Journal of Convex Analysis
Volume 7 (2000), No. 2, 413–426

Existence of a Continuous Solution of Parametric Nonlinear Equation with Constraints

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Received February 26, 1999
Revised manuscript received February 7, 2000

Combining a consequence of the Michael continuous selection theorem and iterative shem, we prove the existence of a continuous solution of parametric nonlinear equation with constraints. An inverse-function theorem for multivalued functions with continuous selection of inverse images is given.

1. Introduction

Let T be a topological space, X and Y two Banach spaces. Let U be an open subset of $T \times X$ wich countains (t_0, x_0), f a map from U into Y and M be a subset of X such that $x_0 \in M$. Let us consider the following parametric nonlinear equation with constraints:

$$\begin{cases} f(t,x) = y \\ x(t,y) \in M \end{cases}$$

and more generally, given N a subset of Y wich countains $y_0 = f(t_0, x_0)$, we consider the inclusion:

$$\begin{cases} f(t,x) \in y + (N - y_0) \\ x(t,y) \in M \end{cases} \tag{1.1}$$

Several papers have treated the existence of solution of (1.1) and study the lipschitz properties of multi-valued solution maps in a neighborhood of (t_0, x_0) (see for example [2], [3], [5], [6], [7], [8], [12], [15]).

In finite dimensional spaces, one can obtain, under weaker assumptions, a lipschitz stability of the solution maps for broad classes of generalized equations (see [10] and [12]).

Antoine and Zouaki ([1] and [16]) have given the existence of a continuous selection of (1.1) in the particular case where $M - x_0$ and $N - y_0$ are closed convex cones . In this paper, we extend this result when M (resp. N) approximates continuously its Clarke tangent cone at x_0 (resp. y_0) (see Definition 2.2). This property is verified at x_0 for an important class of sets. We get among others: sets $x_0 + K$, where K is a closed cone (not necessarily convex); graphs of strictly differentiable functions with respect to the first component of x_0; sets locally convex at x_0 in finite dimension.

Note that, if we replace in Theorem 5.2 and its corollaries the hypothesis:

ISSN 0944-6532 / $ 2.50

M (resp. N) approximates continuously its Clarke tangent cone at x_0 (resp. y_0)

by the following weak hypothesis:

M (resp. N) approximates strictly its Clarke tangent cone at x_0 (resp. y_0),

then except the continuity of the solution, the same conclusions hold. Which gives similar results that ([2], [3], [5], [6], [7], [8], [12], [15]). In finite dimension, every closed set M approximates strictly its Clarke tangent cone $C_M(x)$ for all x (see Proposition 2.5). This allows us, in this case, to obtain the same results without any restriction on M or N.

2. Sets approximating their Clarke tangent cone

Let X be a Banach space, M a subset of X, and let $x_0 \in M$. We denote $B_M(x_0, r)$ the closed ball of M of radius $r > 0$ and centered at x_0. The closed unit ball of X is denoted by B_X.

Definition 2.1. The contingent cone $T_M(x_0)$ to M at some point $x_0 \in M$, is defined by: $v \in T_M(x_0)$ if and only if there exists a sequence $(v_n, t_n)_{n\in\mathbb{N}} \in X \times \mathbb{R}_*^+$ converging to $(v, 0)$ such that, for all $n \in \mathbb{N}$, $x_0 + t_n v_n \in M$.

Definition 2.2. The Clarke tangent cone $C_M(x_0)$ to M at x_0 is the set of vectors $v \in X$ such that for every sequence $(x_n, t_n)_{n\in\mathbb{N}} \in M \times \mathbb{R}_*^+$ converging to $(x_0, 0)$, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of X converging to v which verifies $x_n + t_n v_n \in M$ for all $n \in \mathbb{N}$.

Let us recall that $T_M(x_0)$ is a closed cone, that $C_M(x_0)$ is a closed convex cone and that:

$$C_M(x_0) + T_M(x_0) = T_M(x_0). \tag{2.1}$$

Definition 2.3. We shall say that M approximates strictly its Clarke tangent cone at x_0 if

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \ \forall (x, v) \in B_M(x_0, \eta) \times (C_M(x_0) \cap (\eta B_X)),\quad \exists y \in M\ :\ \|y - (x + v)\| \le \varepsilon \|v\|$$

and that M approximates continuously its Clarke tangent cone at x_0 if for every $\varepsilon > 0$, there exist $\eta > 0$ and a continuous map g from $B_M(x_0, \eta) \times (C_M(x_0) \cap \eta B_X)$ to M such that:

$$\forall\, (x, v) \in B_M(x_0, \eta) \times (C_M\,(x_0) \cap \eta B_X)\,, \|g(x, v) - (x + v)\| \le \varepsilon \|v\|.$$

An important class of sets approximating continuously their Clarke tangent cone at x_0 is given by the following propositions:

Proposition 2.4. *If $M = x_0 + K$, where K is a closed cone, then M approximates continuously its Clarke tangent cone at x_0.*

Proof. Indeed, by (2.1), we have

$$C_M(x_0) + x_0 + T_M(x_0) = x_0 + T_M(x_0),$$

then

$$C_M(x_0) + M = M.$$

Let $g: \ M \times C_M(x_0) \longrightarrow M$ be defined by $g(x,v) = x + v$. Then g is continuous, and we have for all $(x,v) \in M \times C_M(x_0)$, $g(x,v) - (x+v) = 0$. Therefore M approximates continuously its Clarke tangent cone at x_0. □

Proposition 2.5. *Assume that X has a finite dimension. Then M approximates strictly its Clarke tangent cone at x_0.*

