Journal of Algebraic Combinatorics **9** (1999), 5–24

Lattices of Parabolic Subgroups in Connection with Hyperplane Arrangements

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Received December 1, 1997; Revised

Abstract. Let W be a Coxeter group acting as a matrix group by way of the dual of the geometric representation. Let L be the lattice of intersections of all reflecting hyperplanes associated with the reflections in this representation. We show that L is isomorphic to the lattice consisting of all parabolic subgroups of W. We use this correspondence to find all W for which L is supersolvable. In particular, we show that the only infinite Coxeter group for which L is supersolvable is the infinite dihedral group. Also, we show how this isomorphism gives an embedding of L into the partition lattice whenever W is of type A_n, B_n or D_n. In addition, we give several results concerning non-broken circuit bases (NBC bases) when W is finite. We show that L is supersolvable if and only if all NBC bases are obtainable by a certain specific combinatorial procedure, and we use the lattice of parabolic subgroups to identify a natural subcollection of the collection of all NBC bases.

Keywords: hyperplane arrangement, lattice, Coxeter group

1. Introduction

By an arrangement $\mathcal{A}$, we mean a collection of (possibly infinite) codimension 1 subspaces of a finite dimensional real vector space V. Associated to $\mathcal{A}$ is a lattice which consists of all possible intersections of elements of $\mathcal{A}$, ordered by reverse set inclusion. A rich theory has been developed to study the properties of this lattice when $\mathcal{A}$ is finite (see [7]). If W is a finite group generated by a set of reflections acting on $\mathbb{R}^n$, the reflection arrangement corresponding to W is the arrangement consisting of the reflecting hyperplanes of all possible reflections in W. We call the intersection lattice corresponding to this arrangement a reflection lattice (with group W) and denote it by L_W.

The main purpose of this paper is to establish an isomorphism between this lattice and the lattice consisting of all parabolic subgroups of W, denoted $\mathcal{P}_W$, and to use this correspondence to study the supersolvability of L_W. Because of the strong similarities between this isomorphism and the isomorphism established in the fundamental theorem of Galois theory, we refer to this isomorphism as the "Galois correspondence" for L_W.

In Section 2, we establish our notation and recall some of the basic results we use. The Galois correspondence and the characterization of the groups W for which L_W are supersolvable holds for an arbitrary Coxeter group (using the dual of the geometric representation), so we present the basic facts we need about Coxeter groups here. Also included is standard material about arrangements and their associated lattices. These results may be found in [7] for the case in which the Coxeter group (and hence the arrangement) is finite. The proofs that these results hold for L_W even when W is an infinite Coxeter group are straightforward generalizations of the proofs that may be found in [7], so we have not included them. In Section 3 we give the basic tool of the paper which is Theorem 3.1, the theorem that establishes the Galois correspondence for Coxeter groups. In Section 4 we explain how this theorem may be viewed as a generalization of the correspondence between L_{S_n} and the partition lattice by showing how the Galois correspondence can be used to realize A_n, B_n and D_n as sublattices of the partition lattice.

Section 5 is devoted to our main application of the Galois correspondence for reflection lattices. In Theorem 5.1 we give several different characterizations of when a finite Coxeter group has an associated reflection lattice which is supersolvable. For this characterization, W is assumed to be finite because the proof uses heavily the Poincaré polynomial associated with W. In the infinite case, we are still able to achieve a complete enumeration of all W for which L_W is supersolvable (Theorem 5.3); however, we are not able to give the other characterizations which appeared in Theorem 5.1. While the proof of Theorem 5.1 is relatively straightforward given that the Galois correspondence is known, the proof of Theorem 5.3 makes use of a somewhat more intricate analysis of the parabolic subgroups of W. Finally, in Section 6, we use the Galois correspondence to define a special subcollection of the collection of all non-broken circuit bases in the lattice L_W. Here we also assume W to be finite so that we can use certain characterizations of simple root systems which are only true in the finite case.

2. Preliminaries

First we review a few facts about reflection groups that can be found, for example, in [6]. We are borrowing Humphreys' notation. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space endowed with a certain positive definite symmetric bilinear form (v, u) (for $v, u \in \mathbb{R}^n$). A *reflection* $r_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ sends the nonzero vector α to its negative while fixing pointwise the hyperplane H_α orthogonal to α. Define W to be the group generated by all reflections $r_\alpha,\ \alpha \in \Phi$, where Φ is a *root system* of W. In general roots need not be of unit length, but hereafter we will always choose root systems with roots of length one. It happens that the reflections r_α are *all* the reflections in W, and W is said to be a *real (finite) reflection group*.

Each element $w \in W$ can be expressed in the form:

$$w = r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_k}.$$

The smallest value of k in any such expression for w is denoted $al(w)$, and is called the *absolute length* of w. An expression $r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_k}$ is said to be *totally reduced* if $k = al(r_{\alpha_1} \cdots r_{\alpha_k})$.

Given a simple system of roots Δ for W, the subgroups of W generated by subsets $I \subseteq \Delta$ are of fundamental importance to our work.

Definition 2.1 If Δ is a simple system of roots for W, and if $I < \Delta$, define $W_I = \langle \{r_\alpha : \alpha \in I\} \rangle$. H is a parabolic subgroup if $H = W_I$ for some I.

Let m_i denote the *exponents* of W, and $d_i = m_i + 1$ be the degrees of W. There is a very nice presentation for W in terms of the simple roots of W that is also of importance to us. For any roots $\alpha, \beta \in \Phi$, let $m(\alpha, \beta)$ denote the order of the product $r_\alpha r_\beta$ in W.

Proposition 2.1 ([6, p. 16]) *Fix a simple system Δ in Φ, and let s_α be the reflection corresponding to $\alpha \in \Delta$. Then W is generated by the set $S = \{s_\alpha \mid \alpha \in \Delta\}$, subject only to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \quad (\alpha, \beta \in \Delta).$$

This presentation of W shows that W is determined up to isomorphism by the set of integers $m(\alpha, \beta)$, (for $\alpha, \beta \in \Delta$). Coxeter (see [5]) encoded this information in a labelled graph Γ constructed as follows: Let Γ be a graph whose vertex set is indexed by the elements of Δ; two distinct vertices α, β are joined by an edge, labelled $m(\alpha, \beta)$, whenever $m(\alpha, \beta) \geq 3$. A pair of vertices not joined by an edge implicitly means that $m(\alpha, \beta) = 2$. This graph is called the Coxeter graph of W and uniquely determines (up to isomorphism) W. Note that since simple systems are conjugate, Γ does not depend on the choice of Δ.

This result inspires the following generalization of a finite real reflection group, called a Coxeter group, see for example [6, p. 105].

Definition 2.2 (W, S, m) is called a Coxeter group if the following are true:

(a) S is a finite set.
(b) $m : S \times S \to \mathbb{Z} \cup \{\infty\}$ is a function so that
  (i) $m(s, s) = 1$ for all $s \in S$.
  (ii) $m(s, s') = m(s', s) \geq 2$ for all $s, s' \in S$ with $s \neq s'$.
(c) $W = \langle S \rangle / \langle\langle (ss')^{m(s,s')} : s, s' \in S \rangle\rangle$ where $\langle S \rangle$ indicates the free group generated by S and $\langle\langle (ss')^{m(s,s')} : s, s' \in S \rangle\rangle$ indicates the normal subgroup of $\langle S \rangle$ generated by the elements $(ss')^{m(s,s')}$.

