Journal of Convex Analysis
Volume 4 (1997), No. 2, 381–393

Measure-Differential Inclusions in Percussional Dynamics

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Received September 8, 1995
Revised manuscript received July 7, 1997

We give an existence result for the dynamics of a system of particles moving on a line in a horizontal plane and subjected to friction, to percussional effects, to stiffness and to damping. The novelty in our study is that the normal reaction is expressed by a measure, incorporating a series of Dirac measures. The velocity is a function of bounded variation and the acceleration is its Stieltjes measure. Together with the tangential reaction – which is also a measure – they must satisfy a measure-differential inclusion formulation of friction. Convex analysis, variational inequalities and measure theory are used in the existence proof.

Keywords: Particle dynamics, normal percussions, bounded variation, measure-differential inclusions.

1991 Mathematics Subject Classification: 34A37, 34A60, 70F30

1. Introduction

This paper is concerned with an existence result for the dynamics of a system of N particles moving on a line in a horizontal plane and which are subjected to friction and to percussional effects. The particles' masses $m_1, \ldots, m_N > 0$ form a diagonal mass matrix $M = \operatorname{diag}(m_1, \ldots, m_N)$. The particles are linked to – or by – a system of springs and dashpots, the effects of which are translated mathematically by positive definite symmetric $N \times N$ matrices: the stiffness matrix K and the damping or viscosity matrix V. The necessarily tangential displacement of all particles is described by a function q with values in $\mathbb{R}^N$ and defined on an interval $I(T) := [t_0, t_0 + T[$, where $t_0 \in \mathbb{R}$ is given and $T > 0$ is to be determined. The velocity is denoted $u : I(T) \to \mathbb{R}^N$. The initial position q_0 at $t = t_0$ and the initial velocity u_0 are given. The external tangential forces are known in a fixed interval of time $I(T_0)$ and are expressed by a given function $p : I(T_0) \to \mathbb{R}^N$. The particles also experience friction, of Coulomb type: the friction coefficient $\gamma > 0$ is

* Supported by J.N.I.C.T., PRAXIS XXI, project PRAXIS/2/2.1/MAT/125/94 and FEDER.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

assumed constant and the associated set is denoted $C = [-\gamma, \gamma]$.

The main novelty in our study is that the normal reaction, although known, is not a function. Instead, it is expressed by a measure, which includes Dirac measures at a denumerable set of instants $D = \{t_n : n \geq 0\}$. This means that normal percussions occur. To be precise, two measures are introduced, namely

$$d\theta := dt + d\mu, \quad d\mu := \sum_{i=0}^{\infty} a_i \delta_{t_i} , \tag{1.1}$$

where dt is the Lebesgue measure, δ_t is the Dirac measure at t, $a_i > 0$, for all i, and $\sum a_i < +\infty$. Then the normal reaction is given by $\nu\, d\theta$, where $\nu : I(T_0) \to \mathbb{R}_+^N$ is a Borel measurable function, defined everywhere, with nonnegative components and bounded. Clearly, we should expect that the tangential reaction is also expressed in terms of $d\theta$, that is, as $f\, d\theta$ where f is a $\mathbb{R}^N$–valued function, integrable with respect to $d\theta$ or bounded on $I(T_0)$ or on a smaller interval $I(T)$, in the case of a local existence result.

Notice that in the equation of the balance of forces, we must account for this singular tangential reaction term $f\, d\theta$. It turns out that it can only be compensated by allowing the acceleration $\ddot{q} = \dot{u}$ to be a measure. In other words, we have to consider the velocity u as a function of bounded variation, having expected discontinuities at the percussional instants t_i. The acceleration is then the differential or Stieltjes measure of u, denoted du. Let us just recall here that if the right-limit of u is denoted by u^+ and the left-limit by u^-, then $du(]s,t]) = u^+(t) - u^+(s)$ and $du(\{t\}) = u^+(t) - u^-(t)$.

The problem is formulated as follows:

Problem (P). Find $T > 0$, a function of bounded variation $u : I(T) = [t_0, t_0 + T[\to \mathbb{R}^N$ – the velocity – and a function $f \in L^1(I(T), d\theta; \mathbb{R}^N)$ – the density of tangential reaction – such that, defining the displacement by

$$q(t) = q_0 + \int_{t_0}^{t} u(s)\, ds \quad (t \in I(T)), \tag{1.2}$$

we have:

$$u(t_0) = u_0, \tag{1.3}$$

$$M\, du + V u\, dt + K q\, dt = f\, d\theta + p\, dt, \tag{1.4}$$

$$-u^+(t) \in N_{\nu(t)C}(f(t))\,,\ d\theta\text{–a.e.}. \tag{1.5}$$

Equation (1.4) is just the balance equation written in the form of an equality of measures. In the classical setting, where $d\theta$ is replaced by Lebesgue measure dt and u is absolutely continuous (so that $du = \dot{u}\, dt$), (1.4) would mean $M\, \dot{u}(t) + V\, u(t) + K\, q(t) = f(t) + p(t)$, for Lebesgue almost every t, as usual.