Proof. Assume the contrary: there exist $\varepsilon > 0$, and a sequence $(x_n, v_n)_{n\in\mathbb{N}}$ of $M \times C_M(x_0)$ converging to $(x_0, 0)$ such that:

$$(x_n + v_n + \varepsilon \| v_n \| B_X) \cap M = \emptyset,$$

Which implies that $v_n \neq 0$ and for any $n \in \mathbb{N}$,

$$\left(\frac{v_n}{\| v_n\|} + \varepsilon B_X \right) \cap \frac{1}{\| v_n\|} (M - x_n) = \emptyset. \tag{2.2}$$

Since X is a finite dimensional space, we can suppose, without loss of generality, that $\left(\frac{v_n}{\|v_n\|}\right)_{n\in\mathbb{N}}$ converges to some $v \in C_M(x_0)$. Set $t_n = \| v_n\|$ for all n. Then, by the definition of $C_M(x_0)$, there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in X which converges to v such that $x_n + t_n w_n \in M$ for all n. Then, for n large enough:

$$w_n \in \left(\frac{v_n}{\| v_n\|} + \varepsilon B_X \right) \cap \frac{1}{\| v_n\|} (M - x_n),$$

and this contradicts relation (2.2). □

Corollary 2.6. *Let X be a finite dimensional space. Assume that there exists $r > 0$ such that $B_M(x_0, r)$ is convex and closed. Then M approximates continuously its Clarke tangent cone at x_0.*

Proof. Let $\varepsilon > 0$. From Proposition 2.5, there exists $\eta \in]0, r[$ such that:

$$\forall\, (x,v) \in B_M(x_0,\eta) \times (C_M(x_0) \cap \eta B_X), \ \ \exists\, y \in \ B_M(x_0, r) \ : \ \|y - (x+v)\| \leq \varepsilon \|v\|.$$

Let π be the projection of X to $B_M(x_0,\eta)$ and g be the application from $B_M(x_0,\eta) \times (C_M(x_0) \cap \eta B_X)$ to M defined by $g(x,v) = \pi(x+v)$. It is clear that g is continuous and that:

$$\forall\ (x,v) \in B_M(x_0,\eta) \times (C_M(x_0) \cap \eta B_X),\ \| g(x,v) - (x+v)\| \leq \varepsilon \|v\|.$$

Therefore M approximates continuously its Clarke tangent cone at x_0 and the proof is complete. □

3. Multivalued Functions approximating their Clarke derivative

Definition 3.1. Let F be a multivalued function from X to Y, $x_0 \in \operatorname{Dom} F$ and $y_0 \in F(x_0)$. We call the Clarke derivative of F at (x_0, y_0), denoted by $CF(x_0, y_0)$, the closed convex process whose graph is the Clarke tangent cone to the graph of F at (x_0, y_0).

We shall say that F approximates strictly (resp. continuously) its Clarke derivative at (x_0, y_0) if the graph of F approximates strictly (resp. continuously) its Clarke tangent cone at (x_0, y_0).

An important class of multivalued functions approximating continuously their Clarke derivative at (x_0, y_0) is given by the following proposition

Proposition 3.2. *Let U be an open subset of X, $f : U \longrightarrow Y$ be a strictly derivable map at $x_0 \in U$, N a subset of Y, $y_0 \in N$ and $F : X \longrightarrow Y$ be the multivalued function defined by $F(x) = f(x) + N$ if $x \in U$, and $F(x) = \emptyset$ if $x \notin U$. Then:*

1) *For all $u \in X$, $CF\,(x_0, f(x_0) + y_0)\,u = Df(x_0)u + C_N(y_0)$.*
2) *If N approximates strictly (resp. continuously) its Clarke tangent cone at y_0, then F approximates strictly (resp. continuously) its Clarke derivative at $(x_0, f(x_0) + y_0)$.*

Proof. 1) Let $v \in CF\,(x_0, f(x_0) + y_0)\,u$. For showing that $v - Df(x_0)u \in C_N(y_0)$, we shall prove that for each sequence $(y_n, t_n)_{n\in\mathbb{N}}$ of $N \times \mathbb{R}^+_*$ which converges to $(y_0, 0)$, there exists a sequence $(w_n)_{n\in\mathbb{N}}$ of Y converging to $v - Df(x_0)u$ such that $y_n + t_n w_n \in N$ for all $n \in \mathbb{N}$. We set $(x_n, z_n) := (x_0, f(x_0) + y_n)$. Then $(x_n, z_n)_{n\in\mathbb{N}}$ is a sequence of Graph(F) which converges to $(x_0, f(x_0) + y_0)$. Therefore, there exists a sequence $(u_n, v_n)_{n\in\mathbb{N}}$ of $X \times Y$ which converges to (u, v) such that, for all $n \in \mathbb{N}$,

$$(x_0, f(x_0) + y_n) + t_n(u_n, v_n) \in \text{Graph}(F).$$

This implies that for all $n \in \mathbb{N}$,

$$f(x_0) + y_n + t_n v_n - f(x_0 + t_n u_n) \in N.$$

That is:

$$\forall n \in \mathbb{N},\;\; y_n + t_n\left(v_n - \frac{f(x_0 + \, t_n u_n) - f(x_0)}{t_n}\right) \in N.$$

By setting $w_n := v_n - \frac{f(x_0+\, t_n u_n) - f(x_0)}{t_n}$, we see that $y_n + t_n w_n \in N$ and that $(w_n)_{n\in\mathbb{N}}$ converges to $v - Df(x_0)u$.

