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Boundary Homogenization for a Quasi-Linear Elliptic Problem with Dirichlet Boundary Conditions Posed on Small Inclusions Distributed on the Boundary

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On décrit le comportement asymptotique de la solution d'un problème elliptique quasi-linéaire posé dans un domaine de $\mathbb{R}^n$, $n \geq 3$ et comportant des conditions de Dirichlet homogènes sur de petites zones de diamètre inférieur à ε réparties sur la frontière de ce domaine, lorsque le paramètre ε tend vers 0. On utilise les méthodes d'épi-convergence pour décrire ce comportement limite.

We describe the asymptotic behaviour of the solution of a quasi-linear elliptic problem posed in a domain of $\mathbb{R}^n$, $n \geq 3$ and with homogeneous Dirichlet boundary conditions imposed on small zones of size less than ε distributed on the boundary of this domain when the parameter ε goes to 0. We use epi-convergence arguments in order to establish the limit behaviour.

1. Introduction

Let Ω be a smooth domain of $\mathbb{R}^n$, $n \geq 3$, the boundary $\partial\Omega$ of which is decomposed into the disjoint union $\Gamma_1 \cup \Gamma_2 \cup \Sigma$ of non empty sets. We suppose that Σ is smooth and we dispose on Σ $2n(\varepsilon)+1$ zones of size less than ε assuming that $2n(\varepsilon)+1$ is equivalent to $1/\varepsilon$ as ε goes to 0. We denote by T_ε^k the k-th zone for k in $\{-n(\varepsilon), .., n(\varepsilon)\}$, see figure 1.1 below. We define T_ε as the union $\cup_{k=-n(\varepsilon)}^{n(\varepsilon)} T_\varepsilon^k$ of the zones contained in Σ and we suppose that these zones never touch $\Gamma_1 \cup \Gamma_2$. Finally we define: $\Sigma_\varepsilon = \Sigma \setminus \overline{T_\varepsilon}$, see figure 1.1 below. For p in $]1, +\infty[$ we denote by $W^{1,p}(\Omega)$ the classical Sobolev space:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \frac{\partial u}{\partial x_i} \in L^p(\Omega),\, \forall i \in \{1, ..., n\} \right\}.$$

$W^{1,p}(\Omega)$ is a Banach space when equipped with the norm:

$$\|u\|_{1,p}^p = \int_\Omega |u|^p \, dx + \int_\Omega |\nabla u|^p \, dx.$$

$W^{1/q,p}(\Sigma)$ is the space consisting of the traces on Σ of the functions in $W^{1,p}(\Omega)$, where q is the conjugate exponent of p defined by $1 = \frac{1}{p} + \frac{1}{q}$ and $W^{-1/q,q}(\Sigma)$ is its dual space.

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We consider in Ω the following quasi-linear elliptic problem:

$$\begin{cases} -\operatorname{div}\left(\left|\nabla u_\varepsilon\right|^{p-2}\nabla u_\varepsilon\right) &= f \quad \text{in } \Omega \\ u_\varepsilon &= 0 \quad \text{on } \Gamma_1 \cup T_\varepsilon \\ \left|\nabla u_\varepsilon\right|^{p-2}\dfrac{\partial u_\varepsilon}{\partial n} &= 0 \quad \text{on } \Gamma_2 \cup \Sigma_\varepsilon, \end{cases} \tag{1.1}$$

where f is supposed to belong to $L^q(\Omega)$ and n is the outer normal to Ω.

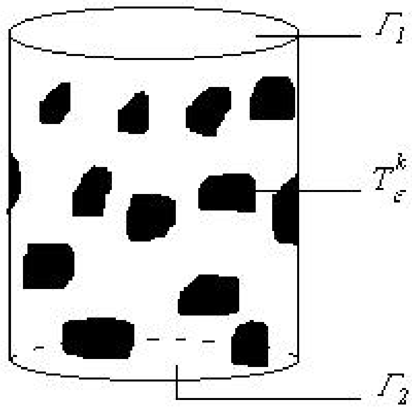

Figure 1.1: The domain Ω and the zones.

If V_ε is the subspace of $W^{1,p}(\Omega)$ defined by:

$$V_\varepsilon = \left\{u \in W^{1,p}(\Omega) \mid u = 0 \text{ on } \Gamma_1 \cup T_\varepsilon\right\}, \tag{1.2}$$

the minimization problem associated to (1.1) is:

$$\min_{u \in V_\varepsilon}\left\{\int_\Omega |\nabla u|^p\,dx - p\int_\Omega fudx\right\},$$

or:

$$\min_{u \in W^{1,p}(\Omega)}\left\{F_\varepsilon(u) - p\int_\Omega fudx\right\}, \tag{1.3}$$

where the functional F_ε is defined on $W^{1,p}(\Omega)$ by:

$$F_\varepsilon(u) = \begin{cases} \int_\Omega |\nabla u|^p\,dx & \text{if } u \in V_\varepsilon \\ +\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

By means of classical arguments we can prove the following:

Lemma 1.1.

(i) *There exists a unique solution u_ε of (1.3) in $W^{1,p}(\Omega)$, which is a weak solution of (1.1).*

(ii) *The sequence $(u_\varepsilon)_\varepsilon$ is bounded in $W^{1,p}(\Omega)$.*

Proof. (i) The functional F_ε is convex, thanks to the convexity of V_ε, continuous for the strong topology of $W^{1,p}(\Omega)$, thanks to the trace theorems in $W^{1,p}(\Omega)$, hence lower semi-continuous for the weak topology of this space. F_ε is trivially coercive on $W^{1,p}(\Omega)$. Moreover F_ε is strictly convex. Hence (1.3) has a unique minimizer u_ε on $W^{1,p}(\Omega)$ which belongs to V_ε and satisfies:

$$\int_\Omega |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon . \nabla v dx = p \int_\Omega f v dx, \qquad \forall v \in V_\varepsilon,$$

hence (1.1) in a weak sense.

(ii) The estimates on u_ε are trivially deduced from (1.3) taking u_ε as a test function by means of Hölder's inequality. □

The purpose of this work is to describe the asymptotic behaviour of the sequence $(u_\varepsilon)_\varepsilon$ when the parameter ε goes to 0. The limit of this sequence will be obtained using epi-convergence arguments. For the description of this variational convergence well-fitted to the asymptotic analysis of minimization problems we refer to [1]. The present work follows the general description of limit obstacle or grid problems written by Dal Maso, De Giorgi and Longo in [6], [7], [9], [10], [8] and by Attouch and Picard in [2], [3], [4]. We will prove that $(F_\varepsilon)_\varepsilon$ epi-converges in the weak topology of $W^{1,p}(\Omega)$ to F_o defined on $W^{1,p}(\Omega)$ by:

$$F_o(u) = \begin{cases} \int_\Omega |\nabla u|^p \, dx + \int_\Sigma a(x) \left| v_{|\Sigma} \right|^p (x) d\mu(x) & \text{if } u \in V_o \\ +\infty & \text{otherwise,} \end{cases} \tag{1.5}$$

where a is a nonnegative, lower semi-continuous and nonincreasing Borel measure, μ is a nonnegative Radon measure which belongs to $W^{-1/q,q}(\Sigma)$ and

$$V_o = \left\{ u \in W^{1,p}(\Omega) \mid u = 0 \text{ on } \Gamma_1 \right\}.$$

In the last part of this work we will discuss the example dealing with the ε-periodic distribution on the lateral boundary of a cylinder of identical strings.

2. Functional Framework

2.1. Capacities

The space $W^{1/q,p}(\Sigma)$ consists of the functions u in $L^p(\Sigma)$ satisfying:

$$\int_\Sigma \int_\Sigma \frac{|u(x) - u(y)|^p}{|x-y|^{n-2+p}} d\sigma(x) d\sigma(y) < +\infty.$$

This is a reflexive Banach space when equipped with the norm:

$$\|u\|_{o,p}^p = \|u\|_{L^p(\Sigma)}^p + \int_\Sigma \int_\Sigma \frac{|u(x) - u(y)|^p}{|x-y|^{n-2+p}} d\sigma(x) d\sigma(y),$$

see [18]. One immediately verifies that if u belongs to $W^{1/q,p}(\Sigma)$ then $u^+ = \max(0, u)$ and $|u|$ also belong to $W^{1/q,p}(\Sigma)$.

For every compact subset K of Σ we define its capacity as:

$$\mathrm{Cap}(K) = \inf\left\{ \|u\|_{o,p}^{p} \mid u \in C_c^o(\Sigma);\ u \geq 0 \text{ in } \Sigma;\ u \geq 1 \text{ in } K \right\}.$$

We can extend this definition to the case of every open subset ω of Σ by:

$$\mathrm{Cap}(\omega) = \sup\{\mathrm{Cap}(K) \mid K \text{ compact; } K \subset \omega\}$$

and finally to every subset A of Σ as:

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(\omega) \mid \omega \text{ open; } A \subset \omega\}.$$

Thus defined, Cap is a Choquet capacity, as can be proved adapting the results of [14, Theorem 1.1] and [13, Lemma 3.4]. A property will be called "$q-e$ true" (quasi everywhere true) if it is true except possibly on a set A satisfying $\mathrm{Cap}(A)$ =0. A function u of $W^{1/q,p}(\Sigma)$ is said Cap-quasi-continuous if there exists a nonincreasing sequence $(\omega_n)_n$ of open subsets of Σ such that $\lim_{n\to+\infty} \mathrm{Cap}(\omega_n) = 0$ and $u_{|(\Sigma\setminus\omega_n)}$ is continuous for every n. Lemma 2.10 of [2] implies that every u in $W^{1/q,p}(\Sigma)$ admits a unique Cap-quasi-continuous representant $\widetilde{u}$.

2.2. Integral representation of functionals

Let $O(\Sigma)$ (resp. $B(\Sigma)$) be the family of all open subsets of Σ (resp. the set of borelian subsets of Σ). We denote by $\mathbf{F}$ the family of functionals F from $W^{1/q,p}(\Sigma) \times O(\Sigma)$ into $[0, +\infty]$ satisfying (2.1), (2.2) and (2.3) with:

$$\forall \omega \in O(\Sigma) : u \mapsto F(u, \omega) \text{ is monotone and lower semi-continuous on } W^{1/q,p}(\Sigma), \text{ equipped with its strong topology,} \tag{2.1}$$

$$\forall u \in W^{1/q,p}(\Sigma) : \omega \mapsto F(u, \omega) \text{ is the restriction to } O(\Sigma) \text{ of some Borel measure,} \tag{2.2}$$

$$\forall u, v \in W^{1/q,p}(\Sigma),\ \forall \omega \in O(\Sigma),\ u_{|\omega} = v_{|\omega} \Rightarrow F(u, \omega) = F(v, \omega). \tag{2.3}$$

We denote by $\mathbf{F}_o$ the subfamily of $\mathbf{F}$ consisting of the functionals F from $W^{1/q,p}(\Sigma) \times O(\Sigma)$ into $[0, +\infty]$ such that : $u \mapsto F(u, \omega)$ is monotone nonincreasing for every ω.

An integral representation theorem has been proved by Dal Maso [6], when the functionals F belonging to the class $\mathbf{F}_o$ are defined from $W^{m,p}(\mathbb{R}^n) \times O(\mathbb{R}^n)$ into $[0, +\infty]$ for some m in $\mathbf{N}^*$. The following result can be proved in the same spirit, see [13].