By abuse of language, we will sometimes state that an abstract group, W, is a Coxeter group. By this, we will mean that there is a Coxeter group (W', S, m) such that W is isomorphic to W'. Using this convention, we observe that Proposition 2.1 simply states that every finite real reflection group is a Coxeter group.

If (W, S, m) is a Coxeter group, we can define the concept of a parabolic subgroup as follows.

Definition 2.3 Let (W, S, m) be a Coxeter group.

(a) $G \subset W$ is called a parabolic subgroup if and only if there is a $T \subset W$ and a $w \in W$ such that the following hold:

(i) $T \subset S$
(ii) $G = \langle wTw^{-1} \rangle$.

If G is a parabolic subgroup, define $\mathrm{rank}(G) = |T|$.

(b) $\mathcal{P}_W$ is the partially ordered set whose elements are all the parabolic subgroups of (W, S, m), ordered by set inclusion.

Note that this definition is independent of any representation of W as a matrix group. One can also generalize the concept of a reflection without explicit use of any representation of the Coxeter group. The reflections of (W, S, m) are simply defined to be all elements of W which are conjugate to some element in S. However, for notions such as roots, simple root systems, etc., we will need a linear representation of the group W. There is a natural representation associated with a Coxeter group which is called its geometric representation.

Definition 2.4 Let (W, S, m) be a Coxeter group.

(a) Let $V = \mathrm{span}_{\mathbb{R}}(S)$ be the vector space generated by S (that is, the free $\mathbb{R}$-module generated by S).

(b) Let σ be the representation

$$\sigma : W \to \mathrm{GL}(V)$$

defined in the following way. First, let $s, s' \in S$. Then define

$$\sigma(s)(s') = s' + 2(\cos[\pi/m(s, s')])s.$$

Next, extend $\sigma(s)$ to a function from V to V by requiring that it be linear. Finally, extend σ to a function from W to $\mathrm{GL}(V)$ by requiring that it be a group homomorphism. σ is called the geometric representation of W.

(c) Let σ^* denote the adjoint representation of σ, that is,

$$\sigma^* : W \to \mathrm{GL}(V^*)$$

is defined by

$$[\sigma^*(w)(\theta)](v) = \theta(\sigma(w)^{-1}v)$$

for all $w \in W$, $\theta \in V^* \equiv \mathrm{Hom}(V, \mathbb{R})$ and $v \in V$. We call σ^* the co-geometric representation. Note that

$$[\sigma^*(w)(\theta)](\sigma(w)v) = \theta(v)$$

for all $w \in W, \theta \in V^*$ and $v \in V$.

(d) Define an inner product B on V by defining

$$B(s, s') = -\cos[\pi/m(s, s')]$$

where $s, s' \in S$, and extending B to a function on $V \times V$ by requiring that it be bilinear. Note that $B(\sigma(w)v, \sigma(w)v') = B(v, v')$ for all $w \in W$ and $v, v' \in V$. Note that B defines a natural map b from V to V^* as follows:

$$b(v)(v') = B(v, v')$$

for all $v, v' \in V$. σ (respectively σ^*) gives V (respectively V^*) the structure of a W module, and one can easily verify that b is a morphism of the W modules V and V^*. Hence, when B is non-degenerate (and thus b is an isomorphism), the representations σ and σ^* are equivalent, so there is no need to distinguish between them. In particular, this is true when W is finite because in this case B is positive definite. However, for many Coxeter groups, B is degenerate, and the representations σ and σ^* are not equivalent. Because we make essential use of the results concerning the Tits' cone, we must restrict our attention to σ^*, and not σ. Next, we summarize these results.

Definition 2.5 Let (W, S, m) be a Coxeter group, with co-geometric representation σ^*.

(a) Let $I \subset S$. Define C_I as follows:

$$C_I = \{\theta \in V^* : \theta(s) = 0 \text{ if } s \in I \text{ and } \theta(s) > 0 \text{ if } s \in S - I\}.$$

(b) Let

$$U = \{\sigma^*(w)(v) : w \in W \text{ and } v \in C_I \text{ for some } I \subset S\}.$$

U is called the Tits' cone of W.

Here is the basic theorem concerning Tits' cones (see [6], p. 126, Theorem (a)).

Theorem 2.1 (Tits) *Let $w \in W$ and $I, J \subset S$. If $w(C_I) \cap C_J \neq \emptyset$, then $I = J$ and $w \in \langle I \rangle$ so $w(C_I) = C_I$. In particular, $\langle I \rangle$ is the precise stabilizer of each point of C_I, and wC_I partitions U, for all $w \in W$, $I \subset S$.*

Finally, we make explicit the definition of roots and simple systems of roots for Coxeter groups.

Definition 2.6 Let (W, S, m) be a Coxeter group with geometric representation σ.

(a) A simple system of roots for W is the set $\sigma(w)(S)$ where w is some fixed element of W.
(b) An element r of V is called a root if r is contained in some simple system of roots for W. Φ is used to denote the set of roots, that is,

$$\Phi = \{r \in V : r = \sigma(w)(s) \text{ for some } w \in W \text{ and } s \in S\}.$$

We have now given the information we need concerning Coxeter groups, and we turn to the definition of the arrangement associated with a Coxeter group, together with some basic properties of arrangements. In what follows we have used [7] as the basic reference. We should note that in [7] it is assumed that $\mathcal{A}$ is finite, while the $\mathcal{A}$ we define here will be infinite for infinite Coxeter groups. However, the proofs of all the results we give below follow exactly as presented in [7] for any arrangement, finite or otherwise, as long as the hyperplanes are in a finite dimensional vector space. Hence, we will not reproduce them here. In our case, this vector space will be V^* which is always finite dimensional.

We start with some basic notation and definitions. Let X be a subset of V. Define $X^\perp \subset V^*$ as follows:

$$X^\perp = \{\theta \in V^* : \theta(x) = 0 \text{ for all } x \in X\}.$$

We use $x^\perp$ to indicate $\{x\}^\perp$ when $x \in V$.

Let $\mathcal{A}$ be the set of all *reflecting hyperplanes* associated with W, that is,

$$\mathcal{A} = \{\alpha^\perp \mid \alpha \in \Phi\} = \{H_\alpha \mid \alpha \in \Phi\},$$

and let L_W denote the poset of all possible intersections of hyperplanes in $\mathcal{A}$ ordered by *reverse set inclusion*. Denote the partial order of L_W by $\leq$ ($X \leq Y$ if and only if $Y \subseteq X$). It is a known fact [7, p. 23] that L_W is a *geometric lattice*, with rank function given by $r(X) = \mathit{codim}(X)$ for any $X \in L_W$ whenever $\mathcal{A}$ is a central arrangement. We should note that, in this paper, our arrangements will always consist of linear subspaces of V^*, so our arrangements will always be central. Certain Coxeter groups have associated with them a natural affine representation. While the standard technique of converting this representation to a linear representation in one higher dimension does give the co-geometric representation which is discussed in this paper (see [6, p. 133]), we never directly discuss the arrangement obtained by taking the collection of reflecting hyperplanes from the original affine representation. Hence, we need never consider the problems associated with non-central arrangements.

