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A Basis for the Top Homology of a Generalized Partition Lattice

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Abstract. For a fixed positive integer k, consider the collection of all affine hyperplanes in n-space given by $x_i - x_j = m$, where $i, j \in [n]$, $i \neq j$, and $m \in \{0, 1, \ldots, k\}$. Let $L_{n,k}$ be the set of all nonempty affine subspaces (including the empty space) which can be obtained by intersecting some subset of these affine hyperplanes. Now give $L_{n,k}$ a lattice structure by ordering its elements by reverse inclusion. The symmetric group $\mathfrak{S}_n$ acts naturally on $L_{n,k}$ by permuting the coordinates of the space, and this action extends to an action on the top homology of $L_{n,k}$. It is easy to show by computing the character of this action that the top homology is isomorphic as an $\mathfrak{S}_n$-module to a direct sum of copies of the regular representation, $\mathbb{C}\mathfrak{S}_n$. In this paper, we construct an explicit basis for the top homology of $L_{n,k}$, where the basis elements are indexed by all labelled, rooted, $(k+1)$-ary trees on n-vertices in which the root has no 0-child. This construction gives an explicit $\mathfrak{S}_n$-equivariant isomorphism between the top homology of $L_{n,k}$ and a direct sum of copies of $\mathbb{C}\mathfrak{S}_n$.

Keywords: intersection lattice, partition lattice, homology, regular representation, rooted tree

1. Introduction

The partition lattice, Π_n, has been an object of much study in the past few decades. The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n by permuting the elements of $[n] = \{1, \ldots, n\}$ and thereby permuting the partitions. This action extends to an action of $\mathfrak{S}_n$ on the homology of Π_n. Since Π_n is Cohen-Macaulay, the main point of interest here is the action on the top homology of Π_n. In 1981, Hanlon [5] computed the Möbius function of the sublattice of Π_n fixed by a permutation in $\mathfrak{S}_n$. This result brings to mind the well known fact (see [11]), sometimes referred to as the Lefschetz Fixed Point Theorem, that the character ϕ of the action of a finite group G on the top homology of a Cohen-Macaulay poset P with $\hat{0}$ and $\hat{1}$ is given by $\phi(g) = (-1)^r \mu_{P^g}([\hat{0}, \hat{1}])$, where r is the rank of $\hat{1}$, and P^g is the poset of elements fixed by g. In 1982, Stanley [11] combined this fact with Hanlon's result to show that the top homology of Π_n is isomorphic (up to tensoring with the alternating representation) to a representation IND n, found by taking the one-dimensional representation of the cyclic group C_n given by a primitive nth root of unity, and inducing it from C_n to $\mathfrak{S}_n$. Much earlier, Klyachko [7] proved that IND n has the same character as that of the Free Lie Algebra, $\mathrm{Lie}[a_1 \cdots a_n]$. (For a thorough introduction to the Free Lie Algebra, see [3].) These results together show that the action of $\mathfrak{S}_n$ on the top homology of Π_n is similar to the action of $\mathfrak{S}_n$ on the Free Lie Algebra.

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Joyal [6] was the first to give a direct proof, using the theory of species, of the correspondence between these two representations. Finally, in 1989, Barcelo [1] demonstrated an explicit bijection between a basis for the top homology of Π_n and a basis of Lie$[a_1 \cdots a_n]$ which preserves the action of $\mathfrak{S}_n$. In this paper, we will give a result similar to Barcelo's for a wider class of intersection lattices. (See [8] for an introduction to intersection lattices and [12] for a survey of recent results about the characteristic polynomials of special classes of intersection lattices.) In general, an intersection lattice L is defined by giving a set of affine hyperplanes in n-space, called a *hyperplane arrangement*, and letting L be the set of subspaces which can be obtained by intersecting some subset of affine hyperplanes in the hyperplane arrangement, with these subspaces ordered by reverse inclusion. By a result of Wachs and Walker [13], any such intersection lattice is Cohen-Macaulay, and so as long as there is some action of $\mathfrak{S}_n$ on L, the Lefschetz Fixed Point Theorem is applicable. Notice that the partition lattice is simply the intersection lattice defined by the hyperplane arrangement $\{x_i - x_j = 0 : i, j \in [n], i \neq j\}$. We will focus on a more general class of intersection lattices $L_{n,k}$ defined as follows.

Definition 1.1 Let n and k be positive integers. Then $L_{n,k}$ is the intersection lattice given by the hyperplane arrangement $\{x_i - x_j = m : i, j \in [n], i \neq j, m \in \{0, 1, \ldots, k\}\}$.

The symmetric group $\mathfrak{S}_n$ acts on $L_{n,k}$ in a straightforward manner. Any permutation $\omega \in \mathfrak{S}_n$ defines an invertible transformation of $[n]$-space by permuting the coordinates. This transformation sends every affine subspace in the intersection lattice to another subspace in the lattice. Notice that the arrangement contains affine hyperplanes which are parallel to each other, and thus $L_{n,k}$ has a top element $\hat{1}$ corresponding to the empty subspace. The existence of this top element makes the structure of $L_{n,k}$ significantly more complicated than that of Π_n. (For example, $L_{n,k}$ is not geometric, and so Björner's result [2] about the homology of geometric lattices no longer applies.) Ironically, while it is more difficult to construct a basis for the top homology of $L_{n,k}$, it turns out that the top homology is isomorphic to a representation much more basic than the Free Lie Algebra, namely a direct sum of copies of the regular representation, $\mathbb{C}\mathfrak{S}_n$. We will prove this fact first by a simple computation of the character of the representation, and then by an explicit basis construction.

We will use the typical notation for the homology of $L_{n,k}$. Let $C_r(L_{n,k})$ be the space generated over $\mathbb{C}$ by all r-chains $(\hat{0} - c_1 - \cdots - c_r - \hat{1})$, and let the boundary map $\delta_r : C_r \to C_{r-1}$ act linearly by sending $(\hat{0} - c_1 - \cdots - c_r - \hat{1})$ to $\sum_{i=1}^{r} (-1)^i (\hat{0} - c_1 - \cdots - \hat{c}_i - \cdots - c_r - \hat{1})$. Let $H_r(L_{n,k}) = \mathrm{Ker}(\delta_r)/\mathrm{Im}(\delta_{r+1})$. The top homology, $H_{n-1}(L_{n,k})$, is simply the subspace of $C_{n-1}(L_{n,k})$, the space generated by all maximal chains in $L_{n,k}$, which is mapped to zero by the boundary map.

