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Homomorphisms of Edge-Colored Graphs and Coxeter Groups

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Abstract. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two edge-colored graphs (without multiple edges or loops). A *homomorphism* is a mapping $\phi : V_1 \mapsto V_2$ for which, for every pair of adjacent vertices u and v of G_1, $\phi(u)$ and $\phi(v)$ are adjacent in G_2 and the color of the edge $\phi(u)\phi(v)$ is the same as that of the edge uv.

We prove a number of results asserting the existence of a graph G, edge-colored from a set C, into which every member from a given class of graphs, also edge-colored from C, maps homomorphically.

We apply one of these results to prove that every three-dimensional hyperbolic reflection group, having rotations of orders from the set $M = \{m_1, m_2, \ldots, m_k\}$, has a torsion-free subgroup of index not exceeding some bound, which depends only on the set M.

Keywords: graph, homomorphism, Coxeter group, reflection group

1. Introduction

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two edge-colored graphs (without multiple edges or loops). We define a mapping $\phi : V_1 \mapsto V_2$ to be a *homomorphism* if, for every pair of adjacent vertices u and v of G_1, $\phi(u)$ and $\phi(v)$ are adjacent in G_2 and the color of the edge $\phi(u)\phi(v)$ is the same as that of the edge uv. In Section 2, we prove a number of results asserting the existence of a graph G, edge-colored from a set C, into which every graph from a given class of graphs, also edge-colored from C, maps homomorphically. In each case we also give explicit upper bounds for the number of vertices in G.

Homomorphisms arise naturally when dealing with Coxeter groups. For each Coxeter group G the edges of the corresponding Coxeter graph are "colored" by integers or ∞, and there is a simple relationship between homomorphisms of the Coxeter graph (in a slightly modified form) and those of the associated group. Hence some results about group homomorphisms have a natural restatement in terms of graph homomorphisms.

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We apply these ideas to prove that every hyperbolic reflection group, having rotations of orders from the set $M = \{m_1, m_2, \ldots, m_k\}$, has a torsion-free subgroup of index not exceeding some bound, which depends only on the set M. We compare this theorem with known results about torsion-free subgroups of Fuchsian groups [4], and of arbitrary Kleinian groups [5]. In outline, the method is to reduce statements about torsion-free subgroups to statements about group homomorphisms, to reformulate these in terms of graph homomorphisms, and then to apply the purely graph-theoretic results of Section 2.

We tacitly identify graphs edge-colored from a set of only one color with uncolored graphs. We say that a colored graph is planar, complete etc., if the underlying uncolored graph has the corresponding property.

We denote by $C(Y)$ the Cayley graph obtained from the generating set Y (where the group will be clear from the context). In all the cases we consider, Y will be closed under taking inverses, and we will regard $C(Y)$ as an *undirected* graph.

2. Edge-colored graphs and their homomorphisms

In this section we prove the following.

Theorem 2.1 *For every integer $n \geq 1$ there is a finite graph G_n whose edges are colored by the n colors $1, 2, \ldots, n$ so that every planar graph whose edges are colored with these colors maps homomorphically into G_n.*

It is of interest to know how small the graphs G_n in the above theorem can be made. Let λ_n denote the minimum possible number of vertices of a graph G_n. We have,

Proposition 2.2 *For every positive integer n,*

$$n^3 + 3 \leq \lambda_n \leq 5n^4.$$

We prove the upper bound of this proposition as a consequence of a more general result. For a family of graphs $\mathcal{G}$ and for an integer $n \geq 1$, let $\lambda(\mathcal{G}, n)$ denote the minimum possible number of vertices in an edge colored graph H so that each member of $\mathcal{G}$ whose edges are colored by colors from the set $\{1, 2, \ldots, n\}$ maps homomorphically into it. ($\lambda(\mathcal{G}, n) = \infty$ if there is no such finite H). The *acyclic chromatic number* of a graph G is the minimum number of colors in a proper vertex coloring of G so that the vertices of each cycle receive at least 3 distinct colors. This notion was introduced by Grünbaum and has been studied by various researchers. In particular, it has been proved by Borodin [2] that the acyclic chromatic number of any planar graph is at most 5. Thus, the upper bound of Proposition 2.2 follows from the following more general result, proved below.