Theorem 2.1. *Let F be any element of $\mathbf{F}_o$. There exists a Borel function f from $\Sigma \times \mathbb{R}$ into $[0, +\infty]$, a nonnegative Radon measure μ in $W^{-1/q,q}(\Sigma)$ and a Borel nonnegative measure ν such that:*

(i) *For every u in $W^{1/q,p}(\Sigma)$, for every ω in $O(\Sigma)$:*

$$F(u, \omega) = \int_\omega f(x, \widetilde{u}(x))\, d\mu(x) + \nu(\omega),$$

where $\widetilde{u}$ is the quasi-continuous representant of u.

(ii) *For every x in Σ, the function $t \mapsto f(x,t)$ is nonincreasing and lower semi-continuous on $\mathbb{R}$.*

The measure μ is absolutely continuous with respect to the capacity Cap, *in the sense that if* $\mathrm{Cap}(B)$ *is equal to 0, for some B in $B(\Sigma)$, then $\mu(B)$ is also equal to 0, see* [6, Part 2].

2.3. Compactness theorem

Let us introduce the following definition:

Definition 2.2. A family R of $B(\Sigma)$ is called rich if for every family $(B_t)_t$ of elements of $B(\Sigma)$ such that $\overline{B_s} \subset \overset{o}{B_t}$, $\forall s < t$, the set $E_t = \{t \in]0,1[\mid B_t \notin R\}$ is at most countable.

Then one has:

Theorem 2.3. *Let $(F_\varepsilon)_\varepsilon$ be any family of functionals of $\mathbf{F}$, such that for every B in $B(\Sigma)$ there exists a bounded sequence $(t_\varepsilon)_\varepsilon$ in $W^{1/q,p}(\Sigma)$ satisfying: $\sup_\varepsilon F_\varepsilon(t_\varepsilon, B) < +\infty$. Then there exists a subsequence $(F_{\varepsilon_k})_k$, a functional F in $\mathbf{F}$ and a rich family R of $B(\Sigma)$, such that :*

$$\forall u \in V_o,\ \forall B \in R : \operatorname*{epi-lim}_{k \to +\infty} \left\{ \int_\Omega |\nabla u|^p\, dx + F_{\varepsilon_k}(u_{|\Sigma}, B) \right\} = \int_\Omega |\nabla u|^p\, dx + F(u_{|\Sigma}, B),$$

where epi-lim *denotes the epi-limit of the sequence of functionals, for the weak topology of $W^{1,p}(\Omega)$ restricted to V_o.*

Proof. Let us define the functionals $\widehat{F}^s$ and $\widehat{F}^i$ on $V_o \times B(\Sigma)$ by:

$$\|u\|_{1,p}^p + \widehat{F}^s(u,B) = \inf \left\{ \limsup_{\varepsilon \to 0} \left\{ \|u_\varepsilon\|_{1,p}^p + F_\varepsilon(u_{\varepsilon|\Sigma}, B) \right\} \mid u_\varepsilon \overset{w-W^{1,p}}{\underset{\varepsilon\to 0}{\rightharpoonup}} u \right\}$$

$$\|u\|_{1,p}^p + \widehat{F}^i(u,B) = \inf \left\{ \liminf_{\varepsilon \to 0} \left\{ \|u_\varepsilon\|_{1,p}^p + F_\varepsilon(u_{\varepsilon|\Sigma}, B) \right\} \mid u_\varepsilon \overset{w-W^{1,p}}{\underset{\varepsilon\to 0}{\rightharpoonup}} u \right\},$$

where the convergence of the sequence $(u_\varepsilon)_\varepsilon$ takes place in the weak topology of $W^{1,p}(\Omega)$. It is easily seen that these functionals agree, in our case, by modifying u_ε near the boundary Γ_1 by means of cut-off functions. For every u in $W^{1/q,p}(\Sigma)$ we consider the convex set $K_u = \{v \in V_o \mid v_{|\Sigma} = u\}$ and the functional:

$$J_u(v) = \begin{cases} \int_\Omega |\nabla v|^p\, dx & \text{if } v \in K_u \\ +\infty & \text{otherwise.} \end{cases}$$

The problem $\min_{v \in V_o} J_u(v)$ has a unique solution $r(u)$ which is the image of u through the continuous mapping r from $W^{1/q,p}(\Sigma)$ into V_o. One can then write:

$$\|r(u)\|_{1,p}^p + \widehat{F}^s(r(u),B) = \inf \left\{ \limsup_{\varepsilon \to 0} \left\{ \|u_\varepsilon\|_{1,p}^p + F_\varepsilon(u_{\varepsilon|\Sigma}, B) \right\} \mid u_\varepsilon \overset{w-W^{1,p}}{\underset{\varepsilon\to 0}{\rightharpoonup}} r(u) \right\}$$

$$\|r(u)\|_{1,p}^p + \widehat{F}^i(r(u),B) = \inf \left\{ \liminf_{\varepsilon \to 0} \left\{ \|u_\varepsilon\|_{1,p}^p + F_\varepsilon(u_{\varepsilon|\Sigma}, B) \right\} \mid u_\varepsilon \overset{w-W^{1,p}}{\underset{\varepsilon\to 0}{\rightharpoonup}} r(u) \right\}$$

and we define the functionals F^s and F^i on $W^{1/q,p}(\Sigma) \times B(\Sigma)$ by:

$$F^s(u,B) = \widehat{F}^s(r(u),B) \ ; \ F^i(u,B) = \widehat{F}^i(r(u),B).$$

□

Let us study the properties of these functionals.

Lemma 2.4.
(i) *For every ω in $O(\Sigma)$, $F^s(.,\omega)$ is nondecreasing (resp. nonincreasing) if F_ε is nondecreasing (resp. nonincreasing) for every ε.*
(ii) *For every ω in $O(\Sigma)$, $F^s(.,\omega)$ is lower semi-continuous on $W^{1/q,p}(\Sigma)$.*

Proof. (i) Choose ω in $O(\Sigma)$, u_1 and u_2 in $W^{1/q,p}(\Sigma)$ such that $u_1 \leq u_2$ almost everywhere on Σ. The maximum principle implies that : $r(u_1) \leq r(u_2)$ almost everywhere in Ω. We choose two sequences $(u_{1\varepsilon})_\varepsilon$ and $(u_{2\varepsilon})_\varepsilon$ converging to $r(u_1)$ and $r(u_2)$ respectively in the weak topology of $W^{1,p}(\Omega)$ and such that:

$$\|r(u_i)\|^p_{1,p} + \widehat{F}^s(r(u_i),\omega) = \limsup_{\varepsilon\to 0} \left\{ \|u_{i\varepsilon}\|^p_{1,p} + F_\varepsilon(u_{i\varepsilon},\omega) \right\} ; \ i = 1,2.$$

We then observe that the supremum $u_{1\varepsilon} \vee u_{2\varepsilon}$ (resp. the infimum $u_{1\varepsilon} \wedge u_{2\varepsilon}$) is such that the sequence $(u_{1\varepsilon} \vee u_{2\varepsilon})_\varepsilon$ (resp. $u_{1\varepsilon} \wedge u_{2\varepsilon})$) converges to $r(u_1) \vee r(u_2) = r(u_2)$ (resp. $r(u_1) \wedge r(u_2) = r(u_1)$) in the weak topology of $W^{1,p}(\Omega)$. Moreover because the norm is lower semi-continuous with respect to the weak topology of $W^{1,p}(\Omega)$ we get:

$$\|r(u_2)\|^p_{1,p} \leq \liminf_{\varepsilon\to 0} \|u_{1\varepsilon} \vee u_{2\varepsilon}\|^p_{1,p}.$$

We thus deduce:

$$\begin{aligned} F^s(u_2,\omega) &\geq \limsup_{\varepsilon\to 0} \left\{ \|u_{2\varepsilon}\|^p_{1,p} + F_\varepsilon(u_{2\varepsilon},\omega) \right\} - \liminf_{\varepsilon\to 0} \|u_{1\varepsilon} \vee u_{2\varepsilon}\|^p_{1,p} \\ &\geq \limsup_{\varepsilon\to 0} \left\{ \|u_{2\varepsilon}\|^p_{1,p} - \|u_{1\varepsilon} \vee u_{2\varepsilon}\|^p_{1,p} + F_\varepsilon(u_{2\varepsilon},\omega) \right\}. \end{aligned}$$

Let us suppose that F_ε is nondecreasing for every ε. Then because of the property:

$$\forall u_1,\ u_2 \in W^{1,p}(\Omega) :\ \|u_1 \vee u_2\|^p_{1,p} + \|u_1 \wedge u_2\|^p_{1,p} = \|u_1\|^p_{1,p} + \|u_2\|^p_{1,p} \tag{2.4}$$

and since: $(u_{1\varepsilon} \wedge u_{2\varepsilon})_{|\Sigma} \leq u_{2\varepsilon|\Sigma}$, we get:

$$\begin{aligned} F^s(u_2,\omega) &\geq \limsup_{\varepsilon\to 0} \left\{ \|u_{1\varepsilon} \wedge u_{2\varepsilon}\|^p_{1,p} - \|u_{1\varepsilon}\|^p_{1,p} + F_\varepsilon(u_{1\varepsilon} \wedge u_{2\varepsilon},\omega) \right\} \\ &\geq \|r(u_1)\|^p_{1,p} + F^s(u_1,\omega) - \|r(u_1)\|^p_{1,p} = F^s(u_1,\omega), \end{aligned}$$

which implies that F^s is nondecreasing. The case where F_ε is nonincreasing for every ε in obtained in a similar way exchanging the roles of $u_{1\varepsilon} \wedge u_{2\varepsilon}$ and $u_{1\varepsilon} \vee u_{2\varepsilon}$.

(ii) Let $(u_n)_n$ be any sequence converging to u in the strong topology of $W^{1/q,p}(\Sigma)$. The sequence $(r(u_n))_n$ converges to $r(u)$ in the strong topology of $W^{1,p}(\Omega)$. The functional: $u \mapsto \|r(u)\|_{1,p}^p + \widehat{F}^s(r(u), \omega)$ is lower semi-continuous for the weak topology of $W^{1/q,p}(\Sigma)$ as an upper epi-limit of lower semi-continuous functionals (see [1]), hence is lower semi-continuous for the strong topology of $W^{1/q,p}(\Sigma)$. We thus deduce:

$$\liminf_{\varepsilon \to 0} \widehat{F}^s(r(u_n), \omega) \geq \widehat{F}^s(r(u), \omega) \Rightarrow \liminf_{\varepsilon \to 0} F^s(u_n, \omega) \geq F^s(u, \omega),$$

because of the definition of F^s. □

Lemma 2.5. *Let u be any element of $W^{1/q,p}(\Sigma)$, A and B be any elements of $O(\Sigma)$. Then:*

$$F^s(u, A \cup B) \leq F^s(u, A) + F^s(u, B).$$

Proof. There exists two sequences $(u_{1\varepsilon})_\varepsilon$ and $(u_{2\varepsilon})_\varepsilon$ converging to u in the weak topology of $W^{1,p}(\Omega)$ such that:

$$\|u\|_{1,p}^p + F^s(u, A) = \limsup_{\varepsilon \to 0} \left\{ \|u_{1\varepsilon}\|_{1,p}^p + F_\varepsilon(u_{1\varepsilon}, A) \right\}$$

$$\|u\|_{1,p}^p + \widehat{F}^i(u, B) = \liminf_{\varepsilon \to 0} \left\{ \|u_{2\varepsilon}\|_{1,p}^p + F_\varepsilon(u_{2\varepsilon}, B) \right\}.$$

