Journal of Convex Analysis
Volume 7 (2000), No. 1, 183–195

# On the Second-Order Contingent Set and Differential Inclusions

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Received May 4, 1998
Revised manuscript received December 9, 1999

In this paper, we establish the existence of solutions of a nonconvex second order differential inclusion of the following type:

$$\ddot{x}\ (t) \in F(x(t), \dot{x}\ (t)) \text{ a.e, } x(0) = x_0 \in K,\ \dot{x}\ (0) = v_0 \in \Omega,$$

such that $x(t) \in K$, where $K$ is a closed subset and $\Omega$ is an open subset of $\mathbb{R}^n$. When $K$ is in addition convex, we introduce the contingent cone $T_K$ to prove the existence of solutions of the differential inclusion:

$$\ddot{x}\ (t) \in G(x(t), \dot{x}\ (t)) \text{ a.e, } x(t) \in K \text{ and } \dot{x}\ (t) \in T_K(x(t))$$

*Keywords:* Continuous multifunctions, compact multifunctions, contingent cone, second-order contingent set

*1991 Mathematics Subject Classification:* 34A60

## 1. Introduction

Let $K$ be a nonempty closed subset of $\mathbb{R}^n$. We shall denote by $d_K(.)$ or $d(., K)$ the distance to $K$, by $T_K(x)$ the contingent cone at $x$, by $gr(T_K)$ the graph of the multifunction $x \to T_K(x)$ and by $A_K(x, y)$ the second-order contingent set to $K$, i.e

$$A_K(x,y) = \left\{ z \in \mathbb{R}^n \ : \ \liminf_{t \to 0^+} \frac{d_K(x + ty + \frac{t^2}{2} z)}{t^2} = 0 \right\}$$

where $(x, y) \in K \times \mathbb{R}^n$. For more on properties of the second-order contingent set, we refer to [4, 5, 8, 9].

In the present paper, we are concerned with $A_K(x, y)$ in the case where the set $K$ is, in addition, convex. More precisely, for all $(x, y) \in gr(T_K)$, we prove that $A_K(x, y)$ and the second-order contingent set

$$C_K(x,y) = \left\{ z \in \mathbb{R}^n : \lim_{t \to 0^+} \frac{d_K(x + ty + \frac{t^2}{2} z)}{t^2} = 0 \right\}$$

ISSN 0944-6532 / \$ 2.50 

coincide with $T_K(x)$.

As application, we establish in a special case where $K$ is defined by constraint equalities that the Aubin's contingent derivative set $D(T_K)(x,y)(y)$ at $(x,y)$ in direction $y$ is equal to $T_K(x)$.

Another application consists of studying the existence of second-order viable solution for a class of multivalued differential equation with nonconvex right-hand side.

Throughout this paper, we denote by $\Omega$ a nonempty open subset of $\mathbb{R}^n$, by $D$ a nonempty closed convex subset of $\mathbb{R}^n$, by $gr(T_D)$ the graph of the multifunction $T_D(.)$, by $\mathcal{C}(\mathbb{R}^n)$ the collection of all nonempty compact subsets of $\mathbb{R}^n$ and by $F$ and $G$ two $\mathcal{C}(\mathbb{R}^n)$- continuous multifunctions defined on $K \times \Omega$ respectively. Let $(x_0, v_0)$ be an element of $K \times \Omega$ or $gr(T_D)$. Consider the problems:

$$\begin{cases} \ddot{x}\ (t) \in F(x(t), \dot{x}\ (t)) & \text{a.e on } [0, T[, \\ x(0) = x_0, \quad \dot{x}\ (0)) = v_0, & \\ x(t) \in K & \forall t \in [0, T[. \end{cases} \tag{1.1}$$

and

$$\begin{cases} \ddot{x}\ (t) \in G(x(t), \dot{x}\ (t)) & \text{a.e on } [0, T[, \\ x(0) = x_0, \quad \dot{x}\ (0)) = v_0, & \\ (x(t), \dot{x}\ (t)) \in gr(T_D) & \forall t \in [0, T[. \end{cases} \tag{1.2}$$

By a solution of (1.1) or (1.2) we mean $(T, x(.))$ where $T > 0$ and $x : [0, T] \to \mathbb{R}^n$ is an absolutely continuous trajectory for which $\dot{x}\ (.)$ is also absolutely continuous, which satisfies (1.1) or (1.2).

