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Vexillary Elements in the Hyperoctahedral Group

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Abstract. In analogy with the symmetric group, we define the vexillary elements in the hyperoctahedral group to be those for which the Stanley symmetric function is a single Schur Q-function. We show that the vexillary elements can be again determined by pattern avoidance conditions. These results can be extended to include the root systems of types A, B, C, and D. Finally, we give an algorithm for multiplication of Schur Q-functions with a superfied Schur function and a method for determining the shape of a vexillary signed permutation using jeu de taquin.

Keywords: vexillary, Stanley symmetric function, reduced word, hyperoctahedral group

1. Introduction

The vexillary permutations in the symmetric group have interesting connections with the number of reduced words, the Littlewood-Richardson rule, Stanley symmetric functions, Schubert polynomials and the Schubert calculus. Lascoux and Schützenberger [16] have shown that vexillary permutations are characterized by the property that they avoid any subsequence of length 4 with the same relative order as 2143. Macdonald has given a good overview of vexillary permutations in [18]. In this paper we propose a definition for vexillary elements in the hyperoctahedral group. We show that the vexillary elements can again be determined by pattern avoidance conditions.

We begin by reviewing the history of the Stanley symmetric functions and establishing our notation. We have included several propositions from the literature, which we have used in the proof of the main theorem. In Section 3, we have defined the vexillary elements in the symmetric group and the hyperoctahedral group to be those elements for which the corresponding Stanley symmetric function is a single Schur function or Schur Q-function respectively with coefficient 1. We state and prove that the vexillary elements are precisely those elements which avoid 18 different patterns of lengths 3 and 4. Due to the quantity of cases that need to be analyzed we have used a computer to verify a key lemma in the proof of the main theorem. The definition of vexillary can be extended to cover the root systems of type A, B, C, and D; in all four cases the definition is equivalent to avoiding

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certain patterns. In Section 4, we give an algorithm for multiplication of Schur Q-functions with a superfied Schur function. In Section 5, we describe a method of finding the Stanley symmetric function of a vexillary signed permutation using jeu de taquin. We conclude with several open problems related to vexillary elements in the hyperoctahedral group.

2. The hyperoctahedral group and Stanley symmetric functions

Let S_n be the symmetric group whose elements are permutations written in one-line notation as $[w_1, w_2, \ldots, w_n]$. S_n is generated by the adjacent transpositions σ_i for $1 \le i < n$, where σ_i interchanges positions i and $i+1$ when acting on the right, i.e., $[\ldots, w_i, w_{i+1}, \ldots]\sigma_i = [\ldots, w_{i+1}, w_i, \ldots]$.

Let B_n be the hyperoctahedral group (or signed permutation group). The elements of B_n are permutations with a sign attached to every entry. We use the compact notation where a bar is written over an element with a negative sign. For example, $[\bar{3}, 2, \bar{1}] \in B_3$. B_n is generated by the adjacent transpositions σ_i for $1 \le i < n$, as in S_n, along with σ_0 which acts on the right by changing the sign of the first element, i.e., $[w_1, w_2, \ldots, w_n]\sigma_0 = [\overline{w}_1, w_2, \ldots, w_n]$.

If w can be written as a product of the generators $\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_p}$ and p is minimal, then the concatenation of the indices $a_1a_2\cdots a_p$ is a *reduced word* for w, and p is the *length* of w, denoted $l(w)$. Let $R(w)$ be the set of all reduced words for w. The signed (or unsigned) permutations $[w_1, \ldots, w_n]$ and $[w_1, \ldots, w_n, n+1, n+2, \ldots]$ have the same set of reduced words. For our purposes it will be useful to consider these signed permutations as the same in the infinite groups $S_\infty = \cup S_n$ or $B_\infty = \cup B_n$.

Let s_λ be the Schur function of shape λ and let Q_λ be the Schur Q-function of shape λ. See [17] for definitions of these symmetric functions.

Definition 1 For $w \in S_n$, define the S_n Stanley symmetric function by

$$G_w(X) = \sum_{\mathbf{a}\in R(w)} \sum_{\substack{(i_1\le\cdots\le i_l)\\ \in A(D(\mathbf{a}))}} x_{i_1}x_{i_2}\cdots x_{i_l}, \tag{1}$$

where $A(D(\mathbf{a}))$ is the set of all weakly increasing sequences such that if $a_{k-1} > a_k$, then $i_{k-1} < i_k$.

For $w \in B_n$, define the C_n Stanley symmetric function by

$$F_w(X) = \sum_{\mathbf{a}\in R(w)} \sum_{\substack{(i_1\le\cdots\le i_l)\\ \in A(P(\mathbf{a}))}} 2^{|\mathbf{i}|} x_{i_1}x_{i_2}\cdots x_{i_l}, \tag{2}$$

where $A(P(\mathbf{a}))$ is the set of all weakly increasing sequences such that if $a_{k-1} < a_k > a_{k+1}$ then we don't have $i_{k-1} = i_k = i_{k+1}$, (i.e., we cannot have equality across a peak in the corresponding reduced word), and $|\mathbf{i}|$ denotes the number of distinct values i_j in the admissible sequence, i.e., the number of distinct variables in the monomial.

In [23], Stanley showed that G_w is a symmetric function and used it to express the number of reduced words of a permutation w in terms of f^λ, the number of standard tableaux of shape λ. Namely,

$$\#R(w) = \sum \alpha_w^\lambda f^\lambda, \tag{3}$$

where α_w^λ is the coefficient of s_λ in G_w. Bijective proofs of (3) were given independently by Lascoux and Schützenberger [15] and Edelman and Greene [6]. Reiner and Shimozono [21] have given a new interpretation of the coefficients α_w^λ in terms of $D(w)$-peelable tableaux.