The differential inclusion (1.5) is a well known form of Coulomb's friction law. The notation $N_A(x)$ stands for the outward normal cone to a convex set $A \subset \mathbb{R}^N$ at a point x : if $x \notin A$, then $N_A(x) = \emptyset$ and if $x \in A$, then

$$v \in N_A(x) \Leftrightarrow \forall z \in A, v \cdot (z - x) \leq 0, \tag{1.6}$$

where the dot denotes the scalar product in $\mathbb{R}^N$. The convex set considered here is defined by

$$\nu(t)C := \{(\nu_1(t)c_1, \ldots, v_N(t)c_N) : c_1, \ldots, c_N \in C := [-\gamma, \gamma]\}. \tag{1.7}$$

The main result is the following:

Theorem 1.1. *Let $p \in L^\infty(I, dt; \mathbb{R}^N)$ and $v \in L^\infty(I, d\theta; \mathbb{R}^N_+)$. Let $u_0, q_0 \in \mathbb{R}^N$. Then, for sufficiently small $T > 0$, there is at least one solution (u, f) to Problem* (P) *which is defined on $I(T) = [t_0, t_0 + T[$.*

The proof uses convex analysis, variational inequalities and some elementary measure theory. Equations such as (1.4), (1.5) are sometimes called measure-differential equations or inclusions.

Different versions of the problem may be considered. For instance, we might have added a transport velocity $e(t)$ of the line where the particles move; then, the velocity $u(t)$ must be replaced by $u(t) - E(t)$ where $E(t) := (e(t), \ldots, e(t))$. The proof would proceed much in the same manner.

Previous works in this direction include [5], [6] and [7] by the authors and some references therein. The present work differs from [5] and [6], where the unknown functions are scalar (in both) or the elastic and viscous effects are absent (in [5]). Moreover, although the general appproach is similar, other technical tools are used here, say, in Sections 2, 3 and 5. This work also differs from [7], Section 3.3, where the normal reactions are not known a priori, but the applied forces are usual functions, while here these forces can also be responsible by the given normal percussions, hence they may be expressed more generally by measures.

We can also refer to [3] and [4], which treat (with a simpler apparatus and from the application point of view) the cases of particles and bodies with known or unknown normal reaction forces. We believe that the mathematical tools developed here will prove useful in the general theoretical setting.

This paper is organized as follows: in Section 2, the corresponding problem without percussion is studied. In Section 3, the existence of a solution to the problem with a finite number of percussions is obtained "almost" explicitly. In Section 4, Problem (P) is solved by a technically challenging limit procedure on the previously obtained approximate solutions. Finally, in Section 5, a few auxiliary results are presented for the convenience of the reader.

2. The problem without percussions

Let us solve first the classical problem – that is, in the absence of percussions – in a subinterval $J := [a, b] \subset I(T_0)$. Given v_a, $q_a \in \mathbb{R}^N$, we have to find an absolutely continuous function $v : J \to \mathbb{R}^N$ and a Lebesgue-integrable function $g : J \to \mathbb{R}^N$ which solve the following problem:

Problem (P$_0$). – Defining $q : J \to \mathbb{R}^N$ by

$$q(t) = q_a + \int_a^t v(s)\, ds \quad (t \in J), \tag{2.1}$$

then v, q and g satisfy:

$$v(a) = v_a, \tag{2.2}$$

$$M\frac{dv}{dt}(t) + Vv(t) + Kq(t) = g(t) + p(t),\ dt\text{–a.e. in } J, \tag{2.3}$$

$$-v(t) \in N_{\nu(t)C}(g(t)),\ dt\text{–a.e. in } J. \tag{2.4}$$

Theorem 2.1. *If $p \in L^2(J, dt;\ \mathbb{R}^N)$ and $\nu \in L^2(J, dt;\ \mathbb{R}^N)$, then Problem* (P$_0$) *has a unique solution $(v, g) \in AC(J; \mathbb{R}^N) \times L^2(J, dt;\ \mathbb{R}^N)$.*

The proof is performed in several steps. First, we notice that, if $g \in L^2(J, dt;\ \mathbb{R}^N)$ is already known, then (2.1)–(2.2) may be expressed as a Cauchy problem for a second-order o.d.e. :

$$M\ddot{q}(t) + V\dot{q}(t) + Kq(t) = \varphi(t),\ dt\text{–a.e. in } J, \tag{2.5}$$

$$q(a) = q_a,\ \dot{q}(a) = v_a, \tag{2.6}$$

where $\varphi := g+p \in L^2(J, dt;\ \mathbb{R}^N)$. Classically, (2.5)–(2.6) has a unique solution. Moreover, the map $\varphi \mapsto (q, \dot{q})$ is continuous from $L^2 := L^2(J, dt;\ \mathbb{R}^N)$ into $(L^2)^2$; to obtain this, just write the explicit integral formula for the solution $(q, \dot{q})$. Thus, the map $\mathcal{S}$ defined by

$$g \in L^2(J, dt;\ \mathbb{R}^N) \mapsto \mathcal{S}(g) := v = \dot{q}, \tag{2.7}$$

where v is the solution of (2.1)–(2.3), is continuous from L^2 into L^2; see also Lemma 2.2 below.