Similar problems were investigated by [4, 6]. In the present paper we study problem (1.1) (resp. (1.2)) under the assumption $\mathrm{H}_1 : \forall\ (x,y) \in K \times \Omega$, $F(x,y) \subset C_K(x,y)$ (resp. $\mathrm{H}_2 : \forall\ (x,y) \in gr(T_D)$, $G(x,y) \subset T_K(x)$).

## 2. Notations, definitions and main results

Let $X$, $Y$ be two metric spaces, $R$ be a closed valued multifunction from $X$ to $Y$. We denote by $gr(R)$ the graph of $R$. We say that $R$ is lower semi-continuous if for any open subset $V$ of $Y$ the set $\{x \in X :\ R(x) \cap V \neq \emptyset\}$ is open. Let $(\mathcal{C}(Y), h)$ be the collection of all nonempty closed subsets of $Y$ equipped with the Hausdorff distance $h$ defined by $h(A,B) = \max\{e(A,B), e(B,A)\}$, where $e(A,B) = \sup\{d(x,B) : x \in A\}$. A $\mathcal{C}(Y)$-valued multifunction $R$ defined on $X$ is continuous if the mapping $R : X \to (\mathcal{C}(Y), h)$ is continuous.

For any vector normed space $S$ and a nonempty subset $A$ of $S$, we denote by $\mathrm{cl}\, A, \mathrm{co}\, A, \chi_A$ the closure, the convex hull and the characteristic function of $A$. For all $x \in S$, $\pi_A(x)$ stands for the set of all $y \in A$ for which $\|x - y\| = d(x, A)$.

For $r > 0$, we denote by $B(x, r)$ the closed with center at $x$ and radius $r$.

Assumption $\text{H}_1$

Let $K$ be a nonempty closed subset of $\mathbb{R}^n$, $\Omega$ be a nonempty open subsets of $\mathbb{R}^n$ and $F$ be a multifunction from $K \times \Omega$ to the space of all nonempty subsets of $\mathbb{R}^n$. Let $(x_0, v_0) \in K \times \Omega$. On $F$ we make the following hypotheses:

- $F$ is continuous with compact values.
- $\forall (x, y) \in K \times \Omega, F(x, y) \subset C_K(x, y)$.

Assumption $\text{H}_2$

Let $D$ be a nonempty closed convex subset of $\mathbb{R}^n$, $G$ be a multifunction from $gr(T_D)$ to nonempty subsets of $\mathbb{R}^n$. Let $(x_0, v_0) \in gr(T_D)$. Suppose that:

- $G$ is continuous with compact values.
- $\forall\, (x, y) \in gr(T_D), G(x, y) \subset T_D(x)$.
- $gr(T_D)$ is closed.

Here are the main results.

**Theorem 2.1.** *Assume that* $\text{H}_1$ *is satisfied, and let* $(x_0, v_0) \in K \times \Omega$*, then there exists* $T_0 > 0$ *and an absolutely continuous function* $x(.) : [0, T_0] \to \mathbb{R}^n$ *for which* $\dot{x}\ (.)$ *is also absolutely continuous such that:*

$$\begin{cases} \ddot{x}(t) \in F(x(t), \dot{x}(t)) & a.e \ on \ [0, T_0[ \\ x(0) = x_0 \quad \dot{x}(0)) = v_0 & \\ x(t) \in K & \forall t \in [0, T_0[ \end{cases}$$

**Theorem 2.2.** *Assume that* $\text{H}_2$ *is satisfied and let* $(x_0, v_0) \in gr(T_D)$*, then there exists* $T_1 > 0$ *and an absolutely continuous function* $u(.) : [0, T_1] \to \mathbb{R}^n$ *for which* $\dot{u}\ (.)$ *is absolutely continuous and such that:*

$$\begin{cases} \ddot{u}\ (t) \in G(u(t), \dot{u}\ (t)) & a.e \ on \ [0, T_1[ \\ u(0) = x_0, \quad \dot{u}\ (0)) = v_0 & \\ (u(t), \dot{u}\ (t)) \in gr(T_D) & \forall t \in [0, T_1[ \end{cases}$$

To prove Theorem 2.2, we need the following result:

**Theorem 2.3.** *Let* $C$ *be a nonempty closed convex subset of* $\mathbb{R}^n$*. Then, the second-order contingent set* $A_C(x, y)$ *at* $(x, y) \in gr(T_C)$ *coincides with the contingent cone* $T_C(x)$*.*

## 3. Preliminary results

In this section, we state some definitions and results collected in Aubin and Cellina [1].