Conversely, suppose that $v - Df(x_0)u \in C_N(y_0)$. Let $(x_n, z_n)_{n\in\mathbb{N}}$ be a sequence of Graph(F) converging to $(x_0, f(x_0) + y_0)$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^+_*$ conveging to 0. We show that there exists a sequence (u_n, v_n) of $X \times Y$ which converges to (u, v) such that:

$$\forall n \in \mathbb{N},\; (x_n, z_n) + t_n(u_n, v_n) \;\in \text{Graph}(F).$$

It is clear that the sequence $y_n := z_n - f(x_n)$ of N converges to y_0. Then there exists a sequence $(w_n)_{n\in\mathbb{N}}$ of Y which converges to $v - Df(x_0)u$ such that

$$\forall n \in \mathbb{N} \quad y_n + t_n w_n \in N\;. \tag{3.1}$$

Let us set $v_n := w_n + \frac{f(x_n+\, t_n u) - f(x_n)}{t_n}$. Since f is strictly differentiable at x_0, $\frac{f(x_n+\, t_n u) - f(x_n)}{t_n}$ converges to $Df(x_0)u$. Therefore $(v_n)_{n\in\mathbb{N}}$ converges to v. On the other hand, we are for all $n \in \mathbb{N}$,

$$\begin{aligned} z_n + t_n v_n &= z_n + t_n w_n + f(x_n + t_n u) - f(x_n) \\ &= z_n - f(x_n) + t_n w_n + f(x_n + t_n u) \\ &= f(x_n + t_n u) \;+\; y_n \;+\; t_n w_n \\ &\in f(x_n + t_n u) + N \qquad \text{(from (3.1))}. \end{aligned}$$

So $(x_n, z_n) + t_n(u, v_n) \in \text{Graph}(F)$, and hence $(u, v) \in C_{\text{Graph}(F)}(x_0, f(x_0) + y_0)$.

2) We give the proof in the case where N approximates strictly its Clarke tangent cone at y_0. The proof in the case where N approximates continuously its Clarke tangent cone at y_0 is analogous.

Let $\varepsilon > 0$. We show that there exists $\eta > 0$ such that: for all $(x, z) \in \text{Graph}(F)$, and $(u, v) \in C_{\text{Graph}(F)}(x_0, f(x_0)+y_0)$ satisfying $\|x-x_0\|+\|z-(f(x_0)+y_0)\| \leq \eta$ and $\|u\|+\|v\| \leq \eta$, there exists $(x', z') \in \text{Graph}(F)$ such that $\|x' - x - u\| + \|z' - z - v\| \leq \varepsilon(\|u\| + \|v\|$.

Since N approximates strictly its Clarke tangent cone at y_0, there exists $\eta_1 > 0$ such that:

$$\forall y \in B_N(y_0, \eta_1),\ \forall w \in C_N(y_0) \cap \eta_1 B_Y,$$
$$\exists y' \in N\ : \|y' - y - w\| \leq \frac{\varepsilon}{1 + \|Df(x_0)\|}\|w\|. \quad (3.2)$$

The strict differentiability of f at x_0, implies the existence of $\eta_2 > 0$ such that:

$$\forall x \in B_U(x_0, \eta_2),\ \ \|f(x) - f(x_0)\| \leq \frac{\eta_1}{2}$$

and

$$\forall (x_1, x_2) \in (B_U(x_0, \eta_2))^2,$$
$$\| f(x_1) - f(x_2) - Df(x_0)(x_1 - x_2)\| \leq \frac{\varepsilon}{1 + \|Df(x_0)\|}\|x_1 - x_2\|. \quad (3.3)$$

Since U is open, we can suppose furthermore that $B(x_0, \eta_i) \subset U$ for $i = 1, 2$. We set $\eta := \frac{1}{2(1+\|Df(x_0)\|)} \inf(\eta_1, \eta_2)$. Let $(x, z) \in \text{Graph}(F)$ and $(u, v) \in C_{\text{Graph}(F)}(x_0, f(x_0) + y_0)$ such that $\|x - x_0\| + \|z - (f(x_0) + y_0)\| \leq \eta$ and $\|u\| + \|v\| \leq \eta$.

Let us set $w := v - Df(x_0)u$ and $y = z - f(x)$. Then $y \in N$, $w \in C_N(y_0)$, $\|y - y_0\| \leq \|z - (f(x_0) + y_0)\| + \|f(x) - f(x_0)\| \leq \eta + \frac{\eta_1}{2} \leq \eta_1$ and $\|w\| \leq \|v\| + \|Df(x_0)\|\|u\| \leq (1 + \|Df(x_0)\|)(\|u\| + \|v\|) \leq \eta_1$. Therefore by (3.2),

$$\exists y' \in N\ : \|y' - y - w\| \leq \frac{\varepsilon}{1 + \|Df(x_0)\|}\|w\|. \quad (3.4)$$

We set $x' := x + u$ and $z' = f(x + u) + y'$, then $(x', z') \in \text{Graph}(F)$ and

$$\begin{aligned}
\|x' - (x + u)\| + \|z' - (z + v)\| &= \|f(x + u) + y' - (z + v)\| \\
&= \|f(x + u) + y' - y - f(x) - v\| \quad (\text{since } z = y + f(x)) \\
&\leq \|f(x + u) - f(x) - Df(x_0)u\| + \|y' - y - w\| \quad (w = v - Df(x_0)u) \\
&\leq \frac{\varepsilon}{1 + \|Df(x_0)\|}\|u\| + \frac{\varepsilon}{1 + \|Df(x_0)\|}\|w\| \quad (\text{from (3.3) and (3.4)}) \\
&\leq \frac{\varepsilon}{1 + \|Df(x_0)\|}\|u\| + \frac{\varepsilon}{1 + \|Df(x_0)\|}\|v\| + \frac{\varepsilon}{1 + \|Df(x_0)\|}\|Df(x_0)\|\ \|u\| \\
&\leq \varepsilon\left(\|u\| + \|v\|\right).
\end{aligned}$$

□

4. A continuous selection lemma

Definition 4.1 (Closed Convex Process). Let X and Y be two normed linear spaces. A multivalued function from X to Y is called a convex process if its graph is a convex cone containing the origin. It is said be a closed convex process if its graph is also closed.