Stanley also conjectured that there should be an analog of (3) for B_n. This conjecture was proved independently by Haiman [9] and Kraśkiewicz [10] in the following form:

$$\#R(w) = \sum \beta_w^\lambda g^\lambda, \tag{4}$$

where g^λ is the number of standard tableaux on the shifted shape λ, and β_w^λ is the coefficient of Q_λ when F_w is expanded in terms of the Schur Q-functions.

The Stanley symmetric functions can also be defined using the nilCoxeter algebra of S_n and B_n, see [7, 8]. The relationship between Kraśkiewicz's proof of (4) and B_n Stanley symmetric functions is explored in [12]. See also [2, 13, 27] for other connections to Stanley symmetric functions. The functions F_w are usually referred to as the Stanley symmetric functions of type C because they are related to the root systems of type C. The Weyl groups for the root systems of type B and C are isomorphic, so we can study the group B_n by studying either root system. We consider the root systems of type B and D at the end of Section 3.

The Stanley functions F_w can easily be computed using Proposition 2 below which is stated in terms of special elements in B_n. There are two types of "transpositions" in the hyperoctahedral group. These transpositions correspond to reflections in the Weyl group of the root system B_n. Let t_{ij} be a transposition of the usual type, i.e., $[\ldots, w_i, \ldots, w_j, \ldots]t_{ij} = [\ldots, w_j, \ldots, w_i, \ldots]$. Let $s_{ij}, i < j$ be a transposition of two elements that also switches sign $[\ldots, w_i, \ldots, w_j, \ldots]s_{ij} = [\ldots, \overline{w}_j, \ldots, \overline{w}_i, \ldots]$. We define s_{ii} to be the "transposition" which changes the sign of the ith element, i.e., $[\ldots, w_i, \ldots]s_{ii} = [\ldots, \overline{w}_i, \ldots]$. A signed permutation w is said to have a *descent* at r if $w_r > w_{r+1}$.

Proposition 2 ([1]) *The Stanley symmetric functions of type C have the following recursive formulas*:

$$F_w = \sum_{\substack{0<i<r \\ l(wt_{rs}t_{ir})=l(w)}} F_{wt_{rs}t_{ir}} + \sum_{\substack{0<i \\ l(wt_{rs}s_{ir})=l(w)}} F_{wt_{rs}s_{ir}}, \tag{5}$$

where r is the last descent of w, and s is the largest position such that $w_s < w_r$. The recursion terminates when w is strictly increasing in which case $F_w = Q_\lambda$, where λ is the partition obtained from arranging $\{|w_i| : w_i < 0\}$ in decreasing order.

For example, let $w = [\bar{4}, 1, \bar{2}, 3]$. Then $r = 2$ since $w_2 > w_3$ is a descent and $w_3 < w_4$, and $s = 3$ since $w_3 < w_2 < w_4$. This implies $wt_{rs} = [\bar{4}, \bar{2}, 1, 3]$ and we have

$$F_{[\bar{4},1,\bar{2},3]} = F_{[\bar{4},1,\bar{2},3]t_{23}t_{12}} + F_{[\bar{4},1,\bar{2},3]t_{23}s_{42}} = F_{[\bar{2},\bar{4},1,3]} + F_{[\bar{4},\bar{3},1,2]}. \tag{6}$$

Continuing to expand the right-hand side we see $[\bar{4}, \bar{3}, 1, 2]$ is strictly increasing, so $F_{[\bar{4},\bar{3},1,2]} = Q_{(4,3)}$ and $F_{[\bar{2},\bar{4},1,3]} = F_{[\bar{2},\bar{4},1,3]t_{12}s_{15}} = F_{[\bar{5},\bar{2},1,3,4]} = Q_{(5,2)}$. Hence, $F_{[\bar{4},1,\bar{2},3]} = Q_{(4,3)} + Q_{(5,2)}$.

Note that $l(wt_{rs})$ is always equal to $l(w) - 1$ in Proposition 2 because of the choice of r and s. Hence, if $l(wt_{rs}\tau_{ir}) = l(w)$, then $l(wt_{rs}\tau_{ir}) = l(wt_{rs}) + 1$ where τ_{ij} is a transposition of either type. The reflections which increase the length of wt_{rs} by exactly 1 are characterized by the following two propositions.

Proposition 3 ([19]) *If $w \in S_\infty$ or B_∞ and $i < j$, then $l(wt_{ij}) = l(w) + 1$ if and only if*
- $w_i < w_j$

and no k exists such that
- $i < k < j$ *and* $w_i < w_k < w_j$.

Note that the first condition above guarantees that $l(wt_{ij}) > l(w)$, and the second condition determines when the length is increased by exactly 1. Similarly, in the next proposition, the first two conditions guarantee that $l(ws_{ij}) > l(w)$, and the next two conditions determine when $l(ws_{ij}) - l(w)$ is 1.

Proposition 4 ([1]) *If $w \in B_\infty$, and $i \leq j$, then $l(ws_{ij}) = l(w) + 1$ if and only if*
- $-w_i < w_j$ *and* $-w_j < w_i$
- *if* $i \neq j$, *either* $w_i < 0$ *or* $w_j < 0$,

and no k exists such that either of the following is true:
- $k < i$ *and* $-w_j < w_k < w_i$
- $k < j$ *and* $-w_i < w_k < w_j$.

3. Main results

In this section we give the definition of the vexillary elements in S_n and B_n. Then we present the main theorem. The proof follows after several lemmas.

Definition 5 If $w \in S_n$ then w is vexillary if $G_w = s_\lambda$ for some shape $\lambda \vdash l(w)$. Similarly, if $w \in B_n$ then w is vexillary if $F_w = Q_\lambda$ for some shape $\lambda \vdash l(w)$ with distinct parts.

It follows from Eq. (3) that if w is vexillary then the number of reduced words for w is the number of standard tableaux of a single shape (unshifted for $w \in S_n$ or shifted for $w \in B_n$).