2. Action of the symmetric group on the top homology of $L_{n,k}$

In this section, we will show using characters that $H_{n-1}(L_{n,k})$ is isomorphic as an $\mathfrak{S}_n$-module to a direct sum of R copies of $\mathbb{C}\mathfrak{S}_n$, where $R = \frac{1}{n}\binom{(k+1)n-2}{n-1}$. The computation relies heavily on the structure of the coatoms of the lattice.

A coatom of $L_{n,k}$ is any one-dimensional affine subspace of n-space obtained by intersecting a subset of hyperplanes in the hyperplane arrangement. A coatom g can be thought of as a map $g : [n] \to \mathbb{N}$ such that $g^{-1}(0) \neq \emptyset$ and such that for all $i \geq k$, $g^{-1}(i) \neq \emptyset$ implies that at least one of $g^{-1}(i-1), \ldots, g^{-1}(i-k)$ is nonempty. Such a map corresponds to the affine subspace in which $x_i - x_j = g(i) - g(j)$ for all $i, j \in [n]$. For example, the map g for which

$$g(1) = 0, \quad g(2) = 2, \quad g(3) = 0, \quad g(4) = 2, \quad g(5) = 2, \quad g(6) = 1$$

represents the coatom given by the relations

$$x_1 = x_3 = x_6 - 1 = x_2 - 2 = x_4 - 2 = x_5 - 2.$$

We are now ready to show that $L_{n,k}$ is isomorphic as an $\mathfrak{S}_n$-module to a direct sum of R copies of $\mathbb{C}\mathfrak{S}_n$.

Theorem 2.1 *Let ψ be the character of $H_{n-1}(L_{n,k})$ as an $\mathfrak{S}_n$-module. Then $\psi(id) = (kn)(kn+1)\cdots((k+1)n-2)$, and $\psi(\omega) = 0$ for all $\omega \neq id$.*

Proof: Since $L_{n,k}$ is Cohen-Macaulay, the character $\psi(\omega)$ of any permutation $\omega \in \mathfrak{S}_n$ is equal to $(-1)^n \mu_{L^{\omega}_{n,k}}[\hat{0}, \hat{1}]$, where $L^{\omega}_{n,k}$ is the sublattice of $L_{n,k}$ fixed by ω, and $\mu_{L^{\omega}_{n,k}}$ is its Möbius function. Assume first that $\omega \neq id$. Then ω contains a cycle $(a_1, a_2, \ldots, a_r)$ of length greater than 1. Any coatom g of $L_{n,k}$ is sent by ω to another coatom ωg, with $\omega g(i) = g(\omega^{-1}(i))$ for $i \in [n]$. A coatom g is fixed by ω only if $g(a_1) = g(a_2) = \cdots = g(a_r)$. Let V be the subspace given by $x_{a_1} = x_{a_2} = \cdots = x_{a_r}$. Then $V > \hat{0}$ (since $\hat{0}$ represents the whole n-dimensional space), and any coatom fixed by g is greater than or equal to V. Therefore, the meet of any subset of coatoms fixed by ω must be greater than $\hat{0}$. By Weisner's Theorem (see [9]),

$$\mu_{L^{\omega}_{n,k}}[\hat{0}, \hat{1}] = \sum_{\substack{A \subset C \\ \bigwedge A = \hat{0}}} (-1)^{|A|},$$

where C is the set of coatoms of $L^{\omega}_{n,k}$. Notice that any element of $L_{n,k}$ which is fixed by ω is covered by a coatom g in which $g(i) = g(j)$ if i and j lie in the same cycle in ω. Hence, every element of $L^{\omega}_{n,k}$ lies below a coatom of $L_{n,k}$ which is fixed by ω. Thus, C can also be described as the set of coatoms of $L_{n,k}$ which are fixed by ω. Since $\bigwedge C > \hat{0}$, there are no subsets A satisfying $\bigwedge A = \hat{0}$. Hence $\psi(\omega) = 0$ when $\omega \neq id$.

It remains to compute the character of $\psi(\omega)$ when $\omega = id$. This can be done by finding $(-1)^n \mu_{L_{n,k}}[\hat{0}, \hat{1}]$. In [4], Gill shows that the characteristic polynomial of $L_{n,k} \backslash \hat{1}$ is

$$\sum_{x \in L_{n,k} \backslash \hat{1}} \mu[\hat{0}, x] t^{\dim(x)} = t(t - kn - 1)(t - kn - 2) \cdots (t - (k+1)n + 1).$$

We can find the value of $\mu_{L_{n,k}}[\hat{0}, \hat{1}]$ by setting t equal to 1 in the characteristic polynomial and multiplying the result by (-1). Thus when $\omega = id$,

$$\begin{aligned}\psi(\omega) &= (-1)^n \mu_{L^{\omega}_{n,k}}[\hat{0}, \hat{1}] = (-1)^n \mu_{L_{n,k}}[\hat{0}, \hat{1}] \\ &= (kn)(kn+1)\cdots((k+1)n-2).\end{aligned}$$

This concludes the proof of the theorem. □

Recall that the character ϕ of $\mathbb{C}\mathfrak{S}_n$ is given by

$$\phi(\omega) = \begin{cases} 0: & \omega \neq id, \\ n!: & \omega = id. \end{cases}$$

Since a group representation is uniquely determined by its character, and since ψ is R times ϕ, we may conclude that $H_{n-1}(L_{n,k})$ is isomorphic as an $\mathfrak{S}_n$-module to the direct sum of R copies of $\mathbb{C}\mathfrak{S}_n$.

3. Index set for a basis of the top homology

The index set for the basis of $H_{n-1}(L_{n,k})$ will be defined in terms of rooted $(k+1)$-ary trees.

Definition 3.1 A $(k+1)$-*ary rooted tree* is a tree in which each vertex has at most $(k+1)$ children, where the children of each vertex are connected to the vertex by edges with distinct labels from the set $\{0, 1, \dots, k\}$. An edge labelled i will be called an i-*edge*. The parent of such an edge will be called an i-*parent*, and the child of such an edge will be called an i-*child*.

The basis elements of $H_{n-1}(L_{n,k})$ will be indexed by $(k+1)$-ary rooted trees on the vertex set $[n]$ such that the root has no 0-child. Before showing how to construct the basis, we will prove that this index set has the desired order, $(kn)(kn+1)\cdots((k+1)n-2)$.