Theorem 2.3 *Let $\mathcal{G}_k$ be the family of all graphs with acyclic chromatic number not exceeding k, then, for every odd n, $\lambda(\mathcal{G}_2, n) = (n + 1)$, and for every k and n, $\lambda(\mathcal{G}_k, n) \leq kn^{k-1}$.*

We note that it is not difficult to show that the family $\mathcal{G}'_{k-1}$ of all complete bipartite graphs with $k-1$ vertices in one side consists of graphs with acyclic chromatic number at most k and yet $n \geq 2$,

$$\lambda(\mathcal{G}'_{k-1}, n) = n^{k-1} + k - 1$$

showing that the above theorem is nearly tight. We note also that, by known results about the acyclic chromatic numbers of graphs embeddable on surfaces other than the plane (see [1]) the assertion of Proposition 2.2 may be extended to more complicated surfaces.

As we have learned from J. Neśetŕil during the completion of this paper, a notion similar to the one considered here has been studied by Raspaud and Sopena [8], (see also [7, 11]). In these papers the authors study homomorphisms between *directed* graphs, and show, in particular, that there exists a directed graph H on 80 vertices, with no cycle of length 2, so that every orientation of a planar graph maps homomorphically into H. The proof is based on acyclic colorings, like our proof here, and although we do not see any way to deduce the results here from the results in the papers mentioned above or vice versa, it seems that the same techniques are useful in both cases.

To prove Theorem 2.3 we need the following two simple lemmas.

Lemma 2.4 *If $\mathcal{T}$ is the family of all forests then, for every odd n, $\lambda(\mathcal{T}, n) = n + 1$.*

Proof: A star with n edges of distinct colors shows that $\lambda(\mathcal{T}, n) \geq n + 1$. A complete graph K on $n+1$ vertices with a proper n edge-coloring of its edges shows that $\lambda(\mathcal{T}, n) \leq n+1$. Indeed, the vertices of each forest can be mapped into those of K one by one, always adding a vertex that has at most one neighbor in the previously mapped vertices, and using the fact that an edge of each color is incident with each vertex of K. □

Lemma 2.5 *Let U be a complete bipartite graph on the classes of vertices $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, with a proper coloring of its edges by n colors. Then for any forest T whose edges are colored by the same n colors and for any bipartition of the set of vertices of T into vertex classes V and W, for which no two vertices of V or of W are adjacent, there is a homomorphism of T into U that maps V into A and W into B.*

Proof: It suffices to map any connected component of T. This can be done as in the previous proof, by mapping the vertices of the component into U one by one, starting by mapping a vertex into the appropriate vertex class of U, and always adding a vertex that has a unique neighbor among the previously mapped vertices. Since each color is incident with each vertex of U, the mapping can indeed be completed. □

Proof of Theorem 2.3: The assertion that $\lambda(\mathcal{G}_2, n) = n + 1$ for odd n follows from Lemma 2.4. To prove the main part of the theorem, let U be a complete bipartite graph on the two vertex classes $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, with a proper coloring of its edges by n colors.

Define an edge-colored graph G' as follows:

The vertices of G' are all k-tuples of the form

$$(i, x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$$

where $1 \le i \le k$ and $1 \le x_j \le n$ for all j.

An edge of G' joins the two vertices

$$(i, x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$$

and

$$(j, y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_k)$$

if and only if $i \neq j$. Such an edge, where $i < j$, is colored the same as the edge $a_{x_j} b_{y_i}$ in the graph U.

We claim that every edge-colored graph colored from $\{1, 2, \ldots, n\}$ with acyclic chromatic number not exceeding k, maps homomorphically into G'. To see this, let G be such a graph and let $V_1, \ldots, V_k$ be a partition of the vertices of G defined by an acyclic coloring of it. Each induced subgraph $G_{i,j} = G[V_i \cup V_j] (1 \le i < j \le k)$ is then a forest so that by Lemma 2.5 there is a homomorphism $\phi_{i,j}$ from each $G_{i,j}$ into U, mapping V_i into A and V_j into B. Suppose $\phi_{i,j}(v) = a_{\psi_{i,j}(v)}$ for all $v \in V_i$ and, similarly, $\phi_{i,j}(w) = b_{\psi_{i,j}(w)}$ for all $w \in V_j$.

Define a map ϕ from the vertices of G to those of G' by taking $v \in V_i$ to the vertex

$$(i, \psi_{1,i}(v), \psi_{2,i}(v), \ldots, \psi_{i-1,i}(v), \psi_{i,i+1}(v), \ldots, \psi_{i,k}(v))$$

of G'.

