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# Upright-Quad Drawing of *st*-Planar Learning Spaces

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**Abstract**

We consider graph drawing algorithms for learning spaces, a type of *st*-oriented partial cube derived from an antimatroid and used to model states of knowledge of students. We show how to draw any *st*-planar learning space so all internal faces are convex quadrilaterals with the bottom side horizontal and the left side vertical, with one minimal and one maximal vertex. Conversely, every such drawing represents an *st*-planar learning space. We also describe connections between these graphs and arrangements of translates of a quadrant. Our results imply that an antimatroid has order dimension two if and only if it has convex dimension two.

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# 1 Introduction

A *partial cube* is a graph that can be given the geometric structure of a hypercube, by assigning the vertices bitvector labels in such a way that the graph distance between any pair of vertices equals the Hamming distance of their labels. Partial cubes can be used to describe benzenoid systems in chemistry [13], weak or partial orderings modeling voter preferences in multi-candidate elections [12], integer partitions in number theory [8], flip graphs of point set triangulations [9], and the hyperplane arrangements familiar to computational geometers [15, 7]; see [10] for additional applications. In previous work we found algorithms for drawing arbitrary partial cubes, as well as partial cubes that have drawings as planar graphs with symmetric faces [5].

Here we consider graph drawing algorithms for *learning spaces*, a type of partial cube that can be derived from an antimatroid and that has been used to model the potential states of knowledge of students [4]. These graphs can be large; Doignon and Falmagne [4] write “the number of knowledge states obtained for a domain containing 50 questions in high school mathematics ranged from about 900 to a few thousand.” However, visualization of these graphs may play an important role in human refinement of these types of models. Thus, it is important to have efficient drawing techniques that can take advantage of the special properties of learning spaces.

Our goal in graph drawing algorithms for special graph families is to combine the standard graph drawing aesthetic criteria of vertex separation, area, etc., with a drawing style from which the specific graph structure we are interested in is visible: that is, the drawing should not just be good looking, it should also be informative. The most fundamental piece of information we are looking to convey about these graphs is membership in the chosen graph family, so, ideally, the drawing should be of a type that exists only for the graph family we are concerned with; if this is the case, membership of the graph in the family may be verified by visual inspection of the drawing. For instance, if a tree is drawn in the plane with no edge crossings, the fact that it is a tree may be seen visually from the fact that the drawing is connected and does not separate any region of the plane from any other. In our previous work [5], the existence of a planar drawing in which all faces are symmetric implies that the graph of the drawing is a partial cube, although not all partial cubes have such drawings. Another result of this type is our proof [11] that the graphs having delta-confluent drawings are exactly the distance-hereditary graphs.

The learning spaces considered in this paper are directed acyclic graphs with a single source and a single sink. We would like to draw learning spaces in general, but, as a first step towards this goal, in this work we consider only those learning spaces that can be drawn in the plane without crossings. Although we do not expect most large learning spaces to be planar, our results may nevertheless be useful in allowing planar subsets of a larger nonplanar learning space to be visualized. In particular, as our results show, if $\pi_1$ and $\pi_2$ are any two paths from source to sink in a learning space (corresponding to two different orders in which a set of topics may be learned), the subgraph formed by unions of prefixes of the two paths is itself a learning space, a planar subgraph of the original larger learning space, and our algorithms may be used to find a planar drawing of this subgraph within which alternative paths may be visualized.

Since the source and sink are particularly important vertices of any learning space, we wish them to have a prominent position in the drawing. We confine our attention, therefore, to *st*-planar learning spaces, those for which there exists a planar embedding with the source and sink on the same face. As we show, such graphs can be characterized by drawings of a very specific type: Every *st*-planar learning space has a dominance drawing in which all internal faces are convex quadrilaterals with the bottom side horizontal and the left side vertical. We call such a drawing an *upright-quad drawing*, and we describe linear time algorithms for finding an upright-quad drawing of any *st*-planar learning space. Conversely, every upright-quad drawing comes from an *st*-planar learning space in this way.

## 2 Learning Spaces

Doignon and Falmagne [4] consider sets of concepts that a student of an academic discipline might learn, and define a *learning space* to be a family $\mathcal{F}$ of sets modeling the possible states of knowledge that a student could have. Some concepts may be learnable only after certain prerequisites have been learned, so $\mathcal{F}$ may not be a power set. However, there may be more than one way of learning a concept, and therefore more than one set of prerequisites the knowledge of which allows a concept to be learned. We formalize these intuitive concepts mathematically with the following axioms:

**[L1]** If $S \in \mathcal{F}$ and $S \neq \emptyset$, then there exists $x \in S$ such that $S \setminus \{x\} \in \mathcal{F}$. That is, any state of knowledge can be reached by learning one concept at a time.

**[L2]** If $S$, $S \cup \{x\}$, and $S \cup \{y\}$ belong to $\mathcal{F}$, then $S \cup \{x, y\} \in \mathcal{F}$. That is, learning one concept cannot interfere with the ability to learn a different concept.

These axioms characterize families $\mathcal{F}$ that form *antimatroids* [14, Lemma III.1.2].

A class of examples of set families satisfying axioms L1 and L2 can be constructed from any partial order by considering the *lower sets* of the order: sets $S$ of elements such that, if $x \leq y$ and $y \in S$, then $x \in S$ (Figure 1). Removing any maximal element of a lower set produces another lower set (L1), and whenever two unrelated elements $x$ and $y$ both have all their predecessors in some lower set, both may be added independently to produce another lower set (L2). The partial order defines a *prerequisite* structure on the concepts of the learning space: a given concept $x$ cannot be learned until all of its prerequisites (the concepts that precede it in the partial order) have already been learned.

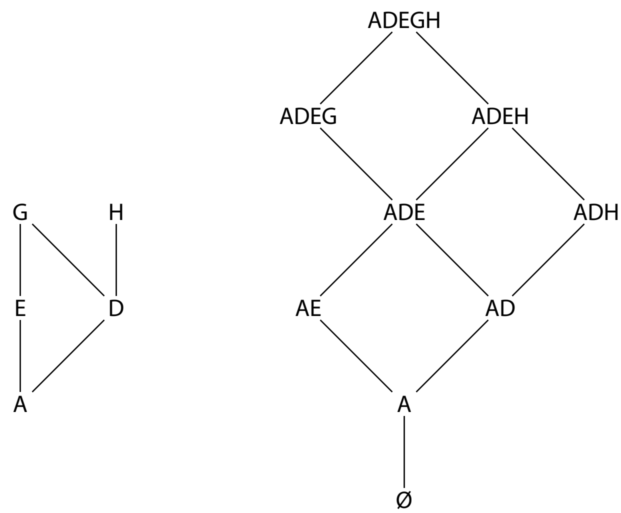



Figure 1: Left: The Hasse diagram of a partial order on five elements. Right: The learning space derived from the lower sets of this partial order.