Suppose that F_ε is nonincreasing on $W^{1,p}(\Omega)$ and define: $u_\varepsilon = u_{1\varepsilon} \vee u_{2\varepsilon}$. We obtain using (2.4) and (2.2):

$$\begin{aligned}
&\|u\|_{1,p}^p + F^s(u, A \cup B) \\
&\quad \leq \limsup_{\varepsilon \to 0} \left\{ \|u_\varepsilon\|_{1,p}^p + F_\varepsilon(u_\varepsilon, A \cup B) \right\} \\
&\quad \leq \limsup_{\varepsilon \to 0} \left\{ \|u_{1\varepsilon}\|_{1,p}^p + \|u_{2\varepsilon}\|_{1,p}^p - \|u_{1\varepsilon} \wedge u_{2\varepsilon}\|_{1,p}^p + F_\varepsilon(u_{1\varepsilon}, A) + F_\varepsilon(u_{2\varepsilon}, B) \right\} \\
&\quad \leq \limsup_{\varepsilon \to 0} \left\{ \|u_{1\varepsilon}\|_{1,p}^p + F_\varepsilon(u_{1\varepsilon}, A) \right\} \\
&\qquad + \limsup_{\varepsilon \to 0} \left\{ \|u_{2\varepsilon}\|_{1,p}^p + F_\varepsilon(u_{2\varepsilon}, B) \right\} + \limsup_{\varepsilon \to 0} \left\{ - \|u_{1\varepsilon} \wedge u_{2\varepsilon}\|_{1,p}^p \right\} \\
&\quad \leq \|u\|_{1,p}^p + F^s(u, A) + \|u\|_{1,p}^p + F^s(u, B) + \limsup_{\varepsilon \to 0} \left\{ - \|u_{1\varepsilon} \wedge u_{2\varepsilon}\|_{1,p}^p \right\} \\
&\quad \leq \|u\|_{1,p}^p + F^s(u, A) + F^s(u, B),
\end{aligned}$$

since: $\limsup_{\varepsilon \to 0} \left\{ - \|u_{1\varepsilon} \wedge u_{2\varepsilon}\|_{1,p}^p \right\} = - \liminf_{\varepsilon \to 0} \|u_{1\varepsilon} \wedge u_{2\varepsilon}\|_{1,p}^p \leq - \|u\|_{1,p}^p$. This proves the result in this case. If F_ε is nonincreasing we use $u_{1\varepsilon} \wedge u_{2\varepsilon}$ instead of $u_{1\varepsilon} \vee u_{2\varepsilon}$. □

Lemma 2.6.

(i) *Let u be any element of $W^{1/q,p}(\Sigma)$, A and B be any elements of $O(\Sigma)$ such that $A \cap B = \emptyset$, A' and B' be any elements of $O(\Sigma)$ such that $\overline{A'} \subset A$, $\overline{B'} \subset B$. Then:*

$$F^i(u, A \cup B) \geq F^i(u, A') + F^i(u, B').$$

(ii) *For every u and v in $W^{1/q,p}(\Sigma)$ and every ω in $O(\Sigma)$ such that $u_{|\omega} = v_{|\omega}$ one has:*

$$\sup_{\overline{\omega'}\subset\omega} F^i(v,\omega') \le F^i(u,\omega).$$

Proof. (i) Let σ be any element of $O(\Sigma)$ and $\Omega_\sigma \subset \Omega$ a smooth and open subset of $\mathbb{R}^n$ such that:

$$\overline{\Omega_\sigma} \cap \Sigma = \sigma\ ;\ \overline{\Omega_\sigma} \cap \Gamma_1 \neq \emptyset.$$

We define:

$$V_\sigma = \{u \in W^{1,p}(\Omega_\sigma) \mid u = 0 \text{ on } \overline{\Omega_\sigma} \cap \Gamma_1\},$$

equipped with the norm: $\|u\|_{1,\sigma,p} = \left(\int_{\Omega_\sigma} |\nabla u|^p\,dx\right)^{1/p}$. For every u in $W^{1/q,p}(\Sigma)$, and σ and ω in $O(\Sigma)$ verifying: $\omega \subset \sigma$, we define $F^i(u,\Omega_\sigma,\omega)$ as:

$$\|u\|^p_{1,\sigma,p} + F^i(u,\Omega_\sigma,\omega) = \inf\left\{\liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|^p_{1,\sigma,p} + F_\varepsilon(u_\varepsilon,\omega)\right\} \mid u_\varepsilon \underset{\varepsilon\to 0}{\overset{w-V_\sigma}{\rightharpoonup}} u\right\}.$$

Let us first prove that:

$$F^i(u,\Omega_\sigma,\omega) \le F^i(u,\Omega_\Sigma,\omega),$$

with: $\Omega_\Sigma = \Omega$. Indeed, there exists $(u_\varepsilon)_\varepsilon$ converging to u in the weak topology of $W^{1,p}(\Omega)$ such that:

$$\|u\|^p_{1,p} + F^i(u,\Omega_\Sigma,\omega) = \liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|^p_{1,p} + F_\varepsilon(u_\varepsilon,\omega)\right\}.$$

Thus, we obtain:

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|^p_{1,p} + F_\varepsilon(u_\varepsilon,\omega)\right\}\\
&\quad\ge \liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|^p_{1,\omega,p} + F_\varepsilon(u_\varepsilon,\omega)\right\} + \liminf_{\varepsilon\to 0}\int_{\Omega\setminus\overline{\Omega_\omega}} |\nabla u|^p\,dx\\
&\quad\ge \|u\|^p_{1,\omega,p} + F^i(u,\Omega_\omega,\omega) + \liminf_{\varepsilon\to 0}\int_{\Omega\setminus\overline{\Omega_\omega}} |\nabla u|^p\,dx\\
&\quad\ge \|u\|^p_{1,p} + F^i(u,\Omega_\omega,\omega).
\end{aligned}$$

Let us choose ω' and ω in $O(\Sigma)$ such that $\overline{\omega'} \subset \omega$ and prove that:

$$F^i(u,\Omega_\Sigma,\omega') \le F^i(u,\Omega_\omega,\omega).$$

There exists a sequence $(u_\varepsilon)_\varepsilon$ converging to u in the weak topology of $W^{1,p}(\Omega_\omega)$ such that:

$$\|u\|^p_{1,\omega,p} + F^i(u,\Omega_\Sigma,\omega) = \liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|^p_{1,p} + F_\varepsilon(u_\varepsilon,\omega)\right\}.$$

We choose $\Omega_{\omega'} \subset \Omega_\omega$ and θ in $C_c^\infty(\mathbb{R}^N)$ with values in $[0,1]$ such that:

$$\theta = \begin{cases} 1 & \text{on } \overline{\Omega_{\omega'}}\\ 0 & \text{on } \Omega\setminus\overline{\Omega_\omega},\end{cases}$$

and define: $v_\varepsilon = \theta u_\varepsilon + (1-\theta)u$. Observing that:

$$\nabla v_\varepsilon = \theta \nabla u_\varepsilon + (u_\varepsilon - u)\nabla\theta + (1-\theta)\nabla u,$$

we write:

$$\|\nabla v_\varepsilon\|_{(L^p(\Omega))^3} \leq \|\theta\nabla u_\varepsilon + (1-\theta)\nabla u\|_{(L^p(\Omega))^3} + \|(u_\varepsilon - u)\nabla\theta\|_{(L^p(\Omega))^3}.$$

Because: $v_{\varepsilon|\omega'} = u_{\varepsilon|\omega'}$, we have: $F_\varepsilon(v_\varepsilon,\omega') = F_\varepsilon(u_\varepsilon,\omega') \leq F_\varepsilon(v_\varepsilon,\omega)$, thanks to (2.2) and (2.3). Because v_ε belongs to V_ε and $(v_\varepsilon)_\varepsilon$ converges to u in the weak topology of $W^{1,p}(\Omega)$ we infer:

$$\begin{aligned}
\|u\|_{1,p}^p + F^i(u,\Omega_\Sigma,\omega') &\leq \liminf_{\varepsilon\to 0}\left\{\|v_\varepsilon\|_{1,p}^p + F_\varepsilon(v_\varepsilon,\omega')\right\}\\
&\leq \liminf_{\varepsilon\to 0}\left\{\|\theta\nabla u_\varepsilon + (1-\theta)\nabla u\|_{(L^p(\Omega))^3}^p + F_\varepsilon(u_\varepsilon,\omega)\right\}\\
&\leq \liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|_{1,\omega,p}^p + F_\varepsilon(u_\varepsilon,\omega)\right\} + \int_{\Omega\setminus\overline{\Omega_\omega}} |\nabla u|^p\,dx\\
&\leq \|u\|_{1,p}^p + F^i(u,\Omega_\omega,\omega),
\end{aligned}$$

because $(\|(u_\varepsilon - u)\,\nabla\theta\|)_\varepsilon$ converges to 0.

Let now u be any element of $W^{1/q,p}(\Sigma)$, A and B be elements of $O(\Sigma)$ such that $A\cap B=\emptyset$. There exist A' and B' in $O(\Sigma)$ such that: $\overline{A'}\subset A$, $\overline{B'}\subset B$, a sequence $(u_\varepsilon)_\varepsilon$ converging to u in the weak topology of $W^{1,p}(\Omega_{A\cup B})$ such that:

$$\begin{aligned}
&\|u\|_{1,A\cup B,p}^p + F^i(u,\Omega_{A\cup B},A\cup B)\\
&\quad = \liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|_{1,A\cup B,p}^p + F_\varepsilon(u_\varepsilon,A\cup B)\right\}\\
&\quad = \liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|_{1,A,p}^p + \|u_\varepsilon\|_{1,B,p}^p + F_\varepsilon(u_\varepsilon,A) + F_\varepsilon(u_\varepsilon,B)\right\}\\
&\quad \geq \|u\|_{1,A,p}^p + F^i(u,\Omega_A,A) + \|u\|_{1,B,p}^p + F^i(u,\Omega_B,B),
\end{aligned}$$

which implies:

$$F^i(u,\Omega_{A\cup B},A\cup B) \geq F^i(u,\Omega_A,A) + F^i(u,\Omega_B,B)$$

and finally:

$$\begin{aligned}
F^i(u,\Omega,A\cup B) &\geq F^i(u,\Omega_{A\cup B},A\cup B)\\
&\geq F^i(u,\Omega_A,A) + F^i(u,\Omega_B,B)\\
&\geq F^i(u,\Omega,A') + F^i(u,\Omega,B').
\end{aligned}$$

(ii) Let u, v be any elements of $W^{1/q,p}(\Sigma)$, ω be any element of $O(\Sigma)$ such that: $u_{|\omega} = v_{|\omega}$. We choose $\Omega_\omega\subset\Omega$ such that $\omega\subset\overline{\Omega_\omega}\cap\Sigma$. We denote u (resp. v) the extension of u (resp. v) in Ω such that: $u_{|\Omega_\omega} = v_{|\Omega_\omega}$. There exists a sequence $(u_\varepsilon)_\varepsilon$ converging to u in the weak topology of V_o such that:

$$\|u\|_{1,p}^p + F^i(u,\omega) = \liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|_{1,p}^p + F_\varepsilon(u_\varepsilon,\omega)\right\}.$$