Lemma 4.2. *Let X and Y be two Banach spaces and P be a closed convex process from X onto Y. Then there exists a continuous positively homogeneous map u from Y to* $\operatorname{dom} P$ *such that:*

(1) $\forall y \in Y, \quad y \in P\,(u(y))\,;$

(2) $\exists A > 0\ :\ \forall y \in Y,\ \ \|u(y)\| \leq A\|y\|.$

For showing this lemma, we need to recall some results.

Let X and Y be two topological spaces. A multivalued function F from X to Y is called lower semi-continuous (l.s c) at $x_0 \in \operatorname{Dom}(F)$ if and only if for any open subset $V \subset Y$ such that $V \cap F(x_0) \neq \emptyset$, there exists a neighborhood U of x_0 such that $V \cap F(x) \neq \emptyset$ for all $x \in U$.

It is said be l.s.c. if it is l.s.c. at every point $x \in \operatorname{Dom} F$. This is equivalent to say that F^{-1} is an open map.

A continuous selection for F is a continuous function $f : X \longrightarrow Y$ such that for all $x \in X$, $f(x) \in F(x)$. The following theorem gives a sufficient condition for the existence of a continuous selection for F.

Theorem 4.3 (Michael [11]). *We suppose that X is paracompact, Y is a Banach space and F is l.s.c. whith non-empty closed convex values. Then F admits a continuous selection.*

Noting that every metrizable space is paracompact, we can therefore replace in the above theorem X paracompact by X metrizable.

Definition 4.4. Let X and Y be two normed linear spaces and let P be a convex process from X to Y. Let us set for all $x \in \operatorname{Dom} P$, $r(x) := \operatorname{Inf}\{\|y\| : y \in P(x)\}$. We define the norm of P by:

$$\|P\| = \operatorname{Sup}\{r(x) : \|x\| \leq 1 \text{ and } x \in \operatorname{Dom} P\}$$

The following theorem gives a necessary and sufficient condition so that P has a finite norm.

Theorem 4.5 (Robinson [14]). *Let X and Y be two normed linear spaces and let P be a convex process from X to Y. Then the following properties are equivalent:*

(i) *P has a finite norm.*

(ii) *P is l.s.c. at 0 as a mapping from $\operatorname{Dom} P$ to Y.*

Theorem 4.6 (Robinson [14]). *Let X and Y be two Banach spaces and let P be a closed convex process from X onto Y. Then P^{-1} is l.s.c.*

Proof of Lemma 4.2. By Theorem 4.6, P^{-1} is l.s.c. and by Theorem 4.5, $\|P^{-1}\|$ has a finite norm.

Let $S_Y = \{y \in Y/\|y\| = 1\}$. Let us set $G(y) = P^{-1}(y) \cap \{x \in X/\|x\| < \|P^{-1}\| + 1\}$ and $F(y) = \operatorname{cl} G(y)$. Then the multivalued map $y \longmapsto F(y)$ from S_Y to X verifies the assumptions of Theorem 4.3. Therefore, there exists a continuous map v from S_Y to X such that, $v(y) \in F(y)$ for all $y \in S_Y$. Consider the map $u : Y \longrightarrow X$ defined by:

$$u(y) = \begin{cases} \|y\| v\left(\frac{y}{\|y\|}\right) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Then u satisfies all conditions. Which completes the proof. $\square$

5. A submersion theorem

Definition 5.1. Let T be a topological space, X and Y two Banach spaces, U an open subset of $T \times X$, f a map from U to Y and $(t_0, x_0) \in U$. We say that f is strictly partially differentiable (s.p.d) with respect to the second variable at $(t_0, x_0) \in U$ if the partial differential $D_2 f(t_0, x_0)$ exists and if for all $\varepsilon > 0$ there exist a neighborhood V of t_0, and $\eta > 0$ such that:

$$\forall t \in V,\ \forall (x, x') \in (B_X(x_0, \eta))^2, \qquad \| f(t, x) - f(t, x') - D_2 f(t_0, x_0)(x - x')\| \leq \varepsilon \| x - x'\|.$$

Theorem 5.2. *Let T be a topological space, X and Y two Banach spaces, U be an open subset of $T \times X$, $f : U \longrightarrow Y$ a continuous map and $(t_0, x_0) \in U$. Let M (resp.N) be a closed subset of X (resp. of Y) which contains x_0 (resp. $y_0 = f(t_0, x_0)$). Assume that:*

(1) *f is s.p.d. with respect to the second variable at (t_0, x_0),*
(2) *M (resp. N) approximates continuously its Clarke tangent cone at x_0 (resp. y_0),*
(3) *$D_2 f(t_0, x_0) C_M(x_0) - C_N(y_0) = Y$.*

Then there exist a neighborhood Ω of (t_0, x_0, y_0) in $T \times M \times N$ and a continuous map φ from Ω to X such that:

(i) $\forall (t, x, y) \in \Omega,\ \varphi(t, x, y) \in M$ *and* $f(t, \varphi(t, x, y)) \in N$;
(ii) $\exists c > 0,\ \forall (t, x, y) \in \Omega,\ \|\varphi(t, x, y) - x\| \leq c\|f(t, x) - y\|$,

By applying the above Theorem to the application $((t, y), x) \rightsquigarrow f(t, x) - y + y_0$, we obtain:

Corollary 5.3. *Under the assumptions of Theorem 5.2, there exist a neighborhood Ω of $((t_0, y_0), (x_0, y_0))$ in $(T \times Y) \times (M \times N)$ and a continuous map ϕ from Ω to X such that:*

(i) $\forall (t, y_1, x, y_2) \in \Omega,\ \phi(t, y_1, x, y_2) \in M$ *and* $f(t, \phi(t, y_1, x, y_2)) \in y_1 + (N - y_0)$;
(ii) $\exists c > 0,\ \forall (t, y_1, x, y_2) \in \Omega,\ \|\phi(t, y_1, x, y_2) - x\| \leq c\|f(t, x) + y_0 - y_1 - y_2\|$.

By setting $\varphi(t, y) = \phi(t, y, x_0, y_0)$, we obtain:

Corollary 5.4. *Under the assumptions of the above theorem, there exist a neighborhood Ω of (t_0, y_0) and a continuous map φ from Ω to X such that:*

(i) $\forall (t, y) \in \Omega,\ \varphi(t, y) \in M$ *and* $f(t, \varphi(t, y)) \in y + (N - y_0)$;
(ii) $\exists c > 0,\ \forall (t, y) \in \Omega,\ \|\varphi(t, y) - x_0\| \leq c\|y - f(t, x_0)\|$.

Taking $N = \{y_0\}$, this corollary gives the following result which is useful for the resolution of nonlinear parametric equations with constraints.

Corollary 5.5. *Take the same notations as in the above theorem and suppose that:*

(1) f *is s.p.d. with respect to the second variable at* (t_0, x_0)

(2) M *approximates continuously its Clarke tangent cone at* x_0 *(resp.* y_0*),*

(3) $D_2 f(t_0, x_0) C_M(x_0) = Y$.

Then there exist a neighborhood Ω *of* (t_0, y_0) *and a continuous map* φ *from* Ω *to* X *such that:*

(i) $\forall (t, y) \in \Omega,\ \varphi(t, y) \in M$ *and* $f(t, \varphi(t, y)) = y$;

(ii) $\exists c > 0,\ \forall (t, y) \in \Omega\ \|\varphi(t, y) - x_0\| \leq c\|y - f(t, x_0)\|$.

In the particular case where $T = \{0\}$, the Corollary 5.3 yields the following result.

Corollary 5.6. *Let* X *and* Y *be two Banach spaces,* U *an open subset of* X, $f : U \longrightarrow Y$ *a continuous map and* $x_0 \in U$*. Let* M *(resp.* N*) a closed subset of* X *(resp. of* Y*) which contains* x_0 *(resp.* $y_0 = f(x_0)$*). Assume that:*

(1) f *is s.p.d. at* x_0,

(2) M *(resp.* N*) approximates continuously its Clarke tangent cone at* x_0 *(resp.* y_0*),*

(3) $Df(x_0) C_M(x_0) - C_N(y_0) = Y$.

Then there exist a neighborhood Ω *of* (x_0, y_0) *in* $M \times Y$ *and a continuous map from* Ω *to* X *such that:*

(i) $\forall (x, y) \in \Omega,\ g(x, y) \in M$ *and* $f(g(x, y)) \in y + (N - y_0)$;

(ii) $\exists c > 0,\ \forall (x, y) \in \Omega,\ \|g(x, y) - x\| \leq c\|y - f(x)\|$

As a consequence of this corollary, we have the following inverse function theorem for multivalued map with a continuous selection of inverse images.

Corollary 5.7. *Let* X *and* Y *be two Banach spaces,* F *be a closed multivalued function from* X *to* Y *and* $(x_0, y_0) \in \mathrm{graph}(F)$*. Assume that:*

(1) F *approximates continuously its Clarke derivative at* (x_0, y_0),

(2) *the Clarke derivative of* F *at* (x_0, y_0) *is surjective.*

Then there exist a neighborhood Ω *of* (x_0, y_0, y_0) *in* $\mathrm{Graph}(F) \times Y$ *and a continuous map* g *from* Ω *to* X *such that:*

(i) $\forall (x, y_1, y_2) \in \Omega,\ y_2 \in F(g(x, y_1, y_2))$;

(ii) $\exists c > 0,\ \forall (x, y_1, y_2) \in \Omega,\ \|g(x, y_1, y_2) - x\| \leq c\|y_2 - y_1\|$.

Proof of Theorem 5.2

(A) Proof of Theorem 5.2 in the particular case where $N = \{y_0\}$.
We have $D_2 f(t_0, x_0) C_M(x_0) = Y$. Therefore, by Lemma 4.2, there exists a continuous positively homogeneous map u from Y to $C_M(x_0)$ such that:

$$\begin{cases} \forall y \in Y, & y = D_2 f(t_0, x_0) u(y) \\ \exists A > 0,\ \forall y \in Y, & \|u(y)\| \leq A\|y\|. \end{cases} \tag{5.1}$$