Another example, that does not come from a partial order in this way, is the set family $\{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$, shown in Figure 2. It can be verified by inspection that all the nonempty sets in this family have an element that can be removed to produce another set in the family (L1). To verify axiom L2, we need only consider the possibilities that $S$ is one of $\emptyset$, $\{a\}$, and $\{c\}$, for there are no two elements $x$ and $y$ disjoint from any other set in the family. If $S = \emptyset$, and $S \cup \{x\}$ and $S \cup \{y\}$ both belong to $\mathcal{F}$, then $\{x,y\}$ can only be $\{a,c\}$, and $S \cup \{x,y\} \in \mathcal{F}$. And, if $|S| = 1$, and $x$ and $y$ are two elements disjoint from $S$, then the conclusion of the axiom must hold, for $S \cup \{x,y\} = \{a,b,c\} \in \mathcal{F}$. Thus, in all cases axiom L2 holds as well. In this example, a strict prerequisite relationship does not hold among the concepts: one may learn concept $b$ either after learning $a$ or after learning $c$, but it is not necessary to learn both $a$ and $c$ before learning $b$.

Antimatroids also arise in other contexts than learning; for instance, let $P$ be a finite set of points in $\mathbb{R}^d$, and let $\mathcal{F}$ be the family of intersections of $P$ with complements of convex bodies. Then $\mathcal{F}$ is an antimatroid [14, p. 20].

We define a *learning space* to be a graph having one vertex for each set in an antimatroid $\mathcal{F}$, and with a directed edge from each set $S \in \mathcal{F}$ to each set $S \cup \{x\} \in \mathcal{F}$. Graphs of this type are shown for the examples described above in Figures 1 and 2. If $U = \bigcup \mathcal{F}$, we say that the graph derived from $\mathcal{F}$ is a learning space *over* $U$.

In the remainder of this section we outline some standard results from antimatroid theory that we will need in later parts of the paper.

**Lemma 1** *If $\mathcal{F}$ satisfies axioms L1 and L2, and $K \subset L$ are two sets in $\mathcal{F}$, with*

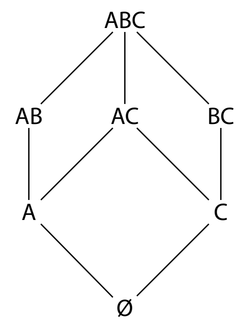



Figure 2: A learning space that cannot be derived from a partial order.

*$|L \setminus K| = n$, then there is a chain of sets $K_0 = K \subset K_1 \subset \cdots \subset K_n = L$, all belonging to $\mathcal{F}$, such that $K_i = K_{i-1} \cup \{q_i\}$ for some $q_i$.*

**Proof:** We use induction on $|K| + |L|$. If $K$ is empty, let $x$ be given by axiom L1 for $S = L$, and combine $q_n = x$ with the chain formed by induction for $K$ and $L \setminus \{x\}$. Otherwise, let $x$ be as given by axiom L1 for $S = K$, and form by induction a chain from $K \setminus \{x\}$ to $L$. By repeatedly applying axiom L2 we may add $x$ to each member of this chain not already containing it, forming a chain with one fewer step from $K$ to $L$. □

Similar repetitive applications of axiom L2 to the chain resulting from Lemma 1 proves the following:

**Lemma 2** *If $\mathcal{F}$ satisfies axioms L1 and L2, and $K \subset L$ are two sets in $\mathcal{F}$, with $K \cup \{q\} \in \mathcal{F}$ and $q \notin L$, then $L \cup \{q\} \in \mathcal{F}$.*

The following lemma may be found in Cosyn and Usun [1]; see also [10, Theorems 4.2.1].

**Lemma 3** *Let $\mathcal{F}$ satisfy the conclusions of Lemmas 1 and 2. Then the union of any two members of $\mathcal{F}$ also belongs to $\mathcal{F}$, and $\mathcal{F}$ is* well-graded*; that is, that any two sets in $\mathcal{F}$ can be connected by a sequence of sets, such that any two consecutive sets in the sequence differ by a single element and the length of the sequence equals the size of the symmetric difference of the two sets.*

Lemma 3 implies that any learning space $G$ is a partial cube when viewed as an undirected graph. However, not every partial cube comes from a learning space in this way; for instance, a four-vertex star $K_{1,3}$ is a partial cube but is not the graph of any learning space.

Recall that a graph is *st-oriented* (or has a *bipolar orientation*) if it is a directed acyclic graph (DAG) with a single source and a single sink.

**Lemma 4** *Any learning space is st-oriented.*

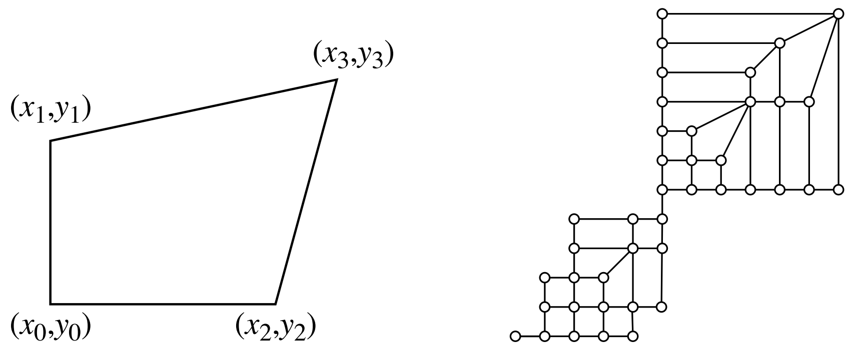



Figure 3: Left: An upright quadrilateral. Right: An upright-quad drawing.

**Proof:** As all edges are oriented from smaller sets to larger ones, there can be no cycle in any learning space; therefore, learning spaces are DAGs. Axiom 1 implies that the empty set belongs to $\mathcal{F}$, and that any other set has an incoming edge; that is, the empty set forms the unique source in $G$. Closure under unions implies that $\bigcup \mathcal{F}$ is the unique sink in $G$. □

In this paper we are particularly concerned with learning spaces for which the *st*-orientation is compatible with a planar drawing of the graph, in that the source and sink can both be placed on the outer face of a planar drawing. We call a graph admitting such a drawing an *st-planar learning space*.

# 3 Upright-Quad Drawings

In any point set in the plane, we say that $(x, y)$ is *minimal* if no point $(x', y')$ in the set has $x' < x$ or $y' < y$, and *maximal* if no point $(x', y')$ in the set has $x' > x$ or $y' > y$.

We define an *upright quadrilateral* to be a convex quadrilateral with a unique minimal vertex and a unique maximal vertex, such that the edges incident to the minimal vertex are horizontal and vertical. That is, the slopes of these two edges must be zero and infinite, respectively. The slopes of the other two edges must be nonnegative, but are otherwise not constrained.

Any upright quadrilateral can be given coordinates as the convex hull of four vertices $\{(x_i, y_i) \mid 0 \leq i < 4\}$ where $x_0 = x_1 < x_2 \leq x_3$ and $y_0 = y_2 < y_1 \leq y_3$ (Figure 3(left)). We define the *bottom edge* of an upright quadrilateral to be the horizontal edge incident to the minimal vertex, the *left edge* to be the vertical edge incident to the minimal vertex, and the *top edge* and *right edge* to be the edges opposite the bottom and left edges respectively.