We now turn our attention to (2.4). By Lemma 5.1, $t \mapsto v(t)C$ is a measurable closed convex-valued multifunction, and by Lemma 5.2, (2.4) is equivalent to the following inclusion, in the Hilbert space $L^2(J, dt;\ \mathbb{R}^N)$:

$$-v \in N_{\mathrm{Sel}(\nu(\cdot)C)}(g), \tag{2.8}$$

the outward normal cone to the set of L^2-selections of $t \mapsto \nu(t)C$. In variational form, this is written:

$$\langle v, \varphi - g\rangle_{L^2} = \int_J v(t) \cdot (\varphi(t) - g(t))\, dt \geq 0, \tag{2.9}$$

for every $\varphi \in L^2(J, dt;\ \mathbb{R}^N)$ such that $\varphi(t) \in \nu(t)C$, dt–a.e. in J.

Coupling this with (2.1)–(2.3) and by definition of $\mathcal{S}$, we obtain the following variational inequality, where $\langle\cdot,\cdot\rangle$ denotes the inner product of L^2:

$$\langle \mathcal{S}(g), \varphi - g\rangle \geq 0,\ \forall \varphi \in \mathrm{Sel}(\nu(\cdot)C). \tag{2.10}$$

By Lemma 5.3, we know that (2.10) has a unique solution provided that $\mathrm{Sel}(\nu(\cdot)C)$ is a nonempty bounded closed convex subset of the Hilbert space $L^2(J, dt;\ \mathbb{R}^N)$ – and this is easy to verify – and that the continuous operator $\mathcal{S}$ is strictly monotone:

Lemma 2.2. *The solution operator $\mathcal{S}$ of* (2.1)–(2.3) *(see* (2.7)*) is strictly monotone and Lipschitz-continuous in $L^2(J, dt;\ \mathbb{R}^N)$.*

Proof. Let $g_i \in L^2$ and $v_i = \mathcal{S}(g_i)$, for $i = 1, 2$, so that $g_i = M\dot{v}_i + Vv_i + Kq_i - p =: \mathcal{L}(v_i)$. We put $w = v_1 - v_2$ and $W(t) = \int_a^t w(s)\,ds$ and we write

$$A := \langle \mathcal{S}(g_1) - \mathcal{S}(g_2),\ g_1 - g_2 \rangle = \int_J (v_1 - v_2) \cdot (g_1 - g_2)\,dt$$
$$= \int_a^b w(t) \cdot (M\dot{w}(t) + Vw(t) + KW(t))dt.$$

But since $\dot{W}(t) = w(t)$ and $w(a) = W(a) = 0$, we obtain that

$$A = \frac{1}{2} Mw(b) \cdot w(b) + \int_a^b Vw(t) \cdot w(t)\,dt + \frac{1}{2} KW(b) \cdot W(b)\ .$$

Since M and K are (definite) positive matrices and V is definite positive (there is $\alpha > 0$ such that $Vx \cdot x \geq \alpha \|x\|^2, \forall x \in \mathbb{R}^N$), it follows that

$$\langle \mathcal{S}(g_1) - \mathcal{S}(g_2), g_1 - g_2 \rangle \geq \alpha \int \|w(t)\|^2 dt = \alpha \|\mathcal{S}(g_1) - \mathcal{S}(g_2)\|^2 \geq 0. \tag{2.11}$$

Hence $\mathcal{S}$ is monotone. Moreover, $\langle \mathcal{S}(g_1) - \mathcal{S}(g_2), g_1 - g_2 \rangle = 0$ implies $\mathcal{S}(g_1) = \mathcal{S}(g_2)$ or $v_1 = v_2$ so that $g_1 = \mathcal{L}(v_1) = \mathcal{L}(v_2) = g_2$ proving that $\mathcal{S}$ is strictly monotone. Finally, by Cauchy-Schwarz inequality, (2.11) implies that $\|\mathcal{S}(g_1) - \mathcal{S}(g_2)\| \leq \frac{1}{\alpha} \|g_1 - g_2\|$. □

3. The case of a finite number of percussions

We solve approximate problems to Problem (P), which we call Problem (P$_n$), for $n \in \mathbb{N}_1$. These are obtained by replacing the percussional measure $d\mu$ with a measure $d\mu_n$ which contains only a finite number of percussions, i.e. a finite number of Dirac measures. To be precise,

$$d\mu_n = \sum_{i=0}^{n} a_i \delta_{t_i}\ . \tag{3.1}$$

The approximate problem (P$_n$) is defined as follows:

Problem (P$_n$). – Define the positive measure $d\theta_n = d\mu_n + dt$ on $I(T_0) = [t_0, t_0 + T_0[$. Let $p \in L^\infty(I(T_0), dt;\, \mathbb{R}^N)$ and $\nu \in L^\infty(I(T_0), dt;\, \mathbb{R}_+^N)$. The initial values $u_0, q_0 \in \mathbb{R}^N$ are given. For any $T \in]0, T_0]$, find $u_n : I(T) = [t_0, t_0 + T[\to \mathbb{R}^N$, a function of bounded variation with Stieltjes measure du_n, and a function $f_n \in L^\infty(I(T), dt;\, \mathbb{R}^N)$ such that, if q_n is the Lipschitz-continuous function defined by

$$q_n(t) = q_0 + \int_{t_0}^{t} u_n(s)\,ds,\ t \in I(T), \tag{3.2}$$

then the following hold:

$$u_n(t_0) = u_0, \tag{3.3}$$

$$M\,du_n + Vu_n\,dt + Kq_n\,dt = f_n\,d\theta_n + p\,dt, \tag{3.4}$$

$$-u_n^+(t) \in N_{\nu(t)C}(f_n(t))\,,\ d\theta_n\text{–a.e.}. \tag{3.5}$$

Let T be fixed. Then the elements of $\{t_0, \ldots, t_n\} \cap I(T)$ may be ordered and we denote them

$$\lambda_0 = t_0 < \lambda_1 < \ldots < \lambda_{n'} < t_0 + T \quad (\text{with } n' \leq n). \tag{3.6}$$

In the interior of every subinterval $I_j := [\lambda_j, \lambda_{j+1}[$, the measure $d\theta_n$ equals the Lebesgue measure, and so the problem is solved as in the previous section, as long as we know how to prescribe the values $q_n(\lambda_j)$ and $u_n(\lambda_j)$; to be precise, we need to know the right-velocity $u_n^+(\lambda_j)$.

Recall that if $\lambda_j = t_{m(j)}$ for some $m(j) \leq n$, then

$$d\theta_n(\{\lambda_j\}) = d\mu_n(\{\lambda_j\}) = (a_{m(j)}\delta_{\lambda_j})(\{\lambda_j\}) = a_{m(j)}, \tag{3.7}$$

$$du_n(\{\lambda_j\}) = u_n^+(\lambda_j) - u_n^-(\lambda_j). \tag{3.8}$$

Here a convention is needed for the left endpoint $\lambda_0 = t_0$:

$$u_n^-(\lambda_0) = u_n(\lambda_0) = u_0, \tag{3.9}$$

by (3.3). Thus the integration of the balance equation (3.4) on the singleton $\{\lambda_j\}$ leads to $M[u_n^+(\lambda_j) - u_n^-(\lambda_j)] = a_{m(j)} f_n(\lambda_j)$ or

$$u_n^+(\lambda_j) = u_n^-(\lambda_j) + a_{m(j)} M^{-1} f_n(\lambda_j). \tag{3.10}$$

In addition, (3.5) has to hold at λ_j (which has a positive measure for $d\theta_n$, by (3.7)). Thus, $f_n(\lambda_j) \in \nu(\lambda_j)C$ and

$$-u_n^+(\lambda_j) \in N_{\nu(\lambda_j)C}(f_n(\lambda_j)). \tag{3.11}$$

By elementary convex analysis, (3.11) is equivalent to: $\forall \rho > 0$, $f_n(\lambda_j)$ is the projection or proximal point of $f_n(\lambda_j) - \rho u_n^+(\lambda_j)$ in the set $\nu(\lambda_j)C$. Combining with (3.10) we obtain:

$$f_n(\lambda_j) = \operatorname{proj}_{\nu(\lambda_j)C}([Id - \rho a_{m(j)} M^{-1}] f_n(\lambda_j) - \rho u_n^-(\lambda_j)), \tag{3.12}$$

where Id is the $N \times N$ identity matrix, $a_{m(j)} M^{-1}$ is a symmetric positive definite matrix and $y = -u_n^-(\lambda_j)$ is assumed known. By Lemma 5.4, there is a unique such $f_n(\lambda_j) \in \mathbb{R}^N$.

Therefore, (3.10) and (3.12) give us the right-velocity at the start of the new subinterval $I_j = [\lambda_j, \lambda_{j+1}[$, while the initial position is obtained by continuity from the solution already known up to λ_j : that is, as the displacement is continuous, we take $q_n(\lambda_j) = q_n^-(\lambda_j)$. In the interior of I_j we do not expect percussions: there, as pointed out above, $d\theta_n$ coincides with dt and the problem reduces to the classical situation considered in Section 2. Thus there is a unique pair $(u_n, f_n) \in AC(I_j, \mathbb{R}^N) \times L^2(I_j, dt; \mathbb{R}^N)$ such that $u_n(\lambda_j) = u_n^+(\lambda_j)$, as given by (3.10), and

$$M\frac{du_n}{dt}(t) + Vu_n(t) + Kq_n(t) = f_n(t) + p(t),\ dt\text{–a.e. in } I_j, \tag{3.13}$$

$$-u_n(t) \in N_{\nu(t)C}(f_n(t)),\ dt\text{–a.e. in } I_j, \tag{3.14}$$

where q_n is the anti-derivative of u_n which satisfies $q_n(\lambda_j) = q_n^-(\lambda_j)$.

We have obtained a solution u_n on I_j (notice that (3.13) may be considered as a definition of f_n). Since the study undertaken in the previous section allows us to work on the closure $\bar{I}_j$, it is clear that the limits $q_n^-(\lambda_{j+1})$ and $u_n^-(\lambda_{j+1})$ exist. Therefore, we may continue with the procedure, so that after a finite number of steps, we shall have defined on the whole of $I(T)$ the unique solution (u_n, f_n) to the problem (P$_n$).