We choose ω' in $O(\Sigma)$ such that $\overline{\omega'} \subset \omega$, $\overline{\Omega_{\omega'}} \subset \Omega_\omega$ and $\overline{\omega'} \subset \overline{\Omega_{\omega'}} \cap \Sigma$ in such a way that $\overline{\Omega_{\omega'}}$ increases to Ω_ω when $\overline{\omega'}$ increases to ω. We then choose a function θ in $C_c^\infty(\mathbb{R}^N)$ with values in $[0,1]$ such that:

$$\theta = \begin{cases} 1 & \text{on } \overline{\Omega_{\omega'}} \\ 0 & \text{on } \Omega \setminus \Omega_\omega \end{cases}$$

and a sequence $(z_\varepsilon)_\varepsilon$ converging to v in the strong topology of $W^{1,p}(\Omega)$, with z_ε in V_ε for every ε. The sequence $(v_\varepsilon)_\varepsilon$ defined by: $v_\varepsilon = \theta u_\varepsilon + (1-\theta) z_\varepsilon$ converges to v in the weak topology of $W^{1,p}(\Omega)$ and: $\nabla v_\varepsilon = \theta \nabla u_\varepsilon + (u_\varepsilon - z_\varepsilon)\nabla\theta + (1-\theta)\nabla z_\varepsilon$. For t in $]0,1[$ we write, thanks to the convexity of $x \mapsto |x|^p$:

$$\begin{aligned} \int_\Omega |t\nabla v_\varepsilon|^p \, dx &\le t \int_\Omega |\theta\nabla u_\varepsilon + (1-\theta)\nabla z_\varepsilon|^p \, dx + (1-t) \int_{\Omega_\omega \setminus \overline{\Omega_{\omega'}}} \left| \frac{t}{1-t}(u_\varepsilon - z_\varepsilon)\nabla\theta \right|^p dx \\ &\le t \int_{\Omega_\omega} |\nabla u_\varepsilon|^p \, dx + \int_{\Omega_\omega \setminus \overline{\Omega_{\omega'}}} |\nabla z_\varepsilon|^p \, dx \\ &\quad + (1-t) \int_{\Omega_\omega \setminus \overline{\Omega_{\omega'}}} \left| \frac{t}{1-t}(u_\varepsilon - z_\varepsilon)\nabla\theta \right|^p dx. \end{aligned}$$

We observe that:

$$\lim_{\varepsilon\to 0} \int_{\Omega_\omega \setminus \overline{\Omega_{\omega'}}} \left| \frac{t}{1-t}(u_\varepsilon - z_\varepsilon)\nabla\theta \right|^p dx = \int_{\Omega_\omega \setminus \overline{\Omega_{\omega'}}} \left| \frac{t}{1-t}(u - v)\nabla\theta \right|^p dx = 0,$$

since: $u_{|\Omega_\omega} = v_{|\Omega_\omega}$. Thus:

$$\begin{aligned} \liminf_{\varepsilon\to 0} \left\{ \int_\Omega |t\nabla v_\varepsilon|^p \, dx + F_\varepsilon(v_\varepsilon, \omega') \right\} &\le \liminf_{\varepsilon\to 0} \left\{ \int_{\Omega_\omega} |\nabla v_\varepsilon|^p \, dx + F_\varepsilon(v_\varepsilon, \omega') \right\} \\ &\quad + \int_{\Omega \setminus \overline{\Omega_{\omega'}}} |\nabla v|^p \, dx. \end{aligned}$$

Furthermore:

$$\|v\|_{1,p}^p + F^i(v, \omega') \le \liminf_{\varepsilon\to 0} \left\{ \|v_\varepsilon\|_{1,p}^p + F_\varepsilon(v_\varepsilon, \omega') \right\}.$$

Letting t increase to 1 in the preceding inequalities we obtain:

$$\begin{aligned} \|v\|_{1,p}^p + F^i(v, \omega') &\le \liminf_{\varepsilon\to 0} \left\{ \int_{\Omega_\omega} |\nabla u_\varepsilon|^p \, dx + F_\varepsilon(u_\varepsilon, \omega) \right\} + \int_{\Omega \setminus \overline{\Omega_{\omega'}}} |\nabla v|^p \, dx \\ &\le \liminf_{\varepsilon\to 0} \left\{ \int_\Omega |\nabla u_\varepsilon|^p \, dx + F_\varepsilon(u_\varepsilon, \omega) \right\} \\ &\quad + \limsup_{\varepsilon\to 0} \left\{ -\int_{\Omega \setminus \overline{\Omega_\omega}} |\nabla u_\varepsilon|^p \, dx \right\} + \int_{\Omega \setminus \overline{\Omega_{\omega'}}} |\nabla v|^p \, dx \\ &\le \|u\|_{1,p}^p + F^i(u, \omega) - \int_{\Omega \setminus \overline{\Omega_\omega}} |\nabla u|^p \, dx + \int_{\Omega \setminus \overline{\Omega_{\omega'}}} |\nabla v|^p \, dx. \end{aligned}$$

We thus deduce:

$$F^i(v,\omega') \leq F^i(u,\omega) + \int_{\Omega_\omega \setminus \overline{\Omega_{\omega'}}} |\nabla v|^p\, dx,$$

because: $\int_{\Omega_\omega} |\nabla v|^p\, dx = \int_{\Omega_\omega} |\nabla u|^p\, dx$. Finally we let $\overline{\omega'}$ increase to ω which implies that $\int_{\Omega_\omega \setminus \overline{\Omega_{\omega'}}} |\nabla u|^p\, dx$ decreases to 0 and we get:

$$\sup_{\overline{\omega'} \subset \omega} F^i(v,\omega') \leq F^i(u,\omega).$$

□

In a second step we intend to build F and R. From [11] we deduce the existence of a subsequence $(\varepsilon_k)_k$ and a countable family D dense in $B(\Sigma)$ such that:

$$\forall u \in V_o,\ \forall B \in D : \operatorname*{epi-lim}_{k\to+\infty} \|u\|^p_{1,p} + F_{\varepsilon_k}(u,B)$$

exists in the weak topology of $W^{1,p}(\Omega)$. This implies that $\widehat{F}^s$ and $\widehat{F}^i$ coincide on $V_o \times D$ and then $F^s = F^i$ on $W^{1/q,p}(\Sigma) \times D$. Let us define F on $W^{1/q,p}(\Sigma) \times B(\Sigma)$ by:

$$F(u,B) = \sup_{\overline{A} \subset \overset{o}{B}} \widehat{F}^s(u,A) = \sup_{\overline{A} \subset \overset{o}{B}} \widehat{F}^i(u,A)$$

and on $V_o \times B(\Sigma)$ by:

$$F(u_{|\Sigma},B) = \sup_{\overline{A} \subset \overset{o}{B}} \widehat{F}^s(u,A) = \sup_{\overline{A} \subset \overset{o}{B}} \widehat{F}^i(u,A).$$

F takes nonnegative values because F^s and F^i take nonnegative values. F is lower semi-continuous on V_o as the upper enveloppe of lower semi-continuous functionals on V_o. Moreover, there exists a positive δ, a bounded sequence $(t_\varepsilon)_\varepsilon$ included in V_o such that:

$$\sup_\varepsilon F_\varepsilon(t_\varepsilon,B) < \delta, \qquad \forall B \in B(\Sigma).$$

Because there exists a subsequence $(t_{\varepsilon_k})_k$ converging to some t in the weak topology of V_o we get:

$$\|t\|^p_{1,p} + F^s(t,\omega) = \limsup_{k\to+\infty} \left\{ \|t_{\varepsilon_k}\|^p_{1,p} + F_{\varepsilon_k}(t_{\varepsilon_k},B) \right\} < +\infty \Rightarrow F^s(t,B) < +\infty,$$

which implies that F is non indentically equal to $+\infty$. From [4], Lemma 3.2, there exists a rich family R in $B(\Sigma)$ such that:

$$\forall u \in W^{1/q,p}(\Sigma),\ \forall B \in R\colon\ F(u,B) = F^s(u,B) = F^i(u,B),$$

hence:

$$\forall u \in V_o,\ \forall B \in R :\ \|u\|^p_{1,p} + F(u,B) = \operatorname*{epi-lim}_{\varepsilon\to 0} \left\{ \|u\|^p_{1,p} + F_{\varepsilon_k}(u,B) \right\},$$

the epi-lim being taken for the weak topology of V_o. We then establish:

Lemma 2.7. *The functional F above defined belongs to the class* $\mathbf{F}$.

Proof. For every ω in $O(\Sigma)$, the functional: $u \mapsto F(u,\omega)$ is lower semi-continuous on $W^{1/q,p}(\Sigma)$, since the functionals $u \mapsto F^s(u,\omega)$ and $u \mapsto F^i(u,\omega)$ are lower semi-continuous on $W^{1/q,p}(\Sigma)$. It is nondecreasing (resp. nonincreasing) on $W^{1/q,p}(\Sigma)$, since the functionals $u \mapsto F_\varepsilon(u,\omega)$ are nondecreasing (resp. nonincreasing) on $W^{1/q,p}(\Sigma)$, see Lemma 2.4. Let us prove that F satisfies (2.2). Let u be any element of $W^{1/q,p}(\Sigma)$, ω_1 and ω_2 be disjoints elements of $O(\Sigma)$. For every Borel set B verifying: $\overline{B} \subset \omega_1 \cup \omega_2$ we write: $B = (B \cap \omega_1) \cup (B \cap \omega_2)$ and $\overline{B \cap \omega_i} = \overline{B} \cap \omega_i$, $i = 1, 2$. From Lemma 2.5, we deduce:

$$F^s(u,B) \le F^s(u, B\cap\omega_1) + F^s(u, B\cap\omega_2) \le F(u,\omega_1) + F(u,\omega_2),$$

which implies:

$$\sup_{\overline{B}\subset\omega_1\cup\omega_2} F^s(u,B) \le F(u,\omega_1) + F(u,\omega_2) \Rightarrow F(u,\omega_1\cup\omega_2) \le F(u,\omega_1) + F(u,\omega_2).$$

Let $\overline{B_1} \subset \omega_1$, $\overline{B_2} \subset \omega_2$, B in $O(\Sigma)$ such that: $\overline{B_1} \cup \overline{B_2} \subset B \subset \overline{B} \subset \omega_1 \cup \omega_2$. One has: $B = (B\cap\omega_1) \cup (B\cap\omega_2)$ and $\overline{B_i} \subset B\cap\omega_i \subset \overline{B\cap\omega_i} \subset \omega_i$, $i = 1, 2$. Moreover $B\cap\omega_1$ and $B\cap\omega_2$ are disjoints. From Lemma 2.6, we deduce:

$$\begin{aligned} F(u,\omega_1\cup\omega_2) &\ge F^i(u,B) = F^i(u,(B\cap\omega_1)\cup(B\cap\omega_2)) \\ &\ge F^i(u,(B\cap\omega_1)) + F^i(u,(B\cap\omega_2)) \\ &\ge F^i(u,B_1) + F^i(u,B_2), \end{aligned}$$

which implies:

$$F(u,\omega_1\cup\omega_2) \ge \sup_{\overline{B_1}\subset\omega_1} F^i(u,B_1) + \sup_{\overline{B_2}\subset\omega_2} F^i(u,B_2) = F(u,\omega_1) + F(u,\omega_2).$$

Let us now verify the σ-additivity property of F. We take any nondecreasing sequence $(\omega_n)_n$ of open Borel subsets of Σ and denote $\omega = \cup_n \omega_n$. Because: $F(u,\omega_n) \le F(u,\omega)$, for every n, we get: $\limsup_{n\to+\infty} F(u,\omega_n) \le F(u,\omega)$. Borel-Lebesgue theorem implies that for every B such that $\overline{B} \subset \omega$ there exists some ω_{n_o} verifying: $\overline{B} \subset \omega_{n_o}$, hence:

$$F^s(u,B) \le F^s(u,\omega_{n_o}) \le \limsup_{n\to+\infty} F(u,\omega_n) \Rightarrow \sup_{\overline{B}\subset\omega} F^s(u,B) \le \limsup_{n\to+\infty} F(u,\omega_n),$$

which finally implies:

$$F(u,\omega) = \lim_{n\to+\infty} F(u,\omega_n).$$

F verifies (2.3). Indeed let u, v be any elements of $W^{1/q,p}(\Sigma)$, ω be any element of $O(\Sigma)$ such that $u_{|\omega} = v_{|\omega}$. From Lemma 2.6 we deduce:

$$F(u,\omega) = \sup_{\overline{\omega'}\subset\omega} F^i(u,\omega') \le F(v,\omega).$$

Exchanging the roles of u and v we get: $F(u,\omega) = F(v,\omega)$.