We define an *upright-quad drawing* of a graph $G$ to be a placement of the vertices of the graph in the plane, with the following properties:

**[U1]** The placement forms a planar straight line drawing. That is, any two vertices are assigned distinct coordinates, and if the edges of $G$ are drawn

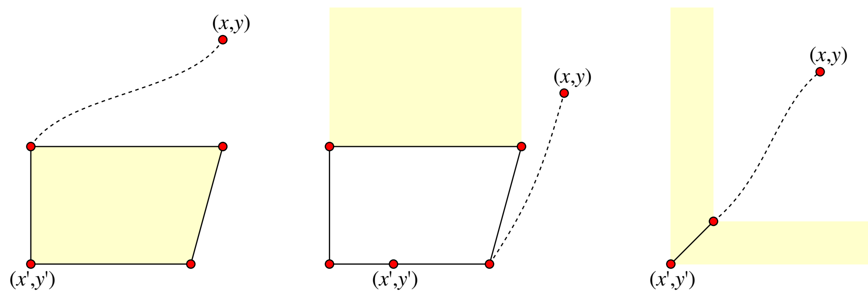



Figure 4: Case analysis for Lemma 6. In each case the shaded region denotes a part of the plane which must be empty of vertices, from which we can deduce by induction the existence of a path (dashed) connecting one of the neighbors of $(x', y')$ to $(x, y)$.

as straight line segments then no two edges intersect except at their endpoints.

**[U2]** There is a unique vertex of $G$ that is the minimal point among the locations of its neighbors in $G$, and a unique vertex of $G$ that is the maximal point among the locations of its neighbors in $G$.

**[U3]** Every interior face of the drawing is an upright quadrilateral, the sides of which are edges of the drawing.

In an upright-quad drawing, all edges connect a pair of points $(x, y)$ and $(x', y')$ with $x' \le x$ and $y' \le y$; if we orient each such edge from $(x', y')$ to $(x, y)$ then the resulting graph is directed acyclic with a unique source and sink. As we now show, with this orientation the drawing is a *dominance drawing*: that is, the dominance relation in the plane and the reachability relation in the graph coincide.

**Lemma 5** *Let $\ell$ be a vertical line, and let $S$ be the sequence of edges of an upright-quad drawing that contain points of $\ell$ in their relative interiors, sorted by y-coordinate. Then the projections of the edges in $S$ onto the $x$ axis form a nested sequence of intervals.*

**Proof:** The convex hull of each two successive edges of the sequence must belong to an internal face of the drawing, for if it were an external face then there would have to be more than one minimal or maximal vertex of the drawing, violating axiom U2. Nesting of the $x$ intervals follows from the fact that the top edge of any upright quadrilateral projects to an interval that equals or contains the projection of the bottom edge. □

**Lemma 6** *For any two vertices $(x', y')$ and $(x, y)$ in an upright-quad drawing, $(x' \le x) \wedge (y' \le y)$ if and only if there exists a directed path in the orientation specified above from $(x', y')$ to $(x, y)$.*

**Proof:** In one direction, if there exists a directed path from $(x', y')$ to $(x, y)$, then each edge in the path steps from a vertex to another vertex that dominates it, and the result holds by transitivity of dominance.

In the other direction, suppose that $(x, y)$ dominates $(x', y')$; we must show the existence of a directed path from $(x', y')$ to $(x, y)$. To do so, we show that we can find an outgoing edge to another vertex dominated by $(x, y)$; the result follows by induction on the number of vertices. First consider the case that $(x', y')$ is the minimal corner of some upright quadrilateral of the drawing (Figure 4, left). In this case there exist both horizontal and vertical outgoing edges from $(x', y')$. $(x, y)$ cannot belong to the bounding rectangle of these two edges, for if it did we could not use them as part of an empty upright quadrilateral. The points dominating at least one of the two neighbors of $(x', y')$ are exactly the points dominating $(x', y')$ that are not in this bounding rectangle, so at least one of the two edges leads to another vertex that is also dominated by $(x, y)$.

In the second case, suppose $(x', y')$ belongs to the bottom side of some upright quadrilateral of the drawing but is not the minimal vertex of that face (Figure 4, center). By Lemma 5, if $(x, y)$ and the top edge of the quadrilateral above $(x', y')$ are projected onto a common horizontal line, the projection of $(x, y)$ cannot lie interior to the projection of the edge, so the horizontal outgoing edge from $(x', y')$ leads to a vertex that is also dominated by $(x, y)$. The case that $(x', y')$ belongs to the left side of an upright quadrilateral but is not its minimal vertex is symmetric to this one.

Finally, suppose that $(x', y')$ is not on the bottom or left side of any upright quadrilateral (Figure 4, right). Then there can only be a single edge outgoing from $(x', y')$, and (again by Lemma 5 and by a symmetric form of the Lemma in which the coordinate axes are exchanged) there can be no interior faces of the drawing directly above or to the right of this edge. Thus, again, $(x, y)$ cannot project into the interior of the projection of this edge in either coordinate axis, so this edge leads to a vertex that is also dominated by $(x, y)$. □

As is well known to the graph drawing community [2, 3], a dominance drawing exists for any *st*-oriented plane graph in which the source $s$ and sink $t$ of the orientation belong to the outer face of the plane embedding; such a graph is known as an *st-planar graph*. However, *st*-planar dominance drawings may have faces that are not upright quadrilaterals, so not every *st*-planar dominance drawing is an upright-quad drawing.

# 4 Arrangements of Quadrants

We follow [10] in defining a *weak pseudoline arrangement* to be a collection of curves in the plane, each of which is the image of a line under some homeomorphism of the plane, such that any two curves have at most one point of intersection and any point of intersection is a proper crossing.

Consider a collection of convex wedges in the plane, all translates of each other. If no two wedges have boundaries on the same line, the boundary curves

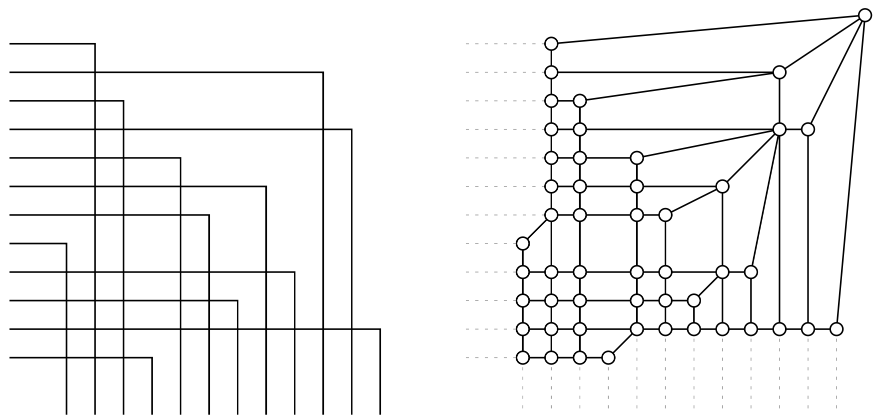


Figure 5: Left: an arrangement of quadrants. Right: the region graph of the arrangement, drawn with each vertex (except the top right one) at the maximal point of its region.

of the wedges form a weak pseudoline arrangement, for any two translates of the same wedge can only meet in a single crossing point.