Now let $v \in V_i$, $w \in V_j$ be adjacent vertices in G, $(i < j)$. Then w is mapped by ϕ to the vertex

$$(j, \psi_{1,j}(w), \psi_{2,j}(w), \ldots, \psi_{j-1,j}(w), \psi_{j,j+1}(w), \ldots, \psi_{j,k}(w))$$

of G'. By the definition of G' the vertices $\phi(v)$ and $\phi(w)$ are adjacent and joined by an edge of the same color as that of the edge $a_{\psi_{i,j}(v)} b_{\psi_{i,j}(w)} = \phi_{i,j}(v)\phi_{i,j}(w)$ in U. Since $\phi_{i,j}$ is a homomorphism, this is also the color of vw in G. We have thus shown that ϕ is a homomorphism. Since G' has kn^{k-1} vertices the proof is complete. $\square$

To complete the proof of Proposition 2.2, we need to establish the lower bound for λ_n. To do this we define a class of graphs, the *triangular graphs*, Δ, inductively as follows:

(1) A triangular circuit is in Δ
(2) If $G \in \Delta$, then the graph obtained by putting a new vertex in one of the faces of G and by joining it to the three existing vertices of this face, is also in Δ.

Clearly all triangular graphs are planar. By a simple counting argument we show,

Lemma 2.6 $\lambda(\Delta, n) \geq n^3 + 3$

Proof: Let H be a graph edge-colored from the colors $\{1, 2, \ldots, n\}$ into which every triangular graph, edge-colored from the same set, maps homomorphically. We suppose for a contradiction that H has fewer than $n^3 + 3$ vertices.

For each $G \in \Delta$, let $h(G)$ be the set of homomorphisms from G to H (ignoring colors), and $c(G)$ the set of edge-colorings of G from the colors $\{1, 2, \ldots, n\}$. Each map $\phi \in h(G)$ induces a unique coloring of G for which ϕ is also a homomorphism of *colored* graphs. This gives a mapping $h(G) \to c(G)$ which, by assumption, is onto. We thus have,

$$|c(G)| \leq |h(G)| \tag{1}$$

Now construct the graph $G' \in \Delta$ by subdividing a face of G. A homomorphism in $h(G)$ can be extended to G' in at most $n^3 - 1$ ways (the image of the new vertex must differ from that of its three neighbors) so that, $|h(G')| \leq (n^3 - 1)|h(G)|$. Each of the three new edges in G' can be colored in n ways so that $|c(G')| = n^3|c(G)|$. Hence by repeatedly subdividing, we obtain a graph $G'' \in \Delta$ for which $|c(G'')| > |h(G'')|$, contrary to (1). □

It is well known that there is a homomorphism of an uncolored graph G into a graph with k vertices if and only if G has a proper vertex coloring by k colors. To see this, observe that given such a coloring, we can form the complete graph whose k vertices are the colors used. The mapping that takes each vertex to its color is then a homomorphism. Conversely given a homomorphism ϕ of G into the complete graph on k vertices, coloring each vertex by its image under ϕ gives a proper vertex coloring. In particular, we have $\lambda_1 = 4$, as a consequence of the four-color theorem.

3. Coxeter groups

3.1. Coxeter groups and homomorphisms

Edge colored graphs arise naturally from *Coxeter groups*. These are groups with a presentation of the form

$$G = \langle X \mid R \rangle$$

where $X = \{a_i : i \in I\}$ and R comprises the relators $a_i^2 (i \in I)$ and possibly some additional relators of the form $(a_i a_j)^{m_{ij}}$ $(i \neq j)$, $m_{ij} \geq 2$. If $a_i a_j$ is of infinite order we set $m_{ij} = \infty$.

We will assume henceforth that the generating set X is finite, $X = \{a_1, \ldots, a_n\}$. We refer to the members of X as *canonical generators* of G.

From G (or more precisely from its presentation) we may form an edge colored graph $\gamma(G)$ by taking as vertices the canonical generators and joining a_i and a_j by an edge colored m_{ij} whenever $m_{ij} \neq \infty$. The graph $\gamma(G)$ is closely akin to the familiar Coxeter diagram, but differs in that a_i and a_j are joined when $m_{ij} = 2$ and not when $m_{ij} = \infty$. If $\gamma(G)$ is disconnected then G can be expressed as a free product of two or more Coxeter groups. We shall suppose henceforth that $\gamma(G)$ is connected.