The solution u_n is made of a finite number of absolutely continuous pieces, "separated" by a finite number of discontinuities (at the λ_j); hence, it has bounded variation. Moreover, by construction it is right-continuous, except at $t_0 = \lambda_0$. In fact, for the sake of (1.3), we take $u_n(t_0) = u_0$, while by (3.10) and (3.12) with $j = 0$,

$$u_n^+(t_0) = u_0 + a_0 M^{-1}\xi, \tag{3.15}$$

where ξ is the unique solution of

$$\xi = \text{proj}_{\nu(t_0)C}([Id - \rho a_0 M^{-1}]\xi - \rho u_0). \tag{3.16}$$

As for f_n, by (3.11) (or (3.12)) and (3.14), we have $f_n(\lambda_j) \in \nu(\lambda_j)C$ and $f_n(t) \in \nu(t)C$, dt–a.e. in the interior of I_j, hence $f_n(t) \in \nu(t)C$, $d\theta_n$–a.e. in $I(T)$. Since ν and C are bounded, we conclude that $f_n \in L^\infty(I(T), d\theta_n; \mathbb{R}^N)$. Furthermore we notice that by defining $f_n = 0$ at the atoms of $d\theta$ which are not included in the definition of $d\mu_n$ (hence of $d\theta_n$ and so this only changes f_n in a $d\theta_n$-null set) we may assume that

$$f_n(t) \in \nu(t)C,\ \theta\text{–a.e.}, \tag{3.17}$$

whence, for all n

$$\|f_n\|_{L^\infty(d\theta)} \leq \gamma\|\nu\|_\infty. \tag{3.18}$$

4. Convergence to a solution

We show that by considering an interval $I = I(T)$ with sufficiently small $T > 0$, it is possible to extract convergent subsequences of (u_n, f_n), in a sense to be specified below, and that any such limit (u, f) is a solution to Problem (P) on I.

By (3.18), (f_n) is a bounded sequence in $L^\infty(I, d\theta; \mathbb{R}^N)$, hence it admits a subsequence, still denoted (f_n), which converges to some $f \in L^\infty(I, d\theta; \mathbb{R}^N)$ in the weak-$*$ topology $\sigma(L^\infty, L^1; d\theta)$. As for (u_n), we have the following result:

Proposition 4.1. *There is $T_1 \in]0, T_0]$ such that, if we consider $I = I(T)$ with $T \in]0, T_1[$, then we may extract from (u_n) a subsequence, still denoted (u_n), which satisfies:*

(a) *(u_n) converges pointwisely to a function of bounded variation $u : I = [t_0, t_0+T[\to \mathbb{R}^N$ which is right-continuous in $]t_0, t_0 + T[$;*

(b) *$u_n^+ \to u^+$ pointwisely and in $L^1(I, d\theta; \mathbb{R}^N)$.*

Proof. Let us point out that the special status of the left endpoint is only due to the fact that there $u(t_0) = (\lim) u_n(t_0) = u_0$ must stand in for the left-behaviour $u^-(t_0)$. For $t \in]t_0, t_0 + T[$, we have $du_n([t_0, t]) = u_n(t) - u_n(t_0)$. Thus, by integrating the balance equation (3.4) we obtain

$$u_n(t) - u_0 + M^{-1}V(\int_{t_0}^{t} u_n(s)ds) + M^{-1}K(\int_{t_0}^{t}(q_0 + \int_{t_0}^{s} u_n(\tau)d\tau)ds) = M^{-1}\int_{[t_0,t]} f_n \, d\theta_n + M^{-1}\int_{t_0}^{t} p(s)\, ds. \tag{4.1}$$

Let us denote by $\|\cdot\|_\infty$ the uniform norm (of u_n and τ) and the $L^\infty(I, dt; \mathbb{R}^N)$ norm (of p). Then the above equation and (3.18) lead to the estimate:

$$\|u_n\|_\infty \leq \|u_0\| + aT\|u_n\|_\infty + bT(\|q_0\| + T\|u_n\|_\infty) + c\gamma\|\nu\|_\infty\|d\theta\| + cT\|p\|_\infty$$

where $\|M^{-1}V\| \leq a$ (the norm of linear operators), $\|M^{-1}K\| \leq b$ and $\|M^{-1}\| \leq c$, with $a, b, c > 0$ and $\|d\theta\| = d\theta(I) = \int_I d\theta$.