By an appropriate linear transformation, we may transform our wedges to any desired orientation and convex angle, without changing the combinatorics of their arrangement. For later convenience, we choose a standard form for such arrangements in which each wedge is a translate of the negative quadrant $\{(x, y) \mid x, y \leq 0\}$ (Figure 5(left)). For such wedges, the condition that no two quadrants share a boundary line is equivalent to all translation vectors having distinct $x$ and $y$ coordinates. We call an arrangement of translated negative quadrants satisfying this distinctness condition an *arrangement of quadrants*. We refer to the curves of the arrangement, and to the wedges they form the boundaries of, interchangeably.

As with any arrangement of curves, we may define a *region graph* that is the planar dual of the arrangement: it has one vertex per region of the arrangement, with two vertices adjacent whenever the corresponding regions are adjacent across a nonzero length of curve of the arrangement. For our arrangements of quadrants, it is convenient to draw the region graph with each region's vertex in the unique maximal point of its region, except for the upper right region which has no maximal point. We draw the vertex for the upper region at any point with $x$ and $y$ coordinates strictly larger than those of any curve in our arrangement. The resulting drawing is shown in Figure 5(right).

**Theorem 1** *The placement of vertices described above produces an upright-quad drawing for the region graph of any arrangement of quadrants.*

**Proof:** The drawing's edges consist of all finite segments of the arrangement curves, together with diagonal segments connecting corners of arrangement curves within a region to the region's maximal point; therefore it is planar.

Each finite region of the arrangement is bounded above and to the right by a quadrant, either a single curve of the arrangement or the boundary of the intersection of two of the wedges of the arrangement. Each finite region is also bounded below and to the left by a staircase formed by a union of wedges of the arrangement; the drawing's edges subdivide this region into upright quadrilaterals by diagonals connecting the concave corners of the region to its maximal point. A similar sequence of upright quadrilaterals connects the staircase formed by the union of all arrangement wedges to the point representing the upper right region, which is the unique maximal vertex of the drawing. The unique minimal vertex of the drawing represents the region formed by the intersection of all arrangement wedges. Thus, all requirements of an upright-quad drawing are met. □

**Theorem 2** *The region graph of any arrangement of quadrants can be oriented to represent an st-planar learning space.*

**Proof:** We associate with each vertex of the region graph the set of wedges that do not contain any point of the region corresponding to the vertex. Each region other than the one formed by intersecting all wedges of the arrangement (associated with the empty set) has at least one arrangement curve on its lower left boundary; crossing that boundary leads to an adjacent region associated with a set of wedges omitting the one whose boundary was crossed; therefore axiom L1 of a learning space is satisfied.

If a region of the arrangement, associated with set $S$, has a single arrangement curve $c$ as its upper right boundary, then all supersets of $S$ associated with other regions are also supersets of $S \cup \{c\}$. Thus, in this case, there can be no two distinct sets $S \cup \{x\}$ and $S \cup \{y\}$ in the family of sets associated with the region graph, and axiom L2 of a learning space is satisfied vacuously.

On the other hand, if a region $r$ of the arrangement, associated with set $S$, has curve $x$ as its upper boundary and curve $y$ as its right boundary, then the only sets in the family formed by adding a single element to $S$ can be $S \cup \{x\}$ and $S \cup \{y\}$. In this case, the region diagonally opposite $r$ across the vertex where $x$ and $y$ meet is associated with the set $S \cup \{x, y\}$ and again axiom L2 of a learning space is met. □

Not all upright-quad drawings have the precise geometric placement as the one described here. For instance, a single square is itself an upright-quad drawing, but not one formed in this way: the method described above would instead place the upper right vertex of the drawing farther above and to the right of the square's upper right corner. Nevertheless, as we describe in the rest of the paper, Theorems 1 and 2 have converses, in that any *st*-planar learning space can be given an upright-quad drawing and any upright-quad drawing is combinatorially equivalent to the region graph of an arrangement of quadrants. Thus, these three seemingly different concepts, *st*-planar learning spaces, upright-quad drawings, and region graphs of arrangements of quadrants, are shown to be three faces of the same underlying mathematical objects.

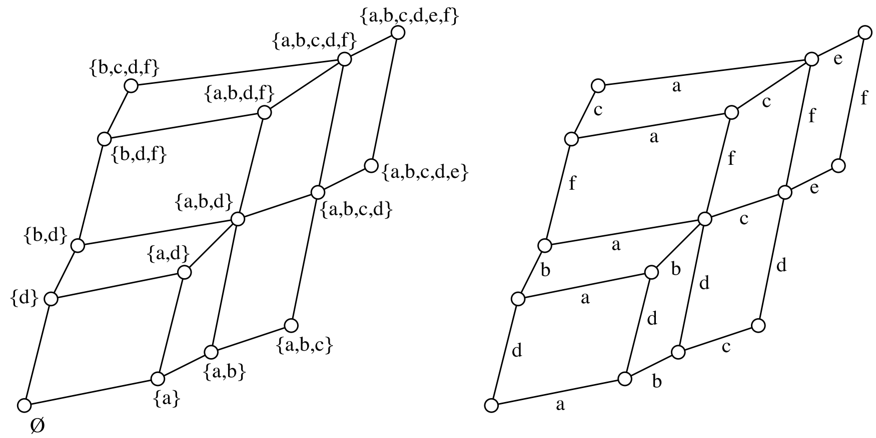



Figure 6: An *st*-planar learning space.

# 5 Drawing *st*-Planar Learning Spaces

As we show in this section, every *st*-planar learning space has an upright-quad drawing. Our algorithmic results will assume that an *st*-planar embedding of the given graph has already been determined; such an embedding may be found in linear time by augmenting the graph by an edge connecting the source and sink, applying any linear-time planar embedding algorithm, inverting the drawing if necessary so that the added edge appears on the outer face of the drawing, and removing the added edge.

An example of an *st*-planar learning space is shown in Figure 6; in the left view, the vertices of a dominance drawing of the graph are labeled by the corresponding sets in the family $\mathcal{F}$, while on the right view, each edge is labeled by the single element by which the sets at the two ends of the edge differ. However, not all planar learning spaces are *st*-planar; Figure 7 shows an example of a learning space that is planar but not *st*-planar.

**Lemma 7** *Let G be an st-planar learning space. Then every interior face of G is a quadrilateral, with equal labels on opposite pairs of edges.*

**Proof:** Let $b$ be the bottom vertex of any interior face $f$, with outgoing edges to $b \cup \{x\}$ and $b \cup \{y\}$. Then by axiom L2 of learning spaces, $G$ must contain a vertex $b \cup \{x, y\}$. This vertex must be the top vertex of $f$, for otherwise the edges from $b \cup \{x\}$ to $b \cup \{x, y\}$ and from $b \cup \{y\}$ to $b \cup \{x, y\}$ would have to pass above the top vertex, implying a subset relationship from the top vertex to $b \cup \{x, y\}$, which is absurd. $\square$

Define a *zone* of a label $x$ in an *st*-planar learning space $G$ to be the set of interior faces containing edges labeled by $x$. (We will later see a parallel definition of a zone in an upright-quad drawing, as depicted in Figure 12(left).) By Lemma 7, zones consist of chains of faces linked by opposite pairs of edges.