2.4. Limit problems

Let us define the functionals F_ε^i, $i = 1,2$ on $W^{1/q,p}(\Sigma) \times O(\Sigma)$ by:

$$F_\varepsilon^1(u,\omega) = \begin{cases} 0 & \text{if } \widetilde{u} \geq 0 \text{ q.e. on } \omega \cap T_\varepsilon \\ +\infty & \text{otherwise;} \end{cases}$$

$$F_\varepsilon^2(u,\omega) = \begin{cases} 0 & \text{if } \widetilde{u} \leq 0 \text{ q.e. on } \omega \cap T_\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

We immediately verify that F_ε^1 belongs to $\mathbf{F}_o$ and F_ε^2 belongs to $\mathbf{F}$ (see [8], Theorem 1.9, for the proof of the lower semi-continuity property for the obstacle functionals). Moreover:

$$\forall u \in W^{1/q,p}(\Sigma),\ \forall \omega \in O(\Sigma): \quad \begin{array}{l} F_\varepsilon^1(u,\omega) = F_\varepsilon^1(-u^-,\omega) \\ F_\varepsilon^2(u,\omega) = F_\varepsilon^2(u^+,\omega) \\ F_\varepsilon^1(u,\omega) = F_\varepsilon^2(-u,\omega), \end{array} \tag{2.5}$$

with $u^+ = \max(u,0)$, $u^- = -\min(0,u)$. □

Proposition 2.8. *There exists a subsequence $(\varepsilon_k)_k$ two functionals F^1 and F^2 in $\mathbf{F}$ and a rich family $R \subset B(\Sigma)$ such that for every u in V_o and for every ω in $R \cap O(\Sigma)$ one has*

(i) $F^2(u^+,\omega) = F^1(-u^+,\omega)$,
(ii) $\|u\|_{1,p}^p + F^1(-u^-,\omega) + F^2(u^+,\omega) = \text{epi-lim}_{\varepsilon\to 0}\left\{\|u\|_{1,p}^p + F_{\varepsilon_k}^1(u,\omega) + F_{\varepsilon_k}^2(u,\omega)\right\}$.

Proof. From the compacity Theorem 2.3, we infer the existence of a nondecreasing functional F^1 and a nonincreasing functional F^2 in $\mathbf{F}$, of a subsequence $(\varepsilon_k)_k$ (we will omit the subscript k in the following) and of a rich family $R \subset B(\Sigma)$ such that:

$$\forall u \in V_o,\ \forall \omega \in R \cap O(\Sigma):\ \|u\|_{1,p}^p + F^i(u,\omega) = \underset{\varepsilon\to 0}{\text{epi-lim}}\left\{\|u\|_{1,p}^p + F_\varepsilon^i(u,\omega)\right\},\ i = 1,2,$$

where the epi-limit is taken for the weak topology of $W^{1,p}(\Omega)$.

(i) Let $(u_\varepsilon)_\varepsilon$ be a sequence converging to u^+ in the weak topology of $W^{1,p}(\Omega)$ such that:

$$\left\|u^+\right\|_{1,p}^p + F^2(u^+,\omega) = \lim_{\varepsilon\to 0}\left\{\|u_\varepsilon\|_{1,p}^p + F_\varepsilon^2(u_\varepsilon,\omega)\right\}.$$

Because: $F_\varepsilon^2(u_\varepsilon,\omega) = F_\varepsilon^1(-(u_\varepsilon)^+,\omega)$, we have:

$$\lim_{\varepsilon\to 0}\left\{\|u_\varepsilon\|_{1,p}^p + F_\varepsilon^1(-(u_\varepsilon)^+,\omega)\right\} \geq \left\|u^+\right\|_{1,p}^p + F^1(-u^+,\omega) \Rightarrow F^1(-u^+,\omega) \leq F^2(u^+,\omega).$$

The reverse inequality is proved in a similar way.

(ii) Because the upper epi-lim always exists, there exists a sequence $(u_\varepsilon)_\varepsilon$ converging to u in the weak topology of V_o such that:

$$\inf\left\{\liminf_{\varepsilon\to 0}\left\{\|v_\varepsilon\|_{1,p}^p + F_\varepsilon^1(v_\varepsilon,\omega) + F_\varepsilon^2(v_\varepsilon,\omega)\right\} \mid v_\varepsilon \underset{\varepsilon\to 0}{\overset{w-V_o}{\rightharpoonup}} u\right\}$$
$$= \liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|_{1,p}^p + F_\varepsilon^1(u_\varepsilon,\omega) + F_\varepsilon^2(u_\varepsilon,\omega)\right\}.$$

Thanks to the equality: $\|u_\varepsilon\|_{1,p}^p = \|(u_\varepsilon)^+\|_{1,p}^p + \|(u_\varepsilon)^-\|_{1,p}^p$, we obtain:

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\left\{\|u_\varepsilon\|_{1,p}^p + F_\varepsilon^1(u_\varepsilon,\omega) + F_\varepsilon^2(u_\varepsilon,\omega)\right\}\\
&\quad\geq \liminf_{\varepsilon\to 0}\left\{\left\|(u_\varepsilon)^-\right\|_{1,p}^p + F_\varepsilon^1(-(u_\varepsilon)^-,\omega)\right\} + \liminf_{\varepsilon\to 0}\left\{\left\|(u_\varepsilon)^+\right\|_{1,p}^p + F_\varepsilon^2((u_\varepsilon)^+,\omega)\right\}\\
&\quad\geq \left\|u^-\right\|_{1,p}^p + F^1(-u^-,\omega) + \left\|u^+\right\|_{1,p}^p + F^2(u^+,\omega)\\
&\quad\geq \|u\|_{1,p}^p + F^1(-u^-,\omega) + F^2(u^+,\omega).
\end{aligned}$$

Conversely, let u be any element of V_o. There exists a sequence $(u_\varepsilon)_\varepsilon$ weakly converging to u such that $((u_\varepsilon)^+)_\varepsilon$ (resp. $((u_\varepsilon)^-)_\varepsilon$) weakly converges to u^+ (resp. u^-) and:

$$\left\|u^-\right\|_{1,p}^p + F^1(-u^-,\omega) = \lim_{\varepsilon\to 0}\left\{\left\|(u_\varepsilon)^-\right\|_{1,p}^p + F_\varepsilon^1(-(u_\varepsilon)^-,\omega)\right\}$$

$$\left\|u^+\right\|_{1,p}^p + F^2(u^+,\omega) = \lim_{\varepsilon\to 0}\left\{\left\|(u_\varepsilon)^+\right\|_{1,p}^p + F_\varepsilon^2((u_\varepsilon)^+,\omega)\right\}$$

hence:

$$\begin{aligned}
&\|u\|_{1,p}^p + F^1(-u^-,\omega) + F^2(u^+,\omega)\\
&\quad= \left\|u^-\right\|_{1,p}^p + F^1(-u^-,\omega) + \left\|u^+\right\|_{1,p}^p + F^2(u^+,\omega)\\
&\quad\geq \lim_{\varepsilon\to 0}\left\{\left\|(u_\varepsilon)^-\right\|_{1,p}^p + F_\varepsilon^1(-(u_\varepsilon)^-,\omega) + \left\|(u_\varepsilon)^+\right\|_{1,p}^p + F_\varepsilon^2((u_\varepsilon)^+,\omega)\right\}\\
&\quad\geq \limsup_{\varepsilon\to 0}\left\{\|u_\varepsilon\|_{1,p}^p + F_\varepsilon^1(u_\varepsilon,\omega) + F_\varepsilon^2(u_\varepsilon,\omega)\right\}\\
&\quad\geq \inf\left\{\limsup_{\varepsilon\to 0}\left\{\|v_\varepsilon\|_{1,p}^p + F_\varepsilon^1(v_\varepsilon,\omega) + F_\varepsilon^2(v_\varepsilon,\omega)\right\} \mid v_\varepsilon \underset{\varepsilon\to 0}{\overset{w-V_o}{\rightharpoonup}} u\right\}.
\end{aligned}$$

□

Proposition 2.9. *There exists a Radon measure μ in $(W^{-1/q,q}(\Sigma))^+$ and a Borel function a from Σ into $[0,+\infty]$ lower semi-continuous and nonincreasing such that on $W^{1/q,p}(\Sigma)\times(R\cap O(\Sigma))$ we have:*

$$F^1(-u^-,\omega) = \int_\omega a(\sigma)\left|u^-(\sigma)\right|^p d\mu(\sigma)\ ;\ F^2(u^+,\omega) = \int_\omega a(\sigma)\left|u^+(\sigma)\right|^p d\mu(\sigma).$$

Proof. Because F^1 belongs to $\mathbf{F}_o$ Theorem 2.1 implies that:

$$F^1(u,\omega) = \int_\omega f(\sigma,\widetilde{u}(\sigma))d\mu(\sigma) + \nu(\omega).$$

Then for every positive real λ one has:

$$\lambda^p \operatorname{epi-lim}_{\varepsilon\to 0}\left\{\|v\|_{1,p}^p + F_\varepsilon^1(v,\omega)\right\} = \operatorname{epi-lim}_{\varepsilon\to 0}\left\{\|\lambda v\|_{1,p}^p + F_\varepsilon^1(\lambda v,\omega)\right\},$$

since F_ε^1 only takes the values 0 or $+\infty$. Hence:

$$F^1(\lambda u,\omega) = \lambda^p F^1(u,\omega), \qquad \text{on } W^{1/q,p}(\Sigma)\times(R\cap O(\Sigma)).$$

For every nonnegative t one has: $F^1(t,\omega) = 0$, since:

$$0 \leq F^1(t,\omega) \leq \liminf_{\varepsilon\to 0} \left\{F^1_\varepsilon(t,\omega)\right\}.$$

This implies: $\int_\omega f(\sigma,t)d\mu(\sigma) + \nu(\omega) = 0$ and thus: $\nu(\omega) = 0$, since f takes nonnegative values and μ and ν are nonnegative measures. For every nonpositive t one has:

$$F^1(t,\omega) = (-t)^p F^1(-1,\omega),$$

from which we deduce:

$$\forall\omega \in R \cap O(\Sigma),\ \forall\sigma\in\omega:\ f\left(\sigma,\widetilde{u}^-(\sigma)\right) = \left(\widetilde{u}^-(\sigma)\right)^p f(\sigma,-1) = a(\sigma)\left(\widetilde{u}^-(\sigma)\right)^p,$$

defining: $a(\sigma) := f(\sigma,-1)$. a is lower semi-continuous and nonincreasing (see Theorem 2.1). This implies:

$$F^1(-u^-,\omega) = \int_\omega a(\sigma)\left|\widetilde{u}^-(\sigma)\right|^p d\mu(\sigma) = \int_\omega a(\sigma)\left|u^-(\sigma)\right|^p d\mu(\sigma),$$

because $\mu(\widetilde{u}\neq u) = 0$. Because: $F^2(u^+,\omega) = F^1(-u^+,\omega)$, we get:

$$F^2(u^+,\omega) = \int_\omega a(\sigma)\left|u^+(\sigma)\right|^p d\mu(\sigma).$$

□

Then we conclude with the following:

Theorem 2.10.