For S_n, this definition is equivalent to the original definition of vexillary given by Lascoux and Schützenberger in [16]. They showed that vexillary permutations w are characterized by the condition that no subsequence $a < b < c < d$ exists such that $w_b < w_a < w_d < w_c$. This property is usually referred to as 2143-*avoiding*. Lascoux and Schützenberger also showed that the Schubert polynomial of type A_n indexed by w is a flagged Schur function if and only if w is a vexillary permutation. One might ask if the Schubert polynomials of type B, C or D indexed by a vexillary element could be written in terms of a "flagged Schur Q-function."

Many other properties of permutations can be given in terms of pattern avoidance. For example, the reduced words of 321-avoiding [3] permutations all have the same content, and a Schubert variety in SL_n/B is smooth if and only if it is indexed by a permutation which avoids the patterns 3412 and 4231 [11]. Also, West [28], Simion and Schmidt [22], Noonan [20], and Bona [4, 5] have studied pattern avoidance more generally and given formulas for computing the number of permutations which avoid combinations of patterns. Recently, Stembridge [27] has described several properties of signed permutations in terms of pattern avoidance as well.

We define pattern avoidance in terms of the following function which *flattens* any subsequence into a signed permutation.

Definition 6 Given any sequence $a_1 a_2 \cdots a_k$ of distinct nonzero real numbers, define $\mathrm{fl}(a_1 a_2 \cdots a_k)$ to be the unique element $b = [b_1, \ldots, b_k]$ in B_k such that

- For all j, both a_j and b_j have the same sign.
- For all i, j, we have $|b_i| < |b_j|$ if and only if $|a_i| < |a_j|$.

For example, $\mathrm{fl}(\bar{6}, 3, \bar{7}, 0.5) = [\bar{3}, 2, \bar{4}, 1]$. Any word containing the subsequence $\bar{6}, 3, \bar{7}$, 0.5 does not avoid the pattern $\bar{3}2\bar{4}1$.

Another way to describe pattern avoidance is with the signed permutation matrices. Namely, a signed permutation matrix w avoids the pattern v if no submatrix of w is the matrix v.

Theorem 7 *An element $w \in B_\infty$ is vexillary if and only if every subsequence of length* 4 *in w flattens to a vexillary element in B_4. In particular, w is vexillary if and only if it avoids the following patterns*:

$$\begin{array}{ccccc} \bar{3}2\bar{1} & \bar{3}21 & 32\bar{1} & 321 & 3\bar{1}2 \\ \bar{2}31 & \bar{1}32 & \bar{4}\bar{1}\bar{2}3 & \bar{4}1\bar{2}3 & \bar{3}\bar{4}\bar{1}\bar{2} \\ \bar{3}\bar{4}1\bar{2} & 3\bar{4}\bar{1}\bar{2} & 3\bar{4}1\bar{2} & 3142 & \bar{2}\bar{3}4\bar{1} \\ 2413 & 2\bar{3}4\bar{1} & 2143 & & \end{array} \tag{7}$$

This list of patterns was conjectured in [13]. Due to the large number of non-vexillary patterns in (7) we have chosen to prove the theorem in two steps. First, we have verified that the theorem holds for B_6 (see Lemma 8). Second, we show that any counterexample in B_∞ would imply a counterexample in B_6.

Lemma 8 *Let $w \in B_6$, then w is vexillary if and only if it does not contain any subsequence of length* 3 *or* 4 *which flattens to a pattern in* (7).

The LISP code used to verify Lemma 8 is available from the first author on request. In summary, we verify that the following two statements are either both true or both false for each element w in B_6:

- Is w vexillary? (This is computed using the recurrence from Proposition 2)
- Does w avoid all of the patterns in (7)? (Compare each flattened subsequence of lengths 3 and 4 with the pattern list).

Lemma 9 *Let w be any signed permutation. Suppose $w_{i_1} w_{i_2} \cdots w_{i_k}$ is a subsequence of w, and let $u \in B_k$ be $\text{fl}(w_{i_1} w_{i_2} \cdots w_{i_k})$. Then the following statements hold*:

1. *If the last descent of $w_{i_1} w_{i_2} \cdots w_{i_k}$ appears in position i_r, then the last descent of u is in position r.*
2. *If in addition, $w_{i_s} < w_{i_r}$ and i_s is the largest index in w such that this is true, then $u_s < u_r$ and s is the largest index in u such that this is true.*
3. *If $v = w\tau_{i_j i_k}$ then $\text{fl}(v_{i_1} \cdots v_{i_k}) = \text{fl}(w_{i_1} \cdots w_{i_k}) \cdot \tau_{jk}$ where $\tau_{i_j i_k}$ and τ_{jk} are transpositions of the same type.*

These facts follow directly from the definition of the flatten function.

Lemma 10 *For any $v \in B_\infty$ and any $0 < i < r$, if $l(vt_{ir}) - l(v) > 0$ then there exists an index k such that $i \le k < r$, $v_i \le v_k < v_r$ and $l(vt_{kr}) - l(v) = 1$. Similarly, if $l(vs_{ir}) - l(v) > 0$ then there exists an index k such that either*

- *$k < r$, $v_k < v_r$, and $l(vt_{kr}) - l(v) = 1$, or*
- *$k \le i$, $-v_r < v_k \le v_i$, and $l(vs_{kr}) - l(v) = 1$.*

Proof: If $l(vt_{ir}) - l(v) > 0$, consider the set $\{v_j : i \le j < r \text{ and } v_j < v_r\}$. This set is nonempty since v_i is a member. Pick k such that v_k is the largest value in this set. Then no j exists such that $k < j < r$ and $v_k < v_j < v_r$, hence by Proposition 3, $l(vt_{kr}) - l(v) = 1$.