Let us set $k := 1+\|D_2 f(t_0,x_0)\|$, $\varepsilon := \min\left(\frac{A}{2},\frac{1}{4(1+k)}\right)$, and $q := \varepsilon(1+k)$. Since U is open, f is s.p.d with respect to the second variable at (t_0,x_0) and M approximates continuously its Clarke tangent cone at x_0, there exist a neighborhood W of t_0, $\eta > 0$ and a continuous map g from $B_M(x_0,\eta) \times (C_M(x_0) \cap (\eta B_X))$ to M such that:

$$\begin{cases} \text{(i)} & W \times B_X(x_0,\eta) \subset U; \\ \text{(ii)} & \forall t \in W,\ \forall (x,x') \in [B_X(x_0,\eta)]^2, \\ & \qquad \|f(t,x) - f(t,x') - D_2 f(t_0,x_0)(x-x')\| \le \dfrac{\varepsilon}{A}\|x-x'\|; \\ \text{(iii)} & \forall (x,v) \in B_M(x_0,\eta) \times (C_M(x_0) \cap (\eta B_X)),\ \|g(x,v) - (x+v)\| \le \frac{\varepsilon}{A}\|v\|. \end{cases} \tag{5.2}$$

Which implies that:

$$\forall t \in W,\ \forall (x,x') \in [B_X(x_0,\eta)]^2,\ \|f(t,x) - f(t,x')\| \le k\|x-x'\|. \tag{5.3}$$

The continuity of f implies the existence of a neighborhood V of t_0 contained in W, and of $r \in \left]0,\frac{\eta}{4}\right[$ such that:

$$\forall (t,x) \in V \times B(x_0,r), \quad \|f(t,x) - f(t_0,x_0)\| \le \frac{\eta}{8A}. \tag{5.4}$$

Consider the sequence $(\varphi_n)_{n\in\mathbb{N}}$ defined from $V \times B_M(x_0,r)$ to M, by:

$$\forall n \in \mathbb{N},\ \forall (t,x) \in V \times B_M(x_0,r)$$
$$\begin{cases} \varphi_0(t,x) &= x \\ \varphi_{n+1}(t,x) &= g\left[\varphi_n(t,x), u\left(f(t_0,x_0) - f\left(t,\varphi_n(t,x)\right)\right)\right] \end{cases} \tag{5.5}$$

(1) Let us prove that φ_n is well defined, takes their values in $B_M(x_0,\frac{\eta}{2})$, and that

$$\forall (t,x) \in V \times B_M(x_0,r),\ \|f\left(t,\varphi_n(t,x)\right) - f(t_0,x_0)\| \le q^n\|f(t,x) - f(t_0,x_0)\|. \tag{5.6}$$

The result holds for $n = 0$. Suppose that it is verified for $k \le n$, and let us prove that it is still true for $n+1$.

(a) For all $(t,x) \in V \times B_M(x_0,r)$, we have $\varphi_n(t,x) \in B_M(x_0,\eta)$ and

$$\begin{aligned} \|u\left(f(t_0,x_0) - f\left(t,\varphi_n(t,x)\right)\right)\| &\le A\,\|f(t_0,x_0) - f\left(t,\varphi_n(t,x)\right)\| \\ &\le Aq^n\|f(t,x) - f(t_0,x_0)\| \quad \text{(from (5.6))} \\ &\le Aq^n\frac{\eta}{8A} \quad \text{(from (5.4))} \\ &\le \eta. \end{aligned}$$

Therefore

$$\left(\varphi_n(t,x)\,,\ u\left(f(t_0,x_0) - f\left(t,\varphi_n(t,x)\right)\right)\right) \in B_M(x_0,\eta) \times (C_M(x_0) \cap (\eta B_X))$$

which proves that φ_{n+1} is well defined.

(b) Let $(t, x) \in V \times B_M(x_0, r)$. Set for all $k \leq n$,

$$\overline{\varphi}_{k+1}(t, x) = \varphi_k(t, x) + u\left(f(t_0, x_0) - f\left(t, \varphi_k(t, x)\right)\right). \tag{5.7}$$

Using (5.2)(iii) and (5.1), we get

$$\|\varphi_{k+1}(t, x) - \overline{\varphi}_{k+1}(t, x)\| \leq \varepsilon \|f(t_0, x_0) - f\left(t, \varphi_k(t, x)\right)\|, \tag{5.8}$$

and using (5.7) and (5.1), we obtain

$$\|\overline{\varphi}_{k+1}(t, x) - \varphi_k(t, x)\| \leq A\|f(t_0, x_0) - f\left(t, \varphi_k(t, x)\right)\|. \tag{5.9}$$

Therefore

$$\begin{aligned}\|\varphi_{k+1}(t, x) - \varphi_k(t, x)\| &\leq (A + \varepsilon)\|f\left(t, \varphi_k(t, x)\right) - f(t_0, x_0)\| \\ &\leq (A + \varepsilon)q^k\|f(t, x) - f(t_0, x_0)\| \quad \text{(from (5.6))}.\end{aligned}$$

Then,

$$\|\varphi_{k+1}(t, x) - \varphi_k(t, x)\| \leq \frac{3}{2}Aq^k\|f(t, x) - f(t_0, x_0)\|. \tag{5.10}$$

Thus, we can write:

$$\begin{aligned}\|\varphi_{n+1}(t, x) - \varphi_0(t, x)\| &\leq \|\varphi_{n+1}(t, x) - \varphi_n(t, x)\| + ... + \|\varphi_1(t, x) - \varphi_0(t, x)\| \\ &\leq \frac{3}{2}A(q^n + ... + 1)\|f(t, x) - f(t_0, x_0)\| \\ &\leq \frac{3}{2}A(\frac{1}{4^n} + ... + 1)\|f(t, x) - f(t_0, x_0)\| \quad (\text{here } q \in]0, \tfrac{1}{4}[).\end{aligned}$$

Which yields

$$\forall(t, x) \in V \times B_M(x_0, r),\ \|\varphi_{n+1}(t, x) - x\| \leq 2A\|f(t, x) - f(t_0, x_0)\|. \tag{5.11}$$

Thanks to (5.4), we deduce that $\|\varphi_{n+1}(t, x) - x\| \leq \frac{\eta}{4}$, and since $\|x - x_0\| \leq r \leq \frac{\eta}{4}$, we have $\|\varphi_{n+1}(t, x) - x_0\| \leq \frac{\eta}{2}$. Hence φ_{n+1} takes their values in $B_M(x_0, \frac{\eta}{2})$.