**Proposition 3.1.** *Let* $A$ *be a nonempty closed convex subset of* $\mathbb{R}^n$ *and* $x \in A$*. Then*

$$T_A(x) = \text{cl}\left( \cup_{h>0} \frac{1}{h}(A - x) \right)$$

For the proof see [1, p.174].

**Proposition 3.2.** *The multifunction* $D \mapsto 2^{\mathbb{R}^n}, x \to T_D(x)$ *is lower semi-continuous.*

**Proof.** See Aubin and Cellina [1, Th.1, p.220]. □

**Lemma 3.3.** *Let $Q$ be a lower semi-continuous multifunction from a metric space $X$ to the space of nonempty subsets of a metric space $Y$. Then for all $\varepsilon > 0$, the set*

$$\{x \in X \ : Q(x_0) \subset Q(x) + B(0, \varepsilon)\}$$

*is open.*

The proof is straighforward and is omitted.

**Definition 3.4.** Let $I$ be a bounded interval of $\mathbb{R}$, recall that a function $f : I \to \mathbb{R}^n$ is called absolutely continuous if there exists an integrable function $g : I \to \mathbb{R}^n$ such that for all $s, t \in I$ we have $f(t) = f(s) + \int_s^t g(\tau)d\tau$, where $g$ is denoted $\dot{f}$.

**Proposition 3.5.** *Let $x, y \in \mathbb{R}^n$, $\varepsilon(.)$ a mapping with values in $\mathbb{R}^n$ such that $\varepsilon(h) \to 0$ as $h \to 0$. Then*

$$\liminf_{h \to 0^+} \frac{1}{h} \left[d_D(x + h(y + \varepsilon(h))) - d_D(x)\right] \le d(y, T_D(\pi_D(x))).$$

For the proof, see [1, Prop.1, pp. 202 or Prop.3, pp. 178-179].

**Corollary 3.6.** *Let $I$ be a bounded interval of $\mathbb{R}$ and $f : I \mapsto \mathbb{R}^n$ be a lipschitz function. Set $g(t) = d_D(f(t))$. Then $g$ is absolutely continuous and for almost every $t \in I$*

$$\dot{g}\ (t) \le d(\dot{f}\ (t), T_D(\pi_D(f(t)))).$$

**Proof.** Let $t \in I$ and $h$ be sufficiently small such that $t + h \in I$. We have

$$f(t+h) = f(t) + \int_t^{t+h} \dot{f}\ (\tau)\, d\tau. \tag{3.1}$$

Without loss of generality, we may assume that

$$\frac{1}{h}\int_t^{t+h} \dot{f}\ (s)ds \to \dot{f}\ (t) \ \text{ when } \ h \to 0.$$

Then (3.1) implies that

$$f(t+h) = f(t) + h(\dot{f}\ (t) + \varepsilon(h)), \tag{3.2}$$

where $\varepsilon(h) \to 0$ as $h \to 0$. By using Proposition 3.5, from (3.2) we obtain

$$\begin{aligned} \dot{g}\ (t) &= \liminf_{h \to 0^+} \frac{1}{h}(g(t+h) - g(t)), \\ &= \liminf_{h \to 0^+} \frac{1}{h}\left[d_D(f(t) + h(f(t) + \varepsilon(h))) - d_D(f(t))\right], \\ &\le d((\dot{f}\ (t), T_D(\pi_D(f(t))). \end{aligned}$$

□

**Definition 3.7.** Let $J$ be an interval of $\mathbb{R}$, $x(.)$: $J \to \mathbb{R}^n$ be a bounded function. We define the oscillation of $x(.)$ on $J$ by

$$\omega_J(x(.)) = \sup \{\|x(t_1) - x(t_2)\| \ : \ t_1, t_2 \in J\}$$

Let $I$ be an inteval of $\mathbb{R}$ and $\mathcal{B}(\mathcal{I}, \mathbb{R}^n)$ the space of all bounded functions from $I$ to $\mathbb{R}^n$ and $\mathcal{H}$ be a nonempty subset of $\mathcal{B}(\mathcal{I}, \mathbb{R}^n)$.