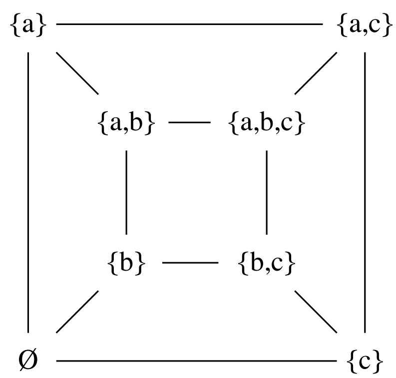



Figure 7: A learning space that is planar but not *st*-planar (the power set on three elements).

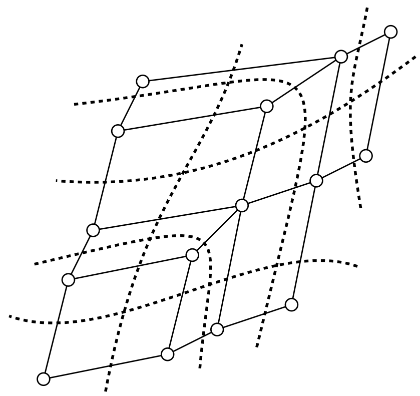
We may form a curve arrangement $\mathcal{A}(G)$ from an *st*-planar graph $G$ by drawing a curve through each face of each zone, crossing only edges of $G$ with the label of the zone. Within each face, there are two curves, which we may draw in such a way that they cross once; they may also be extended to infinity past the exterior edges of the drawing without any crossings in the exterior face (Figure 8(left)). $\mathcal{A}(G)$ can be viewed as a form of planar dual to $G$, in that it has one vertex within each face of $G$, one face containing each vertex of $G$, and one arrangement segment crossing each edge of $G$; however it lacks a vertex dual to the outer face of $G$. We define the *bottom edge* of a zone to be the edge

**Lemma 8** *Let $c$ be a curve in $\mathcal{A}(G)$, labeled by $x$. Then $c$ partitions the plane into two regions; the region above $c$ contains all vertices of $G$ that contain $x$ and the region below $c$ contains all vertices of $G$ that do not contain $x$. The subgraph of $G$ within each region is connected.*

**Proof:** The partition of the plane into two regions follows from the definition of $c$ as a non-self-intersecting curve that extends to infinity in two directions. Consider some edge $e$ of $G$ that crosses $c$; then the endpoint $u$ of $e$ that is below $c$ is a vertex that does not contain $x$ and the endpoint $v$ of $e$ that is above $c$ is a vertex that contains $x$. For any other vertex $w$, if $w$ does not contain $x$, then there exists (by Lemma 3) a path in $G$ from $w$ to $u$ that does not use an edge labeled $x$, and that therefore does not cross $c$; thus, $w$ is like $u$ within the region below $c$. Similarly, if $w$ does contain $x$, there exists a path in $G$ from $w$ to $v$ that does not cross $c$, so $w$ is like $v$ within the region above $c$. □

**Lemma 9** *If $G$ is an st-planar learning space, then $\mathcal{A}(G)$ is a weak pseudoline arrangement.*

**Proof:** The curves in $\mathcal{A}(G)$ are topologically equivalent to lines and meet only at crossings. Suppose for a contradiction that two curves labeled $x$ and $y$ in $\mathcal{A}(G)$ cross more than once; let $x$ be the label of the curve that has its left end



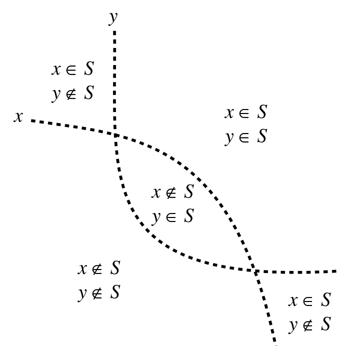



Figure 8: Left: The curve arrangement $\mathcal{A}(G)$ dual to an $st$-planar learning space. Right: Two crossings between the same two curves lead to a contradiction, so $\mathcal{A}$ must be a weak pseudoline arrangement (Lemma 9).

below the other curve, and let $y$ be the label of the curve that has its right end above the other curve. Then (Figure 8(right)) the region of the arrangement above and to the left of the first crossing and the region of the arrangement below and to the right of the second crossing would correspond to two sets $S$ and $S'$ in the learning space that both contain $x$ and both do not contain $y$. By Lemma 3 there exists a path from $x$ to $y$ that does not cross either curve, but no such path can exist. This contradiction completes the proof of the lemma. □

We are now ready to define the vertex coordinates for our upright-quad drawing algorithm. Consider the sequence of labels $x_0, x_1, \dots x_{\ell-1}$ occurring on the right path from the bottom to the top vertex of the external face of the drawing. For any vertex $v$ of our given $st$-planar learning space, let $X(v) = \min\{i \mid x_i \notin v\}$. If $v$ is the topmost vertex of the drawing, define instead $X(v) = \ell$. Similarly, consider the sequence of labels $y_0, y_1, \dots y_{\ell-1}$ occurring on the left path from the bottom to the top vertex of the external face of the drawing. For any vertex $v$ of our given $st$-planar learning space, let $Y(v) = \min\{i \mid y_i \notin v\}$. If $v$ is the topmost vertex of the drawing, define instead $Y(v) = \ell$.

**Lemma 10** *Let $G$ be an st-planar learning space, with $y_i$ as above, let $i < j < k$, and suppose that the curves labeled $y_i$ and $y_k$ both cross the curve labeled $y_j$ in the arrangement $\mathcal{A}(G)$. Then, if one traverses the curve labeled $y_j$ in order from its top left endpoint to its bottom right endpoint, the traversal must pass the crossing with $y_k$ before the crossing point with $y_i$.*

**Proof:** Suppose for a contradiction that the crossing with $y_i$ occurs to the left of the crossing with $y_k$, as depicted in Figure 9(right). In this case, the region of the arrangement immediately above and to the left of the crossing with $y_i$ would correspond to a set $S_{ij}$ that contains $y_i$ but does not contain $y_j$. The

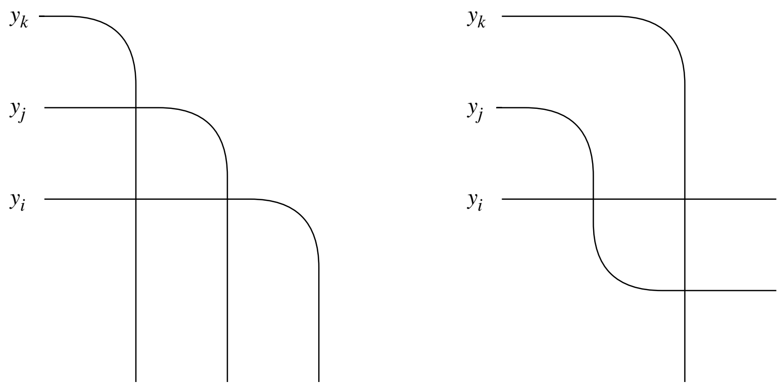



Figure 9: Left: The curve labeled $y_k$ crosses $y_j$ to the left of the crossing with $y_i$, as allowed by Lemma 10. Right: a crossing order that is disallowed by Lemma 10.

region of the arrangement immediately below and to the right of the crossing with $y_k$ would correspond to a set $S_{ik}$ that contains $y_k$ but does not contain $y_j$. Because a learning space must be closed under unions (Lemma 3), there would have to be a region $R$ in the arrangement corresponding to the set $S_{ij} \cup S_{ik}$; region $R$ must be above the curves labeled $s_i$ and $s_k$ but below the curve labeled $s_j$.

