ and end with $\mathbf{a}_3\mathbf{a}_5\mathbf{a}_1$. It follows that for a palindromic central class represented by a word of the form (4.9) to be compatible with σ, it is necessary and sufficient that $\mathbf{a}$ end with $\mathbf{a}_3\mathbf{a}_5\mathbf{a}_1$. Using the decompositions obtained in Section 3.6, we find that

$$\{\mathbf{a}_2\mathbf{a}_4, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_4\}^* \cdot \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_5,$$
$$\{\mathbf{a}_4\mathbf{a}_3\mathbf{a}_5\} \cup \{\mathbf{a}_4, \mathbf{a}_4\mathbf{a}_3, \mathbf{a}_4\mathbf{a}_3\mathbf{a}_4\} \cdot \{\mathbf{a}_2\mathbf{a}_4, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_4\}^* \cdot \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_5,$$
$$\mathbf{a}_6 \cdot \{\mathbf{a}_2\mathbf{a}_4, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3, \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_4\}^* \cdot \mathbf{a}_2\mathbf{a}_4\mathbf{a}_3\mathbf{a}_5$$

are the respective portions of K_2, K_4, and K_6 that end with $\mathbf{a}_3\mathbf{a}_5$; there are no such words in K_5. It follows that

$$\begin{aligned}\bar{C}_{12}(x) &= x^{-1} + x^{-4}\left(x^6 + \frac{x^8 + 2x^{10} + x^{12} + x^{14}}{1 - x^4 - x^6 - x^8} + \frac{x^9}{1 - x^4 - x^6 - x^8}\right) \\ &= x^{-1} + \frac{x^2 + x^4 + x^5 + x^6}{(1 + x^2)(1 - x^2 - x^6)}.\end{aligned}$$

The generating function provided by Theorem 4.3 can be simplified to the form

$$\begin{aligned}\sum_{n\geq 0} \left|\bar{E}^{\mathrm{FC}}_{n+3}\right| x^n = {} & \frac{(2-3x^2)(3+5x-6x^2-9x^3) + Q(x)(1+x-4x^2-3x^3+2x^4)}{1-7x^2+14x^4-9x^6} \\ & - \frac{1+x^2+x^5-x^8}{(1-x)(1+x^2)(1-x^2-x^6)},\end{aligned} \tag{4.10}$$

where $Q(x) = ((1 - 4x^2)^{-1/2} - 1)/2x^2$.

5. Asymptotics

Given the lack of simple expressions for the number of fully commutative members of E_n and F_n, it is natural to consider asymptotic formulas.

Theorem 5.1 *We have*
(a) $|E^{\mathrm{FC}}_n| \sim \frac{1}{31}(25 - 9\beta - 4\beta^2)(\beta^2 + 2)^n$, *where* $\beta \doteq 1.466$ *is the real root of* $x^3 = x^2 + 1$.
(b) $|F^{\mathrm{FC}}_n| \sim (7\gamma - 11)\gamma^{3n}$, *where* $\gamma \doteq 1.618$ *is the largest root of* $x^2 = x + 1$.

Proof: Consider the generating function $G(x) = \sum_{n\geq 0} |W^{\mathrm{FC}}_n| x^n$ of Theorem 2.6.

In the case of F_n, we see from (3.7) that the singularities of $G(x)$ consist of a branch cut at $x = 1/4$, together with simple poles at $x = 1$ and the zeroes of $1 - x - x^2$, $1 - 3x + x^2$, and $1 - 4x - x^2$. The latter are (respectively) $\{1/\gamma, -\gamma\}$, $\{1/\gamma^2, \gamma^2\}$, and $\{1/\gamma^3, -\gamma^3\}$, where $\gamma = (1 + \sqrt{5})/2$ denotes the golden ratio. The smallest of these (in absolute value) is $1/\gamma^3 \doteq 0.236$, a zero of $1 - 4x - x^2$. In particular, since $1/\gamma^3 < 1/4$, the asymptotic behavior of $|F_n^{\mathrm{FC}}|$ is governed by the local behavior of $G(x)$ at $x = 1/\gamma^3$. More specifically, since there is a simple pole at $x = 1/\gamma^3$, it follows that $|F_n^{\mathrm{FC}}| \sim c\,\gamma^{3n}$, where

$$c = \lim_{x\to 1/\gamma^3}(1-\gamma^3 x)x^2 G(x) = \gamma^{-3}\frac{10-5(1+1/\gamma^3)/\gamma}{4+2/\gamma^3} = 7\gamma - 11,$$

using (3.7), together with the relations $\gamma^2 = \gamma + 1$ and $R(1/\gamma^3) - 1 = 1/\gamma$.