Say $l(vs_{ir}) - l(v) > 0$. If there exists $k < r$ such that $v_k < v_r$, choose k such that v_k is the largest value in $\{v_k < v_r : k < r\}$. Then no j exists such that $k < j < r$ and $v_k < v_j < v_r$, hence by Proposition 3, $l(vt_{kr}) - l(v) = 1$.

Otherwise, assume $l(vs_{ir}) - l(v) > 0$ and no k exists such that $k < r$ and $v_k < v_r$. In particular, this means $v_i > v_r$. Recall from Proposition 4, $l(vs_{ir}) - l(v) > 0$ implies that either $v_i < 0$ or $v_r < 0$ and not both. Thus, $v_i > 0 > v_r$. Choose k such that v_k is the smallest value in $\{v_k > -v_r : k \le i\}$. This set is not empty since v_i is in the set from the remarks just before Proposition 4. Furthermore, no $j < r$ exists such that $-v_k < v_j < v_r$ (by assumption), and no $j' < k$ exists such that $-v_r < v_{j'} < v_k$ (by choice of k). Hence $l(vs_{kr}) - l(v) = 1$ by Proposition 4. □

Lemma 11 *Given any $w \in B_\infty$ and any subsequence of w, say $w_{i_1} w_{i_2} \cdots w_{i_k}$, let $v = \text{fl}(w_{i_1} w_{i_2} \cdots w_{i_k}) \in B_k$. If $l(wt_{i_j, i_k}) - l(w) = 1$ then $l(vt_{jk}) - l(v) = 1$. Similarly, if $l(ws_{i_j, i_k}) - l(w) = 1$ then $l(vs_{jk}) - l(v) = 1$.*

Proof: If $l(wt_{i_j, i_k}) - l(w) \ge 1$ then $w_{i_j} < w_{i_k}$ so $v_j < v_k$ since the flatten map preserves the relative order of the elements in the subsequence and signs. Therefore, $l(vt_{jk}) - l(v) \ge 1$. If $l(wt_{i_j, i_k}) - l(w) = 1$ then no $i_j < m < i_k$ exists such that $w_{i_j} < w_m < w_{i_k}$. This in turn implies that no $j < m < k$ exists such that $v_j < v_m < v_k$, hence $l(vt_{jk}) - l(v) = 1$.

If $l(ws_{i_j, i_k}) - l(w) \ge 1$ then $-w_{i_j} < w_{i_k}$ and $-w_{i_k} < w_{i_j}$ so $-v_j < v_k$ and $-v_k < v_j$ since the flatten map preserves the relative order of the elements in the subsequence and signs.

Also, if $i_j \neq i_k$ then either $w_{i_j} < 0$ or $w_{i_k} < 0$ so either $v_j < 0$ or $v_k < 0$. Therefore, $l(vs_{jk}) - l(v) \geq 1$. If $l(ws_{i_j,i_k}) - l(w) = 1$ then no $m < i_k$ exists such that $-w_{i_j} < w_m < w_{i_k}$, and no $m < i_j$ exists such that $-w_{i_k} < w_m < w_{i_j}$. This in turn implies that no $m < k$ exists such that $-v_j < v_m < v_k$, and no $m < j$ exists such that $-v_k < v_m < v_j$, hence $l(vs_{jk}) - l(v) = 1$. □

Lemma 12 *Given any $w \in B_\infty$, if w is non-vexillary then w contains a subsequence of length 4 which flattens to a non-vexillary element in B_4.*

Proof: Since w is non-vexillary then either F_w expands into multiple terms on the first step of the recurrence in (5) or else the recurrence yields $F_w = F_v$ where v is again non-vexillary. Assume the first step of the recurrence gives

$$F_w = F_{wt_{rs}\tau_{ir}} + F_{wt_{rs}\tau_{jr}} + \text{other terms}$$

Let n be the smallest index such that $w_i = i$ for all $i > n$, then $n + 1$ is greater than i, j, r and s. Let $\alpha : \{1, 2, 3, 4\} \to \{i, j, r, s, n+1\}$ be an order preserving map onto the 4 smallest distinct numbers in the range. Let $w' = \text{fl}(w_{\alpha(1)} w_{\alpha(2)} w_{\alpha(3)} w_{\alpha(4)})$. By Lemma 11

$$l\big(w'\big[t_{\alpha^{-1}(r)\alpha^{-1}(s)}\big]\big[\tau_{\alpha^{-1}(i)\alpha^{-1}(r)}\big]\big) = l(w')$$

and

$$l\big(w'\big[t_{\alpha^{-1}(r)\alpha^{-1}(s)}\big]\big[\tau_{\alpha^{-1}(j)\alpha^{-1}(r)}\big]\big) = l(w').$$

Therefore, the recursion implies

$$F_{w'} = F_{w'[t_{\alpha^{-1}(r)\alpha^{-1}(s)}][\tau_{\alpha^{-1}(i)\alpha^{-1}(r)}]} + F_{w'[t_{\alpha^{-1}(r)\alpha^{-1}(s)}][\tau_{\alpha^{-1}(j)\alpha^{-1}(r)}]} + \text{other terms.}$$

Hence, $w' \in B_4$ is not vexillary, and it follows that w contains the non-vexillary subsequence $w_{\alpha(1)} w_{\alpha(2)} w_{\alpha(3)} w_{\alpha(4)}$.