The portion of $y_i$ to the left of its crossing with $y_j$ separates the rest of curve $y_j$ from the points above and to the left of it, including from curve $y_k$. Thus, curve $y_k$ must cross curve $y_i$ above curve $y_j$ in order to reach its crossing with $y_j$, and (by Lemma 9) curves $y_i$ and $y_k$ cannot cross again below $y_j$. Therefore, curves $y_i$ and $y_k$ partition the region below $y_j$ into only three parts: the leftmost part, below $y_k$ and above $y_i$, a central part below both curves, and the rightmost part, below $y_i$ but above $y_k$. None of these three parts can contain the supposed region $R$ that lies below $y_j$ but above both of the other two curves. Thus $R$ cannot exist, and this contradiction completes the proof of the lemma. $\square$

Figure 9(left) shows three curves crossing in the order prescribed by the lemma.

**Lemma 11** *If we place each vertex $v$ of an st-planar learning space $G$ at the coordinates $(X(v), Y(v))$, the result is an upright-quad drawing of $G$.*

**Proof:** It is clear from the definitions that each edge of $G$ connects vertices with monotonically nondecreasing coordinates. We show that each internal face is an upright quadrilateral. Consider any such face $f$, with bottom left vertex $b$, bottom and top edges labeled $x_k$, and left and right edges labeled $y_j$. Note that, because the curve labeled $x_k$ crosses the curve labeled $y_j$ within $f$, $x_k$ must equal $y_{k'}$ for some $k' > j$. Then, for any edge label $y_i$ with $i < j$, $y_i \in b$; for otherwise, the curve for $y_i$ would cross the curve for $y_j$ to the left of $f$,

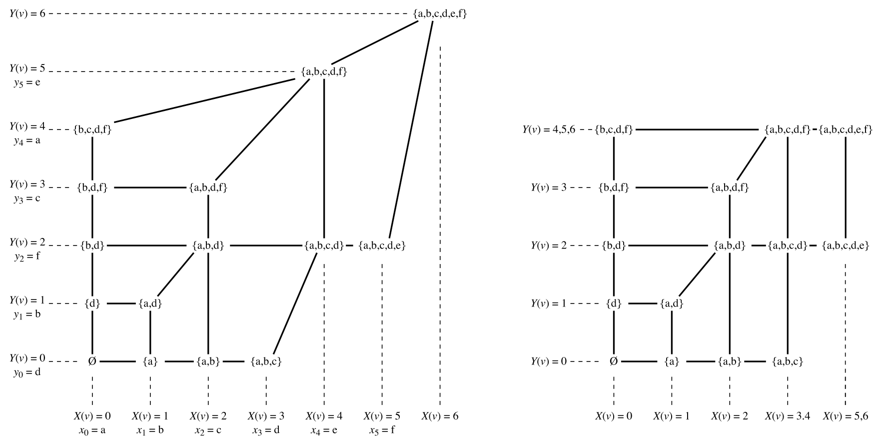



Figure 10: Left: coordinates for conversion of *st*-planar learning space to upright-quad drawing. Right: the same drawing with compacted coordinates.

and curves $y_i$, $y_j$, and $y_{k'}$ would violate Lemma 10. Thus, the vertices $b$ and $b \cup \{x_k\}$ of $f$ are placed at $y$-coordinate value $j$, and the other two vertices have $y$-coordinates larger than $j$. Symmetrically, the vertices $b$ and $b \cup \{y_j\}$ of $f$ have $x$-coordinate $i$, and the other two vertices have $x$-coordinates larger than $i$.

This shows that all edges that are bottom or left edges of an interior face of the drawing are horizontal or vertical. If $e$ is not such an edge, then it belongs to the left or right exterior path of the drawing. If on the left path, it connects a vertex $\{y'_i \mid i' < i\}$ to $\{y'_i \mid i' \leq ii\}$ and thus has strictly increasing $y$ coordinates; symmetrically, if on the right path, it has strictly increasing $y$ coordinates. Thus all such edges also have the correct dominance order for their vertices.

As all edges are oriented correctly, the drawing must have a unique minimal vertex and a unique maximal vertex, the source and sink of $G$ respectively. Together with each face being an upright quadrilateral, this property shows that the drawing is an upright-quad drawing. □

A drawing produced by the technique of Lemma 11 is shown in Figure 10(left). As in standard *st*-planar dominance drawing algorithms [2], we may compact the drawing by merging coordinate values $X(v) = i$ and $X(v) = i + 1$ whenever the merge would preserve the dominance ordering of the vertices; a compacted version of the same drawing is shown on the right of Figure 10.

**Theorem 3** *Every st-planar learning space $G$ over a set $U$, having $n$ vertices, has an upright-quad drawing in an integer grid of area $(|U| + 1)^2$ that may be found in time $O(n)$.*

**Proof:** We construct an *st*-planar embedding for $G$, form from it the dual curve arrangement $\mathcal{A}(G)$, and use the indices of the curves to assign coordinates to vertices as above. The coordinates of each vertex $v$ may be assigned in constant time from the coordinates of some vertex $w$ such that $G$ contains an edge $(v, w)$, To find an upright-quad drawing of an $st$-planar learning space $G$, assuming the input is given as a directed acyclic graph with labeled edges:

1. Find an $st$-planar embedding of $G$, described as a rotation system around each vertex, by adding an edge from the source to the sink of $G$ and applying any linear time planar embedding algorithm.
2. Let $x_0, x_1, x_2, \ldots$ be the sequence of labels on the rightmost path in $G$, and let $y_0, y_1, y_2, \ldots$ be the sequence of labels on the leftmost path in $G$. Build a table that maps each set element to its position in each of these two sequences.
3. For each vertex $v$ of $G$, in depth-first post-order:
   (a) If $v$ is the sink of the graph, set $X(v) = Y(v) = n$.
   (b) Otherwise, let $(v, w)$ be an edge of $G$, labeled with an element $x_i = y_j$. Compute the indices $i$ and $j$ using the table that maps elements to indices. Let $X(v) = \min(X(w), i)$ and $Y(v) = \min(Y(w), j)$.
   (c) Place vertex $v$ at the point $(X(v), Y(v))$.