All the reflecting hyperplanes H_α have rank one and are called the *atoms* of L_W. Moreover, for any two elements X and Y of L_W the *meet* of X and Y is given by

$$X \wedge Y = \bigcap \{Z \in L_W \mid X \cup Y \subseteq Z\},$$

while if $X \cap Y \neq 0$, the *join* of X and Y is defined to be:

$$X \vee Y = X \cap Y.$$

We also need to review the notions of independent set and basis for geometric lattices. Let L be a geometric lattice. Let A denote the set of atoms of L. A subset $B = \{b_1, \ldots, b_m\} \subseteq A$ is said to be *independent* if the rank of the join of its elements $\bigvee B = b_1 \vee \cdots \vee b_m$ satisfies, $r(\bigvee B) = |B|$. Otherwise, B is said to be *dependent*. A subset $B \subseteq A$ is said to be a *base* for an element $X \in L$ if and only if B is independent and if $\bigvee B = X$. A *circuit* is a dependent set $B \subseteq A$ such that all its proper subsets $C \subset B$ are independent. Given a total order $\prec$ on the set of atoms A, we say that $B = \{b_1, \ldots, b_k\} \subseteq A$ is a *broken circuit*, denoted BC, if there is an atom $a \in A$ such that $a \prec b_i$ for all $i = 1, \ldots, k$ and $B \cup \{a\}$ is a circuit.

In other words, the broken circuits are obtained from the circuits by removing the smallest atom. A *non-broken circuit*, *NBC*, is a set of atoms that does not contain any broken circuit. Note that *NBC* sets are independent sets of atoms.

There is a fundamental link between the *NBC* bases of L_W and the elements of W when W is finite. Indeed, the first author together with A. Goupil and A. Garsia established in [2] the following correspondence. Let $\{H_{\alpha_1}, \ldots, H_{\alpha_k}\}$ be an NBC base where $\alpha_i < \alpha_j$ if $i < j$. Let this NBC base correspond to w defined by

$$w = r_{\alpha_1} \cdots r_{\alpha_k}. \tag{2.1}$$

It turns out that Eq. (2.1) is a totally reduced expression for w, and this correspondence is a bijection between W and the set of all *NBC* bases of L_W (for a given total order on $\mathcal{A}$). Moreover, the enumerating polynomial for all the *NBC* bases of L_W

$$\sum_{S \in NBC(W)} t^{|S|} \tag{2.2}$$

has a factorization that involves the exponents of W (see [3]):

$$\sum_{S \in NBC(W)} t^{|S|} = \prod_i (1 + m_i t).$$

We shall return to this factorization in Section 5.

3. The Galois correspondence for the lattice of parabolic subgroups

In this section we will show (Theorem 3.1) that $\mathcal{P}_W$, the partially ordered set of parabolic subgroups of W, is order isomorphic to L_W (and hence is a geometric lattice). This theorem is almost a direct corollary of Tits' theorem stated above. While the proof of Theorem 3.1 primarily uses this well-known basic tool, it appears the fact that $\mathcal{P}_W$ and L_W are isomorphic is not well known. In fact, while there is a huge body of literature devoted to the study of L_W, we are unable to even find the definition of $\mathcal{P}_W$ in the literature. (Frequent reference can be found to the lattice consisting of all parabolic subgroups of the form $\langle T \rangle$ where T is a subset of some fixed S. This lattice is isomorphic to the Boolean lattice of subsets of S and is a proper sublattice of $\mathcal{P}_W$.) This isomorphism is crucial for the results of this paper because our main technique for resolving questions about L_W will be to resolve the corresponding question about $\mathcal{P}_W$.

We start with the definitions of the functions which will turn out to be the lattice isomorphism and its inverse between L_W and $\mathcal{P}_W$. The notation for these functions varies somewhat in the literature. We chose to use the notation from Galois theory because of the very close parallel between this result and the fundamental theorem of Galois theory.

Definition 3.1 Let (W, S, m) be a Coxeter group, and let ρ^* denote the cogeometric representation of (W, S, m).

(a) Let H be a subset of W. Define

$$\mathrm{Fix}(H) = \{\phi \in V^* : \rho^*(h)\phi = \phi \text{ for all } h \in H\}.$$

(b) Let $X \subset V^*$. Define

$$\mathrm{Gal}(X) = \{w \in W : \rho^*(w)(\phi) = \phi \text{ for all } \phi \in X\}.$$

We now give the proof of the basic tool we will use in this paper.

Theorem 3.1 Gal *is an* (*order and rank preserving*) *isomorphism from* L_W *to* $\mathcal{P}_W$ *with inverse* Fix.

Proof: First we observe that $\mathrm{Fix}(G) \in L_W$ if G is a parabolic subgroup. We see this as follows. Let $G = \langle wTw^{-1}\rangle$ where T is a subset of S for some $w \in W$. Then $\mathrm{Fix}(G) = \bigcap_{t\in T}(wt)^\perp$, so $\mathrm{Fix}(G) \in L_W$. Also, since $\{wt : t \in T\}$ is independent (w is a linear isomorphism, and $T \subset S$ is independent), the dimension of $\bigcap_{t\in T}(wt)^\perp$ is $\dim(V^*) - |T|$, which shows the rank of $\mathrm{Fix}(G)$ is $|T|$. By definition, $|T|$ is also $\mathrm{rank}(G)$, so Fix is rank preserving as well.

Next we observe that $\mathrm{Gal}(X)$ is a parabolic subgroup if X is in L_W. To see this, let U denote the Tit's cone in V^* (see Definition 2.5). Let $C = X \cap U$. Notice that the interior of C in X is non-empty, so that $\mathrm{span}(C) = X$. Hence, $\mathrm{Gal}(X) = \mathrm{Gal}(C)$ (using the fact that ρ^* is a linear action). But Theorem 2.1, (in which C is of the form C_I) says that $\mathrm{Gal}(C) = W_I$, which is a parabolic subgroup of W. Moreover, if $C = C_I$, then $\mathrm{rank}(X) = |I|$. Also, $\mathrm{rank}(W_I) = |I|$, thus Gal is rank preserving.

Now, since it is always true that $\mathrm{Fix}(\mathrm{Gal}(X)) \supset X$ and $\mathrm{Gal}(\mathrm{Fix}(G)) \supset G$, the rank preserving properties of these maps show that $\mathrm{Fix}(\mathrm{Gal}(X)) = X$ and $\mathrm{Gal}(\mathrm{Fix}(G)) = G$, which completes the result. □

Observe that in the finite case, if $X \in L_W$ and if Φ is a root system for W then $\Phi \cap X^\perp$ is a root system for $\mathrm{Gal}(X)$.

In the next section we show how this "Galois" correspondence can be viewed as a generalization of the well known correspondence between L_{S_n} and the partition lattice. Indeed when W is the symmetric group S_n, with its usual action by permutation matrices on $\mathbb{R}^n$, the corresponding reflection lattice is isomorphic to the partition lattice, the lattice consisting of all partitions of the set $\{1, \ldots, n\}$ ordered by refinement. We will, in fact, show that when W is of type A_n, B_n or D_n, Theorem 3.1 can be interpreted as giving a correspondence between W and certain sublattices of the partition lattice on the set $[n, \bar{n}] = \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$.