**Definition 3.8.** We say that $\mathcal{H}$ is equioscillating if $\forall \varepsilon > 0$, $\exists$ a finite partition of $I$ into subintervals $J_k$ $(1 \leq k \leq r)$ such that

$$\forall x(.) \in \mathcal{H}, \forall k \in \{1, ..., r\}, \omega_{J_k}(x(.)) \leq \varepsilon.$$

**Theorem 3.9.** *Assume that $\mathcal{H}$ is equioscillating and for all $t \in I$ the set $\mathcal{H}(t) = \{x(t) : x(.) \in \mathcal{H}\}$ is precompact. Then $\mathcal{H}$ is precompact in $\mathcal{B}(\mathcal{I}, \mathbb{R}^n)$.*

For the proof, see Aubin and Cellina [1, Th.5, p.15]

## 4. Proof of the main results

To begin with, let us prove:

$$\forall (x, v) \in gr(T_D), A_D(x, v) = C_D(x, v) = T_D(x).$$

**Proof.** Let $z \in A_D(x, v)$. Since $D$ is closed and convex, there exists $x_n \in D$, $t_n \to 0$ as $n \to +\infty$ and a function $\varepsilon(.)$ with $\varepsilon(t) \to 0$ when $t \to 0^+$ such that:

$$x_n = x + t_n v + \frac{t_n^2}{2} z + t_n^2 \varepsilon(t_n), \quad \forall n \in \mathbb{N},$$

hence

$$\frac{x_n - x}{t_n} - v = \frac{t_n}{2} z + t_n \varepsilon(t_n), \quad \forall n \in \mathbb{N}. \tag{4.1}$$

Since $v$ belongs to $T_D(x)$ which is a closed cone using the Proposition 3.1, we deduce from (4.1) that $z \in T_D(x)$.

Conversely, let $z \in T_D(x)$. For $t > 0$, set $f(t) = x + tv + \frac{t^2}{2} z$. By Corollary 3.6, we have that

$$d(f(t), D) \leq \int_0^t d(v + \tau z, T_D(\pi_D(f(\tau)))) \, d\tau,$$

so that

$$\frac{1}{t^2} d(f(t), D) \leq \frac{1}{t} \int_0^t \frac{1}{\tau} d(v + \tau z, T_D(\pi_D(f(\tau)))) \, d\tau, \tag{4.2}$$

by using Proposition 3.1, it is easy to check that for any $\tau > 0$

$$\frac{1}{\tau} d(v + \tau z, T_D(\pi_D(f(\tau)))) = d(\frac{1}{\tau} v + z, T_D(\pi_D(f(\tau)))).$$

Let $\varepsilon > 0$. Since $\pi_D(.)$ is continuous and $z + \frac{1}{\tau}v \in T_D(x)$, $\forall\ \tau > 0$, by Proposition 3.2 and Lemma 3.3, there exists $t_0 > 0$ such that

$$d(\frac{1}{\tau}v + z, T_D(\pi_D(f(s)))) < \varepsilon, \qquad \forall s \leq t_0, \forall \tau > 0, \tag{4.3}$$

thus by combining (4.2) and (4.3) it follows that

$$\frac{1}{t^2}d(f(t), D) < \varepsilon, \qquad \forall t \leq t_0.$$

This implies that $z \in C_D(x, v)$.

Hence $A_D(x, v) = C_D(x, v) = T_D(x)$. This completes the proof. □

**Remark 4.1.** If $C$ a nonempty compact subset of $gr(A_D)$, then

$$\frac{1}{t^2}d_D(x + ty + \frac{t^2}{2}z) \to 0 \text{ as } t \to 0^+$$

uniformly on $C$.

As application, let us explicit the Aubin's notion of contingent derivative of $T_L$ at point $(x_0, v_0)$ in direction $v_0$ defined by

$$D(T_L)(x_0, v_0)(v_0) = \left\{w \in \mathbb{R}^n \ : \ (x_0, w) \in T_{gr(T_L)}(x_0, v_0)\right\},$$

where $L$ is a set defined by constraint equalities. More precisely, suppose that:

$$L = \{x \in \Omega : f_i(x) = 0, \forall i = 1, ..., m\},$$

where the $f_i$ are real-valued functions defined and $C^2$ on an open subset $\mathbb{R}^n$. Suppose that the gradients $(\nabla f_i(x_0))_{i=1,...,m}$ are linearly independent. Then, it has been proved respectively in [5, Prop.9] and [6, Prop.2.2] that

$$A_L(x_0, v_0) = \left\{w \in \mathbb{R}^n, \langle \nabla f_i(x_0), w\rangle + \left\langle \nabla^2 f_i(x_0)v_0, v_0\right\rangle = 0,\ \forall i = 1, ..., m\right\}$$

and

$$D(T_L)(x_0, v_0)(v_0) = A_L(x_0, v_0),$$

where $\nabla^2 f_i(x_0)$ denotes the Hessian matrix at $x_0$. Hence if $L$ is in addition convex, then by virtue of Theorem 2.3, we have

$$D(T_L)(x_0, v_0)(v_0) = T_L(x_0).$$

We are able to give the proof of Theorem 2.1.