From G we may form the index two subgroup G^0 comprising products of an even number of canonical generators. This subgroup is generated by $T = \{r_{ij} = a_i a_j \mid m_{ij} < \infty\}$ (we use here the assumption that $\gamma(G)$ is connected) and has presentation

$$G^0 = \langle T \mid R' \rangle$$

where R' contains the relations of the form $r_{ij}^{m_{ij}}$ and

$$r_{i_1 i_2} r_{i_2 i_3} \ldots r_{i_m i_1} \tag{2}$$

While the generators of G correspond to vertices of $\gamma(G)$, those of G^0 correspond to its directed edges, r_{ij} being expressed graphically by the directed edge from a_i to a_j. There is a one to one correspondence between directed circuits of $\gamma(G)$ (including those of length two) and relations of the form (2).

Now suppose we have a group H with generating set Y. We color edges of the Cayley graph $C(Y)$ by the orders of the corresponding generators.

Let $\phi : G^0 \to H$ be a homomorphism which maps every generator in R' to a generator in Y of the same order. We define a homomorphism $\tilde{\phi}$ from $\gamma(G)$ to $C(Y)$ as follows. Let $\tilde{\phi}$ be defined arbitrarily at one vertex of $\gamma(G)$. We then extend the definition of $\tilde{\phi}$ to the other vertices of $\gamma(G)$ one by one. If $\tilde{\phi}(a_i)$ is defined and a_j is adjacent to a_i then we set $\tilde{\phi}(a_j) = \tilde{\phi}(a_i)\phi(a_i a_j)$. The map $\tilde{\phi}$ is well defined because ϕ takes relations of the form (2) to the identity.

In the other direction let φ be a graph homomorphism $\gamma(G) \to C(Y)$. We obtain a group homomorphism φ' from G^0 to H, which maps T to Y, as follows. Each generator in T corresponds to a directed edge of $\gamma(G)$, which is mapped to a directed edge in $C(Y)$, which corresponds to a generator in Y. This determines a homomorphism φ' with the required properties. The following theorem is an easy consequence of the definitions.

Theorem 3.1 *If the maps $\phi \to \tilde{\phi}$ and $\varphi \to \varphi'$ are as defined above, then $(\tilde{\phi})' = \phi$ and, if $\tilde{\varphi'} = \varphi$ at one point, then $\tilde{\varphi'} = \varphi$.*

3.2. Reflection groups and their torsion-free subgroups

We now apply the foregoing ideas in a geometrical context. A *Coxeter polyhedron P* in hyperbolic 3-space H^3 is one whose dihedral angles are all integer submultiples of π. Poincaré's polyhedral theorem [6] gives that the group $G = G(P)$ generated by reflections through the faces of P is discrete. Let $x_1, \ldots, x_m$ denote these reflections. Clearly $x_i^2 = 1 (1 \le i \le m)$ and if x_i and x_j are reflections through adjacent faces which meet at an angle of π/p then $(x_i x_j)^p = 1$—since $x_i x_j$ is a rotation through $2\pi/p$. Again by Poincaré's theorem, every relation in $G(P)$ is a consequence of these, so that $G(P)$ is a Coxeter group. A (hyperbolic) *reflection group* is the order two subgroup $G^0(P)$ of such a $G(P)$. In geometric terms, $G^0(P)$ is the subgroup of orientation preserving isometries in $G(P)$, and the graph $\gamma(G(P))$ is simply the dual of the edge skeleton of P. As such it is connected and planar.

According to Selberg's Lemma [9], every finitely generated matrix group has a torsion-free subgroup of finite index. In particular, finitely generated Coxeter groups (being matrix groups) have this property. We consider the problem of determining the smallest index of a torsion-free subgroup of a given group G. We denote this index by $m(G)$.

Let $\ell(G^0)$ denote the least common multiple of the orders of all finite subgroups of G^0. It is easy to show that $m(G^0)$ must be a multiple of $\ell(G^0)$ (see e.g., [4]). When G is a Fuchsian group, Edmonds Ewing and Kulkarni [4] have shown that $m(G^0)/\ell(G^0)$ is either 1 or 2 according to the individual group. By contrast, for Kleinian groups, Jones and Reid [5] have shown that $m(G^0)/\ell(G^0)$ can be made arbitrarily large, even if only cocompact groups are considered. For reflection groups the largest known value of $m(G^0)/\ell(G^0)$ seems to be 4 (e.g., for the group $\Gamma^{(0)}_{1,6}(9)$ of [3]). Now we prove the following theorem.