4. Orbits of parabolic subgroups as partitions

It is a well known fact that the lattice of the braid arrangement A_{n+1} is isomorphic to the partition lattice π_n. Observe that to a partition $\pi = (\pi_1, \ldots, \pi_k)$ of the set $[n]$ there corresponds the parabolic subgroup

$$S_{\pi_1} \times S_{\pi_2} \times \cdots \times S_{\pi_k}$$

where S_{π_i} is the group of permutations of the set π_i. Clearly, the orbit decomposition of this parabolic subgroup is given by the partition π. Thus, one can also interpret the partition lattice as the lattice of orbits of all parabolic subgroups of S_n. It is this latter observation that we wish to generalize to the arrangements of type B_n and D_n. For this entire section, let W stand for S_n, B_n or D_n. Consider the usual action of W on the set $[n, \bar{n}] = \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$. Note that, if $\sigma(i) = j$ for $i, j \in [n, \bar{n}]$ then $\sigma(\bar{i}) = \bar{j}$, where $\bar{\bar{m}} = m$. Let G be any subgroup of W. Its set of orbits, $\mathcal{O}(G) = \{\mathcal{O}_{a_1}, \ldots, \mathcal{O}_{a_k}\}$ where $a_i \in [n, \bar{n}]$, form a partition of the set $[n, \bar{n}]$. We claim that the lattice P_W of parabolic subgroups of W is isomorphic to the poset $\mathcal{O}(P_W)$ of partitions of the set $[n, \bar{n}]$, corresponding to the orbits of the parabolic subgroups of W, ordered by refinement. Even though this correspondence seems very natural we have not encountered it explicitly in the literature. Thus, we will give a detailed listing of the basic lemmas which will be useful in proving Theorem 4.1, without burdening the reader with their detailed proofs.

First, we make some observations. Orbits appear in *pairs*, that is, if $\mathcal{O}_a = \{a_1, \ldots, a_k\}$, with $a_i \in [n, \bar{n}]$ is an orbit of W, then $\bar{\mathcal{O}}_a = \mathcal{O}_{\bar{a}} = \{\bar{a}_1, \ldots, \bar{a}_k\}$ is also an orbit. Note that when $G = S_m \subset S_n$ the orbits are: $\mathcal{O}_1 = \{1, \ldots, m\}$, $\mathcal{O}_{\bar{1}} = \{\bar{1}, \ldots, \bar{m}\}$, $\mathcal{O}_{m+1} = \{m+1\}$, $\mathcal{O}_{\overline{m+1}} = \{\overline{m+1}\}, \ldots, \mathcal{O}_n = \{n\}$ and $\mathcal{O}_{\bar{n}} = \{\bar{n}\}$. In general, for any subgroup G of S_n, the orbits will always be of the form $\mathcal{O}_{b_1} = \{b_1, \ldots, b_k\}$ where $b_i \in [n]$ and $\mathcal{O}_{\bar{b}_1} = \{\bar{b}_1, \ldots, \bar{b}_k\}$ with $\bar{b}_i \in [\bar{n}]$. Clearly the situation for $W = B_m$ or D_m is different. Indeed, in both cases the orbits are $\mathcal{O}_1 = \{1, \bar{1}, \ldots, 2, \bar{2}, \ldots, m, \bar{m}\}$, $\mathcal{O}_{m+1} = \{m+1\}$, $\mathcal{O}_{\overline{m+1}} = \{\overline{m+1}\}$, etc. Since $\mathcal{O}_1 = \mathcal{O}_{\bar{1}}$ we say that $\mathcal{O}_1$ is a *self-barred* part. Next observe that if H is a non-trivial irreducible parabolic subgroup of W then H is itself of type A_m, B_m or D_m. From these observations one can easily conclude the following:

Lemma 4.1 *Let H be a non-trivial parabolic subgroup of W.*

(i) *If H is irreducible, then H has either exactly one non-singleton orbit* (*which is self-barred*), *or exactly* 2 *non-singleton orbits that form a pair.*

(ii) *If $H = H_1 \oplus H_2$, then the non-singleton orbits of H_1 are disjoint from the non-singleton orbits of H_2.*

Corollary 4.1 *Let H and K be two parabolic subgroups of W. Let $H = H_1 \oplus \cdots \oplus H_r$ and $K : K_1 \oplus \cdots \oplus K_s$ where H_i and K_i are irreducible for all i's. Assume that the orbit decomposition of H and K are equal*; *i.e.*, *$\mathcal{O}(H) = \mathcal{O}(K)$. Then $r = s$ and there exist a permutation $\pi \in S_r$ such that $\mathcal{O}(H_i) = \mathcal{O}(K_{\pi(i)})$, for all i's.*

Let $\mathcal{O} : P_W \to \mathcal{O}(P_W)$ be the map that takes a parabolic subgroup of W to its set of orbits; that is, to a partition of $[n, \bar{n}]$. In the next lemma we describe the possible partitions of $[n, \bar{n}]$ occurring in the poset $\mathcal{O}(P_W)$. Let $i, j \in [n]$, then three possible types of reflections $(ij)(\bar{i}\bar{j})$, $(i\bar{j})(j\bar{i})$, $(i\bar{i})$ will be denoted by (ij), $(i\bar{j})$ and $(i\bar{i})$, respectively.

Lemma 4.2 *Let $\pi \in \mathcal{O}(P_W)$; then π has at most one self-barred part.*

Proof:

Case 1. $W = S_n$. If H is a parabolic subgroup of S_n, then H is of the form: $H = H_1 \oplus \cdots \oplus H_k$ where H_i are of type A_{m_i} for all $1 \leq i \leq k$. Thus π contains *no* self-barred parts. Moreover, if $\pi \in \mathcal{O}(P_{s_n})$ then π is of the form

$$\pi = i_1, \ldots, i_k|\bar{i}_1, \ldots, \bar{i}_k| \cdots |i_{k+m}, \ldots, i_n|\bar{i}_{k+m}, \ldots, \bar{i}_n \tag{4.1}$$

where $i_j \in [n]$ and $\bar{i}_j \in [\bar{n}]$ for all j's.

Case 2. $W = D_n$. Note first that in D_n there are no reflections of the form $(i\bar{i})$. Thus, there are no self-barred parts of cardinality 2. The orbits corresponding to the parabolic subgroup generated by a reflection $(i\bar{j})$ are: $1|\bar{1}| \cdots |i\,\bar{j}\,|\,\bar{i}\,j| \cdots |n|\bar{n}$. On the other hand, there can be self-barred part of cardinality $\geq$4. For example, $H = \langle(ij), (i, \bar{j})\rangle$ is a parabolic subgroup of D_n and $\mathcal{O}(H) = 1|\bar{1}| \cdots | \cdots |ij\bar{i}\bar{j}| \cdots |n|\bar{n}$. To see that it is not possible to have more than one self-barred part one needs only to realize that if $H = H_1 \oplus \cdots \oplus H_k$ is a parabolic subgroup of D_n, then there is at most one component H_i which is of type D_{m_i}; the other components are of type A_{m_j}. The components of type A_{m_j} yield the parts occurring in *pairs* while the component of type D_{m_i} yields the self-barred part.