**Proof.** Let $r_0 > 0$ be such that $B(v_0, r_0) \subset \Omega$. Let $w_0 \in F(x_0, v_0)$. Set

$$r = \frac{r_0}{2}\ , \ K_0 = B\left((x_0, v_0), r\right) \cap \left(K \times B(v_0, r)\right).$$

Since $F$ is continuous with compact values, there exists $M > \max(1, \|v_0\| + r)$ such that

$$h\left(F(x,y), \{0\}\right) \le M - 1 \quad \forall (x,y) \in K_0. \tag{4.4}$$

Set

$$K_1 = (K_0 \times B(0, M-1)) \cap gr(C_K), T = \frac{r}{M}$$

and

$$\eta_k = \frac{r}{2^k} \quad \forall k \in \mathbb{N}.$$

Since $K_0$ and $K_1$ are compact there exists $h_k$, $\delta_k$ such that

$$\max\{h_k, \delta_k\} < \min\left\{r, 2\frac{M - \|v_0\| - r}{\|w_0\| + r}\right\}, \tag{4.5}$$

and for all $(x,y), (x',y') \in K_0$ and $(u,v,w) \in K_1$ one has

$$h\left(F(x,y), F(x',y')\right) < \frac{\eta_k}{3}, \quad \forall (x',y') \in B((x,y), \delta_k), \tag{4.6}$$

and

$$d_K(u + tv + \frac{t^2}{2}w) < \frac{\eta_k}{6}t^2, \quad \forall t \in [0, h_k]. \tag{4.7}$$

For any number $k$, choose an integer $n_k$ such that $n_{k+1} > n_k$ and

$$\max\left\{\frac{MT}{2^{n_k}}, \frac{(\|v_0\| + MT)T}{2^{n_k}}\right\} < \min\{h_k, \delta_k\}. \tag{4.8}$$

Set

$$l_k = \frac{T}{2^{n_k}}, \quad t_i^k = il_k, \ k = 0, ..., 2^{n_k}.$$

Observe that

$$\forall i \in \{0, ...2^{n_k}\}, \exists j \in \{0, ..., 2^{n_k}.\}, \quad t_i^{k+1} = t_j^k. \tag{4.9}$$

Let us define a sequence of polygonal approximate solution of the problem (1.1). Indeed, let us consider

$$y \in \pi_K\left(x_0 + t_1^k v_0 + \frac{(t_1^k)^2}{2} w_0\right),$$

and

$$\ddot{x}_1^k(0) = 2\frac{y - x_0 - t_1^k v_0}{(t_1^k)^2}.$$

On $[0, t_1^k]$, define

$$x_1^k(t) = x_0 + tv_0 + \frac{t^2}{2}\ddot{x}_1^k(0),$$

then $x_1^k(t_1^k) \in K$. Moreover for all $s \in [0, t_1^k[$ we have that

$$\begin{aligned} \left\| \ddot{x}_1^k(s) - w_0 \right\| &= \left\| \ddot{x}_1^k(0) - w_0 \right\|, \\ &\le \frac{2}{l_k^2} \left\| y - x_0 - l_k v_0 - \frac{(l_k)^2}{2} w_0 \right\|, \\ &\le \frac{2}{l_k^2} d_K(x_0 + l_k v_0 + \frac{(l_k)^2}{2} w_0). \end{aligned}$$

Then by (4.7), for all $s \in [0, t_1^k[$ it follows that

$$\left\| \ddot{x}_1^k(s) - w_0 \right\| < \frac{\eta_k}{3}, \tag{4.10}$$

and

$$\begin{aligned} d(\ddot{x}_1^k(0), F(x_1^k(0), \dot{x}_1^k(0)) &\le \left\| \ddot{x}_1^k(0) - w_0 \right\|, \\ &< \frac{\eta_k}{3}. \end{aligned}$$

Hence

$$\begin{aligned} \left\| \ddot{x}_1^k(0) \right\| &\le \sup_{(x,y)\in K_0} h\left(F(x,y), \{0\}\right) + \frac{\eta_k}{3}, \\ &\le M. \end{aligned}$$

Consequently, relation (4.8) implies

$$\begin{aligned} \left\| \dot{x}_1^k(t) - v_0 \right\| &\le M l_k, \\ &< \delta_k, \quad \forall t \in [0, t_1^k]. \end{aligned}$$

On the other hand, from (4.5) and (4.8) we deduce

$$\begin{aligned} \left\| x_1^k(t) - x_0 \right\| &\le t_1^k \left\| v_0 \right\| + \frac{(t_1^k)^2}{2} \left\| w_0 \right\| \\ &\le M l_k \\ &< \delta_k \quad \forall t \in [0, t_1^k] \end{aligned}$$

Thus $(x_1^k(t_1^k), \dot{x}_1^k(t_1^k)) \in K_0$.