In the case of E_n, we see from (3.11) that the singularities of $G(x)$ consist of a branch cut at $x = 1/4$, together with simple poles at $x = \pm 1$ and the zeroes of $1 - x - x^3$, $1 - 4x + 5x^2 - 3x^3$, and $1 - 7x + 14x^2 - 9x^3$. These polynomials are related by the fact that if α is any zero of $1 - x - x^3$, then $1 - 4x + 5x^2 - 3x^3$ is the minimal polynomial of $\alpha/(1+\alpha)$, and $1-7x+14x^2-9x^3$ is the minimal polynomial of $\alpha/(1+\alpha)^2$. (The fact that such a simple relationship exists is not coincidental; see Remark 5.3 below.) The smallest of the nine zeroes of these polynomials (in absolute value) is $\delta = \alpha/(1+\alpha)^2 \doteq 0.241$, where $\alpha \doteq 0.682$ is the real zero of $1-x-x^3$. Equivalently, we have $1/\delta = \beta^2+2$, where $\beta = 1/\alpha$ is the real root of $x^3 = x^2 + 1$. Since $\delta < 1/4$, the asymptotic behavior of $|E_n^{\mathrm{FC}}|$ is once again governed by the local behavior of $G(x)$ near a simple pole. In this case, we obtain $|E_n^{\mathrm{FC}}| \sim c\delta^{-n} = c\,(\beta^2+2)^n$, where

$$c = \lim_{x\to\delta}(1-\delta^{-1}x)x^3 G(x) = \delta^2\frac{16-52\delta+45\delta^2-\alpha/\delta}{7-28\delta+27\delta^2} = \frac{1}{31}(25-9\beta-4\beta^2),$$

using (3.11) and the fact that $R(\delta) - 1 = \alpha$. □

Remark 5.2 For the sake of completeness, it is natural also to consider the asymptotic number of fully commutative elements in A_n, B_n, D_n, and H_n. Given the explicit formulas (3.1), (3.2), (3.3), and (3.4), it is easily established that

$$\begin{aligned} \left|A_n^{\mathrm{FC}}\right| &\sim \frac{4}{\sqrt{\pi}}n^{-3/2}4^n, & \left|B_n^{\mathrm{FC}}\right| &\sim \frac{1}{\sqrt{\pi}}n^{-1/2}4^n,\\ \left|D_n^{\mathrm{FC}}\right| &\sim \frac{1}{2\sqrt{\pi}}n^{-1/2}4^n, & \left|H_n^{\mathrm{FC}}\right| &\sim \frac{4}{\sqrt{\pi}}n^{-1/2}4^n, \end{aligned}$$

using Stirling's formula. In each of these cases, the dominant singularity in the corresponding generating function is the branch cut at $x = 1/4$.

Remark 5.3 If α is a pole of $f(x)$, then $\alpha/(1+\alpha)$ is a pole of $f(x/(1-x))$ and $\alpha/(1+\alpha)^2$ is a pole (of some branch) of $f(R(x)-1)$. On the other hand, from Theorem 2.6, we see that aside from the branch cut at $x = 1/4$ and a pole at $x = 1$, the singularities of $G(x) = \sum_{n\geq 0}|W_n^{\mathrm{FC}}|x^n$ are limited to those of $C_2(x)$, $C_i(x/(1-x))$ $(i = 1, 2)$, and $C_i(R(x)-1)$ $(i = 0, 1, 2)$. Thus, unless there is unexpected cancellation, for each pole α of $C_2(x)$, there will be a triple of poles at $\alpha/(1+\alpha)^i$ $(i = 0, 1, 2)$ in $G(x)$.

Now consider the asymptotic enumeration of fully commutative involutions. Again, given the explicit formulas (4.2), (4.3), (4.4), and (4.5), it is routine to show that

$$\left|\bar{A}_n^{\mathrm{FC}}\right| \sim \sqrt{\frac{8}{\pi}}\, n^{-1/2} 2^n, \quad \left|\bar{B}_n^{\mathrm{FC}}\right| \sim 2^n, \quad \left|\bar{D}_n^{\mathrm{FC}}\right| \sim 2^{n-1}, \quad \left|\bar{H}_n^{\mathrm{FC}}\right| \sim 2^{n+1}.$$

In the following, β and γ retain their meanings from Theorem 5.1.

Theorem 5.4 *We have*

(a) $|\bar{E}_{2n}^{\mathrm{FC}}| \sim \frac{1}{31}(20 - \beta + 3\beta^2)(\beta^2 + 2)^n$.

(b) $|\bar{E}_{2n+1}^{\mathrm{FC}}| \sim \frac{3}{31}(9 - 2\beta + 6\beta^2)(\beta^2 + 2)^n$.

(c) $|\bar{F}_{2n}^{\mathrm{FC}}| \sim \gamma^{3n+1}$.

(d) $|\bar{F}_{2n+1}^{\mathrm{FC}}| \sim (2 + \gamma)\gamma^{3n}$.

Proof: Consider the generating series $\bar{G}(x) = \sum_{n\geq 0} |\bar{W}_n^{\mathrm{FC}}| x^n$ of Theorem 4.3.