If, on the other hand, the first step of the recursion gives $F_w = F_v$ then $v = wt_{rs}\tau_{ir}$ and v is not vexillary. Assume by induction on the number of steps until the recurrence branches into multiple terms, that v contains a non-vexillary subsequence say $v_a v_b v_c v_d$. If $i, r, s \notin \{a, b, c, d\}$ then $w_a w_b w_c w_d$ is exactly the same non-vexillary subsequence. So we can assume the set $\{a, b, c, d, i, r, s\}$ has at most 6 elements. Let

$$\phi : \{1, 2, \ldots, 6\} \to \{a, b, c, d, i, r, s\} \cup \{n + 1, n + 2\}$$

be an order preserving map which sends the numbers 1 through 6 to the 6 smallest distinct integers in the range. Let $w' = \text{fl}(w_{\phi(1)} w_{\phi(2)} \cdots w_{\phi(6)})$ and $v' = \text{fl}(v_{\phi(1)} v_{\phi(2)} \cdots v_{\phi(6)})$. By construction, $v' \in B_6$ contains a non-vexillary subsequence, hence v' is not vexillary by Lemma 8. We will use the recursion on $F_{w'}$ to show that w' is not vexillary in B_6. From Lemma 9 it follows that

$$v' = w' t_{\phi^{-1}(r)\phi^{-1}(s)} \tau_{\phi^{-1}(i)\phi^{-1}(r)}.$$

By Lemma 11, $l(v) = l(wt_{rs}) + 1 = l(w)$ implies $l(v') = l(w')$. Therefore,

$$F_{w'} = F_{v'} + \text{possibly other terms.}$$

Regardless of whether there are any other terms in the expansion of $F_{w'}$, w' is not vexillary since v' is not vexillary. Again by Lemma 8, this implies that w' contains a non-vexillary subsequence of length 4, say $w'_e w'_f w'_g w'_h$. Hence, w contains the non-vexillary subsequence $w_{\phi(e)} w_{\phi(f)} w_{\phi(g)} w_{\phi(h)}$. □

This proves one direction of Theorem 7.

Lemma 13 *Given any $w \in B_\infty$, if w contains a subsequence of length 4 which flattens to a non-vexillary element in B_4, then w is non-vexillary.*

Proof: Assume w is vexillary and let $w^{(1)}, w^{(2)}, \ldots, w^{(k)}$ be the sequence of signed permutations which arise in expanding $F_w = F_{w^{(1)}} = F_{w^{(2)}} = \cdots = F_{w^{(k)}}$ using the recurrence (5). This recurrence terminates when the signed permutation $w^{(k)}$ is strictly increasing, hence $w^{(k)}$ does not contain any of the patterns in (7). Replace w by the first $w^{(i)}$ such that $w^{(i)}$ contains a non-vexillary subsequence and $w^{(i+1)}$ does not, and let $v = w^{(i+1)} = wt_{rs}\tau_{ir}$.

Say $w_a w_b w_c w_d$ is a non-vexillary subsequence in w. If $i, r, s \notin \{a, b, c, d\}$, then $v_a v_b v_c v_d$ would be exactly the same non-vexillary subsequence. This contradicts our choice of v. So we can assume that the order of the set $\{a, b, c, d, i, r, s\}$ is less than or equal to 6. As in the proof of Lemma 12, let

$$\phi : \{1, 2, \ldots, 6\} \to \{a, b, c, d, i, r, s\} \cup \{n + 1, n + 2\}$$

be an order preserving map onto the smallest 6 distinct numbers in the range. Let $w' = \mathrm{fl}(w_{\phi(1)} w_{\phi(2)} \cdots w_{\phi(6)})$ and $v' = \mathrm{fl}(v_{\phi(1)} v_{\phi(2)} \cdots v_{\phi(6)})$. To simplify notation, we also let $i' = \phi^{-1}(i)$, $r' = \phi^{-1}(r)$, and $s' = \phi^{-1}(s)$. By construction, $w' \in B_6$ contains a non-vexillary subsequence hence w' is not vexillary by Lemma 8. As in Lemma 12 one can show

$$F_{w'} = F_{v'} + \text{other terms.}$$

Since w' contains a non-vexillary subsequence and v' does not, there must be another term in $F_{w'}$ indexed by a reflection $\tau_{j'r'} \neq \tau_{i'r'}$ with $l(w' t_{r's'} \tau_{j'r'}) = l(w')$. One should note that $i' = j'$ is possible but then $\tau_{i'r'}$ and $\tau_{j'r'}$ must be different types of transpositions. Let $j = \phi(j')$. By Proposition 4 and the definition of the flatten function, we have $l(wt_{rs}\tau_{jr}) - l(wt_{rs}) > 0$. By Lemma 10 there exists a reflection τ_{kr} such that $l(wt_{rs}\tau_{kr}) - l(wt_{rs}) = 1$.

We must have $\tau_{kr} \neq \tau_{ir}$ since $\tau_{i'r'} \neq \tau_{j'r'}$. Hence,

$$F_w = F_{wt_{rs}\tau_{ir}} + F_{wt_{rs}\tau_{kr}} + \text{possibly other terms.}$$

This proves w is not vexillary, contrary to our assumption. □

This completes the proof of Theorem 7.

The definition of vexillary can be extended using Stanley symmetric functions of type B and D. These cover the remaining infinite families of root systems. It was shown in [2, 12] that these Stanley symmetric functions are always nonnegative linear combinations of Schur P-functions. For these cases, we define vexillary by the condition that the Stanley symmetric function is a single Schur P-function with coefficient 1.