For $T \in]0, T_1[$ with $T_1 = \min\{T_0, (\sqrt{a^2 + 4b} - a)/2b\}$ we have $1 - aT - bT^2 > 0$ and

$$\|u_n\|_\infty \leq a_1(T) := \frac{\|u_0\| + bT\|q_0\| + c\gamma\|\nu\|_\infty\|d\theta\| + cT\|p\|_\infty}{1 - aT - bT^2}. \tag{4.2}$$

Moreover, from (3.4), that is, from:

$$du_n = -M^{-1}Vu_n dt - M^{-1}Kq_n dt + M^{-1}f_n \, d\theta_n + M^{-1}p \, dt,$$

we see that the (positive) measures of total variation $|du_n|$ are bounded by a fixed measure $d\beta$:

$$|du_n| \leq d\beta := [aa_1(T) + b\,(\|q_0\| + Ta_1(T)) + cT\|p\|_\infty]\, dt + c\gamma\|\nu\|_\infty d\theta. \tag{4.3}$$

Thus, by [7, Lemma 0.3.5] applied to $u_n^+ = u_n$ in $\mathrm{int}(I)$, we may extract a subsequence, still denoted u_n, that converges pointwisely to a function of bounded variation $u : I \to \mathbb{R}^N$, which is right-continuous in $]t_0, t_0 + T[$. Moreover, $u_n(t_0) = u(t_0)$ and $u_n^+(t_0) = u^+(t_0)$, as in (3.15), (3.16). Thus also $u_n^+ \to u^+$ pointwisely in I.

Since (u_n) and (u_n^+) are bounded, the rest of the proposition is a consequence of Lebesgue's theorem on dominated convergence. □

From this, it follows that

$$q_n(t) = q_0 + \int_{t_0}^{t} u_n(s)\, ds \to q(t) := q_0 + \int_{t_0}^{t} u(s)\, ds, \tag{4.4}$$

uniformly on I. To finish the proof that (u, f) solves Problem (P), we still need to check that both the balance equation (1.4) and the friction law, in the form of the inclusion (1.5), hold.

Balance equation. By (4.1), we have

$$M(u_n(t)-u_0)+V(q_n(t)-q_0)+K\int_{t_0}^t q_n(s)\,ds=\int_{[t_0,t]} f_n\,d\theta_n+\int_{t_0}^t p(s)\,ds. \tag{4.5}$$

We write (with χ denoting a characteristic function, as usual):

$$\int_{[t_0,t]} f_n\,d\theta_n-\int_{[t_0,t]} f\,d\theta=\int f_n\chi_{[t_0,t]}(d\theta_n-d\theta)+\int (f_n-f)\chi_{[t_0,t]}d\theta.$$

The first integral terms in the right-hand side converge to zero, because they are bounded in the norm of $\mathbb{R}^N$ by

$$\gamma\|\nu\|_\infty\|d\theta_n-d\theta\|\le\gamma\|\nu\|_\infty\sum_{i=n+1}^{\infty}a_i\to 0,\ \text{as } n\to\infty.$$

(To be precise, the sum concerns only those a_i corresponding to t_i belonging to $I=I(T)$). The components of the other integral terms in the right-hand side can be written as a duality product $\langle f_n-f,\chi_{[t_0,t]}e_k\rangle$ in $L^\infty(I,d\theta;\mathbb{R}^N)\times L^1(I,d\theta;\mathbb{R}^N)$, where $e_k\in\mathbb{R}^N$ is a vector of the canonical basis. Since $f_n\to f$ weakly-$*$, it follows that those integral terms also converge to zero, as n grows. Hence,

$$\int_{[t_0,t]} f_n d\theta_n\to\int_{[t_0,t]} f\,d\theta. \tag{4.6}$$

Since (u_n) and (q_n) converge pointwisely and uniformly (respectively), we obtain from (4.5) and (4.6)

$$M\,(u(t)-u_0)+V(q(t)-q_0)+K\int_{t_0}^t q(s)\,ds=\int_{[t_0,t]} f\,d\theta+\int_{t_0}^t p(s)\,ds, \tag{4.7}$$

for all $t\in I$. This means that the balance equation (1.4) holds if we apply the measures in both hand-sides to any interval $[t_0,t]$; consequently, it holds as an equality of measures.

Friction law. We have to prove (1.5) or equivalently, as in Section 2 and by Lemma 5.2, that

$$f(t)\in\nu(t)C,\ d\theta\text{–a.e. }, \tag{4.8}$$

$$\int_I u^+(t)\cdot(\varphi(t)-f(t))\,d\theta(t)\ge 0, \tag{4.9}$$

for all $d\theta$-measurable selections φ, with $\varphi(t)\in\nu(t)C$, $d\theta$–a.e. .

By (3.17) and the weak-$*$ convergence of (f_n) to f, it classically follows (4.8). The inequality (4.9) is obtained by taking the limit in the analogous formulation of (3.5):

$$\int_I u_n^+(t)\cdot(\varphi(t)-f_n(t))\,d\theta_n(t)\ge 0 \tag{4.10}$$

By (4.2), (3.18) and the definition of φ we have

$$\left|\int_I u_n^+ \cdot (\varphi - f_n) d\theta_n - \int_I u_n^+ \cdot (\varphi - f_n) d\theta\right| \le 2a_1(T)\gamma\|\nu\|_\infty \|d\theta_n - d\theta\|\,,$$

and the right-hand side converges to zero. Thus (4.9) is a consequence of (4.10) and of the following limit

$$\lim_n \int_I u_n^+ \cdot (\varphi - f_n)\, d\theta = \int_I u^+ \cdot (\varphi - f)\, d\theta. \tag{4.11}$$

By the dominated convergence theorem, thanks to Proposition 4.1 and (3.18), we have:

$$\int_I u_n^+ \cdot \varphi\, d\theta \to \int_I u^+ \cdot \varphi\, d\theta, \tag{4.12}$$

$$\left|\int_I u_n^+ \cdot f_n\, d\theta - \int_I u^+ \cdot f_n\, d\theta\right| \le \gamma\|\nu\|_\infty \|u_n^+ - u^+\|_{L^1(d\theta)} \to 0. \tag{4.13}$$

Therefore (4.11) follows from

$$\int_I u^+ \cdot f_n\, d\theta \to \int_I u^+ \cdot f\, d\theta,$$

by the weak-$*$ convergence of f_n to f. □

5. Auxiliary results

The first result concerns the measurability of a particular multifunction.