In the case of F_n, we see from (4.8) that the singularities of $\bar{G}(x)$ consist of branch cuts at $x = \pm 1/2$, together with simple poles at $x = 1$ and $\pm\gamma^{-1/2}, \pm(-\gamma)^{1/2}$ (the zeroes of $1 - x^2 - x^4$), and $\pm\gamma^{-3/2}, \pm(-\gamma)^{3/2}$ (the zeroes of $1 - 4x^2 - x^4$). In absolute value, the smallest of these occur at $x = \pm\gamma^{-3/2}$. Since $\gamma^{-3/2} < 1/2$, it follows that the asymptotic behavior of $|\bar{F}_n^{\mathrm{FC}}|$ is determined by the local behavior of $\bar{G}(x)$ at $x = \pm\gamma^{-3/2}$. More specifically, we have $|\bar{F}_{2n}^{\mathrm{FC}}| \sim c_+\gamma^{3n}$ and $|\bar{F}_{2n+1}^{\mathrm{FC}}| \sim c_-\gamma^{3n+3/2}$, where

$$c_\pm = \lim_{x\to\gamma^{-3/2}} x^2(1 - \gamma^{3/2}x)(\bar{G}(x) \pm \bar{G}(-x)).$$

Using (4.8) and the fact that $Q(\gamma^{-3/2}) = \gamma^4$, we obtain

$$\begin{aligned} c_+ &= \lim_{x\to\gamma^{-3/2}} x^2(1 - \gamma^{3/2}x)\frac{8 + 4x^2 + x^2(2 + 6x^2)Q(x)}{1 - 4x^2 - x^4} \\ &= \frac{8 + 4/\gamma^3 + (2 + 6/\gamma^3)\gamma}{8 + 4/\gamma^3} = \gamma, \end{aligned}$$

and a similar calculation (details omitted) yields $c_- = (2 + \gamma)\gamma^{-3/2}$.

In the case of E_n, we see from (4.10) that the singularities of $\bar{G}(x)$ consist of branch cuts at $x = \pm 1/2$, together with simple poles at $x = 1, \pm\sqrt{-1}$ and the square roots of the zeroes of $1 - x - x^3$ and $1 - 7x + 14x^2 - 9x^3$. Continuing the notation from the proof of Theorem 5.1, the poles occurring closest to the origin are at $x = \pm\delta^{1/2}$, where $\delta = 1/(\beta^2 + 2)$. Thus we have $|\bar{E}_{2n}^{\mathrm{FC}}| \sim c_+(\beta^2 + 2)^n$ and $|\bar{E}_{2n+1}^{\mathrm{FC}}| \sim c_-(\beta^2 + 2)^{n+1/2}$, where

$$c_\pm = \lim_{x\to\delta^{1/2}} x^3(1 - \delta^{-1/2}x)(\bar{G}(x) \mp \bar{G}(-x)).$$

Using (4.10) and the fact that $Q(\delta^{1/2}) = 1/\delta(\beta - 1)$, we obtain

$$
\begin{aligned}
c_+ &= \lim_{x\to\delta^{1/2}} x^3(1-\delta^{-1/2}x)\frac{(2-3x^2)(10x-18x^3)+Q(x)(2x-6x^3)}{1-7x^2+14x^4-9x^6} \\
&= \frac{(2-3\delta)(10\delta-18\delta^2)+(2-6\delta)/(\beta-1)}{14-56\delta+54\delta^2} = \frac{1}{31}(20-\beta+3\beta^2),
\end{aligned}
$$

and a similar calculation can be used to determine c_-; we omit the details. □

Appendix

Table 1. The number of fully commutative elements.*

n	A_n	B_n	D_n	E_n	F_n	H_n
1	2	(2)				(2)
2	5	7	(4)		(5)	9
3	14	24	(14)	(10)	(24)	44
4	42	83	48	(42)	106	195
5	132	293	167	(167)	464	804
6	429	1055	593	662	2003	3185
7	1430	3860	2144	2670	8560	12368
8	4862	14299	7864	10846	36333	47607
9	16796	53481	29171	44199	153584	182720
10	58786	201551	109173	180438	647775	701349
11	208012	764217	411501	737762	2729365	2695978
12	742900	2912167	1560089	3021000	11496788	10384231

*The parenthetical entries correspond to cases in which the group in question is either reducible or isomorphic to a group listed elsewhere.

Table 2. The number of fully commutative involutions.

n	A_n	B_n	D_n	E_n	F_n	H_n
1	2	(2)				(2)
2	3	5	(4)		(3)	5
3	6	10	(6)	(6)	(10)	12
4	10	21	16	(10)	18	27
5	20	41	25	(25)	48	58
6	35	83	61	42	89	121
7	70	162	98	106	220	248
8	126	325	232	178	405	503
9	252	637	381	443	968	1014
10	462	1275	889	756	1785	2037
11	924	2509	1485	1858	4195	4084
12	1716	5019	3433	3194	7758	8179

Note

1. It should be noted that when $n = -1$, the coefficient of $c_{l,2}$ in (2.1) is zero. Thus the range of summation for this portion of the generating function can be extended to $n \geq -1$.

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