Theorem 3.2 *The number of* $(k+1)$-*ary rooted trees on the vertex set* $[n]$ *in which the root has no* 0-*child is equal to* $(kn)(kn+1)\cdots((k+1)n-2)$.

Proof: Let $G(x) = \sum_{i=1}^{\infty} b_i x^i$, where b_i is the number of unlabelled rooted $(k+1)$-ary trees on i vertices. The generating function $G(x)$ satisfies the equation $G(x) = x(1+G(x))^{k+1}$, or $\frac{G(x)}{(1+G(x))^{k+1}} = x$. Thus $G(x)$ is the left compositional inverse of the formal power series $F(x) = \frac{x}{(1+x)^{k+1}}$. The number of unlabelled rooted $(k+1)$-ary trees on n vertices in which the root has no 0-child is given by $[x^{n-1}](1+G(x))^k$, or $\sum_{i=0}^{k}\binom{k}{i}[x^{n-1}](G(x))^i$. By Lagrange inversion [14, p. 128],

$$[x^{n-1}](G(x))^i = \frac{i}{n-1}[x^{n-i-1}]\left(\frac{x}{F(x)}\right)^{n-1}$$

$$= \frac{i}{n-1}[x^{n-i-1}](1+x)^{(k+1)(n-1)}$$
$$= \frac{i}{n-1}\binom{(k+1)(n-1)}{n-i-1}.$$

Therefore, $\sum_{i=0}^{k}\binom{k}{i}[x^{n-1}](G(x))^i$ is equal to

$$\sum_{i=0}^{k}\binom{k}{i}\binom{kn-k+n-1}{n-i-1}\frac{i}{n-1} = \frac{k}{n-1}\sum_{i=0}^{k}\binom{k-1}{i-1}\binom{kn-k+n-1}{n-i-1}$$
$$= \frac{k}{n-1}\binom{kn+n-2}{n-2}$$
$$= \frac{k(kn+n-2)!}{(n-1)(n-2)!(kn)!}$$
$$= \frac{(kn+n-2)!}{n!(kn-1)!}.$$

Thus, the number of labelled rooted $(k+1)$-ary trees on n vertices in which the root has no 0-child is equal to $\frac{n!(kn+n-2)!}{n!(kn-1)!} = (kn)(kn+1)\cdots((k+1)n-2)$. □

4. The homology beneath a coatom

In this section, we will study the intervals $[\hat{0}, g]$ of $L_{n,k}$, where g is a coatom of $L_{n,k}$. Let G_g be the labelled graph on n vertices with edge set $\{(i,j) : |g(i)-g(j)| \le k\}$. The edges of G_g correspond to the atoms of $L_{n,k}$ which lie below g. In other words, the edge (i,j) (with $g(i) \ge g(j)$) corresponds to the hyperplane $x_i - x_j = g(i) - g(j)$.

Any subgraph S of G_g defines a unique element in the interval $[\hat{0}, g]$ by taking the intersection of the hyperplanes corresponding to the edges of S. Thus, an element x of $L_{n,k}\backslash\hat{1}$ may be denoted by a pair (g, S), where g is any coatom of $L_{n,k}$ lying above x, and S is an appropriate subgraph of G_g. If S is any labelled graph on n vertices, define $\pi(S)$ to be the partition of $[n]$ for which i and j lie in the same part of $\pi(S)$ if and only if vertices i and j are path connected in S. In other words, let the parts of $\pi(S)$ be the vertex sets of the connected components of S. Notice that (g, S) is equal to (g', S') in $L_{n,k}\backslash\hat{1}$ if and only if $\pi(S) = \pi(S')$ and $g(i) - g(j) = g'(i) - g'(j)$ for all i, j in the same part of $\pi(S)$.

For any coatom g, we will denote the maximal chains of $[\hat{0}, g]$ in the following manner. Let $(e_1, \ldots, e_{n-1})$ be a sequence of edges in G_g which form a spanning tree of G_g. For each $i = 0, 1, \ldots, (n-1)$, let S_i be the subgraph of G_g containing edges $e_1, e_2, \ldots, e_i$. Now the chain

$$(\hat{0} - (g, S_1) - (g, S_2) - \cdots - (g, S_{n-2}) - g)$$

is a maximal chain of $[\hat{0}, g]$, since $\pi(S_0) \ne \pi(S_1) \ne \cdots \ne \pi(S_{n-1})$. We will write this chain as $\{g, e_1 e_2 \cdots e_{n-1}\}$. It is clear that any maximal chain of $[\hat{0}, g]$ can be written in this way. We will also define an element $\{g, \rho[e_1e_2\cdots e_{n-1}]\}$ of $C_{n-2}([\hat{0}, g])$ as

$$\{g, \rho[e_1e_2\cdots e_{n-1}]\} = \sum_{\omega\in\mathfrak{S}_{n-1}} (-1)^{\operatorname{sgn}(\omega)}\{g, e_{\omega_1}e_{\omega_2}\cdots e_{\omega_{n-1}}\}.$$

By Björner's work [2] on the homology of geometric lattices, $\{g, \rho[e_1e_2\cdots e_{n-1}]\}$ is an element of $H_{n-2}([\hat{0}, g])$, the top homology of the interval $[\hat{0}, g]$. (It is also possible to show, using a result about supersolvable lattices [10, Proposition 2.8], that these elements generate all of $H_{n-2}([\hat{0}, g])$. However, we will not need that fact here.)

For any chain $\{g, e_1e_2\cdots e_{n-1}\}$, let $\{g, e_1e_2\cdots e_{n-1}\}^1$ be the corresponding maximal chain of $L_{n,k}$ given by appending the element $\hat{1}$ to the end of $\{g, e_1e_2\cdots e_{n-1}\}$. Let $\{g, e_1e_2\cdots e_{n-1}\}^-$ be the chain $\{g, e_1e_2\cdots e_{n-1}\}^1$ with the coatom g removed. In other words, let

$$\{g, e_1e_2\cdots e_{n-1}\}^1 = (\hat{0} - (g, S_1) - (g, S_2) - \cdots - (g, S_{n-2}) - g - \hat{1}),$$

and let

$$\{g, e_1e_2\cdots e_{n-1}\}^- = (\hat{0} - (g, S_1) - (g, S_2) - \cdots - (g, S_{n-2}) - \hat{1}).$$