Lemma 5.1. *Let $\nu : I \subset \mathbb{R} \to \mathbb{R}^N$ be Borel-measurable (respectively, Lebesgue-measurable). Let C be a fixed compact interval of $\mathbb{R}$. For every $t \in I$ define the compact convex set $\nu(t)C \subset \mathbb{R}^N$ by (1.7). Then $t \mapsto \Gamma(t) = \nu(t)C$ is a Borel-measurable (respectively, Lebesgue-measurable) multifunction.*

Proof. Let Q be a countable dense subset of C. For every $(d_1, \ldots, d_N) \in Q^N$, the function

$$t \mapsto (\nu_1(t)d_1, \ldots, \nu_N(t)d_N)$$

is a (Borel or Lebesgue) measurable selection of Γ. Moreover, the set $\mathcal{C}$ of such functions is countable and each $\Gamma(t)$ is the closure of $\{\sigma(t) : \sigma \in \mathcal{C}\}$. Thus, by [1, Chapter III], Γ is a measurable multifunction. □

The second result concerns equivalent formulations of an inclusion. Let $d\eta$ be a complete positive measure on an interval $J = [a, b]$ (and which is defined at least on the Borel subsets of J). Let Γ be a Borel (hence $d\eta$-) measurable multifunction defined on J with closed convex values (contained in a compact subset of $\mathbb{R}^N$).

Lemma 5.2. *Let $v, g : J \to \mathbb{R}^N$ be $d\eta$-measurable functions. Then, the inclusion*

$$-v(t) \in N_{\Gamma(t)}(g(t)),\ d\eta\text{–a.e. in } J, \tag{5.1}$$

is equivalent to the conditions

$$g(t) \in \Gamma(t),\ d\eta\text{–a.e. in } J, \tag{5.2}$$

$$\int_J v(t) \cdot (\varphi(t) - g(t))\, d\eta(t) \geq 0 \tag{5.3}$$

for every $\varphi \in L^2(J, d\eta; \mathbb{R}^N)$ which is a selection of Γ (i.e., $\varphi(t) \in \Gamma(t)$, $d\eta$–a.e.).

Before proving this lemma, let us notice that if $g, v \in L^2$ then (5.2) means that g belongs to the set $\mathrm{Sel}(\Gamma)$ of L^2 selections of Γ. Thus (5.3) means that $\langle v, \varphi - g\rangle_{L^2} \geq 0$ for all $\varphi \in \mathrm{Sel}(\Gamma)$ and so

$$-v \in N_{\mathrm{Sel}(\Gamma)}(g). \tag{5.4}$$

Proof. If (5.1) holds then the outward normal cone is nonempty for $d\eta$-almost every t, and this is only possible if (5.2) is true. Then, there is a $d\eta$-null set N such that, for all $t \notin N, g(t) \in \Gamma(t), -v(t) \in N_{\Gamma(t)}(g(t))$ and $\varphi(t) \in \Gamma(t)$. By definition of normal cone, this implies $v(t) \cdot (\varphi(t) - g(t)) \geq 0$, for $t \notin N$ and so (5.3) holds.

To prove the converse, i.e. that (5.2) and (5.3) imply (5.1), we show that if (5.2) holds but (5.1) does not, then (5.3) cannot hold. Under such assumptions, there is a set of positive measure $J_0 \subset J$ such that, for all $t \in J_0, f(t,z) := v(t) \cdot (z - g(t))$ is negative for some $z \in \Gamma(t)$. Then $m(t) := \inf\{f(t,z) : z \in \Gamma(t)\}$ is a negative $d\eta$-measurable function in J_0, by [1, Lemma III-39]. It follows that $t \mapsto \{z \in \Gamma(t) : f(t,z) \leq m(t)/2\}$ is a $d\eta$-measurable multifunction in J_0 with nonempty closed values, hence it has a $d\eta$-measurable selection $z : J_0 \to \mathbb{R}^N$. Putting $\varphi(t) = z(t)$ if $t \in J_0$ and $\varphi(t) = g(t)$ otherwise, we obtain an L^2-selection of Γ for which

$$\int_J v(t) \cdot (\varphi(t) - g(t))\, d\eta(t) = \int_{J_0} f(t, z(t))\, d\eta(t) \leq \int_{J_0} \frac{m(t)}{2}\, d\eta(t) < 0,$$

contradicting (5.3). □

Next, and for the sake of completeness, we recall an existence result for a classical variational problem.