Case 3. $W = B_n$. Since $(i\bar{i})$ is a reflection of B_n, there are self-barred parts of cardinality 2 among the partitions of $\mathcal{O}(P_{B_n})$ and if $H = H_1 \oplus \cdots \oplus H_k$ is a parabolic subgroup of B_n, then at most one component H_i is of type B_{m_i} while all other components are of type A_{m_j}. Thus, again parts appear in pairs except for at most one part which is self-barred. Observe that B_n has reflection subgroups of type D_m, but those are *not* parabolic subgroups of B_n. □

Our next goal is to show that the map $\mathcal{O} : P_W \to \mathcal{O}(P_W)$ is one to one. If $\pi \in \mathcal{O}(P_W)$ has a self-barred part $\{a_1, \bar{a}_1, \ldots, a_m, \bar{a}_m\}$ (and say for the simplicity of the arguments that all the other parts are singleton) then there are two possible parabolic subgroups yielding π, mainly $H_1 = \langle(a_1, a_2), \ldots, (a_{m-1}, a_m), (a_{m-1}, \bar{a}_m)\rangle \simeq D_m$ and $H_2 = \langle(a_1, a_2), \ldots, (a_{m-1}, a_m), (a_m, \bar{a}_m)\rangle \simeq B_m$. But as we mentioned earlier, H_1 is parabolic in D_n, but not in B_n, thus in this situation once W is fixed the pre-image of π is uniquely determined. It turns out that this example captures the whole complexity of the problem, and we can now state:

Lemma 4.3 *Let $\mathcal{O} : P_W \to \mathcal{O}(P_W)$ be the map that assigns to every parabolic subgroup of W its orbit decomposition. $\mathcal{O}$ is a one-to-one map.*

Proof (Sketch): Let $\mathcal{O}(G) = \mathcal{O}(H)$ where G and H are two irreducible parabolic subgroups of W. Then Lemma 4.1 and Corollary 4.1 imply that $G = H$. If G and H *are not irreducible,* Lemma 4.2 yields the desired result. □

We now describe the join of two partitions π and π' in $\mathcal{O}(P_W)$. Take the usual join in the partition lattice and combine the self-barred parts into a single (thus self-barred) part. This definition of join in $\mathcal{O}(P_W)$ does correspond to the join of parabolic subgroups in P_W.

The meet of two partitions in $\mathcal{O}(P_W)$ is the same as the meet in the partition lattice.

One also easily sees that the rank function in $\mathcal{O}(P_W)$ is given by the following rule. Let k be the number of non-self-barred parts of $\pi \in \mathcal{O}(P_W)$; then the rank of π is $r(\pi) = n - \frac{k}{2}$. Recall that if H is a parabolic subgroup of W with simple system Δ, the rank of H in P_W is $r(H) = |\Delta|$.

Lemma 4.4 *Let $H \in P_W$, and let $\mathcal{O}(H)$ be the corresponding partition of $[n, \bar{n}]$ in $\mathcal{O}(P_W)$. Then $r(H) = r(\mathcal{O}(H))$.*

Proof (Sketch): A straightforward proof by induction on $\text{rank}(H)$ yields the desired result. Indeed, if H is of rank one, the number of non-self-barred parts in $\mathcal{O}(H) = 2(n-1)$, so $r(\mathcal{O}(H)) = n - \frac{2(n-1)}{2} = 1$. To complete the proof, observe that if s is a reflection of W such that $s \not< H$ then $r(H \vee \langle s\rangle) = r(H) + 1$. A study of the different cases, $\pi \vee a_i$ where a_i is an atom of $\mathcal{O}_W$ (and $a_i \not< \pi$) reveals that if the number of non-self-barred parts of π was equal to k, then the number of non-self-barred parts of $\pi \vee a_i = k - 2$. Thus $r(\mathcal{O}\langle H, s\rangle) = n - \frac{(k-2)}{2} = (n - \frac{k}{2}) + 1$ and the proof follows. $\square$

The above lemmas allow us to conclude that

Theorem 4.1 *Let P_W be the lattice of all parabolic subgroups of W ordered by inclusion, and let $\mathcal{O}(P_W)$ be the lattice of orbits of all the elements of P_W ordered by refinement. P_W and $\mathcal{O}_W$ are isomorphic. Moreover*

(a) *$\mathcal{O}(P_{S_n})$ consists of all partitions π of $[n, \bar{n}]$ of the form given in Eq.* (4.1).

(b) *$\mathcal{O}(P_{B_n})$ is the poset of all partitions with at most one self-barred part and with all parts occurring in pairs.*

(c) *$\mathcal{O}(P_{D_n})$ is the poset of all partitions with at most one self-barred part of cardinality ≥ 4 and with all parts occurring in pairs.*

Notice that one could establish the isomorphism between P_W and $\mathcal{O}(P_W)$ as a partially ordered set and then use this correspondence to derive the form of meet, join and rank within $\mathcal{O}(P_W)$.

An interesting corollary is the following criteria for parabolic subgroups. An *admissible* partition of $[n, \bar{n}]$ is a partition with paired parts together with at most one self-barred part.

Corollary 4.7 *If H is a reflection subgroup of W with orbit decomposition yielding a non-admissible partition of $[n, \bar{n}]$ then H is not a parabolic subgroup of W.*

Given that D_n is a reflection subgroup of B_n which is not parabolic, the converse is not true.

5. Supersolvable lattices

An interesting problem concerning the lattices L_W is to determine if they are supersolvable when W is irreducible. For an overview and references regarding this subject see [1].

As we mentioned in the introduction it is not easy, in general, to determine if a lattice is supersolvable. When the reflection group is either S_n, B_n (the group of signed permutations), or $\mathbb{D}_n$ (the dihedral group), it is known that the corresponding lattices are supersolvable. But the supersolvability of L_W for the other reflection groups does not seem to be mentioned in the literature. Through personal communications with G. Ziegler and H. Terao it was suggested that none of the others were supersolvable for finite reflection groups. In this section, we give an elegant combinatorial proof (using the lattice of parabolic subgroups), of the fact that the only supersolvable lattices L_W, when W is finite, are the ones corresponding to either $\mathbb{D}_n$ or the reflection groups of type A_n and B_n. Moreover, we are also able to prove, using more involved arguments, that the only infinite irreducible W for which L_W is supersolvable is $\mathbb{D}_\infty$. Let us first recall the definition of supersolvability. Let L be a geometric lattice of finite rank $r(L) = n$. An element $m \in L$ is called *modular* [8] if

$$r(m) + r(m') = r(m \vee m') + r(m \wedge m')$$

for every $m' \in L$. Let $\hat{0}$ be the minimal element of L and $\hat{1}$ be its maximal element.

A geometric lattice L is said to be *supersolvable* [9] if it has a maximal chain

$$\hat{0} = m_0 < m_1 < \cdots < m_n = \hat{1}$$

of modular elements, (called an M-chain of L). Let A be the set of atoms of L, and let $\sim$ be an equivalence relation on A.

Definition 5.1

(1) Define $\wp(\sim)$ to be

$$\wp(\sim) \;=\; \{S \mid S \subseteq A \text{ and } S \text{ contains at most one element from each equivalence class of } \sim\}.$$

Note that when $\sim$ is equality, then $\wp(\sim) = \wp(A)$, the power set of A.

(2) Let $\prec$ be a total order on A. We say that the *NBC* bases of L, $NBC(L)$, with respect to $\prec$ are obtainable by the hands[1] of $\sim$ if

$$NBC(L) = \wp(\sim).$$

First we restrict our attention to finite reflection groups. In the next theorem we use the classification of all the real finite reflection groups, together with their Coxeter diagrams and lists of degrees. See for example [6, pp. 32, 59].

Theorem 5.1 *Let W be an irreducible real finite reflection group, A be the collection of all its reflecting hyperplanes, and L_W its corresponding lattice. The following are equivalent:*

(a) *L_W is supersolvable.*

(b) *There is a total order $\prec$ and an equivalence relation $\sim$ on A, so that the $NBC(L_W)$ bases with respect to $\prec$ are obtainable from the hands of $\sim$.*

(c) *There is a label of the Coxeter diagram of* W (*other than* 2) *which is a degree of* W.
(d) W *is either of type* A_n, B_n *or is* $\mathbb{D}_n$.