Now define $\{g, \rho[e_1e_2\cdots e_{n-1}]\}^1$ and $\{g, \rho[e_1e_2\cdots e_{n-1}]\}^-$ as

$$\{g, \rho[e_1e_2\cdots e_{n-1}]\}^1 = \sum_{\omega\in\mathfrak{S}_{n-1}} (-1)^{\operatorname{sgn}(\omega)}\{g, e_{\omega_1}e_{\omega_2}\cdots e_{\omega_{n-1}}\}^1,$$

and

$$\{g, \rho[e_1e_2\cdots e_{n-1}]\}^- = \sum_{\omega\in\mathfrak{S}_{n-1}} (-1)^{\operatorname{sgn}(\omega)}\{g, e_{\omega_1}e_{\omega_2}\cdots e_{\omega_{n-1}}\}^-.$$

Unfortunately, it is not the case that $\{g, \rho[e_1e_2\cdots e_{n-1}]\}^1$ is an element of $H_{n-1}(L_{n,k})$. However, the boundary map does act nicely on $\{g, \rho[e_1e_2\cdots e_{n-1}]\}^1$, and we get

$$\delta_{n-1}(\{g, \rho[e_1e_2\cdots e_{n-1}]\}^1) = (-1)^{n-1}\{g, \rho[e_1e_2\cdots e_{n-1}]\}^-.$$

Thus, roughly speaking, the boundary map acts on $\{g, \rho[e_1e_2\cdots e_{n-1}]\}^1$ by simply removing the coatom from each maximal chain.

5. A sublattice of $L_{n,k}$

Consider a sequence $(a_1, \ldots, a_r)$ of distinct elements of $[n]$ with length at least 2. Let $K_{\{a_1,\ldots,a_r\}}$ be the graph with vertex set $[n]$ and edge set $\{(i, j) : i, j \in \{a_1, \ldots, a_r\}\}$. We may define a sublattice $L_{(a_1,\ldots,a_r)}$ of $L_{n,k}$ to be the set of distinct elements $(g, S) \in L_{n,k}$ such that g is a coatom of $L_{n,k}$ and S is a subgraph of $G_g \cap K_{\{a_1,\ldots,a_r\}}$. Let $L^1_{(a_1,\ldots,a_r)} = L_{(a_1,\ldots,a_r)} \cup \hat{1}$. Notice that $L^1_{(a_1,\ldots,a_r)}$ is isomorphic to $L_{r,k}$, since $L^1_{(a_1,\ldots,a_r)}$ is basically just the intersection lattice of the subarrangement consisting of hyperplanes $x_i - x_j = m$ where $i, j \in \{a_1, \ldots, a_r\}$. The maximal chains of $L_{(a_1,\ldots,a_r)}$ can be defined in the same manner as the maximal chains of $L_{n,k}$. Let g be any coatom of $L_{n,k}$ such that the vertices $a_1, \ldots, a_r$ are path connected in $G_g \cap K_{\{a_1,\ldots,a_r\}}$. Finally, let $(e_1, e_2, \ldots, e_{r-1})$ be any sequence of edges forming a tree in $G_g \cap K_{\{a_1,\ldots,a_r\}}$. For each $i = 0, 1, \ldots, (r-1)$, let S_i be the subgraph of $G_g \cap K_{\{a_1,\ldots,a_r\}}$ containing edges $e_1, e_2, \ldots, e_i$. Now the chain

$$\{g, e_1 \cdots e_{r-1}\} = (\hat{0} - (g, S_1) - (g, S_2) - \cdots - (g, S_{r-2}) - (g, S_{r-1}))$$

is a maximal chain of $L_{(a_1,\ldots,a_r)}$. Let

$$\{g, \rho[e_1 \cdots e_{r-1}]\} = \sum_{\omega \in \mathfrak{S}_{r-1}} (-1)^{\mathrm{sgn}(\omega)} \{g, e_{\omega_1} \cdots e_{\omega_{r-1}}\}.$$

Define $\{g, e_1 \cdots e_{r-1}\}^1$ to be the maximal chain of $L^1_{(a_1,\ldots,a_r)}$ attained by adjoining the element $\hat{1}$ to the end of $\{g, e_1 \cdots e_{r-1}\}$, and let $\{g, e_1 \cdots e_{r-1}\}^-$ be the chain $\{g, e_1 \cdots e_{r-1}\}^1$ with the element (g, S_{r-1}) removed. Now define $\{g, \rho[e_1 \cdots e_{r-1}]\}^1$ and $\{g, \rho[e_1 \cdots e_{r-1}]\}^-$ as we did before.

We will now construct an important element of $C_{r-1}(L^1_{(a_1,\ldots,a_r)})$.

Definition 5.1 If q is any integer from 1 to k, let $X(a_1, \ldots, a_r; q)$ be the element of $C_{r-1}(L^1_{(a_1,\ldots,a_r)})$ defined by

$$X(a_1, \ldots, a_r; q) = \sum_{i=1}^{r} (-1)^{i+1} \{g_i, \rho[f_1 f_2 \cdots \hat{f}_i \cdots f_r]\}^1,$$

where g_i is any coatom of $L_{n,k}$ for which

$$g_i(a_j) = \begin{cases} 0: & 1 \le j \le i \le r, \\ q: & 1 \le i < j \le r, \end{cases}$$

and where f_i is the edge (a_i, a_{i+1}) if $1 \le i < r$ or the edge (a_r, a_1) if $i = r$.

It turns out that $X(a_1, \ldots, a_r; q)$ is in fact an element of the top homology of $L^1_{(a_1,\ldots,a_r)}$, as we see in the following theorem.