(i) *The sequence $(F_\varepsilon)_\varepsilon$ epi-converges in the weak topology of V_o to the functional F_o defined by*

$$F_o(v) = \begin{cases} \int_\Omega |\nabla v|^p dx + \int_\Sigma a(\sigma)|v(\sigma)|^p d\mu(\sigma) & \textit{if } v\in V_o \\ +\infty & \textit{otherwise.}\end{cases} \tag{2.6}$$

(ii) *The sequence $(u_\varepsilon)_\varepsilon$ converges in the weak topology of V_o to the solution of the minimization problem associated to this functional F_o, that is:*

$$\min_{v\in V_o}\left\{F_o(v) - p\int_\Omega fv dx\right\}.$$

Proof. We just observe that:

$$F_\varepsilon(v) = \begin{cases} \int_\Omega |\nabla v|^p dx + F^1_\varepsilon(v_{|\Sigma},\Sigma) + F^2_\varepsilon(v_{|\Sigma},\Sigma) & \text{if } v\in V_\varepsilon \\ +\infty & \text{otherwise.}\end{cases}$$

Then we apply the Propositions 2.8 and 2.9 and Theorem 1.10 of [1]. □

Remark 2.11. When we only consider the unilateral constraints: $u \geq 0$ on T_ε or $u\leq 0$ on T_ε, the additional term appearing in the limit problem is:

$$\int_\Sigma a(\sigma)\left|v^+(\sigma)\right|^p d\mu(\sigma) \text{ or } \int_\Sigma a(\sigma)\left|v^-(\sigma)\right|^p d\mu(\sigma).$$

3. Application: periodic distribution of strips on the lateral boundary of a cylinder

We here suppose that Ω is the cylinder:

$$\Omega = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1,\ x_3 \in]-H, H[\right\},$$

where H is positive. Γ_1 and Γ_2 are respectively the upper and lower faces of Ω and Σ is its lateral boundary. Let ε and r_ε be positive parameters with: $0 < r_\varepsilon < \varepsilon$. For every k in $\mathbf{Z}$, we denote by γ_ε^k the circle:

$$\gamma_\varepsilon^k = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1,\ x_3 = k\varepsilon \right\}.$$

Then T_ε^k is the strip of width $2r_\varepsilon$ centered on the circle γ_ε^k:

$$T_\varepsilon^k = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1,\ x_3 \in \left] -\frac{r_\varepsilon}{2} + k\varepsilon, \frac{r_\varepsilon}{2} + k\varepsilon \right[\right\}$$

see figure 3.1 below. We define T_ε as the union $\bigcup_{k=-n(\varepsilon)}^{n(\varepsilon)} T_\varepsilon^k$ of the strips contained in the lateral boundary Σ of Ω. Note that these strips are ε-periodically distributed on Σ and that the total number $2n(\varepsilon) + 1$ of such strips is equivalent to H/ε for small values of ε. Finally we define: $\Sigma_\varepsilon = \Sigma \setminus T_\varepsilon$, see figure 3.1 below.

Because of the periodic repartition of the strips T_ε^k the measure $a(\sigma)d\mu(\sigma)$ which will appear in the limit problem (2.6) will be of the kind $Kd\sigma$ for some constant K in $[0, +\infty]$ and where $d\sigma$ is the Lebesgue measure on Σ. Our purpose is to identify the constant K.

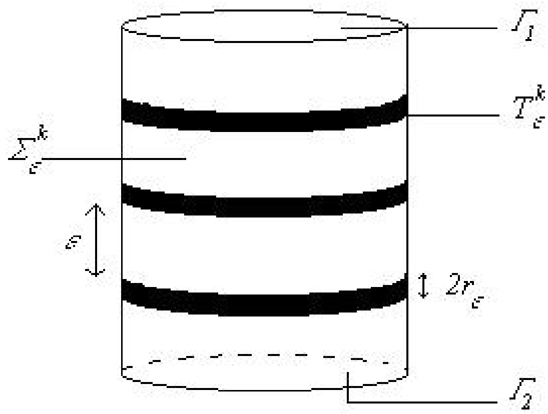

Figure 3.1: The cylinder Ω and the strips T_ε^k.

3.1. Notations

Using the cylindrical coordinates: $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$, $x_3 = x_3$, with r in $[0, 1[$, θ in $[0, 2\pi]$ and x_3 in $]-H, H[$ the prism Q associated to Ω is:

$$Q = [0, 1[\times [0, 2\pi] \times]-H, H[$$

and let us denote $D =]0, 1[\times]-H, H[$. For every positive R, $B(R)$ (resp. $B^+(R)$) is the ball (resp. half-ball) of $\mathbb{R}^2$ defined by:

$$B(R) = \left\{ (r, x_3) \mid r^2 + x_3^2 < R^2 \right\} \ \text{(resp. } B^+(R) = \left\{ (r, x_3) \mid r^2 + x_3^2 < R^2,\ r > 0 \right\} \text{)}.$$

Let us introduce $\rho = 1 - r$ and for every k in $\{-n(\varepsilon), ..., n(\varepsilon)\}$ B_ε^k (resp. B_ε^{k+}) and $B_{r_\varepsilon}^k$ (resp. $B_{r_\varepsilon}^{k+}$) defined by:

$$B_\varepsilon^k = \left\{(\rho, x_3) \mid (1-\rho)^2 + (x_3 - k\varepsilon)^2 < \varepsilon^2/4\right\}, \text{ (resp. } B_\varepsilon^{k+} = B_\varepsilon^{k+} \cap \{\rho > 0\})$$
$$B_{r_\varepsilon}^k = \left\{(\rho, x_3) \mid (1-\rho)^2 + (x_3 - k\varepsilon)^2 < r_\varepsilon^2/4\right\}, \text{ (resp. } B_{r_\varepsilon}^{k+} = B_{r_\varepsilon}^{k+} \cap \{\rho > 0\})$$

and:

$$S_\varepsilon^k = \left\{(\rho, \theta, x_3) \mid (\rho, x_3) \in B_\varepsilon^k \ ; \ \theta \in [0, 2\pi]\right\}, \ S_\varepsilon^{k+} = S_\varepsilon^k \cap \{\rho > 0\}$$

and $\partial S_\varepsilon^{k+} = \partial S_\varepsilon^k \cap \{\rho > 0\}$. If $T = \{0\} \times \left]-\frac{1}{2}, \frac{1}{2}\right[$ the strip T_ε^k can be written in cylindrical coordinates as:

$$T_\varepsilon^k = \{(0, \theta, x_3) \mid x_3 \in r_\varepsilon T + k\varepsilon \ ; \ \theta \in [0, 2\pi]\}.$$

3.2. Test-functions

Let w^ε be the solution of the following local problem written in cylindrical coordinates:

$$\begin{cases} \operatorname{div}\left(|\nabla w^\varepsilon|^{p-2} \nabla w^\varepsilon\right) &= 0 \quad \text{in } B(\varepsilon/2) \setminus r_\varepsilon T \\ w^\varepsilon &= 1 \quad \text{on } r_\varepsilon T \\ w^\varepsilon &= 0 \quad \text{on } \partial B(\varepsilon/2). \end{cases} \tag{3.1}$$

Then we introduce the function w_ε deduced from the preceding function w^ε by means of a periodic process:

$$w_\varepsilon(r, x_3) = \begin{cases} w^\varepsilon(\rho, x_3 - k\varepsilon) & \text{in } B_\varepsilon^{k+} \setminus T_\varepsilon^k \\ 0 & \text{in } D \setminus \bigcup\limits_{-n(\varepsilon)}^{n(\varepsilon)} \overline{B_\varepsilon^{k+}}. \end{cases}$$

Finally we define the test-function w_ε^o in Ω:

$$w_\varepsilon^o(x) = w_\varepsilon(\rho, x_3), \qquad (\rho, x_3) \in D. \tag{3.2}$$

The properties of this test-function are summarized in the following:

Lemma 3.1.
(i) $\lim_{\varepsilon \to 0} \int_\Omega |\nabla w_\varepsilon^o|^p \, dx = ac \ \operatorname{meas}(\Sigma)$, *where a belongs to* $[0, +\infty]$ *and is given by:*

$$a = \begin{cases} \lim\limits_{\varepsilon \to 0} \dfrac{(r_\varepsilon)^{2-p}}{\varepsilon} & \text{if } p \neq 2 \\ \lim\limits_{\varepsilon \to 0} \dfrac{-1}{\varepsilon \ln(r_\varepsilon)} & \text{if } p = 2 \end{cases} \tag{3.3}$$

and c is the $W^{1,p}$-capacity of T with respect to $\mathbb{R}^2$ given by:

$$c = \min_{w \in W_o^{1,p}(\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} |\nabla w|^p dy \mid w = 1 \ on \ T \right\} \tag{3.4}$$

(ii) *If a belongs to* $[0, +\infty[$, *the sequence* $(w_\varepsilon^o)_\varepsilon$ *converges to 0 in the weak topology of* $W^{1,p}(\Omega)$.

(iii) *If a belongs to $[0,+\infty[$:*

(a) *The sequence* $\left(-\frac{1}{2}\sum_k \left(|\nabla w_\varepsilon|^{p-2}\frac{\partial w_\varepsilon}{\partial n}\right)_{|\partial B_\varepsilon^k}\delta_{\partial B_\varepsilon^k}\right)_\varepsilon$ *converges to the measure equal to* $\frac{ac}{2}\,\delta_{\{\rho=0\}\cap\overline{D}}$ *in the strong topology of* $W^{-1,q}(\widetilde{D})$, *for every open subset* $\widetilde{D}$ *of* $\mathbb{R}^2$ *containing* $\overline{D}$, *where* $\delta_{\partial B_\varepsilon^k}$ *(resp.* $\delta_{\{\rho=0\}\cap\overline{D}}$*) is the Dirac measure on* ∂B_ε^k *(resp. on* $\{\rho=0\}\cap\overline{D}$*):*

$$\forall\varphi\in C^\infty(\overline{D}):\quad \langle\delta_{\partial B_\varepsilon^k},\varphi\rangle=\int_{\partial B_\varepsilon^k}\varphi d\sigma;\quad \left\langle\delta_{\{\rho=0\}\cap\overline{D}},\varphi\right\rangle=\int_{-H}^{H}\varphi(0,x_3)dx_3.$$