(c) We prove that, for all $(t, x) \in V \times B_M(x_0, r)$,

$$\|f\left(t, \varphi_{n+1}(t, x)\right) - f(t_0, x_0)\| \leq q^{n+1}\|f(t, x) - f(t_0, x_0)\|.$$

We have

$$\begin{aligned}\|\overline{\varphi}_{n+1}(t, x) - x_0\| &\leq \|\overline{\varphi}_{n+1}(t, x) - \varphi_n(t, x)\| + \|\varphi_n(t, x) - x_0\| \\ &\leq A\|f(t_0, x_0) - f\left(t, \varphi_n(t, x)\right)\| + \frac{\eta}{2} \\ &\leq Aq^n\|f(t, x) - f(t_0, x_0)\| + \frac{\eta}{2} \quad \text{(from (5.6))} \\ &\leq Aq^n\frac{\eta}{8A} + \frac{\eta}{2} \quad \text{(from (5.4))} \\ &\leq \eta. \quad (0 < q < \frac{1}{4})\end{aligned}$$

Then, $\overline{\varphi}_{n+1}(t,x) \in B_X(x_0,\eta)$. On the other hand, we get

$$u\left(f(t_0,x_0) - f\left(t,\varphi_n(t,x)\right)\right) = \overline{\varphi}_{n+1}(t,x) - \varphi_n(t,x),$$

using (5.1), we obtain

$$f(t_0,x_0) = f\left(t,\varphi_n(t,x)\right) + D_2 f(t_0,x_0)\left(\overline{\varphi}_{n+1}(t,x) - \varphi_n(t,x)\right).$$

Now, from (5.2)(ii) and (5.9) we have

$$\begin{aligned}\|f\left(t,\overline{\varphi}_{n+1}(t,x)\right) - f(t_0,x_0)\| &\leq \frac{\varepsilon}{A}\|\overline{\varphi}_{n+1}(t,x) - \varphi_n(t,x)\| \\ &\leq \varepsilon\, \|f\left(t,\varphi_n(t,x)\right) - f(t_0,x_0)\|.\end{aligned}$$

Consequently,

$$\begin{aligned}\|f\left(t,\varphi_{n+1}(t,x)\right) - f(t_0,x_0)\| &\leq \|f\left(t,\varphi_{n+1}(t,x)\right) - f\left(t,\overline{\varphi}_{n+1}(t,x)\right)\| \\ &\quad + \| f\left(t,\overline{\varphi}_{n+1}(t,x)\right) - f(t_0,x_0)\| \\ &\leq \|f\left(t,\varphi_{n+1}(t,x)\right) - f\left(t,\overline{\varphi}_{n+1}(t,x)\right)\| \\ &\quad + \varepsilon\|f\left(t,\varphi_n(t,x)\right) - f(t_0,x_0)\|.\end{aligned}$$

From (5.3) and (5.8) we have

$$\begin{aligned}\|f\left(t,\varphi_{n+1}(t,x)\right) - f\left(t,\overline{\varphi}_{n+1}(t,x)\right)\| &\leq k\|\varphi_{n+1}(t,x) - \overline{\varphi}_{n+1}(t,x) \\ &\leq k\varepsilon\|f(t,\varphi_n(t,x)) - f(t_0,x_0)\|,\end{aligned}$$

then

$$\begin{aligned}\|f\left(t,\varphi_{n+1}(t,x)\right) - f(t_0,x_0)\| &\leq \varepsilon(1+k)\| f\left(t,\varphi_n(t,x)\right) - f(t_0,x_0)\| \\ &\leq q^{n+1}\| f(t,x) - f(t_0,x_0)\|.\end{aligned}$$

(2) From (5.10) and (5.4) we have

$$\|\varphi_{n+1}(t,x) - \varphi_n(t,x)\| \leq \frac{3}{16}\eta q^n.$$

Then $(\varphi_n)_{n\in\mathbb{N}}$ converges uniformly to a continuous function φ taking their values in $B_M(x_0,\eta)$. Moreover from (5.6) we have:

$$\forall (t,x) \in V \times B_M(x_0,r),\ f\left(t,\varphi(t,x)\right) = f(t_0,x_0),$$

and from (5.11) we have:

$$\forall (t,x) \in V \times B_M(x_0,r),\ \|\varphi(t,x) - x\| \leq 2A\|f(t,x) - f(t_0,x_0)\|$$

Thus φ satisfies all the conditions and this completes the proof. □

(B) Proof of Theorem 5.2 in the general case. The map

$$\begin{array}{rccl} \overline{f}: & T \times (X \times Y) & \longrightarrow & Y \\ & (t,(x,y)) & \rightsquigarrow & f(t,x)-y \end{array}$$

is defined and continuous in a neighborhood of $(t_0,(x_0,y_0))$ and s.p.d with respect to the second variable (x,y) at $(t_0,(x_0,y_0))$. On the other hand, $M \times N$ approximates continuously its Clarke tangent cone $C_{M\times N}(x_0,y_0) = C_M(x_0) \times C_N(y_0)$ at (x_0,y_0) and $D_2\overline{f}\,(t_0,(x_0,y_0))\,(C_M(x_0)\times C_N(y_0)) = Y$, then there exist a neigborhood Ω of $(t_0,(x_0,y_0))$ in $T \times (M \times N)$ and a continuous map (φ,ψ) from Ω to $(X \times Y)$ such that:

- $\forall (t,x,y) \in \Omega,\ (\varphi(t,x,y),\psi(t,x,y)) \in M \times N$ and $\overline{f}\,(t,\varphi(t,x,y),\psi(t,x,y)) = 0$
- $\exists c > 0,\ \forall (t,x,y) \in \Omega,\ \|\varphi(t,x,y) - x\| \le c\|\overline{f}\,(t,x,y)\|$ and $\|\psi(t,x,y) - y\| \le c\|\overline{f}\,(t,x,y)\|$.

Which implies that:

- $\forall (t,x,y) \in \Omega,\ \varphi(t,x,y) \in M$ and $f\,(t,\varphi(t,x,y)) \in N$
- $\exists c > 0, \forall (t,x,y) \in \Omega,\ \|\varphi(t,x,y) - x\| \le c\|f(t,x) - y\|$

□

Proof of Corollary 5.7. Let $M = \mathrm{Graph}(F)$ and let π be the map $(x,y) \rightsquigarrow y$ from $X \times Y$ to Y. We have $\pi\,(C_M(x_0,y_0)) = Y$, then by Corollary 5.6, there exists a neighborhood Ω of $((x_0,y_0),y_0)$ in $M \times Y$ and a continuous map (g_1,g_2) from Ω to $X \times Y$ such that:

- $\forall (x,y_1,y_2) \in \Omega,\ (g_1(x,y_1,y_2),g_2(x,y_1,y_2)) \in M$ and $\pi\,(g_1(x,y_1,y_2),g_2(x,y_1,y_2)) = y_2$,
- $\exists c > 0, \forall (x,y_1,y_2) \in \Omega,\ \|g_1(x,y_1,y_2) - x\| \le c\|y_1 - y_2\|$ and $\|g_2(x,y_1,y_2) - y_1\| \le c\|y_1 - y_2\|$.

By setting $g(x,y_1,y_2) = g_1(x,y_1,y_2)$ we deduce that:

- $\forall (x,y_1,y_2) \in \Omega,\ y_2 \in F\,(g(x,y_1,y_2))$
- $\exists c > 0, \forall (x,y_1,y_2) \in \Omega,\ \|g(x,y_1,y_2) - x\| \le c\|y_1 - y_2\|$.

□

Remark 5.8. If we replace in the Theorem 5.2 and its corollaries the hypothesis:

M (resp. N) approximates continuously its Clarke tangent cone at x_0 (resp. y_0)

by the following weak hypothesis:

M (resp. N) approximates strictly its Clarke tangent cone at x_0 (resp. y_0),

then except the continuity of the solution, the same conclusions hold. In particular, we have the following result.

Theorem 5.9. *Let X and Y be two Banach spaces, F a closed multivalued function from X to Y and $(x_0,y_0) \in \mathrm{graph}(F)$. Assume that:*

(1) *F approximates strictly its Clarke derivative at (x_0,y_0),*
(2) *the Clarke derivative of F at (x_0,y_0) is surjective.*

Then there exist a neighborhood Ω of (x_0,y_0,y_0) in $\mathrm{Graph}(F) \times Y$ and a map g from Ω to X such that:

(i) $\forall x, y_1, y_2) \in \Omega,\ y_2 \in F\,(g(x, y_1, y_2))$;
(ii) $\exists c > 0,\ \forall(x, y_1, y_2) \in \Omega,\ \|g(x, y_1, y_2) - x\| \le c\|y_2 - y_1\|$.

We deduce the following corollary:

Corollary 5.10. *Under the assumptions of above theorem, y_0 belongs to the interior of the image of F and F^{-1} is pseudo-lipschitzian around (y_0, x_0) (i.e there exist neighborhoods U of x_0, V of y_0 and a constant $c > 0$ such that: $\forall y_1, y_2 \in V,\ F^{-1}(y_1) \cap V \subset F^{-1}(y_1) + c\|y_2 - y_1\|B_X$).*

Indeed: there exist a neighborhood Ω of (x_0, y_0, y_0) in $(\mathrm{Graph}\, F) \times Y$ and a map g from Ω to X such that

- $\forall(x, y_1, y_2) \in \Omega,\ y_2 \in F\,(g(x, y_1, y_2))$
- $\exists c > 0, \forall(x, y_1, y_2) \in \Omega,\ \|g(x, y_1, y_2) - x\| \le c\|y_1 - y_2\|$.

Let $r > 0$ be such that $\left(B_X(x_0, r) \times [B_Y(y_0, r)]^2\right) \cap ((\mathrm{Graph}\, F) \times Y) \subset \Omega$. We have for all y in $B_Y(y_0, r),\quad y \in F\,(g(x_0, y_0, y))$. Therefore $B_Y(y_0, r)$ is contained in ImF and for all $(y_1, y_2) \in [B_Y(y_0, r)]^2$and $x_1 \in F^{-1}(y_1) \cap B_X(x_0, r),\ g(x_1, y_1, y_2) \in F^{-1}(y_2)$ and $\|g(x_1, y_1, y_2) - x_1\| \le c\|y_2 - y_1\|$. Then $F^{-1}(y_1) \cap B_X(x_0, r) \subset F^{-1}(y_2) + c\|y_2 - y_1\|B_X$.

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