For each integer $p > 1$ and $t \in [0, T[$, set $s_p(t)$ to be the initial point of the $p$-th partition to which $t$ belongs. At each nodal point $t_i^p$ let us define a piecewise function $x_i^p(.)$ on $[0, t_i^p]$ with the following properties:

(i) $x_i^p(.)$ is a continuous function with $x_i^p(0) = x_0, \dot{x}_i^p(0) = v_0$ and its second derivative $\ddot{x}_i^p(.)$ is constant on each interval $[t_j^p, t_{j+1}^p[$, $j = 0, ...., i-1$.

(ii) At each nodal point $t_j^p \in [0, t_i^p[$ , $x_i^p(t_j^p) \in K$ and (for $t_j^p < t_i^p$)

$$\ddot{x}_i^p\ (t_j^p) \in F(x_i^p(t_j^p), \dot{x}_i^p\ (t_j^p)) + \frac{\eta_p}{3} B(0,1).$$

(iii) $\max\{\|x_i^p(t_i^p) - x_0\|, \|\dot{x}_i^p\ (t_i^p) - v_0\|\} < \frac{r\, t_i^p}{T}$

(iv) At each nodal point $t_j^p \in [0, t_i^p[$, $\|\ddot{x}_i^p\ (t_j^p) - \ddot{x}_i^p\ (s_{p-1}(t_j^p))\| < \eta_p$.

Since $s_{p-1}(t_1^p) = 0$, it is clear that $x_1^p(.)$ verifies the previous properties. By induction, assume that has been constructed a function $x_i^p(.)$ on $[0, t_i^p]$ with $i \in \{0, ..., 2^{n_p} - 1\}$ satisfying (i)–(iv). Set $s = s_{p-1}(t_i^p)$. Since $|s - t_i^p| < l_p$ we have by (4.8)

$$\begin{aligned} \|x_i^p(t_i^p) - x_i^p(s)\| &\leq (MT + \|v_0\|)\, l_p, \\ &< \delta_p, \end{aligned}$$

and

$$\begin{aligned} \|\dot{x}_i^p\ (t_i^p) - \dot{x}_i^p\ (s)\| &\leq M l_p, \\ &< \delta_p. \end{aligned}$$

Hence relation (4.6) implies

$$h(F(x_i^p(t_i^p), \dot{x}_i^p\ (t_i^p)), F(x_i^p(s), \dot{x}_i^p\ (s))) < \frac{\eta_p}{3}. \tag{4.11}$$

Moreover, by assumption (ii)

$$d(\ddot{x}_p\ (s), F(x_i^p(s), \dot{x}_i^p\ (s))) < \frac{\eta_p}{3}, \tag{4.12}$$

so that by (4.11), (4.12) and the compactness of the values of $F$, there exists $w \in F(x_i^p(t_i^p), \dot{x}_i^p\ (t_i^p))$ such that

$$\|\ddot{x}_i^p\ (s) - w\| \leq \frac{\eta_p}{3} + \frac{\eta_p}{3}, \tag{4.13}$$

and therefore, assertion (iii) and triangular inequality imply that

$$\|w - w_0\| \leq \frac{\eta_p}{3} + \frac{\eta_p}{3} + \frac{t_i^p}{3}. \tag{4.14}$$

Since $t_i^p \leq T = \frac{r}{M}$ and by the choice of $\eta_p$ and $M$, we have

$$\|w - w_0\| < r, \tag{4.15}$$

hence $\left(x_i^p(t_i^p), \dot{x}_i^p\ (t_i^p), w\right) \in K_1$.