(b) *The sequence* $\left(-\frac{1}{2}\sum_k \left(|\nabla w_\varepsilon^o|^{p-2}\frac{\partial w_\varepsilon^o}{\partial n}\right)_{|\partial S_\varepsilon^k}\delta_{\partial S_\varepsilon^k}\right)_\varepsilon$ *converges to the measure equal to* $\frac{ac}{2}\,\delta_{\{\rho=0\}\cap\overline{Q}}$ *in the strong topology of* $W^{-1,q}(Q')$, *for every open subset* Q' *of* $\mathbb{R}^3$ *containing* $\overline{Q}$, *where* $\delta_{\partial S_\varepsilon^k}$ *(resp.* $\delta_{\{\rho=0\}\cap\overline{Q}}$*) is the Dirac measure on* ∂B_ε^k *(resp. on* $\{\rho=0\}\cap\overline{Q}$*):*

$$\forall\varphi\in C^\infty(\overline{Q}):\quad \langle\delta_{\partial S_\varepsilon^k},\varphi\rangle=\int_{\partial S_\varepsilon^k}\varphi d\sigma$$
$$\left\langle\delta_{\{\rho=0\}\cap\overline{Q}},\varphi\right\rangle=\int_0^{2\pi}\int_{-H}^{H}\varphi(0,\theta,x_3)d\theta dx_3.$$

Proof. (i) We compute:

$$\int_\Omega|\nabla w_\varepsilon^o|^p\,dx=\int_Q|\nabla w_\varepsilon|^p\,(1-\rho)d\rho d\theta dx_3=2\pi\int_D|\nabla_{\rho,x_3}w_\varepsilon|^p\,(1-\rho)d\rho dx_3,$$

using the change of variables: $\rho=1-r$; $t=x_3-k\varepsilon$, with $\nabla_{\rho,x_3}w_\varepsilon=\left(\frac{\partial w_\varepsilon}{\partial\rho},\frac{\partial w_\varepsilon}{\partial x_3}\right)$ and then:

$$\begin{aligned}\int_D|\nabla_{\rho,x_3}w_\varepsilon|^p\,(1-\rho)d\rho dx_3&=\sum_{-n(\varepsilon)}^{n(\varepsilon)}\int_{B_\varepsilon^{k+}}|\nabla_{\rho,x_3}w_\varepsilon|^p\,(\rho,x_3-k\varepsilon)(1-\rho)d\rho dx_3\\&\le\frac{H}{\varepsilon}\int_{B(\varepsilon/2)}|\nabla w^\varepsilon|^p\,(1-\rho)d\rho dt\end{aligned}$$

and because of the symmetry of w_ε in $B(\varepsilon/2)$. Let us now consider three cases according to the values of the exponent p:

First case: $1<p<2$

We compute:

$$\frac{1}{\varepsilon}\int_{B(\varepsilon/2)}|\nabla w^\varepsilon|^p\,d\rho dt=\frac{(r_\varepsilon)^{2-p}}{\varepsilon}\min_{w\in W_o^{1,p}(B(\varepsilon/2r_\varepsilon))}\left\{\int_{B(\varepsilon/2r_\varepsilon)}|\nabla w|^p\,dy\mid w=1\text{ on }T\right\},$$

which implies using the above definition (3.3) of a:

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{B(\varepsilon/2)}|\nabla w^\varepsilon|^p\,d\rho dt=a\lim_{\varepsilon\to0}\min_{w\in W_o^{1,p}(B(\varepsilon/2r_\varepsilon))}\left\{\int_{B(\varepsilon/2r_\varepsilon)}|\nabla w|^p\,dy\mid w=1\text{ on }T\right\}.$$

Because the sequence of convex sets $\{w \in W_o^{1,p}(B(\varepsilon/2r_\varepsilon)) \mid w = 1 \text{ on } T\}$ Mosco-converges to $\{w \in W^{1,p}(\mathbb{R}^2) \mid w = 1 \text{ on } T\}$, see [4], we deduce:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B(\varepsilon/2)} |\nabla w^\varepsilon|^p \, d\rho dt = ac,$$

where c is the $W^{1,p}-$capacity of T with respect to $\mathbb{R}^2$ defined by (3.4). Furthermore, this implies:

$$\frac{1}{\varepsilon}\int_{B(\varepsilon/2)} |\nabla w^\varepsilon|^p \, \rho d\rho dt \leq \frac{1}{2}\int_{B(\varepsilon/2)} |\nabla w^\varepsilon|^p \, d\rho dt \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

because $\rho \leq \varepsilon/2$ and using the preceding argument. We get the following convergence:

$$\lim_{\varepsilon\to 0}\int_Q |\nabla_{\rho,x_3} w_\varepsilon|^p \, r dr d\theta dx_3 = 2\pi H ac = \frac{ac}{2}\,\mathrm{meas}(\Sigma).$$

Second case: $p = 2$

The Appendix of [4] implies:

$$\frac{2\pi}{\ln(\varepsilon/r_\varepsilon) + \ln(2\pi)} \leq \int_{B(\varepsilon/2)} |\nabla w^\varepsilon|^2 \, dr dx_3 \leq \frac{2\pi}{\ln(\varepsilon/r_\varepsilon)}$$

and then:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B(\varepsilon/2)} |\nabla w^\varepsilon|^2 \, dr dx_3 = 2a\pi,$$

from which we deduce:

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla w_\varepsilon^o|^2 \, dx = 2\pi \lim_{\varepsilon\to 0}\int_D |\nabla w_\varepsilon|^2 \, \rho d\rho dx_3 = a\pi \,\mathrm{meas}(\Sigma).$$

Third case: $p > 2$

The test-functions w_ε are equal to 0 far from the boundary $r = 1 - \rho = 1$. One can suppose that $1 - \rho$ belongs to $]1/2, 1[$ hence:

$$2\pi \int_D |\nabla w_\varepsilon|^2 (1-\rho) d\rho dx_3 \geq \pi \int_D |\nabla w_\varepsilon|^2 \, d\rho dx_3.$$

Then we observe that:

$$\lim_{\varepsilon\to 0}\pi \int_D |\nabla w_\varepsilon|^2 \, d\rho dx_3 = \frac{c\,\mathrm{meas}(\Sigma)}{2}\lim_{\varepsilon\to 0}\frac{(r_\varepsilon)^{2-p}}{\varepsilon} = +\infty,$$

because: $2 - p < 0$, from which we get: $\lim_{\varepsilon\to 0}\int_\Omega |\nabla w_o^\varepsilon|^2 \, dx = +\infty$.

(ii) Let us first suppose that a belongs to $[0, +\infty[$ and $1 < p \leq 2$. The sequence $(\nabla w_o^\varepsilon)_\varepsilon$ is bounded in $(L^p(\Omega))^3$, according to the first preceding assertion. The maximum principle, see [12], implies that: $0 \leq w_\varepsilon^o \leq 1$, in Ω. Hence:

$$\int_\Omega |w_\varepsilon^o|^p \, dx = \int_Q |w_\varepsilon|^p \, r dr d\theta dx_3 = \sum_{-n(\varepsilon)}^{n(\varepsilon)} \int_{S_\varepsilon^{k+}} |w_\varepsilon|^p \, (\rho, x_3)(1-\rho) d\rho dx_3$$
$$\leq \sum_{-n(\varepsilon)}^{n(\varepsilon)} \mathrm{meas}(S_\varepsilon^{k+})$$
$$\leq C\varepsilon,$$

for some constant C, which proves that $(w_\varepsilon^o)_\varepsilon$ converges to 0 in the weak topology of $W^{1,p}(\Omega)$.

(iii)(a) Observe first that $(|\nabla w_\varepsilon|^p \, d\rho dx_3)_\varepsilon$ converges to $ac\delta_{\{\rho=0\}\cap\overline{D}}$ in the sense of measures. Indeed, choose δ positive and any function φ in $C_0^\infty(\mathbb{R}^2)$. Green's formula implies:

$$-\int_D \mathrm{div}\left((|\nabla w_\varepsilon| + \delta)^{p-2} \nabla w_\varepsilon \right) \varphi (1 - w_\varepsilon)(1-\rho) d\rho dx_3$$
$$= \int_D (|\nabla w_\varepsilon| + \delta)^{p-2} |\nabla w_\varepsilon|^2 \varphi (1-\rho) d\rho dx_3$$
$$+ \int_D (|\nabla w_\varepsilon| + \delta)^{p-2} \nabla w_\varepsilon . \nabla \left(\varphi (1-\rho) \right) (1 - w_\varepsilon) d\rho dx_3$$
$$- \sum_{-n(\varepsilon)}^{n(\varepsilon)} \frac{1}{2} \int_{\partial B_\varepsilon^k} (|\nabla w_\varepsilon| + \delta)^{p-2} \frac{\partial w_\varepsilon}{\partial n} \varphi d\sigma,$$

because w_ε is equal to 1 on T_ε^k and to 0 on ∂B_ε^k. We then let δ decrease to 0 and get:

$$0 = -\int_D \mathrm{div}\left(|\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \right) \varphi (1 - w_\varepsilon)(1-\rho) d\rho dx_3,$$

because w_ε is independant of θ. Furthermore:

$$\lim_{\varepsilon \to 0} - \sum_{-n(\varepsilon)}^{n(\varepsilon)} \int_{\partial S_\varepsilon^k} |\nabla w_\varepsilon|^{p-2} \frac{\partial w_\varepsilon}{\partial n} \varphi d\sigma = \lim_{\varepsilon \to 0} \int_Q |\nabla w_\varepsilon|^p \, \varphi r dr d\theta dx_3$$
$$= \frac{ac}{2} \int_0^{2\pi} \int_{-H}^{H} \varphi(1, \theta, z) d\theta dx_3,$$

which proves that $(|\nabla w_\varepsilon|^p \, r dr d\theta dx_3)_\varepsilon$ converges to $\frac{ac}{2} \, \delta_{\{\rho=0\}\cap\overline{D}}$ in the sense of measures. Then we apply a classical maximum principle argument see [5] if $p = 2$ and Murat's result in [15] if $1 < p < 2$, in order to prove the announced convergence.

(iii)(b) Observe that for every φ in $C_0^\infty(\mathbb{R}^3)$ we have:

$$\left\langle \delta_{\partial S_\varepsilon^k}, \varphi \right\rangle = \int_0^{2\pi} \int_{\partial B_\varepsilon^k} \varphi d\theta d\sigma.$$

Hence, one can write $\delta_{\partial S_\varepsilon^k}$ as the tensorial product $\delta_{\partial B_\varepsilon^k} \otimes d\theta$. From the preceding computations we infer:

$$-\frac{1}{2}\sum_{-n(\varepsilon)}^{n(\varepsilon)} \left(|\nabla w_\varepsilon^o|^{p-2} \frac{\partial w_\varepsilon^o}{\partial n} \right)_{|\partial S_\varepsilon^k} \delta_{\partial S_\varepsilon^k} \underset{\varepsilon\to 0}{\longrightarrow} \frac{ac}{2} \delta_{\{\rho=0\}\cap\overline{D}} \otimes d\theta,$$

in the strong topology of $W^{-1,q}(Q')$ for every smooth open subset Q' containing $\overline{Q}$. But we finally observe that: $\delta_{\{\rho=0\}\cap\overline{D}} \otimes d\theta = \delta_{\{\rho=0\}\cap\overline{Q}}$. □

3.3. Determination of the constant K

Proposition 3.2. *Let us suppose that:* $1 < p < 2$. *One has* $K = ac$ *where* a *is given by* (3.3) *and* c *is given by* (3.4). *Moreover, when* a *is equal to* $+\infty$ *the limit problem becomes:*

$$\min_{v\in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla v|^p \, dx - p\int_\Omega fv dx \mid v \in V_o;\ v = 0 \text{ on } \Sigma \right\}. \tag{3.5}$$

Proof. Let us first suppose that a is finite. We get:

$$K \operatorname{meas}(\Sigma) = \inf\left\{ \liminf_{\varepsilon\to 0} \int_\Omega |\nabla z_\varepsilon|^p \, dx \mid z_\varepsilon \overset{w-W^{1,p}(\Omega)}{\underset{\varepsilon\to 0}{\rightharpoonup}} 0 \ ;\ z_\varepsilon \in V_\varepsilon \right\}.$$