Theorem 5.2 *The element $X(a_1, \ldots, a_r; q)$ is contained in $H_{r-1}(L^1_{(a_1,\ldots,a_r)})$.*

Proof: Applying the boundary map δ_{r-1} to $X(a_1, \ldots, a_r; q)$ gives

$$
\begin{aligned}
&\delta_{r-1}(X(a_1,\ldots,a_r;q))\\
&\quad= \delta_{r-1}\left(\sum_{i=1}^{r}(-1)^{i+1}\{g_i,\rho[f_1 f_2\cdots \hat{f_i}\cdots f_r]\}^1\right)\\
&\quad= \sum_{i=1}^{r}(-1)^{i+1+(r-1)}\{g_i,\rho[f_1 f_2\cdots \hat{f_i}\cdots f_r]\}^-\\
&\quad= \sum_{i=1}^{r}\sum_{\substack{\omega\in\mathfrak{S}_r\\ \omega_r=i}}(-1)^{\operatorname{sgn}(\omega)}\bigl\{g_i,\ f_{\omega_1} f_{\omega_2}\cdots f_{\omega_{r-1}}\bigr\}^-\\
&\quad= \sum_{i=1}^{r}\sum_{\substack{j=1\\ j\neq i}}^{r}\sum_{\substack{\omega\in\mathfrak{S}_r\\ \omega_{r-1}=j\\ \omega_r=i}}(-1)^{\operatorname{sgn}(\omega)}\bigl\{g_i,\ f_{\omega_1} f_{\omega_2}\cdots f_{\omega_{r-2}} f_j\bigr\}^-\\
&\quad= \sum_{1\le i<j\le r}\ \sum_{\substack{\omega\in\mathfrak{S}_r\\ \omega_{r-1}=j\\ \omega_r=i}}(-1)^{\operatorname{sgn}(\omega)}\Bigl[\bigl\{g_i,\ f_{\omega_1} f_{\omega_2}\cdots f_{\omega_{r-2}} f_j\bigr\}^-\\
&\qquad\qquad\qquad\qquad\qquad - \bigl\{g_j,\ f_{\omega_1} f_{\omega_2}\cdots f_{\omega_{r-2}} f_i\bigr\}^-\Bigr].
\end{aligned}
$$

Notice that $\{g_i,\ f_{\omega_1} f_{\omega_2}\cdots f_{\omega_{r-2}} f_{\omega_j}\}^-$ is equal to $\{g_j,\ f_{\omega_1} f_{\omega_2}\cdots f_{\omega_{r-2}} f_{\omega_i}\}^-$ since $g_i(b) - g_i(c) = g_j(b) - g_j(c)$ for any $b, c \in \{a_1, a_2, \ldots, a_i, a_{j+1}, \ldots, a_r\}$ and any $b, c \in \{a_{i+1}, \ldots, a_j\}$. Therefore, the above sum is equal to 0, and it follows that $X(a_1,\ldots,a_r;q) \in H_{r-1}(L^1_{(a_1,\ldots,a_r)})$. $\square$

6. Gluing together the homologies

Let $A_1, \ldots, A_r$ be a collection of sequences of elements in $[n]$ such that the entries within each sequence are distinct. When it will not cause confusion, we will also use A_i to represent the set of elements in A_i. We will refer to $A_1, \ldots, A_r$ as a *compatible collection of sets* if the graph $K_{A_1} \cup \cdots \cup K_{A_r}$ is a connected graph such that any vertex which appears in more than one set is a cut-point of the graph.

If $A_1, \ldots, A_r$ are sequences forming a compatible collection of sets, with $|A_i| = s_i$ for $1 \le i \le r$, we may define a multilinear map

$$C_{s_1-1}\bigl(L^1_{A_1}\bigr) * \cdots * C_{s_r-1}\bigl(L^1_{A_r}\bigr) \to C_{n-1}(L_{n,k})$$

in the following manner. For $i = 1, \ldots, r$, let $(e_{i,1}, \ldots, e_{i,s_i-1})$ be a sequence of edges in K_{A_i} forming a tree, and let $M_i = \{g_i, e_{i,1}\cdots e_{i,s_i-1}\}^1$ be a maximal chain in $L^1_{A_i}$. By the compatibility of $A_1, \ldots, A_r$, the edges $e_{i,j}$ must form a spanning tree of $K_{[n]}$. Furthermore, there must be a unique coatom g of $L_{n,k}$ which is compatible with all of the g_i's in the sense that $g(p) - g(q) = g_i(p) - g_i(q)$ for any $p, q \in A_i$. Let D be the set of all sequences $f = (f_1, \ldots, f_{n-1})$ of distinct edges in G_g such that each edge $e_{i,j}$ appears exactly once, and such that $e_{i,j}$ appears before $e_{i,l}$ for $1 \le j < l \le s_i - 1$. In other words, let

D be the set of all shuffles of the sequences of edges in $M_1, \ldots, M_r$. Finally, define $\operatorname{sgn}(f)$ to be the sign of f if f is considered as a permutation of the identity shuffle $(e_{1,1}, \ldots, e_{1,s_1-1}, \ldots, e_{r,1}, \ldots, e_{r,s_r-1})$. Now define the gluing operation $*$ on maximal chains as

$$M_1 * \cdots * M_r = \sum_{f \in D} (-1)^{\operatorname{sgn}(f)} \{g,\ f_1 \cdots f_{n-1}\}^1,$$

and extend the operation linearly.

The gluing operation has several interesting properties. It is easy to see that

$$\begin{aligned}&\left\{g_1, \rho\left[e_{1,1} \ldots e_{1,s_1-1}\right]\right\}^1 * \cdots * \left\{g_r, \rho\left[e_{r,1} \ldots e_{r,s_r-1}\right]\right\}^1 \\ &\quad = \left\{g, \rho\left[e_{1,1} \ldots e_{1,s_1-1} \ldots e_{r,1} \ldots e_{r,s_r-1}\right]\right\}^1.\end{aligned}$$

Another important property is given in the following theorem.

Theorem 6.1 *If $Y_i^1 \in H_{s_i-1}(L^1_{A_i})$ for $i = 1, \ldots, r$, then $Y_1^1 * \cdots * Y_r^1 \in H_{n-1}(L_{n,k})$.*

Proof: Let $A = (a_1, \ldots, a_s)$ be any sequence of distinct elements of $[n]$. Fix a coatom g of $L_{n,k}$, and let $e_s, \ldots, e_{n-1}$ be edges in G_g such that $K_{(a_1,\ldots,a_s)} \cup \{e_s, \ldots, e_{n-1}\}$ is a connected graph. Choose any $(s-1)$-element subset B of $[n-1]$. Let $Y \in H_{s-1}(L^1_{(a_1,\ldots,a_s)})$ be written as $Y = \sum_m c_m \{g_m, e_{m,1} \cdots e_{m,s-1}\}^1$, where the c_m's are constants in $\mathbb{C}$. Now consider the element $Z_Y \in C_{n-1}(L_{n,k})$ defined as

$$Z_Y = \sum_m c_m \{g'_m,\ f_{m,1} \cdots f_{m,n-1}\}^1,$$

where g'_m is the unique coatom of $L_{n,k}$ compatible with g and g_m, and $(f_{m,1}, \ldots, f_{m,n-1})$ is the sequence which contains edges $e_{m,1}, \ldots, e_{m,s-1}$ in that order in positions B and which contains the edges $e_s, \ldots, e_{n-1}$ in that order in positions $[n-1] \backslash B$. Let $S_{m,j}$ be the subgraph of $G_{g'_m}$ containing edges $f_{m,1}, \ldots, f_{m,j}$. It is easy to check that if $\delta_{s-1}(Y) = 0$ then $\delta_B(Z_Y) = 0$, where

$$\delta_B(Z_Y) = \sum_m \sum_{j \in B} c_m (-1)^j (\hat{0} - (g'_m, S_{m,1}) - \cdots - \widehat{(g'_m, S_{m,j})} - \cdots - (g'_m, S_{m,n-1}) - \hat{1}).$$