Proof:

$c \Rightarrow d$ An inspection of the list of the Coxeter diagrams [6, p. 32] and of the degrees [6, p. 59] will verify this fact.

$d \Rightarrow a$ This is known (see for example [7]).

$a \Rightarrow b$ This is also a known theorem due to Bjorner and Ziegler in [3, Theorem 2.8].

$b \Rightarrow c$ This result is new and requires a proof. We will use the fact that $\mathcal{P}_W$ and L_W are isomorphic lattices. Assume that we have a total order $\prec$, and an equivalence relation $\sim$ on the set of atoms A so that the *NBC* bases of cardinality 2 with respect to $\prec$ are obtainable by the hands of $\sim$. Let $a \in A$, and $[a]$ denote the equivalence class of a. We first note that $|[a]| + 1$ is a degree of W for all $a \in A$. Indeed, our Definition 5.1 implies that the generating function for the set of all *NBC* bases is

$$\prod_i (1 + |[a_i]|t)$$

where $\{a_i\}$ form a set of representatives for the equivalence classes of $\sim$. But, as we mentioned in the preliminaries this generating function factors out as

$$\prod_i (1 + m_i t) = \prod_i (1 + (d_i - 1)t)$$

where m_i (resp. d_i) are the exponents (resp. degrees) of W. Next we show that

$$[a] = \{b \in A \mid \{a, b\} \text{ is a broken circuit}\} \cup \{a\} \tag{5.1}$$

To this end, first assume that $b \in [a]$ and $b \neq a$. Since, $a \sim b$ then $\{a, b\} \notin \wp(\sim)$, thus $\{a, b\}$ is not an *NBC* basis. This means that $\{a, b\}$ is itself a broken circuit since $\{a, b\}$ is independent and singleton sets are never broken circuits. Hence, we have shown

$$[a] \subseteq \{b \in A \mid \{a, b\} \text{ is a broken circuit}\} \cup \{a\}.$$

Next let $b \in A$ such that $\{a, b\}$ is a broken circuit. Hence $\{a, b\}$ is not an *NBC* basis, which means $\{a, b\} \notin \wp(\sim)$. But this implies that $a \sim b$. Thus showing:

$$[a] \supseteq \{b \in A \mid \{a, b\} \text{ is a broken circuit}\} \cup \{a\}$$

and consequently Eq. (5.1). Now, let a_1 be the smallest atom of A. We have that $\{a_1, b\}$ ($b \in A$) is always an *NBC* basis. Hence $[a_1] = \{a_1\}$, which corresponds to the degree 2 which appears in the list of degrees for each of the real finite reflection groups W. Next, let a_2 be the smallest atom in $A - \{a_1\}$. For which $b \in A$ is $\{a_2, b\}$

a broken circuit? Clearly, $\{a_2, b\}$ is a broken circuit if and only if $\{a_1, a_2, b\}$ is a dependent set. But this is so only when $b < a_1 \vee a_2$ and b is neither a_1 nor a_2. Using the identification between $\wp_W$ and L_W we let $H = a_1 \vee a_2$ be the corresponding parabolic subgroup of W. Viewed this way we realize that $b < H$ if and only if b is a subgroup generated by a single reflection. There are as many such subgroups as there are reflections in H. Hence,

$$\begin{aligned}|\{a_2\}| &= |\{b \in A \mid b < a_1 \vee a_2 \text{ and } b \neq a_1 \text{ or } a_2\}| + 1\\ &= |\{h \in H \mid h \text{ is a reflection}\}| - 1.\end{aligned}$$

But H is a rank 2 reflection group, so it is a dihedral group of order k, for some positive integer k. There are exactly k such reflections in H, so

$$|\{a_2\}| = k - 1.$$

Moreover, k is also the order of the product of any two generating reflections in H. But the orders of such products are labels on the Coxeter diagram of W (using here the fact that H is a parabolic subgroup of W). Hence,

$$|\{a_2\}| + 1 = k$$

must be one of the labels of the diagram of W. On the other hand, we saw earlier that $|[a_2]| + 1$ is also a degree of W. Since the degree 2 occurs only once in the list of degrees of W (for any W), k cannot be 2, and the theorem is complete. $\square$

Now we return to the case in which W is allowed to be infinite. We now start with a definition which will be convenient for our analysis.

Definition 5.1 Let (W, S, m) be a Coxeter group. Define $m(W)$ to be the unordered list of integers $m(s, s')$ for all $s, s' \in S$ with $s \neq s'$. We will use $[n_1, \ldots, n_k]$ to represent such a list. So, for example, $[2, 2, 3] = [2, 3, 2] \neq [2, 3]$.

The following result is the crucial idea which enables us to deal with the infinite case.

Lemma 5.1 *Let W be an infinite Coxeter group for which every proper parabolic subgroup is finite. Let H be a modular element in $\mathcal{P}_W$. Then H has either rank 0, rank 1 or is W.*

Proof: Assume H is a modular parabolic subgroup which does not have rank either 0, 1 or $n \equiv \operatorname{rank}(W)$. Let $j = \operatorname{rank}(H)$. Let H' be any parabolic subgroup with $\operatorname{rank}(H') = n - j + 1$. Such a subgroup exists since $j > 1$ implies that $n - j + 1 < n$. Since both H and H' have rank strictly less than n, they are finite groups by assumption.

First, we claim that for any $w \in W$, there must be a reflection in $H \cap wH'w^{-1}$. To see this, assume that there is a w so that $H \cap wH'w^{-1}$ contains no reflection. Now $wH'w^{-1} \in \mathcal{P}_W$, and hence $H \wedge wH'w^{-1}$ is a parabolic subgroup. Because $H \wedge wH'w^{-1} \subset H \cap wH'w^{-1}$, we have $H \wedge wH'w^{-1}$ also contains no reflections. Since every parabolic subgroup is a reflection group, we have $H \wedge wH'w^{-1}$ is trivial. Now, since H is a modular element, we have

$$r(H \vee H') = r(H) + r(H') - r(H \wedge H') = n + 1$$

which provides our desired contradiction.

Next, for every two (not necessarily distinct) reflections $\sigma \in H$ and $\sigma' \in H'$, define the set

$$A_{\sigma,\sigma'} \equiv \{w \in W : w\sigma w^{-1} = \sigma'\}.$$

We claim every element w of W is in some $A_{\sigma,\sigma'}$. To see this, let σ be a reflection in $wH'w^{-1} \cap H$. So $\sigma \in H$ and $\sigma = w\sigma' w^{-1}$ with σ' a reflection in H', which shows $w \in A_{\sigma,\sigma'}$ for this σ and σ'. Now, there are only a finite number of the sets $A_{\sigma,\sigma'}$ because H and H' are both finite. This means one of the sets $A_{\sigma,\sigma'}$ must be infinite since W is infinite. Let

$$Z(\sigma) \equiv \{w \in W : w\sigma w^{-1} = \sigma\},$$

and let $w' \in A_{\sigma,\sigma'}$, where $A_{\sigma,\sigma'}$ is infinite. Then the function $f : A_{\sigma,\sigma'} \to Z(\sigma)$ defined by $f(w) = w^{-1}w'$ is one-to-one, which means $Z(\sigma)$ is infinite. Now, let v be any eigenvector of σ with eigenvalue -1. We have that $w(v)$ also must be an eigenvector with eigenvalue -1 for σ if $w \in Z(\sigma)$. This means $w(v) = \lambda_w v$. Since W preserves the inner product B, and since v has positive length in B, we have that $\lambda_w^2 = 1$ for every $w \in W$. The function $w \to \lambda_w$ is, therefore, a group homomorphism into $\{1, -1\}$. Hence, its kernel, K, is a subgroup of $Z(\sigma)$ of finite index, and hence is infinite. But K is the stability subgroup of W at v. Now Theorem 3.1 says that any stability subgroup is a parabolic subgroup. Since K is an infinite parabolic subgroup, it must be all of W. This shows that σ commutes with every element of W. Hence, $W = \langle\sigma\rangle \oplus W'$ where W' is the subgroup generated by all reflections distinct from σ in a simple system of reflections containing σ. But W' is a proper parabolic subgroup, and hence finite. This gives us that W is finite, and we have produced the desired contradiction. □