Lemma 5.3. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\ \|$. Assume that $S : H \to H$ is a continuous monotone operator and that K is a nonempty bounded closed convex subset of H. Then there is at least one $x \in H$ such that*

$$x \in K; \quad \langle Sx, y - x\rangle \geq 0,\ \forall y \in K. \tag{5.5}$$

In other words, by definition of outward normal cone, there is $x \in H$ such that

$$-Sx \in N_K(x). \tag{5.6}$$

Moreover, if S is strictly monotone, i.e. if

$$x \neq y \Rightarrow \langle Sx - Sy, x - y \rangle > 0, \tag{5.7}$$

then (5.5) *– i.e.* (5.6) *– has a unique solution.*

Proof. Existence: H is a reflexive Banach space, S is monotone and obviously weakly continuous over all subspaces of finite dimension of H and, since K is a nonempty closed convex set, its indicator function $\varphi = \delta_K$ is a proper lower semicontinuous convex function. Moreover, for any $y_0 \in \operatorname{dom} \varphi = K$:

$$\|y\| \to +\infty \Rightarrow [\langle Sy, y - y_0 \rangle + \varphi(y)] \, \|y\|^{-1} \to +\infty;$$

this is obvious since $\varphi(y) = +\infty$ –i.e. $y \notin K$– for sufficiently large $\|y\|$.

Hence, we may apply Theorem II.3.1 in [2]: it follows that for every $z \in H$ there is $x \in H$ which solves

$$\langle Sx - z, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in H. \tag{5.8}$$

By choosing $y \in K$, we see that $\varphi(x) < +\infty$, whence $x \in K$ and $\varphi(x) = 0$. Therefore, (5.8) is trivial for $y \notin K$ and for $z = 0$ it is certainly equivalent to (5.5).

Uniqueness: Let $x_1, x_2 \in K$ satisfy (5.5). Then $\langle Sx_1, x_2 - x_1 \rangle \geq 0$ (take $x = x_1$ and $y = x_2$) and similarly $\langle Sx_2, x_1 - x_2 \rangle \geq 0$. Hence

$$\langle Sx_1 - Sx_2, x_1 - x_2 \rangle \leq 0,$$

which implies $x_1 = x_2$, by (5.7). □

Finally, let us prove the following

Lemma 5.4. *Let H be a Hilbert space (or simply $H = \mathbb{R}^N$) and $y \in H$. Let $A : H \to H$ be a symmetric strongly positive continuous linear operator and consider the following conditions on $x \in H$:*

$$x = \operatorname{proj}_K((Id - \rho A)x + \rho y); \tag*{$(5.9)_\rho$}$$

$$y - Ax \in N_K(x). \tag{5.10}$$

Then, by elementary convex analysis, (5.10) $\Rightarrow$ $(5.9)_\rho$, *for all $\rho > 0$; and if $(5.9)_\rho$ holds for some $\rho > 0$, then* (5.10) *holds.*
Moreover, $(5.9)_\rho$ and (5.10) *have a unique solution $x \in H$ (where necessarily $x \in K$).*

Proof. To prove $(5.9)_\rho$, hence (5.10), we show that, for a convenient choice of $\rho > 0$, the operator $x \mapsto T_\rho(x) := \operatorname{proj}_K((Id - \rho A)x + \rho y)$ has a unique fixed point in K.

The projection on K is nonexpansive, whence

$$\|T_\rho(x) - T_\rho(z)\| \leq \|(Id - \rho A)(x - z)\|. \tag{5.11}$$

We know that for some $\alpha > 0$, $\langle Aw, w \rangle \geq \alpha \|w\|^2$, $\forall w \in H$. Thus, by developing the square of the r.h.s. of (5.11), we obtain

$$\|T_\rho(x) - T_\rho(z)\| \leq k(\rho) \|x - z\|, \ \forall x, z \in H,$$

where $k(\rho) := (1 - 2\rho\alpha + \rho^2\|A\|^2)^{1/2}$. If $\rho \in]0, 2\alpha\|A\|^{-2}[$, then $k(\rho) \in]0,1[$. Hence, T_ρ is a strict contraction in K and it has a unique fixed point. □

References

[1] C. Castaing, M. Valadier: Convex analysis and measurable multifunctions, Springer-Verlag, Berlin et al., 1977.

[2] I. Ekeland, R. Temam: Convex Analysis and Variational Problems, North-Holland, Amsterdam, 1976.

[3] E. J. Haug, S. C. Wu, S. M. Yang: Dynamics of mechanical systems with Coulomb friction, stiction, impact and constraint addition-deletion-I, Mechanisms and Machine Theory 21 (1986) 407–416.

[4] P. Lötstedt: Coulomb friction in two-dimensional rigid body systems, ZAMM 61 (1981) 605–615.

[5] M. Laghdir, M. D. P. Monteiro Marques: Motion of a particle submitted to dry friction and to normal percussion, Portugaliæ Mathematica 52 (1995) 227–239.

[6] M. Laghdir, M. D. P. Monteiro Marques: Dynamics of a particle with damping, friction and percussional effects, Journal of Mathematical Analysis and Applications 196 (1995) 902–920.

[7] M. D. P. Monteiro Marques: Differential Inclusions in Nonsmooth Mechanical Problems–Shocks and Dry Friction, Birkhäuser Verlag, Basel, 1993.