Now notice that if $Y_i^1 \in H_{s_i-1}(L^1_{A_i})$ for $i = 1, \ldots, r$, then $\delta_{n-1}(Y_1^1 * \cdots * Y_r^1)$ can be written as a linear combination of elements of the form $\delta_B(Z_Y)$ with $\delta(Y) = 0$. Hence $\delta_{n-1}(Y_1^1 * \cdots * Y_r^1) = 0$, and so $Y_1^1 * \cdots * Y_r^1 \in H_{n-1}(L_{n,k})$. □

7. Construction of the basis elements

We are now ready to construct a basis for $H_{n-1}(L_{n,k})$. The basis elements will be denoted X_T, where T is any rooted $(k+1)$-ary tree on the vertices $[n]$ in which the root has no 0-child. For each vertex v, let $g_T(v)$ be the sum of the edge labels on the path from the root of T to v. (The map g_T represents a coatom of $L_{n,k}$ and will play an important role in proving the linear independence of the X_T's.) Let $\{P_1, \ldots, P_r\}$ be the set of all maximal paths $P_i = a_{i,0} - a_{i,1} - \cdots - a_{i,s_i}$ in T with $s_i \geq 1$ which satisfy the condition that $a_{i,j}$ is a 0-child of $a_{i,j-1}$ for $j = 2, \ldots, s_i$, and $a_{i,1}$ is a q_i-child of $a_{i,0}$ for some positive integer q_i. Order the paths $P_1, \ldots, P_r$ in such a way that for any $1 \leq i < j \leq r$, $g_T(a_{i,0}) \geq g_T(a_{j,0})$, and if $g_T(a_{i,0}) = g_T(a_{j,0})$ then $q_i > q_j$. (Ties may be broken in any canonical way, say lexicographically on the sequence of edge labels from the root of T to $a_{i,0}$ or $a_{j,0}$.) It is clear that the vertex sets of $P_1, \ldots, P_r$ form a compatible collection of sets. Thus we may define X_T as

$$X_T = X\big(a_{1,0}, a_{1,s_1}, a_{1,s_1-1}, \ldots, a_{1,1}; q_1\big) * \cdots * X\big(a_{r,0}, a_{r,s_r}, a_{r,s_r-1}, \ldots, a_{r,1}; q_r\big).$$

By Theorems 5.2 and 6.1, $X_T \in H_{n-1}(L_{n,k})$. In the next two sections, we will prove that the X_T's do actually form a basis of $H_{n-1}(L_{n,k})$.

8. A special term in the basis element

In the expansion of X_T, there appears an important element Q_T of $C_{n-1}(L_{n,k})$ defined as follows.

Definition 8.1 Let $e_1, \ldots, e_{n-1}$ be the edges of T given in the order

$$(a_{1,0}, a_{1,1}), (a_{1,1}, a_{1,2}), \ldots, \big(a_{1,s_1-1}, a_{1,s_1}\big), \ldots, (a_{t,0}, a_{t,1}), \ldots, \big(a_{t,s_t-1}, a_{t,s_t}\big),$$

where $a_{i,0}, \ldots, a_{i,s_i}$ are the vertices along the path P_i defined in the construction of X_T, with $P_1, \ldots, P_t$ ordered as in the construction. Then

$$Q_T = \{g_T, \rho[e_1 \cdots e_{n-1}]\}.$$

We will now impose a partial order on the coatoms of $L_{n,k}$.

Definition 8.2 Let $>_c$ be the partial order on the coatoms of $L_{n,k}$ for which $g \geq_c g'$ if and only if $g(i) \geq g'(i)$ for all $i \in [n]$.

The elements Q_T satisfy a very important property, given in the next theorem.

Theorem 8.3 *The element $X_T + (-1)^t Q_T$ of $C_{n-1}(L_{n,k})$ can be expressed as a sum of maximal chains containing only coatoms strictly less than g_T with respect to $>_c$, for some integer t.*

Proof: The expansion of X_T looks like

$$\begin{aligned} X_T &= X\big(a_{1,0}, a_{1,s_1}, a_{1,s_1-1}, \ldots, a_{1,1}; q_1\big) * \cdots * X\big(a_{r,0}, a_{r,s_r}, s_{r,s_r-1}, \ldots, a_{r,1}; q_r\big) \\ &= \sum_{j_1=0}^{s_1} (-1)^{j_1} \big\{g_{1,j_1}, \rho\big[f_{1,0} f_{1,1} \cdots \hat{f}_{1,j_1} \cdots f_{1,s_1}\big]\big\}^1 * \cdots \\ &\quad * \sum_{j_r=0}^{s_r} (-1)^{j_r} \big\{g_{r,j_r}, \rho\big[f_{r,0} f_{r,1} \cdots \hat{f}_{r,j_r} \cdots f_{r,s_r}\big]\big\}^1, \end{aligned}$$

where $f_{i,j}$ is the edge $(a_{i,s_i-j}, a_{i,s_i-j+1})$ if $1 \le j \le s_i$ or the edge $(a_{i,0}, a_{1,s_i})$ if $j = 0$, and where $g_{i,j}$ is any coatom for which

$$g_{i,j}(a_{i,l}) = \begin{cases} 0\colon & l = 0 \text{ or } l > s_i - j, \\ q_i\colon & 1 \le l \le s_i - j. \end{cases}$$