On the other hand, given $y \in \pi_K(x_i^p(t_i^p) + l_p\, \dot{x}_i^p\ (t_i^p) + \frac{(l_p)^2}{2} w)$ and consider

$$\ddot{x}_i^p\ (t_i^p) = 2\frac{y - x_i^p(t_i^p) - l_p\, \dot{x}_i^p\ (t_i^p)}{(l_p)^2}.$$

For $t \in [t_i^p, t_{i+1}^p]$, set

$$y_p(t) = x_i^p(t_i^p) + (t - t_i^p)\ \dot{x}_i^p\ (t_i^p) + \frac{1}{2}(t - t_i^p)^2\ \ddot{x}_i^p\ (t_i^p).$$

Observe that $y_p(t_{i+1}^p) \in K$. Moreover

$$\begin{aligned} d(\ddot{x}_i^p\ (t_i^p), F(x_i^p(t_i^p), \dot{x}_i^p\ (t_i^p)) &\leq \left\| \ddot{x}_i^p\ (t_i^p) - w \right\|, \\ &= \frac{2}{(l_p)^2} d_K(x_i^p(t_i^p) + l_p\ \dot{x}_i^p\ (t_i^p) + \frac{(l_p)^2}{2} w), \\ &< \frac{\eta_p}{3}. \end{aligned} \tag{4.16}$$

Hence the function

$$z_p = x_i^p(.)\chi_{[0,t_i^p]}(.) + y_p(.)\chi_{[t_i^p, t_{i+1}^p]}(.)$$

satisfies (i) and (ii). Let us prove that $z_p$ verifies (iii) and (iv).

$$\begin{aligned} \left\| z_p(t_{i+1}^p) - x_0 \right\| &= \left\| y_p(t_i^p) - x_0 \right\|, \\ &\leq \left\| x_i^p(t_i^p) - x_0 \right\| + l_p\,(\| v_0\| + MT) + \frac{(l_p)^2}{2} \left\| \ddot{x}_i^p\ (t_i^p) \right\|. \end{aligned}$$

Since $MT = r$ and $l_p < 2\frac{M - \| v_0\| - r}{M}$, then

$$\begin{aligned} \left\| z_p(t_{i+1}^p) - x_0 \right\| &\leq \frac{r t_i^p}{T} + \frac{r l_p}{T}, \\ &\leq \frac{r t_{i+1}^p}{T}. \end{aligned} \tag{4.17}$$

Furthermore

$$\dot{z}_p\ (t_i^p) = \dot{x}_i^p\ (t_i^p) + l_p\ \ddot{x}_i^p\ (t_i^p). \tag{4.18}$$

Therefore

$$\begin{aligned} \left\| \dot{z}_p\ (t_{i+1}^p) - v_0 \right\| &\leq \left\| \dot{x}_i^p\ (t_i^p) - v_0 \right\| + M l_p, \\ &\leq \frac{r\ t_i^p}{T} + \frac{r\ l_p}{T}, \\ &\leq \frac{r t_{i+1}^p}{T}. \end{aligned}$$

Hence, relations (4.15) and (4.17) imply (iii). About (iv), we have

$$\left\| \ddot{z}_p\ (t_i^p) - \ddot{z}_p\ (s) \right\| = \left\| \ddot{x}_i^p\ (t_i^p) - \ddot{x}_i^p\ (s) \right\|,$$

from that we obtain by using (4.13) and (4.16)

$$\left\| \ddot{z}_p\ (t_i^p) - \ddot{z}_p\ (s) \right\| \leq \eta_p.$$

This implies (iv). □

Consequently, there exists a sequence $\{u_m(.)\}$ of absolutely continuous functions on $[0,T]$ satisfying (i)–(iv).

**Claim 4.2.** *The sequence* $\{\ddot{u}_m(.)\}$ *is equioscillating.*

**Proof.** Let $p \in \mathbb{N}$ and $I = [t_i^p, t_{i+1}^p[$ an interval of the $p$-th partition of $[0,T]$. Let $q > p$ and $t_j^q$ be a nodal point of $I$. For $i = 0, ..., p$, set $s^i = s_{q-i} \circ s_{q-i+1} \circ ... \circ s_q$. It is clear that $t_j^q = s^0(t_j^q), s^1(t_j^q), ..., s^p(t_j^q) = t_i^p$ are in $I$. We wish to compute the oscillation of $\ddot{u}_m(.)$ on $I$. If $m < p$ then the oscillation is zero. Hence consider $m > p$. Let $t \in I$ and $t_j^m$ be the initial point of the $m$-th partition to which $t$ belongs. Since $\ddot{u}_m(t) = \ddot{u}_m(t_j^m)$ then by (iv) we have that:

$$\left\|\ddot{u}_m(t_j^m) - \ddot{u}_m(s^{m-1}(t_j^m)\right\| \leq \eta_m,$$
$$\left\|\ddot{u}_m(s^{m-1}(t_j^m)) - \ddot{u}_m(s^{m-2}(t_j^m)\right\| \leq \eta_{m-1},$$
$$\vdots$$
$$\left\|\ddot{u}_m(s^{p-1}(t_j^m)) - \ddot{u}_m(t_i^p)\right\| \leq \eta_p.$$