Theorem 14 *An element $w \in B_\infty$ is vexillary for type B if and only if every subsequence of length 4 in w flattens to a vexillary element of type B in B_4. In particular, w is vexillary if and only if it avoids the following patterns*:

$$\begin{array}{rrr} 21 & \bar{3}2\bar{1} & 2\bar{3}4\bar{1} \\ \bar{2}\bar{3}4\bar{1} & 3\bar{4}\bar{1}\bar{2} & \bar{3}\bar{4}1\bar{2} \\ \bar{3}\bar{4}\bar{1}\bar{2} & \bar{4}1\bar{2}3 & \bar{4}\bar{1}\bar{2}3 \end{array} \tag{8}$$

An element $w \in D_\infty$ is vexillary for type D if and only if every subsequence of length 4 avoids the following patterns:

$$\begin{array}{rrrrrr} 132 & \bar{1}32 & 321 & 32\bar{1} & \bar{3}21 & \bar{3}2\bar{1} \\ 2\bar{3}41 & 2\bar{3}4\bar{1} & \bar{2}\bar{3}41 & \bar{2}\bar{3}4\bar{1} & 3412 & 34\bar{1}2 \\ 3\bar{4}1\bar{2} & 3\bar{4}\bar{1}\bar{2} & \bar{3}412 & \bar{3}4\bar{1}2 & \bar{3}\bar{4}1\bar{2} & \bar{3}\bar{4}\bar{1}\bar{2} \\ 41\bar{2}3 & 4\bar{1}\bar{2}3 & \bar{4}1\bar{2}3 & \bar{4}\bar{1}\bar{2}3 & & \end{array} \tag{9}$$

Note, that the patterns that are avoided by vexillary elements of type D are not all type D signed permutations but instead include some elements with an odd number of negative signs. The proof of Theorem 14 is very similar to the proof of Theorem 7 given above. The analogs of Proposition 2 are given in [1]. Again the proof relies on a computer verification that these patterns characterize all vexillary elements in B_6 and D_6.

4. A rule for multiplication

Lascoux and Schützenberger noticed that the transition equation for Schubert polynomials of vexillary permutations can be used to multiply Schur functions [18, p. 62]. This provides an alternative to the Littlewood-Richardson rule. There is an analog of Littlewood-Richardson rule that can be used to multiply Schur Q-functions [25, 30]. L. Manivel asked if the transition equations for Schubert polynomials of types B, C, and D could lead to a rule for multiplying Schur Q-functions. The answer is "sometimes". There are only certain shifted shapes μ which can easily be multiplied by an arbitrary Schur Q-function. Therefore, we have investigated a different problem. In this section we present an algorithm for multiplication of a Schur Q-function by a superfied Schur function, $\phi(s_\lambda)$.

Let ϕ be the homomorphism from the ring of symmetric functions onto the subring generated by odd power sums defined by

$$\phi(p_k) = \begin{cases} 2p_k & \text{for } k \text{ odd,} \\ 0 & \text{for } k \text{ even.} \end{cases} \tag{10}$$

The image of a Schur function under this map, $\phi(s_\lambda)$, is called a *superfied Schur function*. The superfied Schur functions appear in connection with the Lie super algebras [24, 29].

The Stanley symmetric functions of type A and C which are indexed by permutations are related via the superfication operator.

Proposition 15 ([2, 12, 26]) *For $v \in S_n$, we have $F_v = \phi(G_v)$.*

Let $v = [v_1, \ldots, v_d]$ be any signed permutation. We denote the signed permutation $[1, 2, \ldots, n, v_1 + n, \ldots, v_d + n]$ by $1^n \times v$. Also, if $w = [w_1, \ldots, w_n]$ is another signed permutation, let $w \times v$ be $[w_1, \ldots, w_n, v_1 + n, \ldots, v_d + n] \in B_{n+d}$.

Lemma 16 *For $v \in S_\infty$ and $w \in B_n$ we have*

$$F_w F_v = F_w F_{1^n \times v} = F_{w \times v}. \tag{11}$$

Proof: From (2), when $v \in S_\infty$, F_v is equal to $F_{1^n \times v}$ since $a_1 a_2 \cdots a_p \in R(v)$ if and only if $(a_1 + n)(a_2 + n) \cdots (a_p + n) \in R(1^n \times v)$. The reduced words for $w \times v$ are all shuffles of a reduced word for w with a reduced word for $1^n \times v$. It remains to show that the monomials in $F_{v \times w}$ are exactly the product of monomials in F_w and $F_{1^n \times v}$ counted with their coefficients.

Let $\mathbf{a} = a_1 a_2 \cdots a_p \in R(1^n \times v)$ and $\mathbf{i} = i_1 i_2 \cdots i_p$ be an admissible sequence of $\mathbf{a}$. We call $(\mathbf{a}, \mathbf{i})$ an admissible pair of $1^n \times v$. Similarly, let $(\mathbf{b}, \mathbf{j})$ be an admissible pair of w. We now form an admissible pair $(\mathbf{c}, \mathbf{k})$ of $w \times v$. Let $\mathbf{k} = k_1 k_2 \cdots k_{p+q}$ be a rearrangement of $\mathbf{ij}$ in weakly increasing order. To construct $\mathbf{c}$, consider a constant subsequence $k_e = k_{e+1} = \cdots = k_f$ in $\mathbf{k}$ with $k_{e-1} < k_e$ and $k_f < k_{f+1}$. If this subsequence comes entirely from $\mathbf{i}$ (respectively $\mathbf{j}$) the corresponding part of the reduced word $\mathbf{c}$ is made up entirely of the corresponding part of $\mathbf{a}$ (respectively $\mathbf{b}$). If it contains numbers from both $\mathbf{i}$ and $\mathbf{j}$, then there are two choices for $c_e c_{e+1} \cdots c_f$:

$$a_r a_{r+1} \cdots a_s b_t b_{t+1} \cdots b_u a_{s+1} a_{s+2} \cdots a_z$$

or

$$a_r a_{r+1} \cdots a_{s-1} b_t b_{t+1} \cdots b_u a_s a_{s+1} a_{s+2} \cdots a_z$$

where a_s is the smallest number in $a_r a_{r+1} \cdots a_u$. One can check that $\mathbf{k}$ is an admissible sequence for all possible choices of $\mathbf{c}$ described above. Furthermore, each nonempty constant subsequence $k_e \cdots k_f$ contributes a factor to the coefficient of $x_{k_1} x_{k_2} \cdots x_{k_{p+q}}$. Namely, the factor is 2 if the sequence comes entirely from $\mathbf{i}$ or $\mathbf{j}$, and if the sequence comes from both we get a contribution of 2 for each of the two reduced subwords above so the factor is 4. Therefore, the sum of the coefficients of $x_{k_1} x_{k_2} \cdots x_{k_{p+q}}$ for $\mathbf{k}$ and all possible choices of $\mathbf{c}$ will be $2^\alpha 2^\beta 4^\gamma$ where α (and β) is the number of constant sequences only in $|\mathbf{i}|$ (in $|\mathbf{j}|$ respectively), and γ is the number of constant sequences from both. Clearly, $2^\alpha 2^\beta 4^\gamma$ equals $2^{|\mathbf{i}|} 2^{|\mathbf{j}|}$ which is the coefficient of the product corresponding to the admissible pairs $(\mathbf{a}, \mathbf{i})$ and $(\mathbf{b}, \mathbf{j})$. □