Theorem 3.2 *If G^0 is a hyperbolic reflection group then $m(G^0)$ is bounded above by a constant that depends only on $\ell(G^0)$.*

To prove this we require the following well-known lemma (see e.g., [10]).

Lemma 3.3 *If G is a (finitely generated) Coxeter group for which G^0 has a torsion-free subgroup of index n, then there is a homomorphism from G^0 onto a transitive group of permutations of $\{1, 2, \ldots, n\}$ for which every edge relator r_{ij} is mapped to a permutation consisting only of m_{ij}-cycles. If G^0 is a reflection group then the converse holds.*

If the transitivity condition is omitted we have a torsion-free subgroup of index not exceeding n.

Sketch of Proof: Given a torsion free subgroup H of index n, the required permutation representation is obtained by considering the action of G^0 on the n cosets of H.

The converse requires the fact that a finite order element of a reflection group must be conjugate to the power of an edge relator. It then follows that the stabilizer of any point in the set being permuted is torsion free. □

Let $C_n(m_1, m_2, \ldots, m_k)$ denote the Cayley graph generated by permutations of $\{1, 2, \ldots, n\}$ consisting entirely of m_i cycles $(1 \leq i \leq k)$. Using Theorem 3.1, we have the following result, which is essentially a restatement of Lemma 3.3 in terms of graph homomorphisms,

Lemma 3.4 *If G is a finitely-generated Coxeter group with edge relators of orders $m_1, m_2, \ldots, m_k$, and there exists an index n torsion-free subgroup of G^0, then there is a homomorphism from $\gamma(G)$ to $C_n(m_1, \ldots, m_k)$. If G^0 is a reflection group, and such a homomorphism exists, then G^0 has a torsion-free subgroup of index not exceeding n.*

Proof of Theorem 3.2: From Theorem 2.1 there exists a graph U with edges colored $m_1, m_2, \ldots, m_k$ with the property that $\gamma(G)$ maps homomorphically into U whenever G^0

is a reflection group whose edge relators have orders in $\{m_1, \ldots, m_k\}$. Since U can be construed as $\gamma(G)$ for some Coxeter group, Selberg's lemma and Lemma 3.4 give an n for which U maps homomorphically into $C_n(m_1, m_2, \ldots, m_k)$. Since clearly the composition of two homomorphisms is again a homomorphism, every $\gamma(G)$ maps homomorphically into this $C_n(m_1, m_2, \ldots, m_k)$, whenever the orders of the edge relators of G^0 are in $\{m_1, \ldots, m_k\}$. The theorem then follows from Lemma 3.4. $\square$

There are some cases where we can find a precise value for $m(G^0)$. When all the dihedral angles of P are equal to π/m, all the edges of the graph $\gamma(G(P))$ are colored m. In this case $\gamma(G(P))$ maps homomorphically into $C_n(m)$, if and only if it contains an imbedded copy of K_c, where c is the chromatic number of $\gamma(G(P))$. It is readily verified that $C_4(2)$ is isomorphic to K_4, so that, when $m = 2$, $G^0(P)$ has a torsion-free subgroup of index at most 4. Generally (and in all cases where P is bounded), this index will be exactly 4 and, of course, in $G(P)$, the same subgroup has index 8. This result was noted by Vesnin [12], and can be used to construct compact hyperbolic manifolds by glueing together 8 copies of P.

From Andreev's theorem ([13], Chapter 6, Theorem 2.8), the polyhedron P is unbounded when $m \geq 3$, but may have finite volume when $m = 3$. Since $C_6(3)$ contains a copy of K_4 (e.g., the Cayley graph generated by the three permutations (125)(364), (156)(234) and (163)(245)), we conclude, as above, that, when $m = 3$, $G^0(P)$ has a torsion-free subgroup of index at most 6.

We note that one of the main theorems of Edmonds et al. ([4], Theorem 1.4) can be formulated naturally in terms of graph homomorphisms (although this does not seem to lead to any purely combinatorial proof of it). It is equivalent to the statement that every circuit colored from $(m_1, \ldots, m_k)$ maps homomorphically into $C_n(m_1, \ldots, m_k)$ where either $n = \ell(G^0)$ or $n = 2\ell(G^0)$, depending on the individual case. Since, when G is a Dyck group (two-dimensional hyperbolic reflection group), $\gamma(G)$ is a circuit, the existence of index n (or $2n$) torsion-free subgroups follows, for these groups, from Lemma 3.4. This result is relatively easily proved for the other Fuchsian groups.

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