Corollary 5.1 *Assume W is an infinite Coxeter group with $\infty \notin m(W)$. Then $\mathcal{P}_W$ is not supersolvable.*

Proof: Let H be an infinite parabolic subgroup with smallest rank. If $\mathcal{P}_W$ is supersolvable, then $\mathcal{P}_H$ will be supersolvable because it is the lower order ideal $[0, H]$ in $\mathcal{P}_W$. Every proper parabolic subgroup of H is finite, so we can apply Lemma 5.1. Since $\infty \notin m(W)$, we have that $\mathrm{rank}(H) \geq 3$. Thus, no rank 2 element in $\mathcal{P}_W$ can be modular by the lemma, which means that $\mathcal{P}_H$ cannot be supersolvable. □

To complete the study of infinite Coxeter groups, we rely upon a detailed analysis of the rank 3 Coxeter groups for which $\mathcal{P}_W$ is supersolvable. Once this is known, the general infinite case becomes easy to resolve.

Theorem 5.2 *Let W be a rank 3 Coxeter group. $\mathcal{P}_W$ is supersolvable if and only if $m(W)$ is one of the following lists*:

$$[2, 2, n], [2, 3, 3] \text{ or } [2, 3, 4]$$

where n is an integer strictly larger than 1 or $n = \infty$.

Proof: The lists in the theorem represent the Coxeter groups $\mathbb{D}_n \oplus \mathbb{Z}_2$, A_3 and B_3, respectively. Since $\mathcal{P}_{\mathbb{D}_n}$, $\mathcal{P}_{A_n}$ and $\mathcal{P}_{B_n}$ are each supersolvable for all n, the implication "$\Leftarrow$" is true. (Note that $\mathcal{P}_{\mathbb{D}_n}$ is supersolvable for all n, including $n = \infty$, since every rank two lattice is always supersolvable.)

We now show "$\Rightarrow$". Assume $\mathcal{P}_W$ is supersolvable. First, we observe that either 2, or ∞, are in $m(W)$. Indeed, if W is finite, then 2 must be in $m(W)$ by [6, p. 137]; if W is infinite, then ∞ must be in $m(W)$ by Corollary 5.1. Next we perform a calculation. Let $S = \{s, t, u\}$ where s and u are chosen in the following way. If 2 is in $m(W)$, let $m(s, u) = 2$. If not, let $m(s, u) = \infty$. Let ρ denote the geometric representation of W. Define the following matrices:

$$A \equiv \begin{bmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 1 & a & 0 \\ 0 & -1 & 0 \\ 0 & c & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & -1 \end{bmatrix}$$

where $A = \rho(s)^t$, $B = \rho(t)^t$ and $C = \rho(u)^t$. Hence, we have

$$\begin{aligned} a &= 2\cos(\pi/m(s, t)), \\ b &= 2\cos(\pi/m(s, u)), \\ c &= 2\cos(\pi/m(t, u)). \end{aligned}$$

Let $H \equiv \langle A, B\rangle$ and $H' \equiv \langle A, C\rangle$. We will find a conjugate of H which has trivial meet with H'. This will mean that neither H nor H' are modular elements in $\mathcal{P}_W$. Next, notice that our argument will still apply when A and C are interchanged. Hence, we can also conclude that $\langle C, B\rangle$ and $\langle C, A\rangle$ are not modular as well. Since every rank 2 parabolic subgroup is conjugate to one of these three subgroups H, H' or $\langle C, B\rangle$, we will then have shown that there are no rank 2 modular elements in $\mathcal{P}_W$, which shows that $\mathcal{P}_W$ is not supersolvable.

In order to find the desired conjugate of H which will have trivial meet with H', observe that every member of H fixes $e_3 \equiv (0, 0, 1)^t$. So every element of wHw^{-1} will fix the vector we_3.

Case 1. $m(s, u) = 2$ and so $b = 0$. If H' and wHw^{-1} have nontrivial meet, then wHw^{-1} contains a reflection of H'. Since $m(s, u)$ is 2, s and u are the only reflections in H', so either s or u must be in H' and wHw^{-1}. The fixed point sets of these reflections are

$$X_1 \equiv \operatorname{span}\{e_2, e_3\},$$

$$X_2 \equiv \operatorname{span}\{e_1, e_2\},$$

respectively. Thus, if $wHw^{-1} \wedge H' \neq \hat{0}$, then we_3 must be in one of these two sets. First, let $w = BC$ so that

$$we_3 = (ac, -c, c^2 - 1)^t.$$

Since $[2, 2, n]$ is listed in the theorem, we need only deal with the case in which $m(W)$ has only one 2. Hence, a and c are not zero, and we have $we_3 \notin X_1$ and $we_3 \notin X_2$ as long as $c \neq 1$. Thus, it remains to check the case for which $c = 1$. In this situation, we look at

$$BABCe_3 = (a^3 - 2a, -1 - a^2, a^2 - 1)^t.$$

If $a = 1$, then $m(W) = [2, 3, 3]$, which is listed in the theorem. If $a = 2^{1/2}$, then $m(W) = [2, 3, 4]$, which is also listed. Hence we must only consider $a > 2^{1/2}$. Since the roots of $a^3 - 2a$ are 0 and $\pm 2^{1/2}$, we have that $BABCe_3$ is in neither X_1 nor X_2.

Case 2. $m(s, u) = \infty$; that is, $b = 2$. Recall from our notation that if $2 \in m(W)$, then we had $m(s, u) = 2$. Hence, since $m(s, u) \neq 2$, we may assume $2 \notin m(W)$.

We start by identifying the union of the fixed point sets of all the reflections in H' which is $\mathbb{D}_\infty$. Define, for each integer n, the subspace X_n by

$$X_n = \{(x, y, z)^t : x(1 - n) = z(n)\}.$$

Let $Y \equiv \bigcup_n X_n$. Y is the desired union of fixed point spaces, but we only need that Y contains every fixed point space for what follows. To see this, it is convenient to use the following alternative description of the elements of Y : $(x, y, z)^t \in Y$ if and only if either

(a) $x = z = 0$
or
(b) $x + z \neq 0$ and $x/(x + z)$ is an integer.

Using this criterion, it is easy to check that both A and C leave Y invariant so that H' leaves Y invariant. Moreover, Y contains both $\mathrm{Fix}(A)$ and $\mathrm{Fix}(C)$. Hence Y contains any vector fixed by any reflection in H'.