The algorithm for multiplying $F_w F_v$ given above will not carry over for arbitrary elements of B_∞ because $F_v = F_{1^n \times v}$ if and only if $v \in S_\infty$. However, from the algorithm and Proposition 15 we have the following corollary.

Corollary 17 *Let $w \in B_\infty$ such that $F_w = Q_\mu$, and let $v \in S_\infty$ such that $G_v = s_\lambda$. Then*

$$Q_\mu \cdot \phi(s_\lambda) = F_{w\times v} \tag{12}$$

and $F_{w\times v}$ can be determined by the recursive formula in Proposition 2.

Note, that D. Worley has shown that $Q_\mu \cdot \phi(s_\lambda)$ is equal to a certain skew Schur Q-function [29, 7.11].

In the special case when $F_v = \phi(s_\lambda)$ is a single Schur Q-function, say Q_ν, Corollary 17 can be used to multiply Q_μ and Q_ν. This occurs only when v is a vexillary element of type C, and ν is of the form $(m+l-1, m+l-3, m+l-5, \ldots, m-l+1)$ for some positive integers $m \geq l$ [14].

5. The shape of a signed permutation

Given a vexillary element w, for which straight shape λ does $G_w = s_\lambda$ if $w \in S_\infty$? For which shifted shape μ does $F_w = Q_\mu$ if $w \in B_\infty$? For $w \in S_\infty$ there are several ways to determine this shape: the transition equation [18, p. 52], inserting a single reduced word using the Edelman-Greene correspondence [6], or by rearranging the code in decreasing order [23]. For a vexillary element of type C one can find the shape μ for which $F_w = Q_\mu$ by using the recursive formula (5) or by using the Kraśkiewicz insertion [10] or Haiman procedures [9] on a single reduced word.

Definition 18 Given an element v of S_n, the code of v is defined to be the composition $(c_1, c_2, \ldots, c_n)$ where $c_i = \#\{1 \leq j \leq n : v_i > v_j\}$. The shape of v, denoted by $\lambda(v)$, is defined to be the transpose of the partition given by rearranging the code in decreasing order.

It is well known that if v is a vexillary permutation of S_n, then $G_v = s_{\lambda(v)}$. We now describe a procedure to define the shape $\lambda^B(w)$ of a signed permutation w so that when w is vexillary, $F_w = Q_{\lambda^B(w)}$.

Let w be an element of B_n, not necessarily vexillary. Rearrange the numbers in w in increasing order and denote this new signed permutation by u. Let $v \in S_n$ so that $w = uv$. Note that $l(w) = l(u) + l(v)$ and u is vexillary with $F_u = Q_\mu$ where μ is the strictly decreasing sequence given by $\{|u_i| : u_i < 0\}$ which is the same set as $\{|w_i| : w_i < 0\}$. For any standard shifted Young tableau U of shape μ and any standard Young tableau V of straight shape $\lambda(v)$, we form a new standard shifted tableau $U * V$ by jeu de taquin as follows:

1. Embed U into the shifted shape $\delta = (n, n-1, \ldots, 2, 1)$.

2. Obtain a tableau R by filling the remaining boxes of δ with $1', 2', \ldots$ starting from the rightmost column and in each column from bottom to top.
3. Add $|\mu|$ to each entry of V to obtain S.
4. Append R on the left side of S to obtain T.
5. Delete the box containing $1'$ in T. If the resulting tableau is not shifted, apply jeu de taquin to fill in the box. Repeat the procedure for the box containing $2'$ and so on until all the primed numbers are removed.
6. The resulting tableau of shifted shape is denoted $U * V$.

We illustrate the procedure with an example. Suppose $w = [3, -2, -4, 1]$ then $w = uv$ where u is $[-4, -2, 1, 3]$ and v is $[4, 2, 1, 3]$. Let $U = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ & 5 & 6 & \end{array}$ and $V = \begin{array}{cc} 1 & 2 \\ 3 & \\ 4 & \end{array}$ Here, U has shape $(4, 2)$ and V has shape $\lambda([4, 2, 1, 3]) = (2, 1, 1)$. Then, Steps 1 through 4 will produce the following tableau:

$$T = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \\ & 5 & 6 & 3' & 9 & \\ & & 4' & 2' & 10 & \\ & & & 1' & & \end{array}. \tag{13}$$

Deleting the boxes and applying jeu de taquin as in Step 5 gives

$$U * V = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \\ & 5 & 6 & 9 & & \\ & & 10 & & & \end{array} \tag{14}$$

If V is the standard Young tableau with entries filled successively from left to right and then from top row down to the next, we denote the shape of $U * V$ by $\lambda^B(w)$ and call it the shape of w. In the example above, V is of the form described and the result of combining U and V by jeu de taquin in the example gives the shape $\lambda^B([3, -2, -4, 1]) = (6, 3, 1)$. It can be verified that $F_{[3,-2,-4,1]} = Q_{(6,3,1)}$.