Let us choose $z_\varepsilon = w_\varepsilon^o$. Thanks to Lemma 3.1 the sequence $(z_\varepsilon)_\varepsilon$ satisfies the required above properties. Moreover $\left(\int_\Omega |\nabla z_\varepsilon|^p \, dx\right)_\varepsilon$ converges to $ac\operatorname{meas}(\Sigma)$. This implies that: $K \le ac$. Let us now choose any sequence $(z_\varepsilon)_\varepsilon$ converging to 0 in the weak topology of V_o such that $z_{\varepsilon|T_\varepsilon} = 1$ and let us prove that:

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla z_\varepsilon|^p \, dx \ge \liminf_{\varepsilon\to 0} \int_\Omega |\nabla w_\varepsilon^o|^p \, dx,$$

which will imply: $K \ge ac$. We write the subdifferential inequality:

$$\int_\Omega |\nabla z_\varepsilon|^p \, dx \ge \int_\Omega |\nabla w_\varepsilon^o|^p \, dx + p\int_\Omega |\nabla w_\varepsilon^o|^{p-2} \nabla w_\varepsilon^o.\nabla(z_\varepsilon - w_\varepsilon^o) dx$$

and introduce: $\widetilde{\nabla} = \left(\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x_3}\right)$ ..We observe that:

$$\widetilde{\nabla} w_\varepsilon^o.\widetilde{\nabla} z_\varepsilon = \nabla_{\rho,x_3} w_\varepsilon.\nabla_{\rho,x_3} z_\varepsilon,$$

with $\nabla_{\rho,x_3} = \left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial x_3}\right)$ (we will simply write ∇ instead of ∇_{ρ,x_3}). Let:

$$Q_o = Q \cap \left\{0 < \rho < \frac{1}{2}\right\} = Q \cap \left\{\frac{1}{2} < r < 1\right\}.$$

Because w_ε is equal to 0 on $Q \setminus \overline{Q_o}$ we get:

$$\begin{aligned}\int_\Omega |\nabla w_\varepsilon^o|^{p-2} \nabla w_\varepsilon^o.\nabla (z_\varepsilon - w_\varepsilon^o)\, dx &= \int_{Q_o} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon.\nabla (z_\varepsilon - w_\varepsilon)(1-\rho) d\rho d\theta dx_3 \\ &= \int_{Q_o} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon.\nabla ((z_\varepsilon - w_\varepsilon)(1-\rho))\, d\rho d\theta dx_3 \\ &\quad + \int_{Q_o} |\nabla w_\varepsilon|^{p-2} \frac{\partial w_\varepsilon}{\partial \rho} (z_\varepsilon - w_\varepsilon)\, d\rho d\theta dx_3.\end{aligned}$$

Since:

$$\begin{aligned}&\int_{Q_o} |\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon.\nabla ((z_\varepsilon - w_\varepsilon)(1-\rho))\, d\rho d\theta dx_3 \\ &\quad = \sum_{-n(\varepsilon)}^{n(\varepsilon)} \int_{S_\varepsilon^{k+}} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon.\nabla ((z_\varepsilon - w_\varepsilon)(1-\rho))\, d\rho d\theta dx_3 \\ &\quad = \sum^{n(\varepsilon)} -n(\varepsilon) \int_{\partial S_\varepsilon^{k+}} |\nabla w_\varepsilon|^{p-2} \frac{\partial w_\varepsilon}{\partial n} (z_\varepsilon - w_\varepsilon)(1-\rho) d\sigma \\ &\qquad - \sum_{-n(\varepsilon)}^{n(\varepsilon)} \int_{T_\varepsilon^k} |\nabla w_\varepsilon|^{p-2} \frac{\partial w_\varepsilon}{\partial n} (z_\varepsilon - w_\varepsilon)(1-\rho) d\sigma,\end{aligned}$$

using Green's formula. Since $z_\varepsilon - w_\varepsilon = 0$ on T_ε^k the last sum is equal to 0. Moreover:

$$\begin{aligned}&\sum_{-n(\varepsilon)}^{n(\varepsilon)} \int_{\partial S_\varepsilon^{k+}} |\nabla w_\varepsilon|^{p-2} \frac{\partial w_\varepsilon}{\partial n} (z_\varepsilon - w_\varepsilon)(1-\rho) d\sigma \\ &\quad = - \sum_{-n(\varepsilon)}^{n(\varepsilon)} \left\langle \left(|\nabla w_\varepsilon|^{p-2} \frac{\partial w_\varepsilon}{\partial n} \right)_{|\partial S_\varepsilon^{k+}}, (z_\varepsilon - w_\varepsilon)(1-\rho) \right\rangle,\end{aligned}$$

where the last product is interpreted as a duality product between $W^{-1/q,q}(\partial S_\varepsilon^{k+})$ and $W^{1/q,p}(\partial S_\varepsilon^{k+})$. Let us extend by symmetry the term $\psi_\varepsilon := (z_\varepsilon - w_\varepsilon)(1-\rho)$ and choose a smooth function χ in $C_0^\infty(\mathbb{R}^3)$ such that $\chi \equiv 1$ in a neighbourhood of $\rho = 0$ and $\chi \equiv 0$ on $\widetilde{Q}$ where $\widetilde{Q}$ is some smooth open subset containing $\overline{Q_o}$. We define: $\widetilde{\psi_\varepsilon} = \chi\psi_\varepsilon$ and observe that: $\widetilde{\psi_\varepsilon}_{|T_\varepsilon} = \psi_\varepsilon$ and the sequence $\left(\widetilde{\psi_\varepsilon}\right)_\varepsilon$ converges to 0 in the weak topology of $W^{1,p}(\widetilde{Q})$. From Lemma 3.1 we deduce:

$$\begin{aligned}&\lim_{\varepsilon\to 0} \sum_{-n(\varepsilon)}^{n(\varepsilon)} \left\langle \left(|\nabla w_\varepsilon|^{p-2} \frac{\partial w_\varepsilon}{\partial n} \right)_{|\partial S_\varepsilon^{k+}}, (z_\varepsilon - w_\varepsilon)(1-\rho) \right\rangle \\ &\quad = \lim_{\varepsilon\to 0} \frac{1}{2} \sum_{-n(\varepsilon)}^{n(\varepsilon)} \left\langle \left(|\nabla w_\varepsilon|^{p-2} \frac{\partial w_\varepsilon}{\partial n} \right)_{|\partial S_\varepsilon^{k+}}, \widetilde{\psi_\varepsilon} \right\rangle = 0,\end{aligned}$$

where the duality product is now taken between $W^{-1,q}(\widetilde{Q})$ and $W^{1,p}(\widetilde{Q})$. Hölder's inequality implies:

$$\begin{aligned}\left|\int_{Q_o} |\nabla w_\varepsilon|^{p-2} \frac{\partial w_\varepsilon}{\partial \rho}(z_\varepsilon - w_\varepsilon)\, d\rho d\theta dx_3\right| &\le 2^{1/p}\left\{\int_{Q_o} |z_\varepsilon - w_\varepsilon|^p (1-\rho) d\rho d\theta dx_3\right\}^{1/p} \\ &\quad \times \left\{\int_{Q_o} |\nabla w_\varepsilon|^p\, d\rho d\theta dx_3\right\}^{1/q} \\ &\le 2^{1/p}\left\{\int_{\Omega} |z_\varepsilon - w_\varepsilon^o|^p\, dx\right\}^{1/p} \\ &\quad \times \left\{\int_{Q_o} |\nabla w_\varepsilon|^p\, d\rho d\theta dx_3\right\}^{1/q}.\end{aligned}$$

Because $\lim\limits_{\varepsilon\to 0}\int_\Omega |z_\varepsilon - w_\varepsilon^o|^p\, dx = 0$ and $\lim\limits_{\varepsilon\to 0}\int_{Q_o} |\nabla w_\varepsilon|^p\, d\rho d\theta dx_3 = ac\,\mathrm{meas}(\Sigma)$ we infer:

$$\lim_{\varepsilon\to 0}\int_{Q_o} |\nabla w_\varepsilon|^{p-2}\frac{\partial w_\varepsilon}{\partial \rho}(z_\varepsilon - w_\varepsilon)\, d\rho d\theta dx_3 = 0.$$

Finally we get:

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla v_\varepsilon|^p\, dx \ge \liminf_{\varepsilon\to 0}\int_\Omega |\nabla w_\varepsilon^o|^p\, dx = ac\,\mathrm{meas}(\Sigma),$$

which ends the proof of the assertion: $K = ac$.

Let us now suppose that a is equal to $+\infty$ and $1 < p \le 2$.

We define:

$$r_\varepsilon(p) = \begin{cases} (C\varepsilon)^{1/(2-p)} & \text{if } 1 < p < 2 \\ \exp(-1/C\varepsilon) & \text{if } p = 2 \end{cases}$$

and observe that the present situation implies the existence of some positive constant m such that: $r_\varepsilon \ge m r_\varepsilon(p)$. Let $T_{m,\varepsilon}$ be the union of strips of size $m r_\varepsilon(p)$ included in Σ and $F_{m,\varepsilon}$ the functional defined by:

$$F_{m,\varepsilon}(v) = \begin{cases} \int_\Omega |\nabla v|^p\, dx - p\int_\Omega f v dx & \text{if } v \in V_{m,\varepsilon} \\ +\infty & \text{otherwise,} \end{cases}$$

where $V_{m,\varepsilon}$ consists of the functions of $W^{1,p}(\Omega)$ vanishing on Γ_1 and on the strips of size $m r_\varepsilon(p)$ included in Σ. For every v in V_o one has: $F_\varepsilon(v) \ge F_{m,\varepsilon}(v)$. Let $(v_\varepsilon)_\varepsilon$ be any sequence converging to v in the weak topology of V_o and such that v_ε belongs to $V_{m,\varepsilon}$ for every ε. We deduce from the preceding step:

$$\liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \ge \liminf_{\varepsilon\to 0} F_{m,\varepsilon}(v_\varepsilon) \ge \int_\Omega |\nabla v|^p\, dx - p\int_\Omega f v dx + mc\int_\Sigma |v_{|\Sigma}|^p\, d\sigma.$$

If $v_{|\Sigma}$ is not equal to 0 almost everywhere on Σ we get taking the supremum with respect to m:

$$\liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \geq +\infty.$$

If $v_{|\Sigma}$ is equal to 0 almost everywhere on Σ we get taking the supremum with respect to m:

$$\liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) = \int_\Omega |\nabla v|^p \, dx - p\int_\Omega f v dx.$$

We then conclude using Theorem 1.10 of [1]. □

Remark 3.3.

(i) When a is equal to 0 or $+\infty$ we easily prove:

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^p \, dx = \lim_{\varepsilon\to 0}\int_\Omega |\nabla u|^p \, dx.$$

This implies that $(u_\varepsilon)_\varepsilon$ converges to u in the strong topology of $W^{1,p}(\Omega)$.

(ii) When $p > 2$ the asymptotic behaviour of the solution u_ε can describe that of a non-newtonian fluid contained in the cylinder Ω and which is kept fixed along the strips T_ε^k of size r_ε. In this case there is no critical value of r_ε since the energy of the local problems always increases to $+\infty$. We conjecture that K is equal to $+\infty$ in this case and that the limit problem is described by the functional given in (3.5).

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