Consequently

$$\begin{aligned}\left\|\ddot{u}_m(t_i^p) - \ddot{u}_m(s)\right\| &\leq \sum_{i=p}^{m} \eta_i,\\ &\leq \frac{1}{2^{p-1}}.\end{aligned}$$

$$\omega(\ddot{u}_m(.)) \leq \frac{1}{2^{p-1}} \quad \forall m \in \mathbb{N}.$$

On the other hand, since

$$\left\|\ddot{u}_m(t)\right\| \leq M \text{ and } \left\|\dot{u}_m(t)\right\| \leq MT + \|v_0\|, \quad \forall m \in \mathbb{N},\ \forall t \in I,$$

then by Theorem 3.9, $\{\ddot{u}_m(.)\}$ converges uniformly to a function $w(.)$, so that $\{u_m(.)\}$ and $\{\dot{u}_m(.)\}$ converge uniformly to functions $u$ and $v$ respectively. The functions $u$, $v$ and $w$ are related by the formula

$$v(t) = \dot{u}(t) \text{ and } w(t) = \ddot{u}(t) \text{ a.e on } [0,T]$$

Let $t \in [0,T]$, for each $m$ we denote by $t_i^m$ the initial point of the $m$-th partition to which $t$ belongs. We have

$$\begin{aligned} d(w(t), F(u(t), v(t))) \leq & \left\|w(t) - \ddot{u}_m(t)\right\| \\ & + \left\|\ddot{u}_m(t) - \ddot{u}_m(t_i^m)\right\| \\ & + d(\ddot{u}_m(t_i^m), F(u_m(t_i^m), \dot{u}_m(t_i^m)) \\ & + h(F(u_m(t_i^m), \dot{u}_m(t_i^m)), F(u_m(t), \dot{u}_m(t)) \\ & + h(F(u_m(t), \dot{u}_m(t)), F(u(t), v(t)). \end{aligned}$$

Since $\{\ddot{u}_m(t)\}$ converges to $w(t)$, $\ddot{u}_m(t)$ equals $\ddot{u}_m(t_i^m)$, $\ddot{u}(t_i^m)$ belongs to the set $F(u_m(t_i^m), \dot{u}_m(t_i^m)) + \frac{1}{3}\eta_m B(0,1)$, $u_m(.)$, $u(.)$, $\dot{u}_m(.)$ and $\dot{u}(.)$ are lipschitzean, $\{t_i^m : m \in N\}$ converges to $t$ and $F$ is continuous, the right-hand side of the above inequality converges to zero, since $F$ have closed values, we deduce that

$$\ddot{u}(t) = w(t) \in F(u(t), \dot{u}(t)) \text{ a.e on } [0,T[.$$

On the other hand

$$d(u(t), K) \leq \|u_m(t) - u(t)\| + \|u_m(t) - u_m(t_i^m)\| + d(u_m(t_i^m), K),$$

hence

$$u(t) \in K, \forall t \in [0,T[.$$

Finally

$$\begin{cases} \ddot{u}(t) \in F(u(t), \dot{u}(t)) & \text{a.e on } [0,T[ \\ u(0) = x_0, \ \dot{u}(0) = v_0 \in T_K(x_0) & \\ u(t) \in K & \forall t \in [0,T[ \end{cases}$$

To prove Theorem 2.2, by the same reasoning we construct an approximate solution $\{u_m(.)\}$ satisfying at each nodal point $t_i^m$, $\dot{u}_m(t_{i+1}^m) \in T_D(u_m(t_i^m))$.

Relation (4.18) implies that

$$\begin{aligned} \dot{u}_m(t_{i+1}^m) &= \dot{u}_m(t_i^m) + l_m \ddot{u}_m(t_i^m) \\ &= \frac{2}{l_m}\left(y - \dot{u}_m(t_i^m) - u_m(t_i^m)\right). \end{aligned}$$

Since $y \in D$, then by Proposition 3.1

$$(u_m(t_i^m), \dot{u}_m(t_i^m)) \in gr(T_D), \quad \forall m \in \mathbb{N}.$$

Since $gr(T_D)$ is closed then by passing to the limit there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ for which $\dot{x}$ is also absolutely continuous which is a solution of the problem (1.2). This completes the proof of Theorem 2.2. □

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