Table 1: Pseudocode for Theorem 3.

using the pseudocode shown in Table 1; therefore, all coordinates of $G$ may be assigned in linear time. The area bound follows easily. □

**Corollary 1** *Any st-planar learning space over a set $U$ has at most $1 + (|U| + 1)|U|/2$ states.*

**Proof:** Our drawing technique assigns each vertex (other than the topmost one) a pair of coordinates associated with a pair of elements $\{x_i, y_j\} \subset U$ (possibly with $x_i = y_j$), and each pair of elements can supply the coordinates for only one vertex. Thus, there can only be one more vertex than subsets of one or two members of $U$. □

The bound of Corollary 1 is tight, as the family of sets $\mathcal{F}$ that are the unions of a prefix and a suffix of a totally ordered set $U$ (Figure 11) forms an $st$-planar learning space with exactly $1 + (|U| + 1)|U|/2$ states. The state associated with a subset of two members of $U$ consists of the prefix up to but not including the first of the two members and the suffix after but not including the second of the two members. The state associated with a subset of a single member of $U$ consists of all of $U$ except for that single member. Thus, each distinct subset of one or two members of $U$ has a state, and there is one additional state consisting of all of $U$.

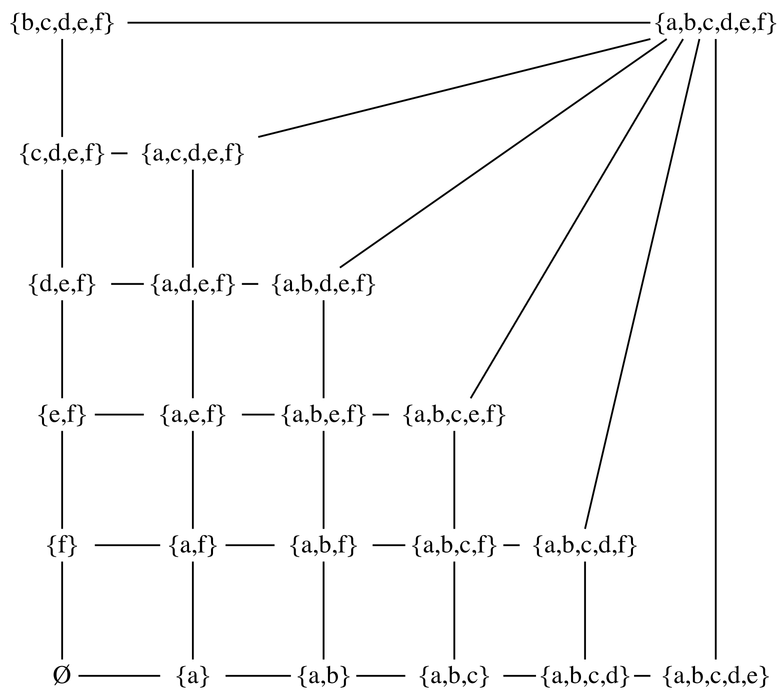



Figure 11: The family of sets formed by the union of a prefix and a suffix of some ordered universe forms an *st*-planar learning space with $1+(|U|+1)|U|/2$ states.

## 6 From Drawings to Quadrant Arrangements

Say that two edges of an upright-quad drawing are *paired* when they are the two opposite edges of some interior face of the drawing. Each edge may be paired with at most two others, one above or to the left and the other below or to the right. The connected components of the pairing relation partition the edges into sequences which we call *zones* (Figure 12(left); this definition is intended to match, but is not the same as, the definition of a zone in an *st*-planar learning space, depicted in Figure 8(left)). Note that an edge that does not belong to any interior face may form an edge by itself, and that a quadrilateral that is not adjacent to any other quadrilateral corresponds to two zones, one containing its top and bottom edges and the other containing its left and right edges.

**Theorem 4** *Each upright-quad drawing is the region graph for an arrangement of quadrants.*

**Proof:** For each zone $z_i$, we choose a coordinate value $x_i$, larger than the $x$-coordinate of the left endpoint of the bottom right edge of the zone and (if the bottom right edge is non-vertical) smaller than the $x$-coordinate of the right endpoint of the edge. We similarly choose a coordinate value $y_i$, larger than

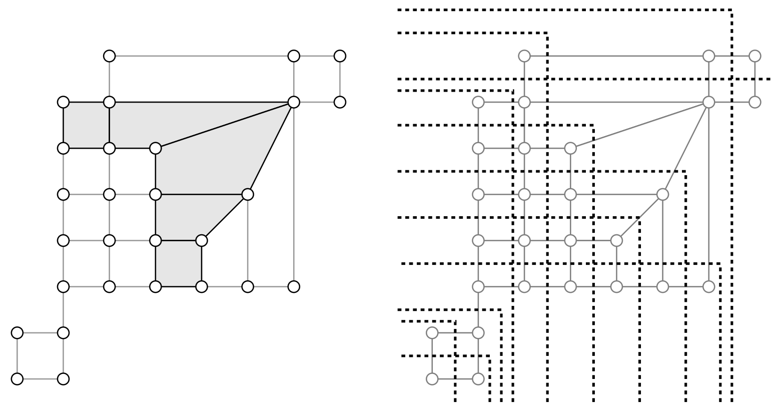


Figure 12: Left: The faces of a zone in an upright-quad drawing. Right: an arrangement of quadrants through each zone.

the $y$-coordinate of the bottom endpoint of the top left edge of the zone and (if the top left edge is non-horizontal) smaller than the $y$-coordinate of the top endpoint of the edge. We choose these coordinates in such a way that, if zone $i$ meets the right exterior path of the drawing prior to zone $j$, then $x_i < x_j$, and, if zone $i$ meets the left exterior path of the drawing prior to zone $j$, then $y_i < y_j$. We then draw a curve for zone $z_i$ by combining a horizontal ray left from $(x_i, y_i)$ with a vertical ray down from $(x_i, y_i)$ (Figure 12(right)). This curve is easily seen to cross all edges and faces of zone $z_i$, and no other interior faces of the drawing; thus, the arrangement $\mathcal{A}$ of these curves forms a planar dual to the drawing (except, as before, that it does not have a vertex representing the external face). □

**Corollary 2** *Each upright-quad drawing represents an st-planar learning space.*

**Proof:** This follows from Theorem 2 and Theorem 4. □

As different sets of translation vectors for quadrants form combinatorially equivalent arrangements if and only if the sorted orders of their $y$-coordinates form the same permutations with respect to the sorted order of their $x$-coordinates, a bound on the number of $st$-planar learning spaces follows.

**Corollary 3** *There are at most $n!$ combinatorially distinct st-planar learning spaces over a set of $n$ unlabeled items.*

We may also give a nearly matching lower bound to this upper bound. If a set of translation vectors $(i, \sigma(i))$ $(1 \leq i \leq n)$ is chosen by letting $\sigma$ be a uniformly random permutation, then the arrangement of quadrants for these vectors corresponds to a learning space with at least two combinatorially distinct upright-quad drawings: one given by the arrangement itself, and the other given by the translation vectors $(\sigma(i), i)$ (equivalently, by flipping the first drawing diagonally). There may be more than two drawings if there exists an interval

$J = [1, j]$ with $1 < j < n - 1$ for which $I = \sigma(I)$; if such an interval exists, one may find additional drawings by using translation vectors $(i, \sigma(i))$ for $i \le j$ and translation vectors $(\sigma(i), i)$ for $i > j$ (flipping part of the drawing). The probability that an interval of length $j$ has this property is $1/\binom{n}{j}$, so the probability that there exists such an interval is at most

$$\sum_{j=2}^{n-2} 1/\binom{n}{j} = O(1/n^2).$$

If no such interval exists, the learning space formed from $\sigma$ in this way has only two drawings. Therefore, the probability that the randomly chosen $\sigma$ has only two drawings is $1 - O(1/n^2)$ and number of distinct $st$-planar learning spaces is $\frac{1}{2}n!(1 - O(1/n^2))$.