By one of the properties of the $*$ operator, the expansion can be expressed as

$$\sum_{(j_1,\ldots,j_r)} (-1)^{j_1+\cdots+j_r} \big\{ g_{j_1,\ldots,j_r}, \rho\big[f_{1,1} \cdots \hat{f}_{1,j_1} \cdots f_{1,s_1} \cdots f_{r,1} \cdots \hat{f}_{r,j_r} \cdots f_{r,s_r} \big]\big\}^1,$$

where the sum is over all sequences $(j_1, \ldots, j_r)$ with $0 \le j_i \le s_i$, and where $g_{j_1,\ldots,j_r}$ is the unique coatom of $L_{n,k}$ compatible with $g_{1,j_1}, \ldots, g_{r,j_r}$. We will show by induction that $g_{j_1,\ldots,j_r} \le_c g_T$, with equality holding only for the sequence $(0, 0, \ldots, 0)$. Let $(j_1, \ldots, j_r)$ be a fixed sequence, and for simplicity let $g = g_{1,j_1}, \ldots, g_{r,j_r}$. Let v be the root of T. Obviously $0 \le g(v) - g(v) \le g_T(v) - g_T(v)$. Now, let p be any vertex of T other than v, and assume that $0 \le g(q) - g(v) \le g_T(q) - g_T(v)$ for any vertex q whose path-length from the root of t is less than p's. Since $p \ne v$, there must be some path $P_i = a_{i,0} - a_{i,1} - \cdots - a_{i,l} - \cdots - a_{i,s_1}$ in the construction of X_T such that $a_{i,0} \ne a_{i,l} = p$. Then $0 \le g(a_{i,0}) - g(v) \le g_T(a_{i,0}) - g_T(v)$ by the inductive assumption. Furthermore, $g_T(p) - g_T(a_{i,0}) = q_i$, whereas

$$g(p) - g(a_{i,0}) = \begin{cases} 0\colon & l = 0 \text{ or } l > s_i - j, \\ q_i\colon & 1 \le l \le s_i - j. \end{cases}$$

Therefore, $0 \le g(p) - g(v) \le g_T(p) - g_T(v)$. It follows by induction that $0 \le g(p) - g(v) \le g_T(p) - g_T(v)$ for all vertices p of T. Hence, $g(v) = g_T(v) = 0$, since 0 is the minimal element of any coatom $g\colon [n] \to \mathbb{N}$. Therefore, $g(p) \le g_T(p)$ for all vertices p. In other words, $g \le_c g_T$. It is clear from the proof that equality holds if and only if $j_1 = j_2 = \cdots = j_r = 0$. But when $(j_1, \ldots, j_r) = (0, \ldots, 0)$, we have

$$\begin{aligned} &(-1)^{j_1+\cdots+j_r} \big\{ g_{j_1\ldots j_r}, \rho\big[f_{1,0} \cdots \hat{f}_{1,j_1} \cdots f_{1,s_1} \cdots f_{r,0} \cdots \hat{f}_{r,j_r} \cdots f_{r,s_r} \big]\big\}^1 \\ &\quad = \big\{ g_T, \rho\big[f_{1,1} \cdots f_{1,s_1} \cdots f_{r,1} \cdots f_{r,s_r} \big]\big\}^1 \\ &\quad = (-1)^t \{ g_T, \rho[e_1 \cdots e_{n-1}]\}^1 \\ &\quad = (-1)^t Q_T, \end{aligned}$$

where t is the sign of the permutation which takes $(f_{1,1}, \ldots, f_{1,s_1}, \ldots, f_{r,1}, \ldots, f_{r,s_r})$ to $(e_1, \ldots, e_{n-1})$. It follows that $X_T + (-1)^t Q_T$ can be written as a sum of maximal chains only involving coatoms strictly less than g_T with respect to $>_c$. □

We may conclude that in order to prove that the X_T's form a basis of $H(L_{n,k})$, it suffices to prove that the Q_T's are linearly independent in $C_{n-1}(L_{n,k})$.

9. Proof that the special terms are linearly independent

We would like to show that the Q_T's form a linearly independent set in $C_{n-1}(L_{n,k})$. We will do this by finding a maximal chain in the expansion of Q_T which cannot appear in the expansion of $Q_{T'}$ for any tree T' other than T.

Theorem 9.1 *Let T and T' be labelled, rooted, $(k+1)$-ary trees in which the roots have no 0-children. Let $e_1, \ldots, e_{n-1}$ be the edges of T ordered as in the definition of Q_T. Let $e'_1, \ldots, e'_{n-1}$ be the edges of T' in any order. Suppose that $\{g_T, e_1 \ldots e_{n-1}\}^1 = \{g_{T'}, e'_1 \ldots e'_{n-1}\}^1$. Then $e_i = e'_i$ for all i, and hence $T = T'$.*

Proof: Recall that the sequence of edges $(e_1, \ldots, e_{n-1})$ defines a sequence of graphs $(S_0, S_1, \ldots, S_{n-1})$ where S_i is the graph on n labelled vertices with edges $e_1, \ldots, e_i$. We may think of the connected components of S_i as rooted subtrees of T, where the root v of a component is the vertex of the component with the shortest path to the root of T, and where the edge labels of the connected components are the same as the corresponding edge labels of T. Notice that $g_T(v) \leq g_T(p)$ for all vertices p in that component. Also recall that this sequence of graphs defines a sequence of partitions $\pi_0, \pi_1, \ldots, \pi_{n-1}$ of $[n]$, where each part of π_i corresponds to the vertices of a connected component of S_i. Similarly, $e'_1, \ldots, e'_{n-1}$ defines a sequence $(S'_0, S'_1, \ldots, S'_{n-1})$ of subgraphs of T' and a sequence $(\pi'_0, \pi'_1, \ldots, \pi'_{n-1})$ of partitions of $[n]$. Since $\{g_T, e_1 \cdots e_{n-1}\}^1 = \{g'_T, e'_1 \cdots e'_{n-1}\}^1$, it must be true that $g_T = g'_T$ and $\pi_i = \pi'_i$ for any i.

Several special properties of the sequence $\pi_0, \ldots, \pi_{n-1}$ follow from the order of the edges $e_1, \ldots, e_{n-1}$.