Theorem 19 *For any $w \in B_n$, $Q_{\lambda^B(w)}$ appears in the expansion of F_w with a nonzero coefficient. In particular, if w is vexillary,*

$$F_w = Q_{\lambda^B(w)}. \tag{15}$$

Proof: Using the notation from the algorithm above, one sees that U corresponds to a reduced word **a** of u under the B_n-Edelman-Greene correspondence (a.k.a. Haiman correspondance) [9]. Also V corresponds to a reduced word **b** of v under the A_n-Edelman-Greene correspondence. Then **ab** is a reduced word of w. The steps described above give a sequence of shifted jeu de taquin moves for the Haiman procedures on **ab**. Hence $Q_{\lambda^B(w)}$ appears with positive coefficient in the expansion of F_w. If w is vexillary, the resulting tableau $U * V$ is independent of the choices of U and V. Since F_w is a single Schur Q-function, the resulting shape $\lambda^B(w)$ must be that of the Schur Q-function. □

6. Open problems

The vexillary permutations in S_n have many interesting properties. We would like to explore the possibility that these properties have analogs for the vexillary elements in B_n.

1. Is there a relationship between smooth Schubert varieties in $Sp(2n)/B$, $SO(2n)/B$ or $SO(2n+1)/B$ and the corresponding vexillary elements? In particular, does smooth imply vexillary as in the case of S_n?
2. Is there a way to define flagged Schur Q-functions so that the Schubert polynomial indexed by w of type B or C is a flagged Schur Q-functions if and only if w is vexillary?
3. Are there other possible ways to define vexillary elements in B_n so that any of the above questions can be answered?

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References

1. S. Billey, "Transition equations for isotropic flag manifolds," *Discrete Math.*, to appear. Special Issue in honor of Adriano Garsia, 1998.
2. S. Billey and M. Haiman, "Schubert polynomials for the classical groups," *J. Amer. Math. Soc.* **8** (1995), 443–482.
3. S. Billey, W. Jockusch, and R. Stanley, "Some combinatorial properties of Schubert polynomials," *J. Alg. Combin.* **2** (1993), 345–374.
4. M. Bona, "Exact enumeration of 1342-avoiding permutations; a close link with labeled trees and planar maps," *J. Combin. Theory Ser. A* **80**(2) (1997), 257–272.
5. M. Bona, "The number of permutations with exactly r 132-subsequences is P-recursive in the size!," *Adv. in Appl. Math.* **18** (1997), 510–522.
6. P. Edelman and C. Greene, "Balanced tableaux," *Adv. Math.* **63** (1987), 42–99.
7. S. Fomin and A.N. Kirillov, "Combinatorial B_n analogues of Schubert polynomials," *Trans. Amer. Math. Soc.* **348** (1996), 3591–3620.
8. S. Fomin and R.P. Stanley, "Schubert polynomials and the NilCoxeter algebra," *Adv. Math.* **103** (1994), 196–207.
9. M. Haiman, "Dual equivalence with applications, including a conjecture of Proctor," *Discrete Math.* **99** (1992), 79–113.
10. W. Kraśkiewicz, "Reduced decompositions in hyperoctahedral group," *C.R. Acad. Sci. Paris Sér. I Math.* **309** (1989), 903–904.
11. V. Lakshmibai and B. Sandhya, "Criterion for smoothness of Schubert varieties in $SL(n)/B$," *Proc. Indian Acad. Sci. Math. Sci.* **100**(1) (1990), 45–52.
12. T.K. Lam, "B and D analogues of stable Schubert polynomials and related insertion algorithms," Ph.D. Thesis, MIT, 1995.
13. T.K. Lam, "B_n Stanley symmetric functions," *Discrete Math.* **157** (1996), 241–270.
14. T.K. Lam, "Superfication of the Stanley symmetric functions," in preparation, 1996.
15. A. Lascoux and M.-P. Schützenberger, "Structure de hopf de l'anneau de cohomologie et de l'anneau de grothendieck d'une variete de drapeaux," *C.R. Acad. Sci. Paris Sér. I Math.* **295** (1982), 629–633.
16. A. Lascoux and M.-P. Schützenberger, "Schubert polynomials and the Littlewood-Richardson rule," *Lett. Math. Phys.* **10** (1985), 111–124.

17. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, Oxford, 1979.
18. I.G. Macdonald, *Notes on Schubert Polynomials*, Vol. 6, Publications du LACIM, Université du Québec à Montréal, 1991.
19. D. Monk, "The geometry of flag manifolds," *Proc. London Math. Soc.* **3**(9) (1959), 253–286.
20. J. Noonan, "The number of permutations containing exactly one increasing subsequence of length three," *Discrete Math.* **152** (1996), 307–313.
21. V. Reiner and M. Shimozono, "Plactification," *J. Alg. Combin.* **4** (1995), 331–351.
22. R. Simion and F.W. Schmidt, "Restricted permutations," *European. J. Combin.* **6** (1985), 383–406.
23. R. Stanley, "On the number of reduced decompositions of elements of Coxeter groups," *European. J. Combin.* **5** (1984), 359–372.
24. R. Stanley, "Unimodality and Lie superalgebras," *Stud. Appl. Math.* **72** (1985), 263–281.
25. J. Stembridge, "Shifted tableaux and the projective representations of symmetric group," *Adv. Math.* **74** (1989), 87–134.
26. J. Stembridge, personal communication, 1993.
27. J. Stembridge, "Some combinatorial aspects of reduced words in finite Coxeter groups," *Trans. Amer. Math. Soc.* **349** (1997), 1285–1332.
28. J. West, "Permutations with forbidden sequences; and, stack-sortable permutations," Ph.D. Thesis, MIT, 1990.
29. D.R. Worley, "A theory of shifted Young tableaux," Ph.D. Thesis, MIT, 1984.
30. D.R. Worley, "The shifted analog of the Littlewood-Richardson rule," preprint, 1987.