# 7 Convex Dimension and Order Dimension

Our results can be viewed as showing that the *convex dimension* of an antimatroid is two if and only if its order dimension is two. To explain this result, define a *chain* antimatroid to be an antimatroid over an $n$-element universe such that, for each $0 \le i \le n$, the antimatroid has exactly one set of size $i$. The *join* of a family of antimatroids $\mathcal{A}_i$ is the antimatroid having as its sets the unions of sets from each of the $\mathcal{A}_i$, and the convex dimension of an antimatroid $\mathcal{A}$ is the minimum number of chains in a family of chains having $\mathcal{A}$ as its join. The convex dimension of any antimatroid may be calculated in polynomial time as the width of a certain partial order associated with the antimatroid [14, Theorem III.6.9].

The *order dimension* of any partially ordered family is the minimum dimension of a Euclidean space into which the elements of the family may be placed so that one element dominates another if and only if it is greater in the partial order. The antimatroids with order dimension one and the antimatroids with convex dimension one are both just the chains. It is known that the order dimension is always upper bounded by the convex dimension [14, Corollary III.6.10]. Thus, if the convex dimension is two, the order dimension is two. As we now show, our techniques can be used to prove the converse:

**Theorem 5** *Let $\mathcal{A}$ be an antimatroid with order dimension two. Then $\mathcal{A}$ has convex dimension two.*

**Proof:** Note that $\mathcal{A}$ must have convex dimension greater than one, for the only antimatroids with convex dimension one are the chains, and those also have order dimension one.

Suppose that $\mathcal{A}$ has order dimension two, and let the sets of $\mathcal{A}$ be placed as points in the plane in such a way that one point dominates another if and only if the set corresponding to the first point is a superset of the set corresponding to the other, and draw an edge from one of these points to another whenever the corresponding two sets differ by a single element. Then (as with any similar

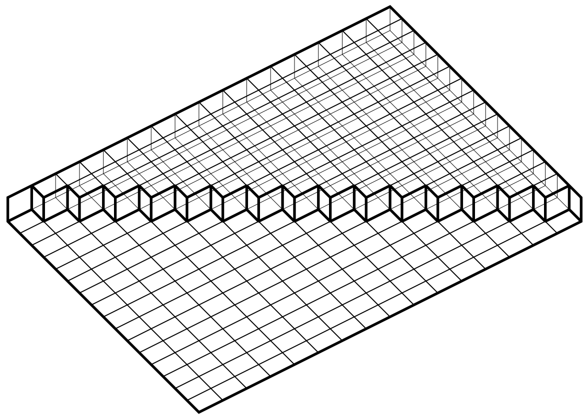


Figure 13: The Hasse diagram of an antimatroid with order dimension three and large convex dimension.

placement of points for a partial order of order dimension two) this drawing is a planar dominance drawing of the Hasse diagram of the order; this Hasse diagram is the learning space corresponding to $\mathcal{A}$. Thus, $\mathcal{A}$ corresponds to an $st$-planar learning space.

We construct an upright-quad drawing for $\mathcal{A}$ by Theorem 3, and let $\mathcal{X}$ and $\mathcal{Y}$ be the sequence of labels along the leftmost and rightmost paths of the drawing. Then $\mathcal{X}$ and $\mathcal{Y}$ are chains, and (by the union closure property of $\mathcal{A}$ any set in the join of $\mathcal{X}$ and $\mathcal{Y}$ belongs to $\mathcal{A}$. Conversely, any set $S$ in $\mathcal{A}$, corresponding to a point $v = (X(v), Y(v))$ in the drawing corresponds to the union of the first $X(v)$ elements in chain $\mathcal{X}$ and the first $Y(v)$ elements in chain $\mathcal{Y}$. Therefore, $\mathcal{A}$ is the join of these two chains and has order dimension two. □

However, for higher values of these two dimensions, the order dimension and convex dimension are not always equal. For instance, the antimatroid over $\{0, 1, 2, 3, 4\}$ formed by the subsets that either don't contain 0 or do contain three or more items has order dimension at most five, because it has five elements. However, it has convex dimension six: there are six sets of the form $\{0, x, y\}$, each such set cannot be formed as a union of smaller sets of the antimatroid, and therefore any representation of this antimatroid as a join of chains must have a chain for each of these six sets.

Figure 13 depicts the Hasse diagram of an antimatroid with a more extreme separation between the convex dimension and the order dimension: its order dimension is three while its convex dimension can be made arbitrarily large. To describe this example more formally, let $N$ be a given number, and let $Z$ consist of the three-dimensional integer lattice points with coordinates $0 \leq x, y < N$, $0 \leq z \leq 1$, and such that, if $z = 1$, then $x + y + z \geq N$. Then $Z$ represents a learning space over a universe of $2N - 1$ elements $x_i$ ($1 \leq i < N$), $y_i$ ($1 \leq i < N$), and $z_1$: a point $(a, b, c)$ contains $x_i$ if $i \geq a$, contains $y_j$ if $j \geq b$, and contains $z_1$ if $c \geq 1$. Thus, unions in the antimatroid correspond to coordinatewise maximization in $Z$. Each of the points $(x, y, 1)$ with $x+y = N-1$ cannot be represented as a coordinatewise maximum of any points in $Z$ with smaller coordinates, so each such point must belong to a separate chain in any representation of the antimatroid as a join of chains.

It would be of interest to determine the algorithmic complexity of calculating the order dimension of a learning space. For arbitrary partial orders, calculating the order dimension is NP-complete [16] but it is unclear whether the reduction proving this can be made to apply to learning spaces.

# 8 Conclusions

We have characterized *st*-planar learning spaces, both in terms of the existence of an upright-quad drawing and as the region graphs of quadrant arrangements. Our technique for drawing these graphs provides good vertex separation and small area, and it is straightforward to verify from its drawing that a graph is an *st*-planar learning space.

It is natural to wonder to what extent these results may apply to learning spaces that are not *st*-planar. We note that the convex dimension is computable in polynomial time, and are hopeful that the ability to represent an antimatroid as a join of a small number of chains could lead to interesting methods of graph drawing for antimatroids with convex dimension higher than two, analogously to the way that our minimum-dimensional lattice embedding technique [6] led to drawing algorithms for arbitrary nonplanar partial cubes [5]. Alternatively, when confronted with the task of drawing large nonplanar learning spaces, it may be helpful to find large *st*-planar subgraphs and apply the techniques described here to those subgraphs. Additionally, it would be of interest to extend our *st*-planar drawing techniques to broader classes of partial cubes that are not derived from antimatroids.

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