With this information, we proceed as we did in the case when $b = 0$. Let $v = BCe_3$. We will show that v is not in the fixed point set of any reflection in H'. Assume this is not true; namely, that $v \in Y$. Observe that $a, c \geq 1$ since $2 \notin m(W)$. Hence $x = 2 + ac$ is not zero, so we may conclude that

$$(2 + ac)/(c^2 + ac + 1) \equiv n \tag{5.1}$$

is an integer. Using that $1 \leq a, c \leq 2$, we find that

$$(2 + ac)/(c^2 + ac + 1) \leq 6/3. \tag{5.2}$$

Since n is a strictly positive integer, we have either that $n = 1$ or $n = 2$. If $n = 2$, then both a and c must be 1, for otherwise (5.2) would be a strict inequality. Replacing a and c by 1 in Eq. (5.1) would then say that $n = 1$, giving a contradiction. If $n = 1$, then $c = 1$.

We now deal with the case $c = 1$ in the same way as we dealt with it before. Consider

$$BABCe_3 = (a^3 + 2a^2 - 2a - 2, -a^2 - 2a + 1, a^2 + 4a + 3).$$

If this vector is in Y, then

$$n \equiv (a^3 + 2a^2 - 2a - 2)/(a^3 + 3a^2 + 2a + 1)$$

is an integer. Note that n is always less than 1. When $a > 1$, we have $a \geq 2^{1/2}$ so that n is positive, giving a contradiction. When $a = 1$, then $n = -1/7$ also giving a contradiction, and the proof is now complete. □

Corollary 5.2 *Let W be a connected Coxeter group with $\infty \in m(W)$. $\mathcal{P}_W$ is supersolvable if and only if* $\mathrm{rank}(W) = 2$.

Proof: If $\mathrm{rank}(W) = 2$, $\mathcal{P}_W$ is a rank 2 lattice which is therefore supersolvable. Assume W is not of rank 2. Then $\mathrm{rank}(W) \geq 3$ since $\infty \in m(W)$ implies W is not of rank 1. Assume $s, s' \in S$ with $m(s, s') = \infty$. There is an s'' with $H \equiv \langle s, s', s''\rangle$ connected since W is connected. Because H is connected, $m(H)$ cannot be $[2, 2, \infty]$. This is the only list in Theorem 5.2 which contains ∞, so $\mathcal{P}_H$ is not supersolvable. Hence, $\mathcal{P}_W$ is also not supersolvable. □

Theorem 5.3 *Let W be a connected Coxeter group. If $\mathcal{P}_W$ is supersolvable, then either $W = \mathbb{D}_\infty$ or W is finite. Hence, $\mathcal{P}_W$ is supersolvable if and only if W is of type A_n or B_n for some $n \in \mathbb{N}$, or W is $\mathbb{D}_n$, the dihedral group, for $n \in \mathbb{N} \cup \{\infty\}$.*

Proof: Combine Corollaries 5.1 and 5.2 with Theorem 5.1. □

6. Non-broken circuit bases

In this section we assume W is finite. As we saw earlier, the *NBC* bases play a fundamental role in many aspects of the theory of reflection groups, and the elements of $NBC(L_W)$ are in one to one correspondence with the elements of W. So now if we consider the lattice of parabolic subgroups $\mathcal{P}_W$ what can be said about its *NBC* bases? Are they easy to characterize? In this section we identify some of the *NBC* bases of $\mathcal{P}_W$ and show that when translated into the L_W lattice they remain *NBC* bases. Unfortunately, this characterization does not yield all *NBC* basis. For this entire section we fix a total order on $\mathbb{R}^n$. Let H_α be an atom of L_W. Then there will be a unique positive root $\alpha \in H_\alpha^\perp$. Thus, the total ordering on $\mathbb{R}^n$, when restricted to the roots of W, gives rise to a total ordering on the atoms of L_W. Using this total ordering one can define *NBC* bases for L_W. Also, for any reflection subgroup $W_I \subseteq W$, we may use this total ordering of $\mathbb{R}^n$ to induce a total ordering on the root space of W_I. This ordering defines a unique system of simple roots for W_I, denoted by Δ_I.

Theorem 6.1 *Let W_I be a parabolic subgroup of W. Then*

$$NBC_{W_I} \equiv \{\alpha^\perp \mid \alpha \in \Delta_I\}$$

is an NBC basis for L_W.

Proof: Since W_I is parabolic, we have that $\Delta_I \subseteq \Delta$, where Δ is a simple system of roots for W. Let T be a subset of NBC_{W_I}. We first show that T is not a broken circuit. Note that T is of the form

$$T = \{\alpha^\perp \mid \alpha \in T'\}$$

where $T' \subseteq \Delta_I \subseteq \Delta$. Hence $\langle T'\rangle$ is a parabolic subgroup. First note that since Δ is an independent set of vectors in $\mathbb{R}^n$, so is T'. This means that in L_W, T is an independent set of atoms. Next, let H_{α_b} be any atom of L_W so that $\{H_{\alpha_b}\} \cup T$ is a dependent set. We must show that H_{α_b} is larger or equal to some atom of T. Now since $\{H_{\alpha_b}\} \cup T$ is dependent, we have that α_b is a linear combination of elements of T'. Thus, $\alpha_b \in (\mathrm{Fix}(\langle T'\rangle))^\perp$. But by the remark following Theorem 3.1, the root system for $\mathrm{Gal}(\mathrm{Fix}(\langle T'\rangle))$ is $(\mathrm{Fix}(\langle T'\rangle))^\perp \cap \Phi$ (where Φ is the root system for W). Since by definition $\alpha_b \in \Phi$, we have that α_b is a root of $\mathrm{Gal}(\mathrm{Fix}(\langle T'\rangle))$. By means of the Gal correspondence and using the fact that $\langle T'\rangle$ is parabolic, this means that α_b is a root of $\langle T'\rangle$. So,

$$\alpha_b = \sum_{t \in T'} \lambda_t \alpha_t$$

where the λ_t are positive real numbers. Since $\alpha_b \neq 0$, one of the λ_t is not equal to 0. We consider two cases. First, assume that exactly one of the λ_t, say λ_{t_0}, is different than 0. In this case $\alpha_b = \pm\alpha_{t_0}$. But both α_b and α_{t_0} are positive so $\alpha_b = \alpha_{t_0}$. Hence $H_{\alpha_b} = \alpha_b^\perp = \alpha_{t_0}^\perp \in T$ as desired. Next, assume that two or more of the λ_i are different than 0. Let α_{t_0} be such that $\alpha_{t_0}^\perp$ is the smallest atom in T. Let α_{t_1} be such that $\alpha_{t_1}^\perp$ is as large as possible subject to the condition that $\lambda_{t_1} \neq 0$. Then $\alpha_b - \alpha_{t_0}$ has λ_{t_1} as the coefficient of α_{t_1} in its expansion in terms of α_{t_i}. Since λ_{t_1} is strictly positive, and since the total order on the atoms of L_W is given in terms of the lexicographic order taking Δ as an ordered basis, we have that $\alpha_b - \alpha_{t_0}$ is strictly greater than 0. This shows that H_{α_b} is strictly greater than $H_{\alpha_{t_0}}$. Hence, T is not a broken circuit, and the proof is complete. □

Note

1. By analogy to previous work of the first author, the equivalence classes of $\sim$ are called the hands of $\sim$.

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