Lemma 9.2 *For any part A of a partition π_i, let $\tau(A) = \min\{g_T(p) : p \in A\}$. If $|\{g_T(p) : p \in A\}| \geq 2$, let $\sigma(A)$ be the second smallest element of $\{g_T(p) : p \in A\}$, i.e., $\sigma(A) = \min(\{g_T(p) : p \in A\} \backslash \tau(A))$. Then $\pi_0, \ldots, \pi_{n-1}$ satisfy the following three properties:*

1. *If parts A and B of π_{i-1} are joined to get π_i, then $\tau(A) \neq \tau(B)$. Without loss of generality, assume that $\tau(A) < \tau(B)$. Then $\sigma(A) \geq \tau(B)$.*
2. *If A is any part of a partition π_i, then there is a unique element $m(A)$ of A such that $g_T(m(A)) = \tau(A)$.*
3. *If A is any part of a partition π_i and if A has more than one element, then $\sigma(A)$ is defined (by property (2)), and there is a unique element $z(A)$ of A such that $g_T(z(A)) = \sigma(A)$ and such that $z(A)$ is not the parent of a 0-edge in $\{e_1, \ldots, e_i\}$.*

Proof: These properties can be shown by induction on i. They are clearly true for $i = 1$ since e_1 is an edge with label greater than 0. Now assume that they are true for $i-1$. Let $e_i = (p, v)$ where v is the child of p. Let D be the maximal subtree of S_{i-1} containing p, and let F be the maximal subtree of S_{i-1} containing v. Let $A = \pi(D)$ and $B = \pi(F)$. Since all vertices of F except for the root have a parent in F, v must be the root of F.

Consider the path $P = a_0 - a_1 - \cdots - p - v - \cdots - a_s$, where (a_0, a_1) is an r-edge with $r > 0$, and all other edges are 0-edges. Since the edges of P appear in order in $e_1, \ldots, e_{n-1}$, a_0 must be a vertex of D. Furthermore, for any edge in $e_1, \ldots, e_{i-1}$ with parent c and child d, $g_T(c) \geq g_T(a_0)$ and $g_T(d) > g_T(a_0)$. Hence a_0 does not appear as a child in any edge in $e_1, \ldots, e_i$, and so a_0 must be the root of D. Finally, notice that all edges in D with parent a_0 (except possibly (a_0, a_1) itself) have label strictly greater than r, by the way the paths of T were ordered. Hence, $\tau(A) = g_T(a_0) < g_T(a_1) = g_T(v) = \tau(B)$, and $\sigma(A) \geq g_T(a_1) = g_T(v) = \tau(B)$. Thus, property (1) holds for i. Property (2) follows by the inductive assumption and by property (1) for i. Property (1) also implies that $(g_T|_{A\cup B})^{-1}(\sigma(A \cup B)) = \{a_1, a_2, \ldots, v\}$. Thus, v is the unique element of $A \cup B$ such that $g_T(v) = \sigma(A \cup B)$ and such that v does not appear as the parent of a 0-edge in $\{e_1, \ldots, e_i\}$. This proves property (3) for i. $\square$

Now we will prove by induction that $e_i = e'_i$ for $i = 1, \ldots, n-1$. Clearly $e_1 = e'_1$ since $\pi_1 = \pi'_1$ implies that $S_1 = S'_1$, where S_1 and S'_1 contain only the edges e_1 and e'_1, respectively. Now suppose that $e_j = e'_j$ for $j = 1, \ldots, i-1$. Then $S_{i-1} = S'_{i-1}$. Let D and F be the subtrees of S_{i-1} which are joined by edge e_i to obtain S_i, and let $A = \pi(D)$ and $B = \pi(F)$. Since $\pi'_{i-1} = \pi_{i-1}$ and $\pi'_i = \pi_i$, A and B are also parts of π'_{i-1}, and π'_i is obtained by joining A and B. Thus S'_i is obtained by joining the subtrees D and F of S'_{i-1} by edge e'_i.

The only vertex in D without a parent in D is the root, $m(A)$. Similarly, $m(B)$ is the only vertex in F without a parent in F. Thus e'_i must have either $m(A)$ or $m(B)$ as a child. By property (1), we may assume that $\tau(A) < \tau(B)$. Thus, $m(B)$ must be the child of e'_i. If D contains only one vertex, then $m(A)$ must be the parent of e'_i. Otherwise, $\sigma(A)$ is defined because of property (2), and $\sigma(A) \geq \tau(B)$ by property (1). If $\sigma(A) > \tau(B)$, then $g_{T'}(p) > g_{T'}(m(B))$ for all $p \in A$ except for $m(A)$, by property (2), and so $m(A)$ must be the parent of e'_i. If $\sigma(A) = \tau(B)$ then $m(A)$ already has a child in D with label $g_{T'}(m(B)) - g_{T'}(m(A))$, and so $m(A)$ cannot be the parent of $m(B)$. In this case, $m(B)$ must be the 0-child of some element p of A with $g_{T'}(p) = \sigma(A)$. But by property (3), there is only one choice for the parent, namely $z(A)$.

We see that e'_i is uniquely determined by $e_1, \ldots, e_{i-1}$, π'_i, and the properties of the sequence of partitions. Since e_i satisfies all of the same conditions as e'_i, it follows that $e_i = e'_i$. By induction, $e_i = e'_i$ for all $i = 1, \ldots, n-1$. Therefore, $T = T'$. $\square$

We have just shown that there is a maximal chain in the expansion of Q_T which does not appear in the expansion of $Q_{T'}$ for any tree T' other than T. Therefore, the Q_T's form a linearly independent set in $C_{n-1}(L_{n,k})$.

10. Bijection between the set $\{X_T\}$ and a basis of $\oplus\mathbb{C}\mathfrak{S}_n$

By Theorem 3.2, Theorem 9.1, and the comments at the end of Sections 8 and 9, the set $\{X_T\}$ forms a basis of $H_{n-1}(L_{n,k})$. It is now a simple matter to show that $H_{n-1}(L_{n,k})$ is isomorphic as an $\mathfrak{S}_n$-module to a direct sum of copies of $\mathbb{C}\mathfrak{S}_n$. Let G be any *unlabelled* $(k+1)$-ary rooted tree for which the root has no 0-child, and let $B(G)$ be the set of all X_T's such that G is the underlying graph of T. It is clear from the construction of the X_T's that $\omega X_T = X_{\omega T}$ for any $\omega \in \mathfrak{S}_n$, where ωT is the tree obtained by permuting the vertex labels of T by ω. Hence, the linear span of the elements of $B(G)$ is a submodule of $H_{n-1}(L_{n,k})$

isomorphic to $\mathbb{C}\mathfrak{S}_n$. Furthermore, $H_{n-1}(L_{n,k})$ is isomorphic to $\oplus_G B(G)$, where the sum is over all possible unlabelled trees G. Therefore, $H_{n-1}(L_{n,k})$ is isomorphic to a direct sum of copies of $\mathbb{C}\mathfrak{S}_n$.

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